J. OPERATOR THEORY
29(1993), 43–56

AN EXTENSION OF BEREZIN'S APPROXIMATION METHOD

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1. INTRODUCTION

This work is intended as an attempt to extend the approximation theorem due to Berezin [1]. Let us recall briefly his result. Suppose we are given a selfadjoint operator $B = B^*$ in a Hilbert space K such that $B > I$. In what follows I always means the indentity operator in a given space. Let $H \subset K$ be a closed subspace of K such that $D(B) \cap H$ is dense in H, where $D(B)$ denotes the domain of B. Define the symmetric operator A in H by

$$Af = PBf, \quad f \in H \cap D(B),$$

where $P\colon K \to H$ is the orthogonal projection. Let $D(A) := H \cap D(B)$.

Fix $t > 0$. Berezin defined in [1] the sequence $A_n = A_n(t)$ of operators in H by

$$A_n = \int_0^1 P\mathrm{e}^{-\frac{stB}{n}} BP\mathrm{d}s,$$

and proved about it two facts

a) A_n^{-1} is strongly convergent to $\underline{A}^{-1}$, where $\underline{A}$ is a selfadjoint extension of A independent of t,

b) for any fixed $t > 0$ the sequence $\left(P\mathrm{e}^{-\frac{tB}{n}}P\right)^n$ is strongly convergent to $\mathrm{e}^{-t\underline{A}}$.

Motivated by our previous study of Toeplitz operators in the Bargmann-Segal space [4], we are interested in extension of the above result to a more general context. We shall find below such an extension. Moreover, our proof is based on different ideas than Berezin's and seems to be simpler even in the selfadjoint case, $B = B^*$.

In what follows for a given $0 \leqslant \Theta < \frac{\pi}{2}$, $S_\Theta = \{z \in \mathbb{C},\ |\mathrm{Arg} z| < \Theta\}$ denotes the sector centered at the origin.

We can now formulate the assumptions and the statement of our generalization.

Let B be a normal operator in K, see [5, p. 276] for the definition. Suppose that its spectrum $\sigma(B)$ is contained in a sector of the form $1+S_\Theta$, $0 \leqslant \Theta < \frac{\pi}{2}$. Consider the compression A of B to H given by

$$Af = PBf, \quad f \in H \cap D(B), \tag{1}$$

where, as above $H \cap D(B)$ is assumed to be dense in H. Put $D(A) = H \cap D(B)$. It is clear that A is a densely defined, closable operator in H.

Write

$$B = B_1 + \mathrm{i}\, B_2,\ B_k^* = B_k,\ k = 1, 2.$$

Denote by $|B_2|$ the absolute value of B_2. The above mentioned generalization of Berezin's result says as follows

THEOREM. *If $P|B_2|P$ is bounded in H and $(2\cos\Theta - 1)\cos\Theta > \sin\Theta$ then*

i) *there exists a closed extension $\tilde{A}$ of A wich generates a C_0-semigroup,*

ii) *for any fixed $t > 0$ the sequence $\left(P\mathrm{e}^{-\frac{tB}{n}}P\right)^n$ is strongly convergent to $\mathrm{e}^{-t\tilde{A}}$.*

Later we shall give an application of this theorem to Toeplitz operators in the Bargmann-Segal space.

In what follows for an operator S we denote by $W(S)$, $R(z,S)$ the numerical range, the resolvent of S, respectively.

2. APPROXIMATION RESULTS

Fix $t > 0$. Following [1] we define the sequece $A_n = A_n(t)$ of operators in H by

$$A_n f = \frac{n}{t}\left(f - P\mathrm{e}^{-\frac{tB}{n}}f\right). \tag{2}$$

It turns out that the sequence A_n is in a sense convergent to A.

PROPOSITION 1. *Let A_n be given by (2). For any $f \in D(A)$ the sequences $A_n f$ and $A_n^* f$ are convergent to Af and $A^* f$, respectively.*

Proof. Let $B = \int z\mathrm{d}E_z$. For $f \in D(A)$ we have

$$A_n f = \int_0^1 P\mathrm{e}^{-\frac{tsB}{n}}Bf\mathrm{d}s =$$

$$= P\int_0^1\int \mathrm{e}^{-\frac{tsz}{n}}\mathrm{d}E_z Bf\mathrm{d}s = P\int F_n(t,z)\mathrm{d}E_z Bf,$$

where

$$F(t,z) = \frac{n}{tz}\left(1 - e^{-\frac{tz}{n}}\right).$$

Note that $F_n(t,z) \underset{n\to\infty}{\longrightarrow} 1$ for every $z \in 1 + S_\Theta$ and can be majorized as follows

$$|F_n(t,z)| \leqslant \begin{cases} (e^{-t\delta}+1)(t\delta)^{-1}, & |z| \geqslant n\delta \\ e^{t\delta}, & |z| \leqslant n\delta. \end{cases}$$

Hence the Lebesgue dominated convergence theorem implies that

$$P\int F_n(t,z)\mathrm{d}E_z Bf \underset{n\to\infty}{\longrightarrow} PBf = Af.$$

The same reasoning shows that $A_n^* f$ is also convergent to $PB^*f = A^*f$ and this completes the proof.

