J. OPERATOR THEORY
28(1992), 309–320

ON THE DIFFERENTIAL STRUCTURE OF PRINCIPAL SERIES REPRESENTATIONS

A.F.M. TER ELST

1. INTRODUCTION

Each continuous representation of a Lie group has natural differential and analytic structures which are to a large extent determined by the comparable structures of the strongly elliptic operators affiliated with the representations. In particular the analytic structures coincide for all continuous representations and the differential structures coincide for unitary representations, Lipschitz representations, and the regular representations on the reflexive L^p-spaces. Nevertheless the differential structures do differ for the regular representations on $L^1(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$.

Let U be a representation of a Lie group G in a Banach space $\mathcal{X}$. For $x \in \mathcal{X}$ let $\tilde{x}: G \to \mathcal{X}$ be defined by $\tilde{x}(g) := U(g)x$ $(g \in G)$. For $n \in \mathbb{N}$ and $\lambda \geqslant 1$ let $\mathcal{X}_n(U)$, $\mathcal{X}_\infty(U)$, $\mathcal{X}_\omega(U)$ and $\mathcal{X}^\lambda(U)$ be the space of all $x \in \mathcal{X}$ such that $\tilde{x}$ is a C^n-function, a C^∞-function, an analytic function and a Gevrey function of order λ, respectively. There exists an infinitesimal description of these spaces.

For X in the Lie algebra $\mathfrak{g}$ of G let $\mathrm{d}U(X)$ be the infinitesimal generator of the one-parameter group $t \mapsto U(\exp(tX))$. Let $\partial U(X)$ be the restriction of $\mathrm{d}U(X)$ to the space $\mathcal{X}_\infty(U)$. Let $X_1, \ldots, X_d$ be an arbitrary basis in $\mathfrak{g}$. Then

$$\mathcal{X}_n(U) = D^n(\mathrm{d}U(X_1), \ldots, \mathrm{d}U(X_d)),$$

$$\mathcal{X}_\infty(U) = D^\infty(\mathrm{d}U(X_1), \ldots, \mathrm{d}U(X_d)),$$

$$\mathcal{X}_\omega(U) = S_1(\mathrm{d}U(X_1), \ldots, \mathrm{d}U(X_d)),$$

$$\mathcal{X}^\lambda(U) = S_\lambda(\mathrm{d}U(X_1), \ldots, \mathrm{d}U(X_d)),$$

where in general for operators $A_1, \ldots, A_d$ in $\mathcal{X}$

$$D^n(A_1, \ldots, A_d) := \bigcap_{i_1, \ldots, i_n \in \{1, \ldots, d\}} D(A_{i_1} \circ \ldots \circ A_{i_n}),$$

$$D^\infty(A_1,\ldots,A_d) := \bigcap_{n=1}^\infty D^n(A_1,\ldots,A_d),$$

$$S_\lambda(A_1,\ldots,A_d) := \{x \in D^\infty(A_1,\ldots,A_d) : \exists_{c,t>0}\forall_{n\in\mathbb{N}_0}\forall_{i_1,\ldots,i_n\in\{1,\ldots,d\}}$$
$$[\|A_{i_1}\circ\cdots\circ A_{i_n}x\| \leqslant ct^n n!^\lambda]\}.$$

(see [5, Proposition 1.1] and [7, Proposition 1.5].) With the norm

$$\|x\|_n = \max_{k\leqslant n}\max_{i_1,\ldots,i_k\in\{1,\ldots,d\}}\|\mathrm{d}U(X_{i_1})\circ\cdots\circ\mathrm{d}U(X_{i_k})x\| \quad (x\in\mathcal{X}_n),$$

$(\mathcal{X}_n, \|\ \|_n)$ becomes a Banach space.

Let C_m be a strongly elliptic form of order m over $\mathbb{R}^d$. Let $c_\alpha \in \mathbb{C}$, $|\alpha| \leqslant m$ be such that $C_m(\xi) = \sum_{|\alpha|\leqslant m} c_\alpha\xi^\alpha$ for all $\xi \in \mathbb{R}^d$. Let H_m be the corresponding strongly elliptic operator. So

$$H_m := \sum_{|\alpha|\leqslant m} c_\alpha \mathrm{d}U(X_1)^{\alpha_1}\circ\cdots\circ\mathrm{d}U(X_d)^{\alpha_d}$$

with domain $D(H_m) := \mathcal{X}_m(U)$. Langlands, [12, Theorem 2] or [11, Theorem 8], has proved that $\overline{H_m}$ generates a holomorphic semigroup. (For a proof, see [16, Theorem I.5.1].) In this paper we shall prove that if U is a standard principal series representation, then H_m determines the C^n-structure of U, i.e. $\mathcal{X}_n(U) = D((\overline{H_m}+\lambda I)^{\frac{n}{m}})$ for all $n \in \mathbb{N}$ and all $\lambda > 0$ such that $\overline{H_m}+\lambda I$ generates a bounded semigroup. Moreover, the norms $\|\ \|_n$ and $x \mapsto \|x\| + \|(\overline{H_m}+\lambda I)^{\frac{n}{m}}x\|$ are equivalent. These kind of equalities have been proved before for unitary representations and Hermitian elliptic Hermitian operators by [6, Theorem 3.1 and Corollary 4.1], for general strongly elliptic operators in unitary representations by [16, Example II.5.10], for Lipschitz representations by [16, Theorem II.5.8] and for the Laplacian corresponding to left and right translations in all reflexive $L^p(G)$ spaces by [2, Theorem 2.3]. However, it fails for the Laplacian corresponding to the translations in $L^1(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$. (See [14] and [13].) Moreover, in general the validity for $n = m$ does not imply the validity for lower n.

