J. OPERATOR THEORY
29(1993), 57–67

ON AN INEQUALITY OF HAAGERUP-PISIER

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Dedicated to the memory of Marshall H. Stone, teacher and friend

1. INTRODUCTION

In [5; Section 3], Haagerup gives a simplified proof (with improved constant) of some inequalities of Pisier [12]. Pisier, in turn, was proving a non-commutative version of an inequality of Grothendieck [4,6], and as a consequence, affirmed a conjecture of Ringrose [13].

The key to Haagerup's results is his Theorem 3.2 of [5]:

Let $\mathfrak{A}$ be a C^*-algebra, and η be a bounded linear mapping of $\mathfrak{A}$ into a Hilbert space. Then there are states ρ and ρ' of $\mathfrak{A}$ such that

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}). \tag{1}$$

These inequalities and the circle of ideas associated with them have found important application in the work of Haagerup and E. Christensen, notably in moving toward a solution [5] of the similarity problem [7], and the main cohomology problem [3,8]. Especially for Christensen's techniques [2,3], it is important to have a version of (1) that applies to a von Neumann algebra or a represented C^*-algebra and produces *normal* states ρ and ρ' when η has appropriate weak continuity properties. One purpose of this note is to prove such a result.

THEOREM A. *If $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$ and η is a linear mapping of $\mathfrak{A}$ into a Hilbert space $\mathcal{K}$ continuous from $\mathfrak{A}$ in its ultraweak topology to $\mathcal{K}$ in its weak topology, then there are normal states ρ and ρ' of $\mathfrak{A}$ such that*

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}).$$

Haagerup's proof of (1) proceeds by establishing the inequality when η assumes its norm at some unitary operator in $\mathfrak{A}$. This argument produces normal states ρ and ρ' with the hypotheses of Theorem A. The second stage reduces the argument to the case where η assumes its norm at a unitary operator in $\mathfrak{A}$ by a clever use of Russo-Dye [14] and "ultrapowers". With the ultrapower argument, we no longer produce normal states.

In the next section, Haagerup's simplification of the Pisier argument is simplified further; among other things, the need for ultrapowers is removed. (See Theorem B.) We follow the proof of the Theorem B with Haagerup's proof of the Pisier-Ringrose inequality (with improved constant).

2. THE GENERAL CASE

Haagerup proves his result for unital C^*-algebras and then uses properties of the universal representation to pass to the non-unital case [5; Theorem 3.2]. The following simplified proof of the unital case has a form that allows us to be reasonably explicit about the states ρ and ρ'. The description of those states is included in the statement.

THEOREM B. *If η is a non-zero, bounded, linear mapping of a unital C^*-algebra $\mathfrak{A}$ into a Hilbert space $\mathcal{H}$, there is a sequence of unitary elements $\{U_n\}$ in $\mathfrak{A}$ such that $\|\eta(U_n)\| \to \|\eta\|$. There are weak* limit points of $\{\|\eta\|^{-2}\rho_n\}$ and $\{\|\eta\|^{-2}\rho'_n\}$, where*

$$\rho_n(A) = \langle \eta(U_nA), \eta(U_n)\rangle, \ \rho'(A) = \langle \eta(AU_n), \eta(U_n)\rangle \qquad (A \in \mathfrak{A}).$$

If ρ and ρ' are such limit points, then they are states of $\mathfrak{A}$ and

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}).$$

Proof. We may assume that $\|\eta\| = 1$. Let $\{A_n\}$ be a sequence of elements of norm less than 1 in $\mathfrak{A}$ such that

$$\frac{n^3-1}{n^3} \leqslant \|\eta(A_n)\|^2.$$

From [10], there are unitary elements $V_1, \ldots, V_m$ in $\mathfrak{A}$ such that $A_n = \frac{1}{m}\sum_{j=1}^{m} V_j$. Since

$$\left(\frac{n^3-1}{n^3}\right)^{\frac{1}{2}} \leqslant \|\eta(A_n)\| \leqslant \frac{1}{m}\sum_{j=1}^{m}\|\eta(V_j)\|,$$

there is some V_j such that $(n^3-1)n^{-3} \leqslant \|\eta(V_j)\|^2$. Let U_n be V_j.