REMARK. We don't know whether $A_n^* h$ is convergent to A^*h for every $h \in D(A^*) \supseteq D(A)$.

Before proceeding further let us introduce the following notations:

$$B = I + \tilde{B},\ A = I + \tilde{A}_0,\ B_n(t) = \frac{n}{t}\left(I - e^{-\frac{tB}{n}}\right),$$

where $\operatorname{Re}\tilde{B} \geqslant 0$, $\tilde{A}_0 := P\tilde{B}P$. We have

$$A_n(t) = \alpha_n + e^{-\frac{t}{n}}\tilde{A}_n(t),\ B_n(t) = \alpha_n + e^{-\frac{t}{n}}\tilde{B}_n(t)$$

with $\alpha_n = \frac{n}{t}\left(1 - e^{-\frac{t}{n}}\right)$ and $\tilde{A}_n$, $\tilde{B}_n$ are defined by analogous formulas as above. It turns out that the numerical range of $\tilde{A}_n$ is contained in S_Θ.

PROPOSITION 2. *The numerical range of $\tilde{B}_n$ is contained in S_Θ, for $n = 1, 2, \ldots$.*

Proof. Let $c := \tan\Theta$. Since $W(\tilde{B}_n) = \operatorname{conv}\sigma(\tilde{B}_n)$ it suffices to show that $\sigma(\tilde{B}_n) \subset S_\Theta$. Let $z \in \sigma(\tilde{B}_n)$. We have

$$z = \frac{n}{t}\left(1 - e^{-\frac{t\lambda}{n}}\right), \quad \text{where } \lambda \in \sigma(\tilde{B}_n).$$

Write $\lambda = x + \mathrm{i}y$. Then $|y| \leqslant cx$ and

$$|\operatorname{Im} z|(\operatorname{Re} z)^{-1} = \left|\sin\frac{ty}{n}\right|\left(e^{\frac{tx}{n}} - \cos\frac{ty}{n}\right)^{-1} \leqslant$$

$$\leqslant \left|\sin\frac{ty}{n}\right|\left(e^{\frac{t|y|}{cn}} - 1\right)^{-1} \leqslant \left|\sin\frac{ty}{n}\right|\left(\frac{t|y|}{nc}\right)^{-1} \leqslant c.$$

Since z is arbitrary the proof is complete.

It turns out that the sequence $A_n(t)$ has another crucial property

PROPOSITION 3. *The sequence*

$$\operatorname{Re} A_n(t) := \frac{A_n(t) + A_n^*(t)}{2}$$

is ingreasing for $n > (1-c)^{-1}\max(c, 2c^2)$, $c = \tan\Theta < 1$.

Proof. Since $(\operatorname{Re} A_n(t)f, f) = (\operatorname{Re} B_n(t)f, f)$, $f \in H$, it is enough to show that the sequence $f_{nt}(\cdot)$ of functions given by

$$f_{nt}(z) = \frac{n}{t}\left(1 - \operatorname{Re} e^{-\frac{zt}{n}}\right)$$

is increasing in $1 + S_\Theta$.

For the simplicity of notation we assume that $t = 1$ (but the reasoning for arbitrary t is similar).

Let $f_n(z) := f_{n1}(z)$. We have

$$f_{n+1}(z) - f_n(z) = 1 - \operatorname{Re}\left[(n+1)e^{-\frac{z}{n+1}} - ne^{-\frac{z}{n}}\right].$$

Therefore we have to prove that

$$\text{(3)} \qquad \operatorname{Re}\left[(n+1)e^{-\frac{z}{n+1}} - ne^{-\frac{z}{n}}\right] \leqslant 1, \quad z \in 1 + S_\Theta, \quad n > (1-c)^{-1}\max(c, 2c^2).$$

Since $\tilde{H}_n(z) := \operatorname{Re}\left[(n+1)e^{-\frac{z}{n+1}} - ne^{-\frac{z}{n}}\right]$ is harmonic in $1 + S_\Theta$ is suffices to prove (3) on the boundary of $1 + S_\Theta$. But $\tilde{H}_n(z) = \tilde{H}_n(\overline{z})$ and so (3) is equivalent to

$$\text{(4)} \qquad H_n(x) := \tilde{H}_n(x, c(x-1)) \leqslant 1$$

for $x \geqslant 1$ and $n > (1-c)^{-1}\max(c, 2c^2)$.

