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ELEMENTARY ROTATIONS OF LINEAR OPERATORS IN KREIN SPACES

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1. INTRODUCTION

The elementary rotation of a contraction in a Hilbert space is a unitary operator extending the given contraction. Under suitable minimality conditions this unitary operator is essentially unique and it plays a central role in dilation theory, as emphasized by P. R. Halmos [13] and B. Sz.-Nagy and C. Foiaş [21].

Beginning with the theorem of B. Sz.-Nagy [20], the existence of minimal unitary dilations has been proved for more and more general classes of operators (cf. C. Davis [9], P. Sorjonen [19]) this culminating with the result of T. Ya. Azizov [3] (see also [4]) which states that any bounded linear operator in a Krein space has a minimal unitary dilation). Implicitly, this result contains the existence of elementary rotations of any bounded operator in Krein space.

Motivated by investigations in lifting of operators, the possibility of using an elementary rotation which can be described explicitly in terms of the given operator was pointed out in [6] for contractions in Pontryagin spaces, while in [2] such a description is obtained for any operator in Krein spaces.

The purpose of this article is to illustrate a technique of induced Krein spaces and an abstract scattering theoretical interpretation of elementary rotations in Krein space. Briefly speaking this means that first one associates a certain selfadjoint operator A to the given operator T, then considering two dual indefinite factorizations of A one obtains two unitary operators Ω_+ and Ω_- and $S = \Omega_+\Omega_-^{-1}$ is an elementary rotation of T (see Theorem 3.3 and its lemmas).

In [2], the existence of elementary rotation follows as a consequence of the so-called link operators. We can show that the converse is also true, once the elementary rotation is obtained, the existence of link operators and their properties follow.

In Theorem 3.12 we obtain a spectral characterization of those operators which possess unique elementary rotation, up to unitary equivalence. In connection with this we should mention that, during a short visit to Bucharest that he paied at the end of July 1990, M. A. Dritschel informed us about a geometric characterization of operators which have unique Julia operators, this being a result from a paper, which at that time was in preparation. Since Julia operators, as introduced in [12], define the same object as elementary rotations, Theorem 3.12 can be considered as a counterpart of their result.

Elementary rotations and unitary dilations are closely related. Using the elementary rotation $R(T)$ we gave in [7] the Schäffer form of the minimal unitary dilation. Due to the nonuniqueness of minimal dilation (early remarked by C. Davis [9]), the problem of characterizing those minimal unitary dilations produced by elementary rotations is natural. We have considered this problem in Section 4. Here the usual difficulties encountered in the geometry of Krein spaces, illustrated by the savage behaviour of shifts on Krein spaces (see paper of B. McEnnis [17]) show up. Finally, a discussion on characteristic functions, from the point of view of the approach used in this paper, is considered.

In Section 2 we present preliminary results concerning the geometry of Krein spaces and their linear operators, a boundedness criterion for isometric operators, the construction and the basic properties of induced Krein spaces, as well as of indefinite factorizations which produce unitary operators. For basic results concerning linear operators on Krein spaces we recommend, T. Ando [1], J. Bognár [5] and T. Ya. Azizov and I. S. Iokhvidov [4].

2. NOTATION AND SOME PRELIMINARY RESULTS

2.1. Geometry in Krein spaces. Let $\mathcal{K}$ be a complex vector space and $[\cdot,\cdot]$ an inner product on $\mathcal{K}$ (i.e. $[\cdot,\cdot]$ is linear with respect to the first variable and antisymmetric). $\mathcal{K}$ is called a *Krein space* if one of the following equivalent conditions holds:

(i) There exists a linear operator $J:\mathcal{K}\to\mathcal{K}$ such that $J^{-1}=J$ and denoting

$$(x,y)_J=[Jx,y],\quad x,y\in\mathcal{K}, \tag{2.1}$$

$(\cdot,\cdot)_J$ is a positive definite inner product on $\mathcal{K}$ such that $(\mathcal{K},(\cdot,\cdot)_J)$ is a Hilbert space.

(ii) There exist two subspaces $\mathcal{K}^\pm\subseteq\mathcal{K}$ such that $\mathcal{K}=\mathcal{K}^+ +\mathcal{K}^-$, $\mathcal{K}^+\perp\mathcal{K}^-$ (i.e. $[x,y]=0$, $x\in\mathcal{K}^+$, $y\in K^-$) and $(\mathcal{K}^+,[\cdot,\cdot])$, $(\mathcal{K}^-,-[\cdot,\cdot])$ are Hilbert spaces.