If H is a self-adjoint element in the unit ball $(\mathfrak{A})_1$ of $\mathfrak{A}$, then from spectral theory, for each real t,

$$\left\| I - \frac{t^2}{2}H^2 \pm \mathrm{i}tH \right\|^2 \leqslant 1 + \frac{t^4}{4}.$$

Let $\eta_n(A)$ be $\eta(U_nA)$ for A in $\mathfrak{A}$. Then $\|\eta_n\| \leqslant 1$ and

$$\left\| \eta_n(I) - \frac{t^2}{2}\eta_n(H^2) \pm \mathrm{i}t\eta_n(H) \right\|^2 \leqslant 1 + \frac{t^4}{4}.$$

From the parallelogram law,

$$\left\| \eta_n(I) - \frac{t^2}{2}\eta_n(H^2) \right\|^2 + t^2\|\eta_n(H)\|^2 \leqslant 1 + \frac{t^4}{4}$$

and

$$\|\eta_n(I)\|^2 - t^2 \mathrm{Re}\,\langle \eta_n(H^2), \eta_n(I)\rangle + \frac{t^4}{4}\left\|\eta_n(H^2)\right\|^2 + t^2\|\eta_n(H)\|^2 \leqslant 1 + \frac{t^4}{4}.$$

Since $\|\eta_n(I)\|^2 = \|\eta(U_n)\|^2 \geqslant (n^3-1)n^{-3}$, it follows that

$$\frac{t^4}{4}\left\|\eta_n(H^2)\right\|^2 + \|\eta_n(H)\|^2 \leqslant \frac{1}{t^2n^3} + \frac{t^2}{4} + \mathrm{Re}\,\langle \eta_n(H^2), \eta_n(I)\rangle$$

for each non-zero, real t. Choosing $\frac{1}{n}$ for t, we have that

$$\|\eta_n(H)\|^2 \leqslant \frac{2}{n} + \mathrm{Re}\,\rho_n(H^2).$$

For arbitrary A in $(\mathfrak{A})_{\frac{1}{2}}$, $A + A^*$ and $\mathrm{i}(A - A^*)$ are self-adjoint elements of $(\mathfrak{A})_1$. Thus

$$\|\eta_n(A)\|^2 + \|\eta_n(A^*)\|^2 = \frac{1}{2}\left[\, \|\eta_n(A+A^*)\|^2 + \|\eta_n(\mathrm{i}(A-A^*))\|^2 \,\right] \leqslant$$
$$\leqslant \frac{1}{2}\left[\frac{4}{n} + \mathrm{Re}\,(\rho_n((A+A^*)^2 + (\mathrm{i}(A-A^*))^2))\right] =$$
$$= \mathrm{Re}\,\rho_n(A^*A + AA^*) + \frac{2}{n}.$$

It follows that

$$\|\eta(U_nA)\|^2 \leqslant \mathrm{Re}\,\langle \eta(U_nA^*A + U_nAA^*), \eta(U_n)\rangle + \frac{2}{n}$$

for each A in $(\mathfrak{A})_{\frac{1}{2}}$. Replacing A by U_n^*A, we have that

$$\|\eta(A)\|^2 \leqslant \mathrm{Re}\,\langle \eta(U_nA^*U_nU_n^*A + U_nU_n^*AA^*U_n), \eta(U_n)\rangle + \frac{2}{n} =$$
$$= \mathrm{Re}\,\left[\, \rho_n(A^*A) + \rho_n'(AA^*) \,\right] + \frac{2}{n}.$$

Under the assumption that $\|\eta\| = 1$, the functionals ρ_n and ρ'_n lie in the unit ball $(\mathfrak{A}^{\#})_1$ of the norm dual of $\mathfrak{A}$. Since $(\mathfrak{A}^{\#})_1$ is weak* compact, there are cofinal subnets of $\{\rho_n\}$ and $\{\rho'_n\}$ with the same indices tending to functionals ρ and ρ', respectively, in $(\mathfrak{A}^{\#})_1$. As

$$1 \geqslant \rho_n(I) = \|\eta(U_n)\|^2 \geqslant \frac{n^3 - 1}{n^3},$$

we have that $1 = \rho(I) = \rho'(I)$. It follows that ρ and ρ' are states of $\mathfrak{A}$.