As a result, we prove that $\mathcal{X}_n(U) = \bigcap_{k=1}^d D([\mathrm{d}U(X_k)]^n)$ for all $n \in \mathbb{N}$ for each standard principal series representation. This kind of equality is proved for unitary representations by [6, Theorem 5.2]. We also prove that $\mathcal{X}_\omega(U) = \bigcap_{k=1}^d S_1(\mathrm{d}U(X_k))$ (separate and joint analyticity) and more general, $\mathcal{X}^\lambda(U) = \bigcap_{k=1}^d S_\lambda(\mathrm{d}U(X_k))$ for all $\lambda \geqslant 1$. The first equality is proved for unitary representations of reductive Lie groups by [17, Proposition 2].

Let K be a maximal compact subgroup of a semisimple Lie group G with finite center. Let U be a standard principal series representation of G. In [3, Corollaries 4 and 7 and the Added in proof] we proved that $\mathcal{X}_\infty(U) = \mathcal{X}_\infty(U|_K)$ and $\mathcal{X}^\lambda(U) = \mathcal{X}^\lambda(U|_K)$ for all $\lambda \geqslant 1$. Now we prove the same kind of equality for the C^n-spaces: $\mathcal{X}_n(U) = \mathcal{X}_n(U|_K)$.

2. C^n-VECTORS FOR RESTRICTIONS TO K

Let G be a connected semisimple Lie group with finite center. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. Let K, A and N be connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively. By [8, Theorem VI.1.1], K is compact. Let $M := \{k \in K : \mathrm{Ad}(k)|\mathfrak{a} = I\}$. Then M is a compact group. Let σ be a fixed unitary irreducible representation of M in a Hilbert space $\mathcal{H}^\sigma$. Let $(\mathfrak{a}')^{\mathbf{C}}$ be the vector space of all linear functions from $\mathfrak{a}$ into $\mathbf{C}$. Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$. Let $\rho := \frac{1}{2}\sum_{\Gamma+}(\dim\mu)\mu$, where Γ^+ is the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for N. Let U^ν be the principal series representation corresponding to MAN, σ and ν, realized in the "compact picture". (Cf. [10, p. 169].) So let

$$\mathcal{H}_0 := \{f : K \to \mathcal{H}^\sigma \text{ continuous} : \forall_{k\in K}\forall_{m\in M}[f(km) = \sigma(m)^{-1}f(k)]\},$$

$$\|f\|^2 := \int_K \|f(k)\|^2 \mathrm{d}k \quad (f \in \mathcal{H}_0),$$

$\mathcal{H}$ is the completion of $(\mathcal{H}_0, \|\ \|)$ and

$$[U^\nu(g)f](k) := a(g^{-1}k)^{-(\nu+\rho)}f(\kappa(g^{-1}k)) \quad (g \in G,\ f \in \mathcal{H}_0,\ k \in K).$$

Here $\kappa: G \to K$, $a: G \to A$ and $n: G \to N$ are the analytic functions such that $g = \kappa(g)a(g)n(g)$ in the Iwasawa decomposition of G. (See [8, Theorem VI.5.1].) Let V be the restriction of U^ν to K. Note that V is independent of ν. So $\mathcal{H}_\infty(U^\nu) = \mathcal{H}_\infty(V)$ is independent of ν. (See [3, Corollary 4].) Let $Y_1, \ldots, Y_{d_1}$ be a basis in $\mathfrak{k}$ and $X_1, \ldots, X_d$ be a basis in $\mathfrak{g}$.

LEMMA 1. *Let $X \in \mathfrak{g}$ and $\nu \in (\mathfrak{a}')^{\mathbf{C}}$. Then there exist continuous operators $B_{X,\nu}, B_{X,1}, \ldots, B_{X,d_1}$ such that for all $f \in \mathcal{H}_\infty(U^\nu)$*

$$\mathrm{d}U^\nu(X)f = B_{X,\nu}f + \sum_{j=1}^{d_1} B_{X,j}\mathrm{d}V(Y_j)f.$$

More explicitly, $B_{X,\nu}$ can be taken as the multiplication operator with the function

$$k \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 a(\exp(-tX)k)^{-(\nu+\rho)}$$

and the operators $B_{X,1}, \ldots, B_{X,d_1}$ are independent of ν.

Proof. For $t \in \mathbb{R}$ let $g_t := \exp(tX)$. There exists an open set $W_0 \subset \mathfrak{k}$, $0 \in W_0$ and an open set $W \subset K$ such that $\exp|_{W_0}: W_0 \to W$ is a diffeomorphism. The map $(t,k) \mapsto \kappa(g_t^{-1}k)k^{-1}$ from $\mathbb{R} \times K$ into K is continuous and maps $\{0\} \times K$ into the identity element of K, so by the compactness of K there exists $\varepsilon > 0$ such that $\kappa(g_t^{-1}k)k^{-1} \in W$ for all $t \in (-\varepsilon, \varepsilon)$ and all $k \in K$. For $t \in (-\varepsilon, \varepsilon)$ and $k \in K$ let $X_{t,k} \in W_0$ be such that

$$\exp X_{t,k} = \kappa(g_t^{-1}k)k^{-1}.$$

Then $(t,k) \mapsto X_{t,k}$ is infinitely differentiable from $(-\varepsilon,\varepsilon) \times K$ into $\mathfrak{k}$. Let $x_{t,k,1}, \ldots, x_{t,k,d_1} \in \mathbb{R}$ be such that

$$X_{t,k} = \sum_{j=1}^{d_1} x_{t,k,j} Y_j$$

for all $t \in (-\varepsilon,\varepsilon)$ and all $k \in K$.