(iii) There exists a positive definite inner product $(\cdot,\cdot)$ on $\mathcal{K}$ such that $(\mathcal{K},(\cdot,\cdot))$ is a Hilbert space and, denoting by $\|\cdot\|$ the associated norm, we have

$$\|x\| = \sup_{\|y\|\leqslant 1} |[x,y]|, \quad x \in \mathcal{K}. \tag{2.2}$$

Let $(\mathcal{K},[\cdot,\cdot])$ be a Krein space. An operator $J : \mathcal{K} \to \mathcal{K}$ satisfying the property (i) is called a *fundamental symmetry* (in brief f.s.). With respect to the Hilbert space $(\mathcal{K},(\cdot,\cdot)_J)$ J is a symmetry, i.e. $J^* = J = J^{-1}$. If $\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^-$ is a decomposition of $\mathcal{K}$ as in (ii), then it is called a *fundamental decomposition* (in brief f.d.). Fundamental symmetries and fundamental decompositions of the Krein space are in bijective correspondence: If J is a f.s. let $J = J^+ - J^-$ be its Jordan decomposition and $\mathcal{K}^+ = J^+\mathcal{K}$, $\mathcal{K}^- = J^-\mathcal{K}$. Then $\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^-$ is a f.d. Conversely, if $\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^-$ is a f.d. then define $J : \mathcal{K} \to \mathcal{K}$ by

$$J(x^+ + x^-) = x^+ - x^-, \quad x^\pm \in \mathcal{K}^\pm. \tag{2.3}$$

J is a f.s. of $\mathcal{K}$.

A norm on $\mathcal{K}$ satisfying the property (iii) is called a *unitary norm* on $\mathcal{K}$. Any unitary norm $\|\cdot\|$ on $\mathcal{K}$ is of the form

$$\|x\| = [Jx,x]^{\frac{1}{2}}, \quad x \in \mathcal{K}, \tag{2.4}$$

where J is a f.s. of $\mathcal{K}$. Any two unitary norms on $\mathcal{K}$ are equivalent. The *strong topology* of the Krein space $\mathcal{K}$ is the topology defined by an arbitrary unitary norm on $\mathcal{K}$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspaces of the Krein space $\mathcal{K}$ (i.e. $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed linear submanifolds of $\mathcal{K}$). If $\mathcal{L}_1 \perp \mathcal{L}_2$ and the algebraic sum $\mathcal{L}_1 + \mathcal{L}_2$ is direct and closed then we use the notation $\mathcal{L}_1[+]\mathcal{L}_2$. In particular, a f.d. of $\mathcal{K}$ will be written $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$.

If $\mathcal{L}$ is a subspace of the Krein space $\mathcal{K}$ we denote by $\mathcal{L}^\perp = \{x \in \mathcal{K} | [x,y] = 0,\ y \in \in \mathcal{L}\}$ the *orthogonal companion* of $\mathcal{L}$ and by $\mathcal{L}^\circ = \mathcal{L} \cap \mathcal{L}^\perp$ the *isotropic subspace* of $\mathcal{L}$. The subspace $\mathcal{L}$ is called *nonnegative (positive)* if $[x,x] \geqslant 0$, $x \in \mathcal{L}([x,x] > 0,\ x \in \mathcal{L} \setminus \{0\})$. The subspace $\mathcal{L}$ is called *uniformly positive* if for some unitary norm $\|\cdot\|$ (equivalently, for any unitary norm) on $\mathcal{K}$, there exists $\alpha > 0$ such that

$$[x,x] \geqslant \alpha\|x\|^2, \quad x \in \mathcal{L}. \tag{2.5}$$

Similarly one defines *nonpositive* subspaces, *negative* subspaces and *uniformly negative* subspaces.

Let $\mathcal{L}$ be a nonnegative subspace of the Krein space $\mathcal{K}$, $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ be a f.d. of $\mathcal{K}$, J the corresponding f.s. and $J = J^+ - J^-$ be its Jordan decompoition. Then $\mathcal{D}_+ = J^+\mathcal{L}$ is a closed linear manifold in $\mathcal{K}^+$, the operator $K \in \mathcal{L}(\mathcal{D}_+, \mathcal{K}^-)$ defined by

$$K(J^+x) = J^-x, \quad x \in \mathcal{L}, \tag{2.6}$$

is a Hilbert space contraction, and $\mathcal{L}$ is the graph of $\mathcal{K}$

$$\mathcal{L} = G(K) = \{x + Kx | x \in \mathcal{D}_+\}. \tag{2.7}$$

The operator K is called the *angular operator* of the nonnegative subspace $\mathcal{L}$. Moreover, the subspace $\mathcal{L}$ is positive (uniformly positive) if and only if its angular operator K is a *strict contraction*, i.e. $\|Kx\| < \|x\|$, $x \in \mathcal{D}_+$ (respectively, K is a *uniform contraction*, i.e. $\|K\| < 1$), where $\|\cdot\|$ denotes the unitary norm associated to J. Similar statements hold for nonpositive, negative and uniformly negative subspaces.