Since ρ and ρ' are states of $\mathfrak{A}$ and are weak* limit points of $\{\rho_n\}$ and $\{\rho'_n\}$, respectively, we conclude that

$$\|\eta(A)\|^2 \leqslant \mathrm{Re}\ [\ \rho(A^*A) + \rho'(AA^*)\ ] = \rho(A^*A) + \rho'(AA^*) \qquad (A \in (\mathfrak{A})_{\frac{1}{2}}).$$

Dividing by $4\|A\|^2$, we have the same inequality for all A in $\mathfrak{A}$. ■

Corollary C. *With $\mathfrak{A}$, η, and $\mathcal{H}$, as in Theorem B, there is a state ρ_0 of $\mathfrak{A}$ such that*

$$\|\eta(A)\|^2 \leqslant 2\|\eta\|^2 \rho_0(A^*A + AA^*) \qquad (A \in \mathfrak{A}).$$

Proof. From Theorem B, there are states ρ and ρ' of $\mathfrak{A}$ such that

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}).$$

Let ρ_0 be $\frac{1}{2}(\rho + \rho')$. Then $\rho \leqslant 2\rho_0$ and $\rho' \leqslant 2\rho_0$. Thus

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \leqslant 2\|\eta\|^2\rho_0(A^*A + AA^*)\ (A \in \mathfrak{A}).$$ ■

Theorem D. (Pisier-Ringrose) *If γ is a bounded, linear mapping of one C^*-algebra $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$, then*

$$\left\|\sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) + \gamma(A_j)\gamma(A_j)^*\right\| \leqslant 4\|\gamma\|^2 \left\|\sum_{j=1}^{n} A_j^*A_j + A_jA_j^*\right\|$$

for each finite set $\{A_1, \ldots, A_n\}$ of elements A_j of $\mathfrak{A}$.

Proof. We may assume that $\mathfrak{B}$ acts faithfully on a Hilbert space $\mathcal{H}$. Let x be a unit vector in $\mathcal{H}$ and $\eta(A)$ be $\gamma(A)x$. With A in $\mathfrak{A}$ and $\|A\| \leqslant 1$,

$$\|\eta(A)\| = \|\gamma(A)x\| \leqslant \|\gamma(A)\|\,\|x\| \leqslant \|\gamma\|.$$

Thus $\|\eta\| \leqslant \|\gamma\|$. From Corollary C, there is a state ρ_x of $\mathfrak{A}$ such that

$$\|\eta(A_j)\|^2 = \langle \gamma(A_j)^*\gamma(A_j)x, x\rangle \leqslant 2\|\eta\|^2\rho_x(A_j^*A_j + A_jA_j^*).$$

Summing, we have

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j)^* \gamma(A_j) \right) x, x \right\rangle \leqslant 2\|\eta\|^2 \rho_x \left(\sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right) \leqslant$$

$$\leqslant 2\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|$$

This result applied to the mapping η^* of $\mathfrak{A}$ into $\mathcal{H}$, where $\eta^*(A) = \gamma^*(A)x$ (and $\gamma^*(A) = \gamma(A^*)^*$), and to the elements $A_1^*, \dots, A_n^*$ yields

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j) \gamma(A_j)^* \right) x, x \right\rangle \leqslant 2\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|.$$

Thus

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j)^* \gamma(A_j) + \gamma(A_j)\gamma(A_j)^* \right) x, x \right\rangle \leqslant 4\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|.$$

As the preceding inequality holds for each unit vector in $\mathcal{H}$ and $\sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j)+$ $+\gamma(A_j)\gamma(A_j)^*$ is positive, we have that

$$\left\| \sum_{j=1}^{n} \gamma(A_j)^* \gamma(A_j) + \gamma(A_j)\gamma(A_j)^* \right\| \leqslant 4\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|. \qquad \blacksquare$$

3. WEAK CONTINUITY

We establish Theorem A in this section. Three proofs are given. The first uses Haagerup's result in conjunction with a direct appeal to the properties of the universal representation. The second, suggested by Christensen and Haagerup, employs a technique used in similar circumstances by each of Christensen and Haagerup. We extract an assertion, which we call the "Christensen-Haagerup Principle", from this technique and then apply it to the present situation to give our second proof. The third proof (again suggested by Christensen and Haagerup) follows from an application of the (much harder) result of Haagerup [6] proving the Grothendieck conjecture.

Before beginning the proof, we note that the Kaplansky density theorem [11] (see also [9; Theorem 5.3.5]) can be reformulated to state that the unit ball of the von Neumann algebra generated by a $*$-algebra $\mathfrak{A}_0$ of operators on a Hilbert space is the strong-operator $*$-closure of operators in the unit ball of $\mathfrak{A}_0$. That is, with T in the unit ball of $\mathfrak{A}_0^-$ and strong operator neighborhoods of T and T^* assigned, there is a T_0 in the unit ball of $\mathfrak{A}_0$ such that T_0 lies in the given strong-operator neighborhood of T and T_0^* lies in the given strong-operator neighborhood of T^*. In the proofs of [11] (and [9; Theorem 5.3.5]), the approximation is first made with T self-adjoint; the case of general T is dealt with by (Halmos's suggestion to Kaplansky of) considering the *algebra of 2×2 matrices with entries in $\mathfrak{A}_0$ together with the matrix having 0 diagonal entries and T and T^* at the off-diagonal entries. The strong-operator *density can be read from this.