Clearly the function $(t,k) \mapsto (a(g_t^{-1}k))^{-(\nu+\rho)}$ is infinitely differentiable. Now let $f \in \mathcal{H}_\infty(U^\nu)$. By [15, Theorem 5.1] and the correspondence between the "induced picture" and the "compact picture" ([10, §VII.1]) we may assume that f is a C^∞-function from K into $\mathcal{H}^\sigma$. Moreover, for all $Y \in \mathfrak{g}$ we have for a.e. $k \in K$:

$$[\mathrm{d}U^\nu(Y)f](k) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 [U^\nu(\exp(tY))f](k).$$

Then for a.e. $k \in K$:

$$\begin{aligned}
[\mathrm{d}U^\nu(X)f](k) &= \frac{d}{dt}\bigg|_0 [U^\nu(\exp(tX))f](k) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 a(g_t^{-1}k)^{-(\nu+\rho)} f(\kappa(g_t^{-1}k)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 a(g_t^{-1}k)^{-(\nu+\rho)} f(\exp X_{t,k} k) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 a(g_t^{-1}k)^{-(\nu+\rho)} f(k) + \sum_{j=1}^{d_1} [\mathrm{d}V(Y_j)f](k) \cdot \frac{\partial}{\partial t}\bigg|_0 x_{t,k,j}.
\end{aligned}$$

The lemma follows. ∎

For $n \in \mathbb{N}$ let $\|\ \|_{G,\nu,n}$ and $\|\ \|_{K,n}$ be the C^n-norms of U^ν and V corresponding to the basisses $X_1, \ldots, X_d$ and $Y_1, \ldots, Y_{d_1}$, respectively.

THEOREM 2. *Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $n \in \mathbb{N}$. Then $\mathcal{H}_n(U^\nu) = \mathcal{H}_n(V) = \mathcal{H}_n(U^\nu|_K)$. So $\mathcal{H}_n(U^\nu)$ is independent of ν. Moreover, the norms $\|\ \|_{G,\nu,n}$ and $\|\ \|_{K,n}$ are equivalent.*

Proof. By Lemma 1 for all $X \in \mathfrak{g}$ there exists $c > 0$ such that $\|\mathrm{d}U^\nu(X)f\| \leqslant$ $\leqslant c(\|f\| + \sum_{j=1}^{d_1} \|\mathrm{d}V(Y_j)f\|)$ for all $f \in \mathcal{H}_\infty(U^\nu)$. So by induction there exists $c_n > 0$

such that $\|f\|_{G,\nu,n} \leqslant c_n\|f\|_{K,n}$ for all $f \in \mathcal{H}_\infty(U^\nu)$. Since $\mathcal{H}_\infty(U^\nu) = \mathcal{H}_\infty(U^\nu|_K) = \mathcal{H}_\infty(V)$ is dense in $\mathcal{H}_n(V)$ and $\mathcal{H}_n(U^\nu)$ is complete, it follows that $\mathcal{H}_n(V) \subset \mathcal{H}_n(U^\nu)$. Hence $\mathcal{H}_n(U^\nu) = \mathcal{H}_n(V)$. By continuity, $\|f\|_{G,\nu,n} \leqslant c_n\|f\|_{K,n}$ for all $f \in \mathcal{H}_n(U^\nu)$. ∎

3. DIFFERENTIAL STRUCTURE AND STRONGLY ELLIPTIC OPERATORS

Let C_m be a strongly elliptic form. Let H_m^ν be the strongly elliptic operator associated with the form C_m in the representation $(\mathcal{H}, G, U^\nu)$ for all $\nu \in (\mathfrak{a}')^{\mathbf{C}}$. For $X \in \mathfrak{g}$ and $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ let $B_{X,\nu}$ be as in Lemma 1.

LEMMA 3. *Let $X \in \mathfrak{g}$, $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $n \in \mathbf{N}_0$. Then $B_{X,\nu}$ and $B_{X,\nu}^*$ map $\mathcal{H}_n(U^\nu)$ continuously in $\mathcal{H}_n(U^\nu)$.*

Proof. Let all $f \in \mathcal{H}_\infty(V)$. Again we may assume that f is a C^∞-function from K into $\mathcal{H}^\sigma$. Then the function $k \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 a(\exp(-tX)k)^{-(\nu+\rho)}f(k)$ is a C^∞-function from the compact group K into $\mathcal{H}^\sigma$, so by [15, Theorem 5.1] this function is an element of $\mathcal{H}_\infty(V)$, i.e. $B_{X,\nu}f \in \mathcal{H}_\infty(V)$. Moreover, for all $Z_1, \ldots, Z_n \in \mathfrak{k}$ we have for a.e. $k \in K$:

$$
\begin{aligned}
&[\mathrm{d}V(Z_1) \circ \cdots \circ \mathrm{d}V(Z_n)B_{X,\nu}f](k) = \\
&= \left.\frac{\partial}{\partial t_1}\right|_0 \cdots \left.\frac{\partial}{\partial t_n}\right|_0 [B_{X,\nu}f](\exp(-t_nZ_n)\cdots\exp(-t_1Z_1)k) = \\
&= \left.\frac{\partial}{\partial t_1}\right|_0 \cdots \left.\frac{\partial}{\partial t_n}\right|_0 \left.\frac{\partial}{\partial t}\right|_0 [a(\exp(-tX)\exp(-t_nZ_n)\cdots\exp(-t_1Z_1)k)^{-(\nu+\rho)} \cdot \\
&\qquad \cdot f(\exp(-t_nZ_n)\cdots\exp(-t_1Z_1)k)].
\end{aligned}
$$