Now we find (by the Mean Value Theorem) that

$$H_n(x) = ne^{-\frac{x}{n+1}}\left[\cos\frac{c(x-1)}{n+1} - \cos\frac{c(x-1)}{n}\right] +$$

$$\text{(5)} \qquad +n\left[e^{-\frac{x}{n+1}} - e^{-\frac{x}{n}}\right]\cos\frac{c(x-1)}{n} + e^{-\frac{x}{n+1}}\cos\frac{c(x-1)}{n+1} =$$

$$= \frac{c(x-1)}{n+1}e^{-\frac{x}{n+1}}\sin x_n + \frac{x}{n+1}e^{-w_n}\cos\frac{c(x-1)}{n} + e^{-\frac{x}{n+1}}\cos\frac{c(x-1)}{n+1},$$

where $\frac{c(x-1)}{n+1} < x_n < \frac{c(x-1)}{n}$, $\frac{x}{n+1} < w_n < \frac{x}{n}$.

We may consider three cases

1°) $c(x-1) \geqslant \pi n$

Since xe^{-x} is decreasing for $x > 1$ and $2xe^{-x} + e^{-x} < 1$ for $x > 2$ (5) implies that

$$H_n(x) < \frac{c(x-1)}{n+1} e^{-\frac{c(x-1)}{n+1}} + \frac{c(x-1)}{n+1} e^{-\frac{c(x-1)}{n+1}} + e^{-\frac{c(x-1)}{n+1}} < 1.$$

2°) $\frac{\pi n}{2} + c \leqslant cx < c + \pi n$

If x satisfies 2° then $\cos \frac{c(x-1)}{n} \leqslant 0$. Thus

$$H_n(x) \leqslant \frac{c(x-1)}{n+1} e^{-\frac{c(x-1)}{n+1}} < 1.$$

3°) $c \leqslant cx \leqslant c + \frac{\pi n}{2}$

Let

$$P_n(x) := e^{\frac{x}{n+1}},$$

$$L_n(x) := \frac{c(x-1)}{n+1} \sin \frac{c(x-1)}{n} + \frac{x}{n+1} \cos \frac{c(x-1)}{n} + \cos \frac{c(x-1)}{n+1}.$$

Note that $P_n(x) - L_n(x) \geqslant 0$ implies that $H_n(x) \leqslant 1$, for x satisfying 3°). Let $T_n := P_n - L_n$. Then

$$T_n(1) = e^{\frac{1}{n+1}} - 1 - \frac{1}{n+1} > 0$$

We claim that $T_n'(x) \geqslant 0$ for x satisfying 3°). We have

$$(n+1)T_n'(x) = e^{\frac{x}{n+1}} - \cos \frac{c(x-1)}{n} - \frac{c^2(x-1)}{n} \cos \frac{c(x-1)}{n} -$$

$$-c \left(\sin \frac{c(x-1)}{n} - \sin \frac{c(x-1)}{n+1} \right) + \frac{cx}{n} \sin \frac{c(x-1)}{n}.$$

Denote $\frac{c(x-1)}{n} = \alpha$, $\frac{c(x-1)}{n+1} = \beta$, $\frac{1}{c} = 1 + r$, $r > 0$. Then the last equality can be written as follows:

$$(n+1)T_n'(x) = e^{\frac{x}{n+1}} - \cos\alpha - c\alpha\cos\alpha - 2c \sin \frac{\alpha}{n+1} \cos \frac{\alpha+\beta}{2} + \left(\alpha + \frac{c}{n}\right) \sin\alpha.$$

Hence

$$(n+1)T_n'(x) > 1 + \frac{x}{n+1} - \cos\alpha - c\alpha\cos\alpha - 2c\sin\frac{\alpha}{n+1} >$$

$$> 1 - \cos\alpha + \alpha\left[\frac{n}{n+1} - c\cos\alpha\right] + \frac{n}{n+1}\alpha r - 2c\sin\frac{\alpha}{n+1}.$$

Taking $n \geqslant (1-c)^{-1} \max(c, 2c^2)$ we check that the above expression is positive. This completes the proof.

COROLLARY 4. *For any fixed $t > 0$ the sequence $[\operatorname{Re} A_n(t)]^{-1}$ is strongly convergent to a bounded operator $S(t)$. Moreover, $S(t)$ does not depend on t.*

Proof. Since $\alpha_n \to 1$ and $\operatorname{Re}\tilde{A}_n(t) \geqslant 0$ it follows that $\operatorname{Re} A_n(t) \geqslant bI$, for a certain $b = b(t) > 0$ and n sufficiently large. Applying Proposition 3 we know that $[\operatorname{Re} A_n(t)]^{-1}$ is decreasing for n sufficiently large. Hence there exists $S(t)$ in $L(H)$ such that $[\operatorname{Re} A_n(t)]^{-1} \underset{n\to\infty}{\longrightarrow} S(t)$ strongly.