Proof of Theorem A. We note, first, that η is bounded. To see this, let x be a vector in $\mathcal{K}$. By hypothesis, $A \to \langle \eta(A), x \rangle$ is an ultraweakly continuous linear functional on $\mathfrak{A}$. Since the norm topology on $\mathfrak{A}$ is stronger than the ultraweak topology, this functional is norm continuous and, therefore, bounded. Thus for each continuous linear functional f on $\mathcal{K}$, the set $\{ f(\eta(A)) \ : \ A \in (\mathfrak{A})_1 \}$ is bounded in $\mathbb{C}$. From the uniform boundedness principle (cf. [9; Corollary 1.8.11]), $\{ \eta(A) \ : \ A \in (\mathfrak{A})_1 \}$ is a bounded subset of $\mathcal{K}$, whence η is bounded.

At the same time, the functional $A \to \langle \eta(A), x \rangle$ has an unique ultraweakly continuous extension ρ_x to $\mathfrak{A}^-$ which has the same norm (since $(\mathfrak{A})_1$ is ultraweakly dense in $(\mathfrak{A}^-)_1$ by the Kaplansky density theorem). With x and y in $\mathcal{K}$, a a scalar, and A in $\mathfrak{A}$, we have that

$$\rho_{ax+y}(A) = \langle \eta(A), ax + y \rangle = (\bar{a}\rho_x + \rho_y)(A),$$

whence $\rho_{ax+y} = \bar{a}\rho_x + \rho_y$ by ultraweak continuity of $\rho_{ax+y} - \bar{a}\rho_x - \rho_y$ and the ultraweak density of $\mathfrak{A}$ in $\mathfrak{A}^-$. Moreover,

$$|\rho_x(A)| \leqslant \|\eta(A)\| \, \|x\| \leqslant \|\eta\| \, \|A\| \, \|x\|;$$

thus $|\rho_x(T)| \leqslant \|T\| \, \|\eta\| \, \|x\|$ for T in $\mathfrak{A}^-$. It follows that $x \to \rho_x(T)$ is a bounded conjugate-linear functional on $\mathcal{K}$ and corresponds to a vector $\tilde{\eta}(T)$ such that $\rho_x(T) = \langle \tilde{\eta}(T), x \rangle$. From this equality, $\tilde{\eta}$ induces a mapping from weakly continuous functionals on $\mathcal{K}$ (e.g. the one corresponding to x) into the ultraweakly continuous functionals on $\mathfrak{A}^-$ (in the case of x, the functional ρ_x). Thus $\tilde{\eta}$ is continuous from $\mathfrak{A}^-$ in its ultraweak topology to $\mathcal{K}$ in its weak topology. With a a scalar and T and S in $\mathfrak{A}^-$,

$$\langle \tilde{\eta}(aT + S), x \rangle = \rho_x(aT + S) = a\langle \tilde{\eta}(T), x \rangle + \langle \tilde{\eta}(S), x \rangle$$

for each x in $\mathcal{K}$. Thus

$$\tilde{\eta}(aT + S) = a\tilde{\eta}(T) + \tilde{\eta}(S).$$

Since $\rho_x(A) = \langle \eta(A), x \rangle$, for each x in $\mathcal{K}$, when $A \in \mathfrak{A}$, we have that $\tilde{\eta}(A) = \eta(A)$. Of course, $\|\eta\| \leqslant \|\tilde{\eta}\|$. With A in $(\mathfrak{A})_1$, we have that

$$|\langle \tilde{\eta}(A), x \rangle| = |\langle \eta(A), x \rangle| \leqslant \|\eta(A)\| \, \|x\| \leqslant \|\eta\| \, \|x\|$$

for each x in $\mathcal{K}$. Since $(\mathfrak{A})_1$ is ultraweakly dense in $(\mathfrak{A}^-)_1$ and $\tilde{\eta}$ is ultraweakly continuous, this same inequality is valid for each T in $(\mathfrak{A})_1$ (and each x in $\mathcal{K}$). Thus $\|\tilde{\eta}(T)\| \leqslant \|\eta\|$ for each T in $(\mathfrak{A}^-)_1$, and $\|\tilde{\eta}\| \leqslant \|\eta\|$. Hence $\|\tilde{\eta}\| = \|\eta\|$. (Arguing as in [9; Lemma 10.1.10], another proof of these extension and bound conclusions can be given. That proof relies on general results extending uniformly continuous mappings from uniform structures to their completions and makes use of the ultraweak compactness of $(\mathfrak{A}^-)_1$ and the weak compactness of $(\mathcal{K})_1$ and, hence, their completeness in the associated uniform structures. The Kaplansky density theorem remains the key to the proof.) Let Φ be the universal representation of $\mathfrak{A}$ on $\mathcal{H}_u$. Since Φ is a *isomorphism (hence, an isometry) and each bounded linear functional on $\Phi(\mathfrak{A})$ is (weak-operator, hence) ultraweakly continuous [9; Proposition 10.1.1], the mapping