With the notation stated above, the nonnegative subspace $\mathcal{L}$ is *maximal nonnegative* (i.e. there exist no proper nonnegative extensions of $\mathcal{L}$) if and only if $J^+\mathcal{L} = \mathcal{K}^+$. Also, $\mathcal{L}$ is maximal nonnegative if and only if $\mathcal{L}^\perp$ is maximal nonpositive.

A subspace $\mathcal{L}$ of the Krein space $\mathcal{K}$ is called *regular* if $\mathcal{K} = \mathcal{L}[+]\mathcal{L}^\perp$. A nonnegative subspace is regular if and only if it is uniformly positive. The subspace $\mathcal{L}$ is maximal uniformly positive if and only $\mathcal{K} = \mathcal{L}[+]\mathcal{L}^\perp$ is a f.d. of $\mathcal{K}$.

LEMMA 2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of the Krein space $\mathcal{K}$ such that $\mathcal{M}$ is uniformly positive, $\mathcal{N}$ is nonpositive, $\mathcal{M} \perp \mathcal{N}$, and $\mathcal{M} + \mathcal{N}$ is dense in $\mathcal{K}$. Then $\mathcal{M}$ is a maximal uniformly positive subspace and $\mathcal{N} = \mathcal{M}^\perp$.*

Proof. Let us first notice that the subspace $\mathcal{N}$ is negative. Indeed, let $x \in \mathcal{N}$ such that $x \perp \mathcal{N}$. Then $x \perp \mathcal{M} + \mathcal{N}$, hence $x \perp \mathcal{K}$. Since $\mathcal{K}$ is nondegenerate this implies $x = 0$.

Using the extension theorem of R. S. Philips [5], it follows that there exist $\tilde{\mathcal{M}}$ a maximal positive subspace, and $\tilde{\mathcal{N}}$ a maximal negative subspace such that $\tilde{\mathcal{M}} \supseteq \mathcal{M}$, $\tilde{\mathcal{N}} \supseteq \mathcal{N}$ and $\tilde{\mathcal{M}} \perp \tilde{\mathcal{N}}$.

Using the same extension theorem of R. S. Philips, there exists a f.d. $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ such that $\mathcal{M} \subseteq \mathcal{K}^+$. Let J be the corresponding f.s. and $K \in \mathcal{L}(\mathcal{K}^+, \mathcal{K}^-)$ be the angular operator of $\tilde{\mathcal{M}}$. If $x \in \mathcal{K}^+ \ominus \mathcal{M}$, then $x + Kx \in \tilde{\mathcal{M}} \cap \mathcal{M}^\perp$ hence $x + Kx \perp \mathcal{M} + \mathcal{N}$. Since $\mathcal{M} + \tilde{\mathcal{N}} \supseteq \mathcal{M} + \mathcal{N}$ is dense in $\mathcal{K}$, from here we obtain $x = 0$. We have proved in this way that $\mathcal{M} = \mathcal{K}^+$ is a maximal uniformly subspace. This yields $\mathcal{N} \subseteq \mathcal{K}^-$ is uniformly negative, in particular $\mathcal{M}[+]\mathcal{N} = \mathcal{K}$, hence $\mathcal{N} = \mathcal{M}^\perp = \mathcal{K}^-$. ∎

REMARK 2.2. In order to prove that $\mathcal{M}$ is a maximal positive subspace in $\mathcal{K}$, the assumption in Lemma 2.1 that $\mathcal{M}$ be uniformly positive is essential, as shown by the example of H. Langer [15] of two subspaces $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M}$ positive, $\mathcal{N}$ negative, $\mathcal{M} \perp \mathcal{N}$, $\mathcal{M} + \mathcal{N}$ dense in $\mathcal{K}$ but neither $\mathcal{M}$ is maximal positive nor $\mathcal{N}$ is maximal negative.