The proof of independence of $S(t)$ on t is the same as in [1] and therefore it is omitted here.

What can be said about $\operatorname{Im} A_n(t)$? We don't know the answer in general. However, under additional assumption on B one can prove that $\operatorname{Im} A_n(t)$ is strongly convergent to A_2.

Namely, suppose that

$$(*) \qquad D(A) \ni f \longrightarrow P|B_2|f \text{ extends to a bounded operator in } H.$$

PROPOSITION 5. *If B_2 satisfies the condition $(*)$ then for any $t > 0$ $\operatorname{Im} A_n(t)$ is strongly convergent to A_2, where $A_2 = PB_2P$.*

Proof. Let $h \in D(A)$. We know (by Proposition 1) that $A_n(t)h \underset{n\to\infty}{\longrightarrow} Ah$ and $A_n(t)^*h \underset{n\to\infty}{\longrightarrow} A^*h$. Hence

$$\operatorname{Im} A_n(t)h \underset{n\to\infty}{\longrightarrow} A_2 h, \tag{6}$$

where

$$A_2 h := \frac{1}{2i}(A - A^*)h.$$

Now

$$(\operatorname{Im} A_n(t)h, h) = \frac{n}{t}\int e^{-\frac{tx}{n}} \sin\frac{ty}{n} \mathrm{d}(E_{x,y}h, h) = \int_{\{(x,y), y\neq 0\}} G_{nt}(x,y)\mathrm{d}(E_{x,y}h,h),$$

where

$$G_{nt}(x,y) = e^{-\frac{tx}{n}} y \frac{\sin\frac{ty}{n}}{\frac{ty}{n}}.$$

Fix $\varepsilon > 0$ take $\delta > 0$ so small that $\dfrac{\sin a}{a} \leqslant 1 + \varepsilon$ for $|a| \leqslant \delta$. For the above δ define the sets

$$Z_{n\delta} = \left\{(x,y) \in \sigma(B),\ \frac{t|y|}{n} \leqslant \delta\right\}.$$

If $(x,y) \in Z_{n\delta}$ then

$$|G_{nt}(x,y)| \leqslant |y|(1+\varepsilon), \quad n = 1, 2, \ldots \tag{7}$$

On the other hand for

$$(x_1, y_1) \in \sigma(B) \cap (C \backslash Z_{n\sigma})$$

$$|G_{nt}(x_1, y_1)| \leqslant \frac{|y_1|}{\delta}, \quad n = 1, 2, \ldots \tag{8}$$

Both (7) and (8) imply that

$$|G_{nt}(x, y)| \leqslant M|y|, \quad n = 1, 2, \ldots, \quad (x, y) \in \sigma(B), \tag{9}$$

where $M = \max(1 + \varepsilon, \delta^{-1})$. Hence

$$|(\operatorname{Im} A_n(t)h, h)| \leqslant M \int |y| \mathrm{d}(Eh, h) =$$

$$= M(|B_2|h, h) = M(P|B_2|h, h) \leqslant M\|P|B_2|h\| \cdot \|h\|.$$

If follows that $\|\operatorname{Im} A_n(t)\|$ is uniformly bounded. But we know that $\operatorname{Im} A_n(t)h$ is convergent to $A_2 h$, for any $h \in D(A)$ (by (6)) and the result follows easily.

Now we are going to prove that the whole sequence $A_n(t)^{-1}$ is strongly convergent and to find its limit.

PROPOSITION 6. *The sequence $A_n(t)^{-1}$ is strongly convergent to an operator T such that*

i) $\operatorname{Ker} T = \{0\}$

ii) $T^{-1} \supset A$

iii) $(-T^{-1})$ *generates a C_0-semigroup in H.*

Proof. Let

$$A_n(t) = S_n(t) + \mathrm{i}\, R_n(t), \tag{10}$$

where $S_n(t)^* = S_n(t)$, $R_n(t)^* = R_n(t)$.

The sequence $S_n(t)$ is increasing for $n > (1-c)^{-1} \max(c, 2c^2)$ (by Proposition 3). Hence

$$S_n(t)^{\frac{1}{2}} \leqslant S_{n+1}(t)^{\frac{1}{2}}$$

and so

$$S_n(t)^{-\frac{1}{2}} \geqslant S_{n+1}(t)^{-\frac{1}{2}}.$$

Since $S_n(t)^{-\frac{1}{2}}$ is bounded from below (by zero) it must be strongly convergent to $S(t)$. Repeating the reasoning given in [1] one can prove that $S(t) \equiv S$ i.e. $S(t)$ does not depend on t. In what follows we omit the t-variable. Rewrite (10) as

$$A_n = S_n^{\frac{1}{2}} \left(I + \mathrm{i}\, S_n^{-\frac{1}{2}} R_n S_n^{-\frac{1}{2}}\right) S_n^{\frac{1}{2}}.$$

Thus

$$A_n^{-1} = S_n^{-\frac{1}{2}} \left(I + \mathrm{i}\, S_n^{-\frac{1}{2}} R_n S_n^{-\frac{1}{2}}\right)^{-1} S_n^{-\frac{1}{2}}.$$

Now

$$S_n^{-\frac{1}{2}} R_n S_n^{-\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} SA_2S$$

in the strong topology.