Since a is a C^∞-function, we see that there exists $c > 0$, independent of f, such that $\|\mathrm{d}V(Z_1) \circ \cdots \circ \mathrm{d}V(Z_n)B_{X,\nu}f\| \leqslant c\|f\|_{K,n}$. So there exists c_1 such that for all $f \in \mathcal{H}_\infty(V)$ we have $\|B_{X,\nu}f\|_{K,n} \leqslant c_1\|f\|_{K,n}$. Since $\mathcal{H}_\infty(V)$ is dense in $\mathcal{H}_n(V)$, the first part of the lemma follows by Theorem 2. Similarly, $B_{X,\nu}^*$ maps $\mathcal{H}_n(U^\nu)$ continuously in $\mathcal{H}_n(U^\nu)$. ∎

THEOREM 4. *Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$. Then $\mathcal{H}_m(U^\nu) = D(\overline{H_m^\nu})$. Moreover, there exists $c > 0$ such that $\|f\|_{G,\nu,m} \leqslant c(\|\overline{H_m^\nu}f\| + \|f\|)$ for all $f \in \mathcal{H}_m(U^\nu)$.*

Proof. Let $c_\alpha \in \mathbf{C}$, $|\alpha| \leqslant m$ be such that $C_m(\xi) = \sum_{|\alpha|\leqslant m} c_\alpha\xi^\alpha$. Let $B_{X,\nu}$ be as in Lemma 1. For $j \in \{1, \ldots, d\}$ let $B_j := B_{X_j,0} - B_{X_j,\nu}$. Then $\mathrm{d}U^0(X_j) = \mathrm{d}U^\nu(X_j) + B_j$ and B_j maps $\mathcal{H}_n(U^0)$ continuously in $\mathcal{H}_n(U^0)$ for all $n \in \mathbf{N}_0$ by Lemma 3. By expanding on $\mathcal{H}_\infty(U^0)$

$$
H_m^0 = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_d)\\|\alpha|\leqslant m}} c_\alpha(\mathrm{d}U^\nu(X_1) + B_1)^{\alpha_1} \circ \cdots \circ (\mathrm{d}U^\nu(X_d) + B_d)^{\alpha_d}
$$

as a large sum, consisting of terms without a B_i and terms with at least one B_i, we obtain that there exists $c_1 > 0$ such that

$$\|H_m^0 f\| \leqslant \|H_m^\nu f\| + c_1 \|f\|_{G,0,m-1}$$

for all $f \in \mathcal{H}_\infty(U^0)$. Now U^0 is a unitary representation. (See [10, §VII.2].) So by [16, Corollary I.6.5] there exists $c_2 > 0$ such that

$$\|f\|_{G,0,m} \leqslant c_2(\|H_m^0 f\| + \|f\|)$$

for all $f \in \mathcal{H}_\infty(U^0)$. By Theorem 2 and [11, proof of Theorem 7] or [16, Theorem I.4.1] there exists $c_3 > 0$ such that

$$\|f\|_{G,0,m-1} \leqslant c_3(\|H_m^\nu f\| + \|f\|)$$

for all $f \in \mathcal{H}_\infty(U^0)$. Finally, by Theorem 2 there exists $c_4 > 0$ such that

$$\|f\|_{G,\nu,m} \leqslant c_4 \|f\|_{G,0,m}$$

for all $f \in \mathcal{H}_\infty(U^\nu)$. So

$$\|f\|_{G,\nu,m} \leqslant c_2 c_4 (1 + c_1 c_3)(\|H_m^\nu f\| + \|f\|)$$

for all $f \in \mathcal{H}_\infty(U^\nu)$. Hence $D(\overline{H_m^\nu}) \subset \mathcal{H}_m(U^\nu)$ and $\|f\|_{G,\nu,m} \leqslant c(\|\overline{H_m^\nu} f\| + \|f\|)$ for all $f \in \mathcal{H}_m(U^\nu)$, where $c := c_2 c_4 (1 + c_1 c_3)$. ■

COROLLARY 5. *Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $\lambda \geqslant 1$. Then $\mathcal{H}^\lambda(U^\nu) = S_{m\lambda}(\overline{H_m^\nu})$.*

Proof. By [16, Theorem I.4.1], $\mathcal{H}_\infty(U^\nu) = D^\infty(\overline{H_m^\nu})$. Now the corollary follows from Theorem 4 and [7, Theorem 1.7]. ■

Not only C^m-norm is determined by $\overline{H_m^\nu}$, also every C^n-norm is determined by $(\overline{H_m^\nu} + \lambda I)^{\frac{n}{m}}$ for large λ.

THEOREM 6. *Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $n \in \mathbb{N}$. Then for large $\lambda > 0$, $\mathcal{H}_n(U^\nu) = D((\overline{H_m^\nu} + \lambda I)^{\frac{n}{m}})$. Moreover, the norms $\| \ \|_{G,\nu,n}$ and $f \mapsto \|(\overline{H_m^\nu} + \lambda I)^{\frac{n}{m}} f\| + \|f\|$ are equivalent.*

Proof. Let B_j, $j \in \{1, \ldots, d\}$ be as in the proof of Theorem 4. By [1, Theorem 1.2] (Gårding's inequality) there exist $p, q > 0$ such that

$$\operatorname{Re}(H_m^0 f, f) \geqslant p\|f\|^2_{G,0,\frac{m}{2}} - q\|f\|^2$$

for all $f \in \mathcal{H}_\infty(U^0)$.