2.2. Linear operators in Krein spaces. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces and T densely defined in $\mathcal{K}_1$ and valued in $\mathcal{K}_2$. One defines the adjoint operator of T, denoted $T^\#$, as follows

$$\mathcal{D}(T^\#) = \{y \in \mathcal{K}_2 \,|\, \mathcal{D}(T) \ni x \mapsto [Tx, y] \text{ is bounded } \} \tag{2.8}$$

$$[Tx, y] = [x, T^\# y], \quad x \in \mathcal{D}(T), \ y \in \mathcal{D}(T^\#).$$

Let J_1 and J_2 be f.s. of $\mathcal{K}_1$ and, respectively, $\mathcal{K}_2$. Considering T^* the adjoint of the densely defined operator $T : \mathcal{D}(T)(\subseteq \mathcal{K}_1) \to \mathcal{K}_2$ with respect to the Hilbert spaces $(\mathcal{K}_1, (\cdot,\cdot)_{J_1})$ and $(\mathcal{K}_2, (\cdot,\cdot)_{J_2})$, we have $\mathcal{D}(T^\#) = J_2\mathcal{D}(T^*)$ and

$$T^\# = J_1 T^* J_2. \tag{2.9}$$

In the following we denote by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ the set of bounded (with respect to arbitrary unitary norms on the Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$) linear operators $T : \mathcal{K}_1 \to \mathcal{K}_2$. An operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is *contractive* if

$$[Tx, Tx] \leqslant [x, x], \quad x \in \mathcal{K}_1, \tag{2.10}$$

equivalently $I - T^\# T$ is a *nonnegative operator*, i.e.

$$[(I - T^\# T)x, x] \geqslant 0, \quad x \in \mathcal{K}_1. \tag{2.11}$$

T is called *doubly contractive* if both of T and $T^\#$ are contractive. T is called *expansive* if $T^\# T - I$ is nonnegative, and it is called *doubly expansive* if both of T and $T^\#$ are expansive.

A (possibly unbounded) operator $V : \mathcal{D}(V)(\subseteq \mathcal{K}_1) \to \mathcal{K}_2$ is called *isometry* if

$$[Vx, Vy] = [x, y], \quad x, y \in \mathcal{D}(V). \tag{2.12}$$

If $V \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ then V is isometry if and only if $V^\# V = I_1$. An operator $U \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ is called *unitary* if it is isometric and surjective, equivalently $U^\# U = I_1$ and $UU^\# = I_2$.

Let $\mathcal{K}$ be a Krein space and $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ be a f.d. of $\mathcal{K}$. The cardinal numbers $\kappa^+(\mathcal{K}) = \dim(\mathcal{K}^+)$ and $\kappa^-(\mathcal{K}) = \dim(\mathcal{K}^-)$ are called, respectively, the *positive signature* and the *negative signature* of $\mathcal{K}$. They are independent on the f.d. The cardinal member $\kappa(\mathcal{K}) = \min\{\kappa^+(\mathcal{K}), \kappa^-(\mathcal{K})\}$ is called the *rank of indefiniteness* of the Krein space $\mathcal{K}$. If $\kappa(\mathcal{K})$ is finite then $\mathcal{K}$ is called a *Pontryagin space.*

Given two Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, in order to exist unitary operators $U : \mathcal{K}_1 \to \to \mathcal{K}_2$ it is necessary and sufficient that $\kappa^+(\mathcal{K}_1) = \kappa^+(\mathcal{K}_2)$ and $\kappa^-(\mathcal{K}_1) = \kappa^-(\mathcal{K}_2)$.

LEMMA 2.3. *Let $V : \mathcal{D}(V)(\subseteq \mathcal{K}_1) \to \mathcal{K}_2$ be an isometry with dense domain and dense range. Assume that there exists a f.s. J_1 of $\mathcal{K}_1$ such that $J_1\mathcal{D}(V) \subseteq \mathcal{D}(V)$. If at least one the linear manifolds $J_1^+\mathcal{D}(V)$ and $J_1^-\mathcal{D}(V)$ is closed, then V is bounded and thus, it can be uniquely extended to a unitary operator in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$.*

Proof. Let $V : \mathcal{D}(V)(\subseteq \mathcal{K}_1) \to \mathcal{K}_2$ be an isometry such that $\mathcal{D}(V)$ is dense in $\mathcal{K}_1$ and $R(V)$ is dense in $\mathcal{K}_2$. We prove first that V is injective.