Applying [5, Corollary 1.6, p. 489] we have

$$\left(I + \mathrm{i}\, S_n^{-\frac{1}{2}} R_n S_n^{-\frac{1}{2}}\right)^{-1} \underset{n\to\infty}{\longrightarrow} (I + \mathrm{i}\, SA_2S)^{-1}$$

strongly. It follows that

$$A_n^{-1} \underset{n\to\infty}{\longrightarrow} T := S(I + \mathrm{i}\, SA_2S)^{-1}S.$$

The same reasoning shows that $A_n^{-1^*}$ converges strongly to T^*. We claim that $\mathrm{Ker}T = \{0\}$. In fact, for $f \in H$ and $g \in D(A)$, by applying Proposition 1, we have

$$(f,g) = (A_nA_n^{-1}f, g) = (A_n^{-1}f, A_n^* g) \underset{n\to\infty}{\longrightarrow} (Tf, A^*g),$$

and the claim holds.

Moreover, $T^{-1} \supset A$. This is immediate by the following identities.

Let $h \in D(A)$ and $k \in H$. Then

$$(h,k) = (A_n^{-1}A_nh, k) = \left(A_nh, A_n^{-1^*}k\right) \underset{n\to\infty}{\longrightarrow} (Ah, T^*k).$$

Finally, $T = (S^{-2} + \mathrm{i}\, A_2)^{-1}$ and so $T^{-1} = S^{-2} + \mathrm{i}\, A_2$.

Since $S^* = S \geqslant 0$ and $A_2 \in L(H)$ it is clear that $-T^{-1}$ generates a C_0 - semigroup in H.

REMARK. Note that $-A_n$ and $-T^{-1}$ belong to the $\mathcal{G}(1,\beta)$ class of generators of C_0 - semigroups, for some $-1 < \beta < 0$, see [5, p. 487]. Indeed, by Proposition 2, $W(A_n) \subset 1 + S_\Theta$ and so for any $\gamma > \beta$ we have

$$\|(A_n+\gamma)^{-k}\| \leqslant \frac{1}{\mathrm{dist}(\gamma, W(-A_n))^k} < \frac{1}{(\gamma-\beta)^k}, \quad k = 1,2,\dots$$

Now, for any $\alpha > 0$ and $n \in \mathbb{N}$

$$\|(I+\alpha A_n)^{-1}\| \leqslant \frac{1}{\mathrm{dist}(1, W(-\alpha A_n))} < 1.$$

Hence

$$\lim_{\alpha\searrow 0} \|(I+\alpha A_n)^{-1}f - f\| = 0, \quad f \in H$$

uniformly in n. Th. 2.17 given in [5, p. 505] yields that $-T^{-1} \in \mathcal{G}(1,\beta)$ and this proves our Remark.

The above Remark and Proposition 6 enable us apply Theorem 2.16 from [5, p. 504] and we have

COROLLARY 7. *For any $f \in H$ and $t > 0$*

$$\lim_{n\to\infty} e^{-tA_n} f = e^{-tT^{-1}} f.$$

However, we need to know more, namely, whether $\left(I - \frac{tA_n}{n}\right)^n f \xrightarrow[n\to\infty]{} e^{-tT^{-1}} f$. It turns out to be true under some additional assumption on Θ (it should not be too large).

PROPOSITION 8. *Fix $t > 0$. Under the above assumptions we have*

$$\lim_{n\to\infty} \left\| e^{-tA_n} - \left(I - \frac{tA_n}{n}\right)^n \right\| = 0$$

provided that

$$(W) \qquad (2\cos\Theta - 1)\cos\Theta > \sin\Theta.$$

Proof. Fix $t > 0$ and take $\Theta < \Theta_1 < \frac{\pi}{2}$.
Let $c = \tan\Theta_1$. Note that

$$(\operatorname{Re} A_n(t)f, f) \leqslant (1+c)\frac{n}{t}\|f^2\|.$$

In fact, $F_n(x,y) = 1 - e^{-\frac{tx}{n}} \cos\frac{ty}{n}$ is harmonic in $1 + S_{\Theta_1}$ and so

$$\sup_{1+S_{\Theta_1}} F_n(x,y) = \sup_{x\geqslant 1}\left[1 - e^{-\frac{tx}{n}}\cos\frac{ct(x-1)}{n}\right] \leqslant 1 + ce^{-\frac{tx_n}{n}} \leqslant 1 + c,$$

where

$$\cos\frac{ct(x_n-1)}{n} + c\sin\frac{ct(x_n-1)}{n} = 0 \quad \text{and} \quad \sin\frac{ct(x_n-1)}{n} > 0.$$