$$\Phi(A) \to \langle\, (\eta \circ \Phi^{-1})(\Phi(A)), x \,\rangle = \langle\, \eta(A), x \,\rangle \qquad (A \in \mathfrak{A})$$

is ultraweakly continuous. Thus $\eta \circ \Phi^{-1}$ is continuous from $\Phi(\mathfrak{A})$ in its ultraweak topology to $\mathcal{K}$ in its weak topology. From the preceding paragraph, $\eta \circ \Phi^{-1}$ has a (unique) linear extension $\overline{\eta}$ to $\Phi(\mathfrak{A})^-$ that is continuous from $\Phi(\mathfrak{A})^-$ in its ultraweak topology to $\mathcal{K}$ in its weak topology and $\|\eta \circ \Phi^{-1}\| = \|\overline{\eta}\|$. Since Φ is an isometry, $\|\eta \circ \Phi^{-1}\| = \|\eta\|$. Thus $\|\eta\| = \|\overline{\eta}\| = \|\tilde{\eta}\|$.

Let P be the central projection in $\Phi(\mathfrak{A})^-$, which is described in [9; Theorem 10.1.12], such that $A \to \Phi(A)P$ extends to a $*$-isomorphism Ψ of $\mathfrak{A}^-$ onto $\Phi(\mathfrak{A})^- P$ and $\tilde{\eta}'(T)$ be $\tilde{\eta}(\Psi^{-1}(TP))$ for each T in $\Phi(\mathfrak{A})^-$. Since Ψ is a *isomorphism of $\mathfrak{A}^-$ onto $\Phi(\mathfrak{A})^- P$, Ψ is ultraweakly continuous (see [9; Remark 7.4.4]) and $\tilde{\eta}'$ is an ultraweakly-weak continuous linear mapping of $\Phi(\mathfrak{A})^-$ into $\mathcal{K}$. Note that, for each A in $\mathfrak{A}$,

$$\tilde{\eta}' \left(\Phi(A) \right) = \tilde{\eta} \left(\Psi^{-1}(\Phi(A)P) \right) = \tilde{\eta}(A) = \eta(A) = \left(\eta \circ \Phi^{-1} \right) \left(\Phi(A) \right) = \overline{\eta} \left(\Phi(A) \right).$$

Combining this equality with the ultraweak-weak continuity of $\tilde{\eta}'$ and $\overline{\eta}$ on $\Phi(\mathfrak{A})^-$, we have that $\tilde{\eta}' = \overline{\eta}$. It follows that

$$\overline{\eta} \left(T(I-P) \right) = \tilde{\eta}' \left(T(I-P) \right) = \tilde{\eta}(\Psi^{-1} \left(T(I-P)P \right)) = 0.$$

Hence $\overline{\eta}(T) = \overline{\eta}(TP)$, for each T in $\Phi(\mathfrak{A})^-$.

From [5; Theorem 3.2] (or Theorem B), there are states of $\Phi(\mathfrak{A})$ and, hence, unit vectors x and y in $\mathcal{H}_u$ such that

$$\|\overline{\eta} \left(\Phi(A) \right) \|^2 \leqslant \|\eta\|^2 \left[\omega_x \left(\Phi(A^*A) \right) + \omega_y \left(\Phi(AA^*) \right) \right] \tag{2}$$

for each A in $\mathfrak{A}$. We shall show that

$$\|\bar{\eta}(T)\|^2 \leqslant \|\eta\|^2[\omega_x(T^*T) + \omega_y(TT^*)] \tag{3}$$

for each T in $\Phi(\mathfrak{A})^-$. By Kaplansky density and the ultraweak-weak continuity of $\bar{\eta}$, given T of norm 1 in $\Phi(\mathfrak{A})^-$ and a positive ε, we can choose T_0 in the unit ball of $\Phi(\mathfrak{A})$ so close to T in the strong-operator $*$-topology that

$$\|\eta\|\,[\|T_0x\|^2 + \|T_0^*y\|^2]^{\frac{1}{2}} \leqslant \|\eta\|\,[\|Tx\|^2 + \|T^*y\|^2]^{\frac{1}{2}} + \frac{1}{2}\varepsilon$$

and at the same time, so close to T in the ultraweak topology that

$$\langle\bar{\eta}(T), \bar{\eta}(T)\rangle \leqslant |\langle\bar{\eta}(T_0), \bar{\eta}(T)\rangle| + \frac{1}{2}\varepsilon\|\bar{\eta}(T)\|.$$