Indeed, let $x \in \mathcal{D}(V)$ be such that $Vx = 0$. Then

$$0 = [Vx, Vy] = [x, y], \quad y \in \mathcal{D}(V),$$

hence, since $\mathcal{D}(V)$ is dense in $\mathcal{K}_1$, from here we obtain $x = 0$.

We can consider now the linear operator $V^{-1} : \mathcal{R}(V)(\subseteq \mathcal{K}_2) \to \mathcal{K}_1$, which is also an isometry. Let $z \in \mathcal{R}(V)$ and denote $y = V^{-1}z$. Then we have

$$[Vx, z] = [Vx, Vy] = [x, y] = [x, V^{-1}z], \quad x \in \mathcal{D}(V),$$

hence $V^{-1} \subseteq V^\#$. Since $\mathcal{D}(V^{-1}) = \mathcal{R}(V)$ is dense in $\mathcal{K}_2$, it follows that $\mathcal{D}(V^\#)$ is also dense in $\mathcal{K}_2$, hence V is closable.

Let now J_1 be a f.s. of $\mathcal{K}_1$ such that $J_1\mathcal{D}(V) \subseteq \mathcal{D}(V)$. Then the following decompositions holds

$$\mathcal{D}(V) = J_1^+\mathcal{D}(V) + J_1^-\mathcal{D}(V), \tag{2.13}$$

where $J_1 = J_1^+ - J_1^-$ is the Jordan decomposition of J_1. Since $J_1^+\mathcal{D}(V) \subseteq \mathcal{K}_1^+$ and $J_1^-\mathcal{D}(V) \subseteq \mathcal{K}_2^-$, where $\mathcal{K}_1 = \mathcal{K}_1[+]\mathcal{K}_1^-$ is the f.d. corresponding to J_1^-, then $J_1^+\mathcal{D}(V)$ is uniformly positive, $J_1^-\mathcal{D}(V)$ is uniformly negative, and $J_1^+\mathcal{D}(V) \perp J_1^+\mathcal{D}(V)$.

If, let us say, the linear manifold $J_1^+\mathcal{D}(V)$ is closed, it follows that $V|J_1^+\mathcal{D}(V)$ is bounded (since V is closable and $J_1^+\mathcal{D}(V) \subseteq \mathcal{D}(V)$). We claim that $VJ_1^+\mathcal{D}(V)$ is a maximal uniformly positive subspace of $\mathcal{K}_2$.

Indeed, consider on $\mathcal{K}_1$ the unitary norm associated to J_2 and on $\mathcal{K}_2$ we consider an arbitrary unitary norm. Then, for any vector $x \in \mathcal{D}(V)$ we have

$$[VJ_1^+x, VJ_1^+x] = [J_1^+x, J_1^+x] = \|J_1^+x\|^2 \geqslant \frac{1}{\|VJ_1^+\|}\|VJ_1^+x\|^2,$$

hence $VJ_1^+\mathcal{D}(V)$ is uniformly positive. Since $J_1^+\mathcal{D}(V)$ is closed and $V|J_1^+\mathcal{D}(V)$ is isometric and bounded it follows that $VJ_1^+\mathcal{D}(V)$ is also closed, hence $VJ_1^+\mathcal{D}(V)$ is a uniformly positive subspace of $\mathcal{K}_2$.

On the other hand, since V is isometric we have

$$\mathcal{R}(V) = VJ_1^+\mathcal{D}(V) + VJ_1^-\mathcal{D}(V),$$

where $VJ_1^+\mathcal{D}(V) \perp VJ_1^-\mathcal{D}(V)$ and $VJ_1^-\mathcal{D}(V)$ is negative. Since $\mathcal{R}(V)$ is dense in $\mathcal{K}_2$ and $VJ_1^+\mathcal{D}(V)$ is a uniformly positive subspace, application of Lemma 2.1 proves that $VJ_1^+\mathcal{D}(V)$ is a maximal uniformly positive subspace and, in addition, $VJ_1^-\mathcal{D}(V) = (VJ_1^+\mathcal{D}(V))^\perp$ is a maximal uniformly negative subspace. The claim is proved.