As in the proof of Theorem 4 we can write $H_m^\nu = H_m^0 + R$ with R a sum of terms of the form $c_\alpha T$ where T is the product of at most $m-1$ operators from the set

$$\{\mathrm{d}U^0(X_1), \ldots, \mathrm{d}U^0(X_d), B_1, \ldots, B_d\}$$

and with one factor in $\{B_1, \dots, B_d\}$. Consider one such therm $c_\alpha T$. Write $T = T_1 T_2$ with each T_i consists of at most $\frac{m}{2}$ operators from the set $\{\mathrm{d}U^0(X_1), \dots, \mathrm{d}U^0(X_d), B_1, \dots, B_d\}$. Now T_1 or T_2 contains at least one factor B_i, suppose T_1 contains a B_i. Since each B_i and B_i^* maps $\mathcal{H}_l(U^0)$ continuously in $\mathcal{H}_l(U^0)$ for all $l \in \mathbb{N}_0$ and U^0 is unitary, there exists $c_1 > 0$ such that for all $f \in \mathcal{H}_\infty(U^0)$:

$$|(c_\alpha T f, f)| = |c_\alpha (T_2 f, T_1^* f)| \leqslant |c_\alpha| \, ||T_2 f|| \, ||T_1^* f|| \leqslant c_1 ||f||_{G,0,\frac{m}{2}} ||f||_{G,0,\frac{m}{2}-1}.$$

So there exists $c_2 > 0$ such that for all $f \in \mathcal{H}_\infty(U^0)$:

$$|(Rf, f)| \leqslant c_2 ||f||_{G,0,\frac{m}{2}} ||f||_{G,0,\frac{m}{2}-1}.$$

For all $\varepsilon_1 > 0$ we have

$$||f||_{G,0,\frac{m}{2}} ||f||_{G,0,\frac{m}{2}-1} \leqslant \varepsilon_1 ||f||^2_{G,0,\frac{m}{2}} + \frac{1}{4\varepsilon_1} ||f||^2_{G,0,\frac{m}{2}-1}.$$

Moreover, by [16, Lemma III.3.3] there exists $c_3 > 0$ such that

$$||f||_{G,0,\frac{m}{2}-1} \leqslant \varepsilon_2 ||f||_{G,0,\frac{m}{2}} + c_3 \varepsilon_2^{-\frac{m}{2}+1} ||f||$$

for all $\varepsilon_2 > 0$. So

$$|(Rf, f)| \leqslant \left(c_2 \varepsilon_1 + \frac{c_2}{2\varepsilon_1} \varepsilon_2^2 \right) ||f||^2_{G,0,\frac{m}{2}} + \frac{c_2 c_3^2}{2\varepsilon_1 \varepsilon_2^{m-2}} ||f||^2$$

for all $f \in \mathcal{H}_\infty(U^0)$. Hence for all $\lambda \geqslant 0$ and $f \in \mathcal{H}_\infty(U^0)$

$$\begin{aligned} \operatorname{Re}((H_m^\nu + \lambda I) f, f) &\geqslant \operatorname{Re}(H_m^0 f, f) + \lambda ||f||^2 - |(Rf, f)| \geqslant \\ &\geqslant \left(p - c_2 \varepsilon_1 - \frac{c_2}{2\varepsilon_1} \varepsilon_2^2 \right) ||f||^2_{G,0,\frac{m}{2}} + \left(\lambda - \frac{c_2 c_3^2}{2\varepsilon_1 \varepsilon_2^{m-2}} - q \right) ||f||^2. \end{aligned} \tag{1}$$

By choosing ε_1 and ε_2 small and λ large, it follows that $\operatorname{Re}((H_m^\nu + \lambda I) f, f) \geqslant 0$ for all $f \in \mathcal{H}_\infty(U^0)$. Hence $\operatorname{Re}((\overline{H_m^\nu} + \lambda I) f, f) \geqslant 0$ for all $f \in D(\overline{H_m^\nu} + \lambda I)$. So $\overline{H_m^\nu} + \lambda I$ is a maximal closed accretive operator. Also $\Delta_0^{\frac{m}{2}}$ is a closed maximal accretive operator, where $\Delta_0 := -\sum_{j=1}^{d} dU^0(X_j)^2$. By Theorem 4 and [5, Proposition 1.3], $D(\overline{H_m^\nu} + \lambda I) = D(\Delta_0^{\frac{m}{2}})$ and the operator norms on the two spaces are equivalent. Then by [9, Corollary following Theorem 1], $D((\overline{H_m^\nu} + \lambda I)^{\frac{n}{m}}) = D(\Delta_0^{\frac{n}{2}})$ and the norms $f \mapsto ||(\overline{H_m^\nu} + \lambda I)^{\frac{n}{m}} f|| + ||f||$ and $f \mapsto ||\Delta_0^{\frac{n}{2}} f|| + ||f||$ are equivalent for all $n \in \{1, \dots, m\}$. So again by [5, Proposition 1.3] and Theorem 4, $D(\Delta_0^{\frac{n}{2}}) = \mathcal{H}_n(U^0) = \mathcal{H}_n(U^\nu)$ and the norms are equivalent. This proves the theorem for all $n \in \{1, \dots, m\}$. Since H_m^ν is an operator of order m, the theorem follows for all values of n. ∎

Let S be the holomorphic semigroup generated by $\overline{H^\nu_m}$. By general semigroup theory there exist $\omega, \theta > 0$ and $M \geqslant 1$ such that $||S_z|| \leqslant M e^{\omega|z|}$ for all $z \in \mathbb{C}$ with $|\arg z| < \theta$. Now for principal series representations we prove that we can take $M = 1$, as is in the case for unitary representations. (See [1, Corollary 4.1].)