Choose $\Theta_1 > \Theta$ so close to Θ that $(2\cos\Theta_1 - 1)\cos\Theta_1 > \sin\Theta_1$. This is possible by (W). For this Θ_1 fix $r > 0$ so large that

$$\left[2\cos\Theta_1 - \left(1 + \frac{1}{r}\right)\right]\cos\Theta_1 > \sin\Theta_1.$$

It follows that

$$(W_1) \qquad 2\cos\Theta_1 > 1 + c + \frac{1}{r},$$

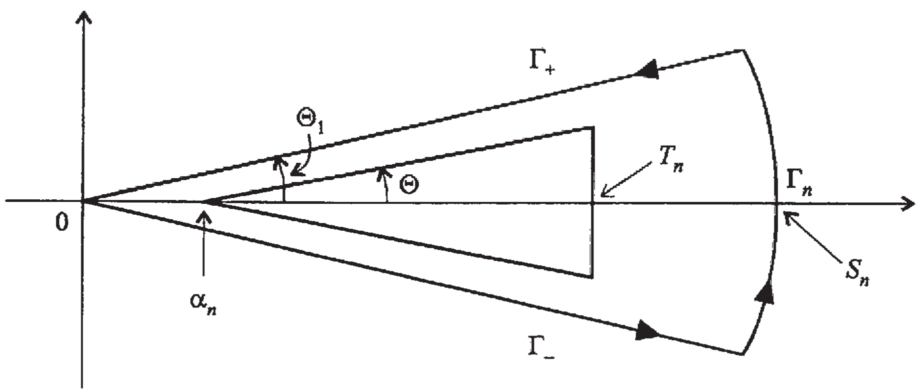
where $c = \tan\Theta_1$.

Appling Proposition 5 we know that $\|\operatorname{Im} A_n\| \leqslant C$, for a certain $C > 0$, $n = 1, 2 \ldots.$

Choose $n_0 = n_0(C, r, t)$ such that

$$\frac{1}{2}\frac{n}{tr} > C + c + 1, \quad \text{for } n \geqslant n_0. \tag{11}$$

Let $T_n := \frac{n}{t}(1+c)$, $S_n := T_n + \frac{n}{rt}$. If $n \geqslant n_0$, we define the contour $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_n$ by the following picture



By Proposition 2 and the above choice of n_0 we know that $\sigma(A_n)$ is contained in the set $\{z \in \alpha_n + S_\Theta,\ z = x + \mathrm{i}\,y,\ x \leqslant T_n,\ |y| \leqslant C\}$, where $\alpha_n = \frac{n}{t}\left(1 - \mathrm{e}^{-\frac{t}{n}}\right)$, $n \geqslant n_0$, and $C > 0$. Hence, by a direct computation and using the choice of n_0 we have

$$\operatorname{dist}(z, W(A_n)) > \frac{n}{2rt}, \quad z \in \Gamma_n,\ n \geqslant n_0. \tag{12}$$

Let $z \in \Gamma_\pm$. By the definition of $\Gamma_\pm$ we easily obtain the following estimate

$$\operatorname{dist}(z, W(A_n)) \geqslant \alpha_n \sin\Theta_1 > (1-\rho)\sin\Theta_1, \tag{13}$$

for a certain $\rho > 0$ and $n \geqslant \tilde{n}_0$.

Fix ε and choose $R > 0$ so large that

$$[2\pi\sin(\Theta_1 - \Theta)]^{-1}\int_R^\infty \mathrm{e}^{-pt\cos\Theta_1}\frac{\mathrm{d}p}{p} < \varepsilon \tag{14}$$

and

$$4[\pi\sin(\Theta_1 - \Theta)Rt\cos\Theta_1]^{-1} < \varepsilon. \tag{15}$$

Write

$$\mathrm{e}^{-tA_n} - \left(1 - \frac{tA_n}{n}\right)^n = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \left[\mathrm{e}^{tz} - \left(1 - \frac{tz}{n}\right)^n\right] R(z, A_n)\mathrm{d}z.$$

Note that

$$\|R(z, A_n)\| \leqslant [\mathrm{dist}(z, W(A_n))]^{-1}, \quad z \in \Gamma.$$

Therefore we want to estimate

$$\left| \mathrm{e}^{-tz} - \left(1 - \frac{tz}{n}\right)^n \right| \text{ for } z \in \Gamma.$$

Since $\left(1 - \frac{tz}{n}\right)^n$ is uniformly convergent to e^{-tz} on compact sets, there exists $n_1 = n_1(\varepsilon, R) \geqslant \max(n_0, \tilde{n}_0)$ such that

$$\sup_{\substack{|z| \leqslant R \\ z \in \Gamma_\pm}} \left| \mathrm{e}^{-tz} - \left(1 - \frac{tz}{n}\right)^n \right| \leqslant \frac{\pi\varepsilon(1-\rho)\sin\Theta_1}{2R} \tag{16}$$

for $n \geqslant n_1$.