Consider now the f.d. of $\mathcal{K}_2$

$$\mathcal{K}_2 = VJ_1^+\mathcal{D}(V)[+]\overline{VJ_1^-\mathcal{D}(V)}, \tag{2.14}$$

and denote by J_2 the corresponding f.s. From (2.13) and (2.14) it follows that

$$VJ_1 = J_2V. \tag{2.15}$$

We change now the unitary norm on $\mathcal{K}_2$ to be that induced by J_2. Then, using (2.15), it follows that

$$\|Vx\|^2 = (Vx, Vx)_{J_2} = [J_2Vx, Vx] = [VJ_1x, Vx] = [J_1x, x] = \|x\|^2, \quad x \in \mathcal{D}(V).$$

This shows that V is bounded, hence it can be (uniquely) extended to a unitary operator in $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$.

In case $J_1^-\mathcal{D}(V)$ is closed, the reasoning is similar. ∎

2.3. The Krein space $\mathcal{H}_A$. Let $\mathcal{K}$ be a Krein space and $A \in \mathcal{L}(\mathcal{K})$ be selfadjoint, i.e. $A = A^\#$. If J is a f.s. of $\mathcal{K}$ then JA is a selfadjoint operator on the Hilbert space $(\mathcal{K}, (\cdot,\cdot)_J)$, hence we can consider its polar decomposition

$$JA = S_{JA}|JA|, \tag{2.15}$$

where $S_{JA} = \operatorname{sgn}(JA)$ is a selfadjoint partial isometry such that $\ker S_{JA} = \ker A$. Then S_{JA} is a symmetry on the Hilbert space $(\overline{\mathcal{R}(JA)}, (\cdot,\cdot)_J)$. We denote by $\mathcal{H}_A$ the Krein space $(\overline{\mathcal{R}(A)}, [\cdot,\cdot])$ where the indefinite inner product $[\cdot,\cdot]$ is induced by the symmetry S_{JA},

$$[x, y] = (S_{JA}x, y)_J, \quad x, y \in \mathcal{H}_A. \tag{2.16}$$

Let us remark that the linear manifolds $\mathcal{R}(|JA|)$ and $\mathcal{R}(|JA|^{\frac{1}{2}})$ are dense in $\mathcal{H}_A$ and that the strong topology on the Krein space $\mathcal{H}_A$ is inherited from the strong toplogy of the original Krein space $\mathcal{K}$.

Apparently, the definition of the Krein space $\mathcal{H}_A$ depends on the f.s. J. We postpone the proof of the fact that if a different $f.s.$ is used, the Krein space obtained by a construction similar with that of $\mathcal{H}_A$ is actually unitary equivalent to $\mathcal{H}_A$ (see Corollary 2.8), in particular, this observation justifying our notation.

A particular case of this construction occurs when starting a Hilbert space $\mathcal{H}$ and a selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$. An even more particular situation is in case the selfadjoint operator A is a symmetry. This kind of construction is usually used in the definition of direct sum of Krein spaces.

Let $(\mathcal{K}_i)_{i\in\mathcal{J}}$ be a family of Krein spaces. Fix on each $\mathcal{K}_i$ a f.s. J_i and consider the direct sum Hilbert space $\bigoplus_{i\in\mathcal{J}} \mathcal{K}_i$. Let J be the symmetry $\bigoplus_{i\in\mathcal{J}} J_i$ on this Hilbert space. We denote by $\underset{i\in\mathcal{J}}{[+]} \mathcal{K}_i$ the Krein space induced by this symmetry.

A special case of this construction is the Krein space of the type $l^2(\mathcal{K})$ (i.e. in case the index set $\mathcal{J}$ is $\mathbb{N}$). On the Krein space $l^2(\mathcal{K})$ the forward shift S of multiplicity $\mathcal{K}$ acts.

2.4. The Krein space $\mathcal{K}_A$. Let $\mathcal{K}$ be a Krein space and $A \in \mathcal{L}(\mathcal{K})$, $A = A^\#$. Define an inner product on $\mathcal{K}$,

$$[x, y]_A = [Ax, y], \quad x, y \in \mathcal{K}, \tag{2.17}$$

where $[\cdot,\cdot]$ denotes the inner product of the Krein space $\mathcal{K}$. Notice that $\ker A$ is the isotropic subspace of the inner product space $(\mathcal{K}, [\cdot,\cdot]_A)$. Fix J a f.s. of $\mathcal{K}$ and denote $\hat{\mathcal{K}} = J(\ker A)^\perp$ (i.e. $\hat{\mathcal{K}}$ is the orthogonal of $\ker A$ with respect to the inner product $(\cdot,\cdot)_J$). Then consider the Jordan decomposition of the selfadjoint operator JA with respect to the Hilbert space $(\mathcal{K}, (\cdot,\cdot)_J)$

$$JA = (JA)_+ - (JA)_- \tag{2.18}$$

and denoting $\hat{\mathcal{K}}_+ = \overline{(JA)_+\mathcal{K}}$ and $\hat{\mathcal{K}}_- = \overline{(JA)_-\mathcal{K}}$, we have the decomposition