COROLLARY 7. *Let $\nu \in (\mathfrak{a}')^{\mathbb{C}}$. Let S denote the holomorphic semigroup generated by $\overline{H^\nu_m}$. Then there exist $\theta > 0$ and $\omega > 0$ such that S is holomorphic in $\{z \in \in \mathbb{C}: |\arg z| < \theta\}$ and $||S_z|| \leqslant e^{\omega|z|}$.*

Proof. By Inequality (1) there exist $p, q > 0$ such that $\mathrm{Re}(H^\nu_m f, f) \geqslant p||f||^2_{G,0,\frac{m}{2}} - -q||f||^2$ for all $f \in \mathcal{H}_\infty(U^0)$. As in the proof of the previous theorem, it follows that there exist $c_1, c_2 > 0$ such that for all $z \in \mathbb{C}$ and all $f \in \mathcal{H}_\infty(U^0)$

$$\mathrm{Re}(zH^\nu_m f, f) \geqslant \mathrm{Re}z\mathrm{Re}(H^\nu_m f, f) - |\mathrm{Im}z|c_1||f||^2_{G,0,\frac{m}{2}} - |\mathrm{Im}z|c_2||f||^2.$$

Hence

$$\mathrm{Re}\left(\frac{z}{|z|}H^\nu_m f, f\right) \geqslant \frac{1}{|z|}(p\mathrm{Re}z - c_1|\mathrm{Im}z|)||f||^2_{G,0,\frac{m}{2}} - (q + c_2)||f||^2.$$

Let $\theta := \arctan \dfrac{p}{c_1}$ and $\omega := q + c_2$. We may assume that S is holomorphic in $\{z \in \in \mathbb{C}: |\arg z| < \theta\}$. Then for all $f \in H$ and all $z \in \mathbb{C}$ with $|\arg z| < \theta$

$$\frac{d}{d|z|}||S_z f||^2 e^{-2\omega|z|} = -2\mathrm{Re}\left(\left(\frac{z}{|z|}\overline{H^\nu_m} + \omega I\right) S_z f, S_z f\right) e^{-2\omega|z|} \leqslant 0.$$

So $||S_z f|| \leqslant e^{\omega|z|}||f||$ and $||S_z|| \leqslant e^{\omega|z|}$. ∎

Next we want to prove that $\mathcal{H}_n(U^\nu) = \bigcap_{j=1}^{d} D([dU^\nu(X_j)]^n)$. Therefore we need to consider duals of fractional operators. Let $H^{\nu,\dagger}_M$ be the strongly elliptic operator associated with the dual form $C^\dagger_m$ in the dual represetation $(\mathcal{H}, G, U^\nu_*)$. (See [16].)

LEMMA 8. *Let $\nu \in (\mathfrak{a}')^{\mathbb{C}}$. Let $\alpha \in (0, 1)$. Let $\lambda > 0$ be so large that $\overline{H^\nu_m} + \lambda I$ and $\overline{H^{\nu,\dagger}_m} + \lambda I$ generate bounded semigroups. Then $\left((\overline{H^{\nu,\dagger}_m} + \lambda I)^\alpha\right)^* = (\overline{H^\nu_m} + \lambda I)^\alpha$.*

Proof. Let $f \in D(\overline{H^\nu_m} + \lambda I)$. Then for all $g \in D(\overline{H^{\nu,\dagger}_m} + \lambda I)$

$$(f, (\overline{H^{\nu,\dagger}_m} + \lambda I)^\alpha g) = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^{\alpha-1}(f, (\overline{H^{\nu,\dagger}_m} + (\lambda + \mu)I)^{-1}\overline{H^{\nu,\dagger}_m} g) d\mu$$

$$= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \mu^{\alpha-1}((\overline{H^\nu_m} + (\lambda + \mu)I)^{-1}\overline{H^\nu_m} f, g) d\mu$$

$$= ((\overline{H^\nu_m} + \lambda I)^\alpha f, g).$$

Since $D(\overline{H_m^{\nu,\dagger}}+\lambda I)$ is a core for $(\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha$, it follows that $f\in D(((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*)$ and $((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^* f=(\overline{H_m^\nu}+\lambda I)^\alpha f$. Because $((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*$ is a closed operator and $D(\overline{H_m^\nu}+\lambda I)$ is a core for $(\overline{H_m^\nu}+\lambda I)^\alpha$, then also $(\overline{H_m^\nu}+\lambda I)^\alpha\subset((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*$.

Now let $f\in D(((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*)$. Let $\mu>0$. Then $\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}f\in$ $\in D(\overline{H_m^\nu}+\lambda I)\subset D(((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*)$. So for all $g\in D(\overline{H_m^{\nu,\dagger}}+\lambda I)$

$$(((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}f,g)=$$

$$=(\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}f,(\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha g)=$$

$$=(f,\mu(\overline{H_m^{\nu,\dagger}}+(\lambda+\mu)I)^{-1}(\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha g)=$$

$$=(f,(\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha\mu(\overline{H_m^{\nu,\dagger}}+(\lambda+\mu)I)^{-1}g)=$$

$$=(\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*f,g).$$

Hence $\lim_{\mu\to\infty}(\overline{H_m^\nu}+\lambda I)^\alpha\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}f=\lim_{\mu\to\infty}((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*\mu(\overline{H_m^\nu}+$ $+(\lambda+\mu)I)^{-1}f=\lim_{\mu\to\infty}\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*f=((\overline{H_m^{\nu,\dagger}}+\lambda I)^\alpha)^*f$. Since also $\lim_{\mu\to\infty}\mu(\overline{H_m^\nu}+(\lambda+\mu)I)^{-1}f=f$ and $(\overline{H_m^\nu}+\lambda I)^\alpha$ is a closed operator, $f\in D((\overline{H_m^\nu}+\lambda I)^\alpha)$. ∎

THEOREM 9. *Let $n\in\mathbb{N}$ and $\nu\in(\mathfrak{a}')^{\mathbb{C}}$. Then* $\mathcal{H}_n(U^\nu)=\bigcap_{j=1}^{d}D([\mathrm{d}U^\nu(X_j)]^n)$.