With this choice of T_0, we have, from (2) that

$$\|\bar{\eta}(T)\|^2 - \frac{1}{2}\varepsilon\|\bar{\eta}(T)\| \leqslant |\langle\bar{\eta}(T_0), \bar{\eta}(T)\rangle| \leqslant \|\bar{\eta}(T)\|\,\|\bar{\eta}(T_0)\| \leqslant$$
$$\leqslant \|\bar{\eta}(T)\|\,\|\eta\|\,[\|T_0x\|^2 + \|T_0^*y\|^2]^{\frac{1}{2}} \leqslant \|\bar{\eta}(T)\|\,\|\eta\|\,[\|Tx\|^2 + \|T^*y\|^2]^{\frac{1}{2}} + \frac{1}{2}\varepsilon\|\bar{\eta}(T)\|.$$

Thus

$$\|\bar{\eta}(T)\| \leqslant \|\eta\|\,[\|Tx\|^2 + \|T^*y\|^2]^{\frac{1}{2}} + \varepsilon$$

for each positive ε, from which (3) follows for each T in $\Phi(\mathfrak{A})^-$.

From (3), for each T in $\Phi(\mathfrak{A})^-$, we have that

$$\|\bar{\eta}(T)\|^2 = \|\bar{\eta}(TP)\|^2 \leqslant \|\eta\|^2[\omega_{Px}(T^*T) + \omega_{Py}(TT^*)].$$

If either of Px or Py is 0, this inequality is valid with an arbitrary unit vector (in particular, one in $P(\mathcal{H}_u)$) in place of that 0 vector. If neither Px nor Py is 0, both have norm not greater than 1 (since x and y are unit vectors). Thus, with $u = \|Px\|^{-1}Px$ and $v = \|Py\|^{-1}Py$,

$$\omega_{Px} \leqslant \|Px\|^{-2}\omega_{Px} = \omega_u, \qquad \omega_{Py} \leqslant \|Py\|^{-2}\omega_{Py} = \omega_v.$$

It follows that the states $\omega_u \mid \Phi(\mathfrak{A})^-$ and $\omega_v \mid \Phi(\mathfrak{A})^-$ satisfy

$$\|\bar{\eta}(T)\|^2 \leqslant \|\eta\|^2[\omega_u(T^*T) + \omega_v(TT^*)] \qquad (T \in \Phi(\mathfrak{A})^-)$$

and $Pu = u$, $Pv = v$.

Let $\rho(S)$ be $\omega_u(\Psi(S))$ and $\rho'(S)$ be $\omega_v(\Psi(S))$ for S in $\mathfrak{A}^-$. Then ρ and ρ' are normal states of $\mathfrak{A}^-$, since Ψ is a *isomorphism of $\mathfrak{A}^-$ onto $\Phi(\mathfrak{A})^-P$ and

$$\rho(I) = \omega_u(P) = 1 = \omega_v(P) = \rho'(I).$$

Moreover, for each A in $\mathfrak{A}$,

$$\|\eta(A)\|^2 = \|\tilde{\eta}(A)\|^2 = \|\tilde{\eta}\ (\ \Psi^{-1}(\Phi(A)P)\ )\ \|^2 = \|\tilde{\eta}'(\Phi(A))\|^2 = \|\overline{\eta}(\Phi(A))\|^2 \leqslant$$
$$\leqslant \|\eta\|^2\ [\ \omega_u\ (\ \Phi(A^*A)\ ) + \omega_v\ (\ \Phi(AA^*)\ )] =$$
$$= \|\eta\|^2\ [\ \omega_u\ (\ \Phi(A^*A)P\ ) + \omega_v\ (\ \Phi(AA^*)P\ )] =$$
$$= \|\eta\|^2\ [\ \omega_u\ (\ \Psi(A^*A)\ ) + \omega_v\ (\ \Psi(AA^*)\ )] = \|\eta\|^2\ [\ \rho(A^*A) + \rho'(AA^*)\ ]\ . \qquad \blacksquare$$

The next proof we give of Theorem A is based on a technique of Christensen [1] and Haagerup [6]. It relies on universal representation techniques to the extent that it cites properties of that representation in connection with the decomposition of a bounded functional on a von Neumann algebra as a sum of a singular component and an ultraweakly-continuous component. We begin with a codification of that technique in the following result.