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}_+ + \hat{\mathcal{K}}_-. \tag{2.19}$$

Notice that $(\hat{\mathcal{K}}_+, [\cdot,\cdot]_A)$ and $(\hat{\mathcal{K}}_-, -[\cdot,\cdot]_A)$ are pre-Hilbert spaces and denote by $\mathcal{K}_A^+$ and, respectively, $\mathcal{K}_A^-$ their completions to Hilbert spaces. Define

$$\mathcal{K}_A = \mathcal{K}_A^+[+]\mathcal{K}_A^-, \tag{2.20}$$

where the inner product is the extension by continuity of the inner product $[\cdot,\cdot]_A$. Then $(\mathcal{K}_A, [\cdot,\cdot]_A)$ is a Krein space and (2.20) is a f.d. of $\mathcal{K}_A$.

LEMMA 2.4. *Let $\|\cdot\|$ be the unitary norm assciated to the f.s. J. Then the unitary norm on $\mathcal{K}_A$, corresponding to the f.d. (2.20) is the extension by continuity of norm*

$$\hat{\mathcal{K}} \ni x \mapsto \|\,|JA|^{\frac{1}{2}}x\|.$$

Proof. Let x be a vector in $\hat{\mathcal{K}}$. According to (2.19) we represent $x = x_+ + x_-$, where $x_+ \in \hat{\mathcal{K}}_+ \subseteq \mathcal{K}_A^+$. Then

$$[x_+, x_+]_A - [x_-, x_-]_A = ((JA)_+x_+, x_+)_J + ((JA)_-x_-, x_-)_J =$$

$$= (((JA)_+ + (JA)_-)(x_+ + x_-), (x_+ + x_-))_J = (|JA|(x_+ + x_-), (x_+ + x_-))_J =$$

$$= (|JA|^{\frac{1}{2}}x, |JA|^{\frac{1}{2}}x)_J = \|\,|JA|^{\frac{1}{2}}x\|^2.$$

This shows that the unitary norm, corresponding to the f.d. (2.20), when restricted to $\hat{\mathcal{K}}$ coincides with the norm $\|\,|JA|^{\frac{1}{2}}x\|$. The rest follows from the density of $\hat{\mathcal{K}}$ in $\mathcal{K}_A$. ∎

We can now clarify the relation between Krein spaces $\mathcal{H}_A$ and $\mathcal{K}_A$.

PROPOSITION 2.5. *If $A \in \mathcal{L}(\mathcal{K})$ is selfadjoint, $\mathcal{K}$ a Krein space, the the Krein spaces $\mathcal{H}_A$ and $\mathcal{K}_A$ are unitary equivalent, more precisely, if J is a f.s. used in the definitions of $\mathcal{H}_A$ and $\mathcal{K}_A$, the linear operator*

$$\mathcal{K}_A \supseteq \hat{\mathcal{K}} \ni x \mapsto |JA|^{\frac{1}{2}}x \in \mathcal{R}(|JA|^{\frac{1}{2}}) \subseteq \mathcal{H}_A,$$

extends uniquely to a unitary operator $\mathcal{K}_A \to \mathcal{H}_A$.

Proof. Let V denote the operator defined by (2.21). We first prove that V is isometric, considered as an operator $V : \mathcal{D}(V)(\subseteq \mathcal{K}_A) \to \mathcal{H}_A$. Indeed, for any $x, y \in \hat{\mathcal{K}}$ we have

$$[Vx, Vy] = [|JA|^{\frac{1}{2}}x, |JA|^{\frac{1}{2}}y] = (S_{JA}|JA|^{\frac{1}{2}}x, |JA|^{\frac{1}{2}})_J =$$

$$= (JAx, y)_J = [Ax, y] = [x, y]_A.$$

Using Lemma 2.4 it follows that V is bounded hence, since $\mathcal{D}(V) = \hat{\mathcal{K}}$ is dense in $\mathcal{K}_A$ and $\mathcal{R}(V) = \mathcal{R}(|JA|^{\frac{1}{2}})$ is dense in $\mathcal{H}_A$, V extends uniquely to a unitary operator in $\mathcal{L}(\mathcal{K}_A, \mathcal{H}_A)$. ∎

We record now an important result of M. G. Krein [14], W. T. Reid [18], P. D. Lax [16], and J. Dieudonne [10]. In this paper we shall use a slightly more general variant of this result, equivalent with that considered by A. Dijksma, H. Langer, and H. S. V. de Snoo [11]. For the reader's convenience we give a proof following the original one.