Proof. Let $C_m(\xi):=(-1)^n\sum_{j=1}^{d}\xi_j^{2n}$ and let H_m^ν and $H_m^{\nu,\dagger}$ be the associated strongly elliptic operators. Note that the contragredient representation of U^ν is also a principal series representation, in fact $U_*^\nu=U^{-\overline{\nu}}$. Let $\lambda>0$ be so large that $\overline{H_m^\nu}+(\lambda-1)I$ and $\overline{H_m^{\nu,\dagger}}+(\lambda-1)I$ generate bounded semigroups.

Let $f\in\bigcap_{j=1}^{d}D([\mathrm{d}U^\nu(X_j)]^n)$. Let $c_1:=\lambda\|f\|+\sum_{j=1}^{d}\|[\mathrm{d}U^\nu(X_j)]^n f\|$. Then for all $g\in\mathcal{H}_\infty(U^{-\overline{\nu}})$

$$\left|(f,(\overline{H_m^{\nu,\dagger}}+\lambda I)g)\right|=\left|(-1)^n\left(f,\sum_{j=1}^{d}[\mathrm{d}U^{-\overline{\nu}}(X_j)]^{2n}g\right)+\lambda(f,g)\right|=$$

$$=\left|\sum_{j=1}^{d}([\mathrm{d}U^\nu(X_j)]^n f,\mathrm{d}U^{-\overline{\nu}}(X_j)]^n g)+\lambda(f,g)\right|\leqslant$$

$$\leqslant c_1\|g\|_{G,-\overline{\nu},n}.$$

By Theorem 6 there exists $c_2 > 0$ such that

$$\|g\|_{G,-\overline{\nu},n} \leqslant c_2 \|(\overline{H_m^{\nu,\dagger}} + \lambda I)^{\frac{1}{2}} g\|$$

for all $g \in \mathcal{H}_\infty(U^{-\overline{\nu}})$. Since $(\overline{H_m^{\nu,\dagger}}+\lambda I)^{\frac{1}{2}}$ is a bijection from $\mathcal{H}_\infty(U^{-\overline{\nu}})$ onto $\mathcal{H}_\infty(U^{-\overline{\nu}})$,

$$|(f, (\overline{H_m^{\nu,\dagger}} + \lambda I)^{\frac{1}{2}} g)| \leqslant c_1 c_2 \|g\|$$

for all $g \in \mathcal{H}_\infty(U^{-\overline{\nu}})$, and by continuity, for all $g \in D((\overline{H_m^{\nu,\dagger}} + \lambda I)^{\frac{1}{2}})$. So it follows that $f \in D(((\overline{H_m^{\nu,\dagger}} + \lambda I)^{\frac{1}{2}})^*)$. Therefore, $f \in D((\overline{H_m^{\nu}} + \lambda I)^{\frac{1}{2}})$ by Lemma 8 and hence $f \in \mathcal{H}_n(U^\nu)$ by Theorem 6. ∎

By the closed graph theorem we immediately obtain the following corollary.

COROLLARY 10. *Let $\nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $n \in \mathbf{N}$. Then there exists $c > 0$ (depending on n) such that for all $f \in \mathcal{H}_n(U^\nu)$*

$$\|f\|_{G,\nu,n} \leqslant c \left(\|f\| + \sum_{j=1}^{d} \|[\mathrm{d}U^\nu(X_j)]^n f\| \right).$$

4. SEPARATE AND JOINT ANALYTIC OR GEVREY VECTORS

Again we have to examine the operators $B_{X,\nu}$ of Lemma 1.

LEMMA 11. *Let $\mu, \nu \in (\mathfrak{a}')^{\mathbf{C}}$ and $X \in \mathfrak{g}$. For $n \in \mathbf{N}_0$ let $B_{X,\mu}^{(n)}$ be the multiplication operator on $\mathcal{H}$ with the function $h_{X,\mu,n}$, where $h_{X,\mu,n}\colon K \to \mathbf{C}$ is defined by*

$$h_{X,\mu,n}(k) := \left.\frac{\partial^n}{\partial t^n}\right|_0 \left.\frac{\partial}{\partial s}\right|_0 a(\exp(-sX)\kappa(\exp(-tX)k))^{-(\mu+\rho)}.$$

Then $\mathrm{d}U^\nu(X) B_{X,\mu}^{(n)} f = B_{X,\mu}^{(n+1)} f + B_{X,\mu}^{(n)} \mathrm{d}U^\nu(X) f$ for all $f \in \mathcal{H}_\infty(U^\nu)$ and all $n \in \mathbf{N}_0$. Moreover, there exists $M > 0$ such that $|h_{X,\mu,n}(k)| \leqslant M^{n+1} n!$ for all $k \in K$ and all $n \in \mathbf{N}_0$ uniformly.

Proof. Let $f \in \mathcal{H}_\infty(U^\nu)$. We may assume that f is a C^∞-function from K into $\mathcal{H}^\sigma$. Then for all $n \in \mathbf{N}_0$ we obtain by Leibnitz' rule for a.e. $k \in K$:

$$[[\mathrm{d}U^\nu(X)]^n B_{X,\mu}^{(0)} f](k) =$$

$$= \left.\frac{\partial^n}{\partial t^n}\right|_0 [U^\nu(\exp(tX)) B_{X,\mu}^{(0)} f](k) =$$

$$= \left.\frac{\partial^n}{\partial t^n}\right|_0 [a(\exp(-tX)k)^{-(\nu+\rho)} f(\kappa(\exp(-tX)k))][h_{X,\mu,0}(\kappa(\exp(-tX)k))] =$$

$$= \sum_{j=0}^{n} \binom{n}{j} [B_{X,\mu}^{(n-j)}[\mathrm{d}U^\nu(X)]^j f](k).$$