CHRISTENSEN-HAAGERUP PRINCIPLE. *Let f and g be continuous, real-valued functions on $\mathbb{C}^{4m}$ and $\mathbb{C}^{4n}$, respectively, $\sigma_1, \ldots, \sigma_m$ be ultraweakly continuous, linear functionals on a von Neumann algebra $\mathcal{R}$ acting on the Hilbert space $\mathcal{H}$, and $\rho_1, \ldots, \rho_n$ be bounded linear functionals on $\mathcal{R}$ such that, for each A in $\mathcal{R}$,*

$$(4)\quad \begin{aligned} &f(\sigma_1(A), \sigma_1(A^*), \sigma_1(AA^*), \sigma_1(A^*A), \ldots, \sigma_m(A), \sigma_m(A^*), \sigma_m(AA^*), \sigma_m(A^*A)) \leqslant \\ &\leqslant g(\rho_1(A), \rho_1(A^*), \rho_1(AA^*), \rho_1(A^*A), \ldots, \rho_n(A), \rho_n(A^*), \rho_n(AA^*), \rho_n(A^*A)). \end{aligned}$$

Then (4) holds when each ρ_j is replaced by its ultraweakly continuous component ρ_j^{u}.

Proof. Let $\rho_j^{\mathrm{u}} + \rho_j^{\mathrm{s}}$ be the (unique) decomposition of ρ_j as the sum of a linear functional ρ_j^{u} ultraweakly continuous on $\mathcal{R}$ and a linear functional ρ_j^{s} singular on $\mathcal{R}$ (cf. [9; Theorem 10.1.15]). Each non-zero projection E in $\mathcal{R}$ has a non-zero subprojection E_1 in $\mathcal{R}$ such that $\rho_1^{\mathrm{s}}(E_1A) = 0$ for each A in $\mathcal{R}$. To see this, express ρ_1^{s} as a linear combination of singular states of $\mathcal{R}$ [9; Proposition 10.1.17] and apply the result of [9; Exercise 10.5.15] to these states (in succession starting with E). Now find E_2, a non-zero subprojection of E_1 in $\mathcal{R}$ such that $\rho_2(E_2A) = 0$ for each A in $\mathcal{R}$. Continuing in this way, we arrive at a non-zero subprojection E_0 of E such that $\rho_j^{\mathrm{s}}(E_0A) = 0$ for each A in $\mathcal{R}$ and each j in $\{1, \ldots, m\}$. Let $\{E_a\}_{a\in\mathbb{A}}$ be a maximal orthogonal family of non-zero projections in $\mathcal{R}$ such that $\rho_j^{\mathrm{s}}(E_aA) = 0$ for each A in $\mathcal{R}$, each a in $\mathbb{A}$, and each j in $\{1, \ldots m\}$. By maximality, $\sum_{a\in\mathbb{A}} E_a = I$. Let $\mathbb{B}$ be the set of finite subsets of $\mathbb{A}$ ordered by inclusion. With b in $\mathbb{B}$, let F_b be $\sum_{a\in b} E_a$. Then $\{F_b\}_{b\in\mathbb{B}}$ is an increasing net of projections in $\mathcal{R}$ with strong-operator limit I such that $\rho_j^{\mathrm{s}}(F_bA) = 0$ for each b in $\mathbb{B}$, each A in $\mathcal{R}$, and each j in $\{1, \ldots, m\}$.

In (4), replace each ρ_j by $\rho_j^{\mathrm{u}} + \rho_j^{\mathrm{s}}$ and A by F_bAF_b to arrive at

$$(5)\quad f\ (\ \sigma_1(F_bAF_b), \ldots, \sigma_m(F_bA^*F_bAF_b)\ ) \leqslant g\ (\ \rho_1^{\mathrm{u}}(F_bAF_b), \ldots, \rho_n^{\mathrm{u}}(F_bA^*F_bAF_b)\ )\ .$$