Denote $H_n(z) := \mathrm{e}^{-tz} - \left(1 - \frac{tz}{n}\right)^n$. Direct computation and (12) prove that

$$\frac{1}{2\pi} \int_{\Gamma_n} |H_n(z)| \, \|R(z, A_n)\| \, |\mathrm{d}z| \leqslant \tag{17}$$

$$\leqslant \frac{1}{2\pi} \left(\int_{-\Theta_1}^{\Theta_1} \mathrm{e}^{-tS_n \cos\omega} \mathrm{d}\omega + 2\Theta_1 \alpha^n \right) \cdot \frac{2tr}{n} \cdot S_n,$$

where

$$\alpha := \left[1 - 2\left(1 + c + \frac{1}{r}\right)\cos\Theta_1 + \left(1 + c + \frac{1}{r}\right)^2 \right]^{\frac{1}{2}} < 1.$$

But $\frac{2tr}{n} \cdot S_n \leqslant 2r\left(1 + c + \frac{1}{r}\right)$, and so the above integral is less than ε for n sufficiently large, say $n \geqslant n_2 = n_2(\varepsilon, \Theta_1)$.

On the other hand,

$$\frac{1}{2\pi} \int_{\Gamma_+} |H_n(z)| \, \|R(z, A_n)\| \, |\mathrm{d}z| = \frac{1}{2\pi} \left[\int_{\{z \in \Gamma_+, |z| \leqslant R\}} |H_n(z)| \, \|R(z, A_n)\| \, |\mathrm{d}z| \right] +$$

$$+ \frac{1}{2\pi} \left[\int_{\{z \in \Gamma_+, |z| \leqslant R\}} |H_n(z)| \, \|R(z, A_n)\| \, |\mathrm{d}z| \right].$$

The first of the integrals is less than $\frac{\varepsilon}{2}$ (by (13) and (16)). The second integral of the right-hand side can be estimated as the sum

$$[2\pi \sin(\Theta_1 - \Theta)]^{-1} \left(\int_R^{S_n} \left\{ \mathrm{e}^{-pt\cos\Theta_1} + \left[\left(1 - \frac{pt\cos\Theta_1}{n}\right)^2 + \frac{(pt)^2 \sin^2\Theta_1}{n^2} \right]^{\frac{n}{2}} \right\} \frac{\mathrm{d}p}{p} \right)$$

By applying (14) the first integral is less than ε. The second integral may be written as $I_{1n} + I_{2n}$, where

$$I_{1n} := [2\pi \sin(\Theta_1 - \Theta)]^{-1} \int_R^{s_n} \left[\left(1 - \frac{st\cos\Theta_1}{n}\right)^2 + \frac{(st)^2 \sin^2\Theta_1}{n^2} \right]^{\frac{n}{2}} \frac{ds}{s},$$

$$I_{2n} := [2\pi \sin(\Theta_1 - \Theta)]^{-1} \int_{s_n}^{S_n} \left[\left(1 - \frac{st\cos\Theta_1}{n}\right)^2 + \frac{(st)^2 \sin^2\Theta_1}{n^2} \right]^{\frac{n}{2}} \frac{ds}{s},$$

$$s_n := \frac{n}{t}\cos\Theta_1.$$

Here we take n so large that $s_n > R$.

Note that

$$\frac{s^2t^2}{n^2} - \frac{st\cos\Theta_1}{n} \leqslant 0, \quad R \leqslant s \leqslant s_n.$$

Hence

$$I_{1n} \leqslant \int_R^{s_n} \left(1 - \frac{st\cos\Theta_1}{n}\right)^{\frac{n}{2}} \frac{ds}{s} [2\pi\sin(\Theta_1 - \Theta)]^{-1}.$$

Put

$$p := 1 - \frac{st\cos\Theta_1}{n}.$$

We have

$$\int_R^{s_n} \left(1 - \frac{st\cos\Theta_1}{n}\right)^{\frac{n}{2}} \frac{ds}{s} = \int_{P_1}^{P_n} p^{\frac{n}{2}} \frac{dp}{1-p},$$

where

$$P_1 := 1 - \cos^2\Theta_1, \quad P_n := 1 - \frac{tR\cos\Theta_1}{n}.$$

Now

$$[2\pi\sin(\Theta_1 - \Theta)] I_{1n} \leqslant \int_{P_1}^{P_n} p^{\frac{n}{2}} \frac{dp}{1-p} < \int_0^{P_n} p^{\frac{n}{2}} \frac{dp}{1-p} < \frac{2}{Rt\cos\Theta_1},$$

and so by (15)

$$I_{1n} \leqslant \varepsilon. \tag{18}$$

If $s_n \leqslant s \leqslant S_n$ then the function

$$s \longrightarrow 1 - 2\frac{st\cos\Theta_1}{n} + \frac{s^2t^2}{n^2}$$

is majorized by $\alpha^2 < 1$ (see the line below (17)).