So $[\mathrm{d}U^\nu(X)]^n B_{X,\mu}^{(0)} f = \sum_{j=0}^{n} \binom{n}{j} B_{X,\mu}^{(n-j)}[dU^\nu(X)]^j f$ for all $f \in \mathcal{H}_\infty(U^\nu)$. For $n \in \mathbb{N}_0$ define $C_0 f := B_{X,\mu}^{(0)} f$ and inductively $C_{n+1} f := \mathrm{d}U^\nu(X) C_n f - C_n \mathrm{d}U^\nu(X) f$ ($f \in \mathcal{H}_\infty(U^\nu)$). Then it follows by induction that $[\mathrm{d}U^\nu(X)]^n C_0 f = \sum_{j=0}^{n} \binom{n}{j} C_{n-j} [\mathrm{d}U^\nu(X)]^j f$ for all $f \in \mathcal{H}_\infty(U^\nu)$ and all $n \in \mathbb{N}_0$ and again by induction it follows that $C_n = B_{X,\mu}^{(0)}$ for all $n \in \mathbb{N}_0$.

For the proof of the last part of the lemma, the function

$$(t,k) \mapsto \left.\frac{\partial}{\partial s}\right|_0 a(\exp(-sX)\kappa(\exp(-tX)k))^{-(\mu+\rho)}$$

is a real analytic function from $\mathbb{R} \times K$ into $\mathbb{C}$, so for all $k_0 \in K$ there exist an open subset W in K and $M > 0$ such that $k_0 \in W$ and

$$\left| \left.\frac{\partial^n}{\partial t^n}\right|_0 \left.\frac{\partial}{\partial s}\right|_0 a(\exp(-sX)\kappa(\exp(-tX)k))^{-(\mu+\rho)} \right| \leqslant M^{n+1} n!$$

for all $k \in W$ and $n \in \mathbb{N}_0$. Since K is compact, we may assume that $W = K$. ■

THEOREM 12. *Let $\nu \in (\mathfrak{a}')^{\mathbb{C}}$ and let $\lambda \geqslant 1$. The* $\mathcal{H}^\lambda(U^\nu) = \bigcap_{j=1}^{d} S_\lambda(\mathrm{d}U^\nu(X_j))$.

Proof. Let $f \in \bigcap_{j=1}^{d} S_\lambda(\mathrm{d}U^\nu(X_j))$. Then $f \in \bigcap_{j=1}^{d} D^\infty(\mathrm{d}U^\nu(X_j)) = \mathcal{H}_\infty(U^\nu)$ by [4, Theorem 1.1]. For $X \in \mathfrak{g}$, $\mu \in (\mathfrak{a}')^{\mathbb{C}}$ and $n \in \mathbb{N}_0$ let $h_{X,\mu,n}$ and $B_{X,\mu}^{(n)}$ be as in Lemma 11. Let $j \in \{1, ..., d\}$. For $n \in \mathbb{N}_0$ let $B^{(n)} := B_{X_j,0}^{(n)} - B_{X_j,\nu}^{(n)}$ and $h_n := h_{X_j,\nu,n} - h_{X_j,0,n}$. By Lemma 11 there exists $M \geqslant 1$ such that $|h_n(k)| \leqslant M^{n+1} n!$ for all $n \in \mathbb{N}_0$ and $k \in K$. Then $\mathrm{d}U^0(X_j) = \mathrm{d}U^\nu(X_j) + B^{(0)}$. For all $g \in \mathcal{H}_\infty(U^\nu)$ we have

$$||\mathrm{d}U^0(X_j)g|| \leqslant ||\mathrm{d}U^\nu(X_j)g|| + ||h_0||_\infty ||g|| \leqslant M(||\mathrm{d}U^\nu(X_j)g|| + ||g||).$$

Also $[\mathrm{d}U^0(X_j), \mathrm{d}U^\nu(X_j)]g = [B^{(0)}, \mathrm{d}U^\nu(X_j)]g = -B^{(1)}g$ and, by induction for all $n \in \mathbb{N}$, $[\mathrm{ad}\, \mathrm{d}U^0(X_j)]^n(\mathrm{d}U^\nu(X_j))g = -B^{(n)}g$. So $||[\mathrm{ad}\, \mathrm{d}U^0(X_j)]^n(\mathrm{d}U^\nu(X_j))g|| \leqslant ||h_n||_\infty ||g|| \leqslant M^{n+1} n! ||g||$. Hence by "analytic" domination ([7, Theorem 1.1]) we obtain that $f \in S_\lambda(\partial U^0(X_j))$. (Note that $D(\partial U^0(X_j)) = \mathcal{H}_\infty(U^0) = \mathcal{H}_\infty(U^\nu)$.)

So $f \in \bigcap_{j=1}^{d} S_\lambda(\partial U^0(X_j)) = \bigcap_{j=1}^{d} S_\lambda(\mathrm{d}U^0(X_j))$. Now U^0 is a unitary representation, so by [17, Proposition 2], $f \in \mathcal{H}^\lambda(U^0)$. Finally, by [3, Corollary 7 and the Added in proof], $f \in \mathcal{H}^\lambda(U^0) = \mathcal{H}^\lambda(U^0|_K) = \mathcal{H}^\lambda(U^\nu|_K) = \mathcal{H}^\lambda(U^\nu)$. ■

ACKNOWLEDGEMENT. The author wishes to thank C. Meaney and D. W. Robinson for their suggestions and comments.

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A. F. M. TER ELST
Mathematics, IAS,
The Australian National University,
G. P. O. Box 4,
Canberra, ACT 2601,
Australia.

Received January 3, 1991.