Since $\{F_b\}$ is strong-operator convergent to I and multiplication is (jointly) strong-operator continuous on bounded sets, we have that

$$F_bAF_b \to A, \qquad F_bAF_bA^*F_b \to AA^*,$$
$$F_bA^*F_b \to A^*, \qquad F_bA^*F_bAF_b \to A^*A,$$

where convergence is in the strong-operator and, hence, the ultraweak topologies over the respective (bounded) nets. The continuity of f and g and the ultraweak continuity of each σ_j and each ρ_k^{u} yields

$$f(\sigma_1(A), \ldots, \sigma_m(A^*A)) \leqslant g(\rho_1^{\mathrm{u}}(A), \ldots, \rho_m^{\mathrm{u}}(A^*A))$$

on passing to the limit, $F_b \to I$, in (5). ■

To apply the Christensen-Haagerup Principle in our situation, we construct $\tilde{\eta}$, as in the first proof of Theorem A, and note that for each unit vector x in $\mathcal{K}$ and each A in $\mathfrak{A}^-$, we have

$$|\langle\tilde{\eta}(A), x\rangle|^2 \leqslant \|\tilde{\eta}(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)].$$

From the ultraweak continuity of $\tilde{\eta}$ on $\mathfrak{A}^-$, we have that the linear functional $A \to \to\langle\ \tilde{\eta}(A), x\ \rangle$ $(= \sigma(A))$ on $\mathfrak{A}^-$ is ultraweakly continuous. If $\rho^{\mathrm{u}} + \rho^{\mathrm{s}}$ is the decomposition of ρ as the sum of an ultraweakly continuous and a singular linear functional, then ρ^{u} and ρ^{s} are positive linear functionals on $\mathfrak{A}^-$ [9; Theorem 10.1.15] since ρ is a state. Thus $\rho^{\mathrm{u}}(I) \leqslant \rho(I) = 1$ and $\rho^{\mathrm{u}} \leqslant \rho^{\mathrm{u}}(I)^{-1}\rho^{\mathrm{u}}$ $(= \rho_1)$. (If $\rho^{\mathrm{u}}(I) = 0$, let ρ_1 be any normal state of $\mathfrak{A}^-$.) With similar notation, we have that $\rho'^{\mathrm{u}} \leqslant \rho'_1$. Both ρ_1 and ρ'_1 are normal states of $\mathfrak{A}^-$. From the Christensen-Haagerup Principle, we see that, for each A in $\mathfrak{A}^-$, and each unit vector x in $\mathcal{H}$,

$$|\langle\tilde{\eta}(A), x\rangle|^2 \leqslant \|\eta\|^2[\rho^{\mathrm{u}}(A^*A) + \rho'^{\mathrm{u}}(AA^*)] \leqslant \|\eta\|^2[\rho_1(A^*A) + \rho'_1(AA^*)].$$

Since this inequality holds for each unit vector x in $\mathcal{H}$, we have that, for each A in $\mathfrak{A}$,

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho_1(A^*A) + \rho'_1(AA^*)].$$

The third proof that we note, again proposed by Christensen and Haagerup, follows directly from Proposition 2.3 of Haagerup's [6] with a small amount of work. We proceed as in the first proof of Theorem A to the point where η has been shown to be bounded and extended (uniquely) to $\tilde{\eta}$ on $\mathfrak{A}^-$ with the same norm as η. We then define the bilinear form V on $\mathfrak{A}^- \times \mathfrak{A}^-$ with values in $\mathbb{C}$ by means of the equation

$$V(A, B) = \langle\ \tilde{\eta}(A), \tilde{\eta}(B^*)\ \rangle,$$

and note that V is separately ultraweakly continuous with norm equal to $\|\eta\|^2$. From Haagerup's [6; Proposition 2.3], there are normal states $\rho_1, \rho_2, \sigma_1, \sigma_2$ on $\mathfrak{A}^-$ such that

$$|V(A,B)| \leqslant \|V\| \left(\rho_1(A^*A) + \rho_2(AA^*) \right)^{\frac{1}{2}} \left(\sigma_1(B^*B) + \sigma_2(BB^*) \right)^{\frac{1}{2}} \leqslant$$
$$\leqslant \frac{1}{2}\|V\| \left(\rho_1(A^*A) + \rho_2(AA^*) + \sigma_1(B^*B) + \sigma_2(BB^*) \right)$$

for all A and B in $\mathfrak{A}^-$. To complete this proof of Theorem A, choose ρ to be $\frac{1}{2}(\rho_1 + \sigma_2)$, ρ' to be $\frac{1}{2}(\rho_2 + \sigma_1)$, and B^* to be A. The drawback to this approach is that [6; Proposition 2.3] is proved by specializing the Christensen-Haagerup Principle to [6; Theorem 1.1] which has a more difficult proof than Theorem B (even more difficult than the proof Haagerup gives of his [5; Theorem 3.2]).

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Received: August 10, 1990; revised: July 11, 1991.