Thus

$$I_{2n} \leqslant \alpha^{\frac{n}{2}} \int_{s_n}^{S_n} \frac{\mathrm{d}s}{s} \cdot [2\pi \sin \Theta_1 - \Theta)]^{-1} =$$

$$= [2\pi \sin(\Theta_1 - \Theta)]^{-1} \alpha^{\frac{n}{2}} \ln \left[\left(1 + c + \frac{1}{r} \right) (\cos \Theta_1)^{-1} \right].$$

Consequently, applying (18) we have

$$I_{1n} + I_{2n} \leqslant \varepsilon + \varepsilon, \quad \text{for } n \geqslant n_3(\Theta_1, R, \varepsilon) \geqslant n_2. \tag{19}$$

Finaly, by combining (17) and (19) we can write

$$\left\| \mathrm{e}^{-tA_n} - \left(I - \frac{tA_n}{n} \right)^n \right\| \leqslant \varepsilon + 2 \left(\frac{\varepsilon}{2} + \varepsilon + \varepsilon \right) = 6\varepsilon$$

for $n \geqslant n_3$. The proof is complete.

In this way we have proved our Theorem for $\tilde{A} := T^{-1}$.

PROBLEM. We don't know whether the above theorem holds without the assumption (W).

3. AN APPLICATION

In this section we shall give a straightforward application of our Theorem. Let F_2 be the Bargmann-Segal space of entire functions in $\mathbb{C}^n$ square integrable with respect to the Gaussian measure $\mathrm{d}\mu(z) = \pi^{-n}\mathrm{e}^{-|z|^2}\mathrm{d}V(z)$, $\mathrm{d}V(z)$ denoting the Lebesgue measure in $\mathbb{C}^n$. Denote by $P: L^2(\mu) \longrightarrow F_2$ the orthogonal projection of $L^2(\mu)$ onto F_2. For a measurable function φ on $\mathbb{C}^n$, the multiplication operator M_φ in $L^2(\mu)$ is defined by $M_\varphi f = \varphi f$. The Toeplitz operator T_φ is defined in F_2 by

$$T_\varphi f = PM_\varphi f, \quad f \in F_2.$$

The Berezin transform $\tilde{T}_\varphi$ of T_φ (see [1], [2]) is given by

$$\tilde{T}_\varphi(\lambda) = (\varphi k_\lambda, k_\lambda) = \tilde{\varphi}(\lambda),$$

where

$$k_\lambda(z) = \mathrm{e}^{(z,\lambda) - |\lambda|^2/2}, \quad (z, \lambda) = \sum_{s=1}^{n} z_s \overline{\lambda}_s.$$

To apply our Theorem we have to impose on φ the following conditions.
(a) $\varphi = \varphi_1 + \mathrm{i}\varphi_2$, where $\varphi_1(z) > 1$, $|\varphi_2(z)| \leqslant \varphi_1(z)\tan\Theta$, $0 \leqslant \Theta < \frac{\pi}{2}$;

(b) the Toeplitz operator $T_{|\varphi_2|}$ is bounded and $(2\cos\Theta - 1)\cos\Theta > \sin\Theta$.

Recall that $T_{|\varphi_2|}$ is bounded if and only if its Berezin symbol $|\tilde{\varphi}_2|(\cdot)$ is bounded in $\mathbb{C}^n$, [2], [3]. Thus $|\varphi_2|$ need not to be bounded but may induce bounded $T_{|\varphi_2|}$!

We are now in position to apply our Theorem. In fact, the conditions (a) and (b) guarantee that all the assumptions of the Theorem hold. If additionally T_{φ_1} is selfadjoint then as we know (by Proposition 5) $\tilde{A} = T_\varphi$. Hence in this case we have

COROLLARY 9. *Let φ satisfy (a) and (b). Suppose that T_{φ_1} is selfadjoint. Then*

$$e^{-tT_\varphi} = \lim_{N\to\infty} (T_{e^{\frac{-t\varphi}{N}}})^N.$$

REMARK. Proposition 3.7 from [4] implies that T_{φ_1} is surely selfadjoint for φ_1 satisfying

$$|\varphi_1(z) - \varphi_1(w)| \leqslant C(1 + |z - w|).$$

Acknowledgment. This work was started when the author visited Macquarie University in the fall of 1989. We are deeply indebted Professor Alan McIntosh for his hospitality and numerous discussions.

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Received July 31, 1990; revised December 3, 1991.