LEMMA 2.6. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces and $A \in \mathcal{L}(\mathcal{K}_1)$, $A = A^\#$, $B \in \mathcal{L}(\mathcal{K}_2)$, $B = B^\#$, $T_1 \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$, and $T_2 \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ be such that*

$$[T_1 x, y]_B = [x, T_2 y]_A, \quad x \in \mathcal{K}_1,\ y \in \mathcal{K}_2,$$

or equivalently,

$$T_2^\# A = B T_1. \tag{2.22}$$

Then T_1 and T_2 induce uniquely determined operators $\tilde{T}_1 \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ and $\tilde{T}_2 \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$ such that

$$[\tilde{T}_1 x, y]_B = [x, \tilde{T}_2 y]_A, \quad x \in \mathcal{K}_A,\ y \in \mathcal{K}_B. \tag{2.23}$$

Proof. Fix f.s. J_1 and J_2 on $\mathcal{K}_1$ and, respectively $\mathcal{K}_2$. Then (2.22) becomes

$$T_2^* J_1 A = J_2 B T_1. \tag{2.24}$$

Considering the unitary norms associated to J_1 and J_2, we shall prove that for any $x \in \mathcal{K}_1$ the following inequality holds

$$\| |J_2 B|^{\frac{1}{2}} T_1 x\| \leqslant \| |J_2 B|^{\frac{1}{2}} T_1 S_{J_1 A} T_2 S_{J_2 B} T_1\|^{\frac{1}{2}} \cdot \| |J_1 A|^{\frac{1}{2}} T_1 x\|. \tag{2.25}$$

Indeed, using (2.24) and Schwarz inequality for the nonegative operators $|J_1 A|$ and $|J_2 B|$ on the Hilbert spaces $(\mathcal{K}_1, (\cdot,\cdot)_{J_1})$ and, respectively, $(\mathcal{K}_2, (\cdot,\cdot)_{J_2})$, it follows easily

$$\| |J_2 B|^{\frac{1}{2}} T_1 x\| \leqslant \| |J_1 A|^{\frac{1}{2}} x\| \cdot \| |J_2 B|^{\frac{1}{2}} T_1 x\|^{\frac{1}{2}} \cdot \| |J_2 B|^{\frac{1}{2}} T_1 S_{J_1 A} T_2 S_{J_2 B} T_1 x\|^{\frac{1}{2}}$$

Iterating this inequality it follows that for arbitrary $n \in \mathbb{N}$ it holds

$$\begin{aligned} \| |J_2 B|^{\frac{1}{2}} T_1 x\| \leqslant \| |J_1 A|^{\frac{1}{2}} x\|^{2-(1/2^{2n})} \cdot \| |J_2 B|^{\frac{1}{2}} T_1 x\|^{(1/2^{2n+1})} \cdot \\ \| |J_2 B|^{\frac{1}{2}} T_1 S_{J_1 A} T_2 S_{J_2 B} T_1 x\|^{1-(1/2^{2n+1})}. \end{aligned} \tag{2.26}$$

Further, if $|J_2 B|^{\frac{1}{2}} T_1 x = 0$, the inequality (2.25) is clearly true, so let us assume $|J_2 B|^{\frac{1}{2}} T_1 x \neq 0$. If $\|x\| \leqslant 1$ then we obtain (2.25) by letting $n \to \infty$ in (2.26). If $\|x\| > 1$ then use (2.25) for the vector $x/\|x\|$. Thus (2.25) holds for any $x \in \mathcal{K}_1$.

Now, from (2.22) it follows that $T_1 \ker A \subseteq \ker B$, hence T_1 factors to an operator $\hat{T}_1 : \hat{\mathcal{K}}_1 (\subseteq \mathcal{K}_A) \to \mathcal{K}_2 \subseteq \mathcal{K}_B$. Using Lemma 2.4, from (2.25) it follows that $\hat{T}_1$ extends by continuity to an operator $\tilde{T}_1 \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$. Similarly it can be proved that T_2 induces an operator $\tilde{T}_2 \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$, while the property (2.23) is clear. ∎

2.5. Indefinite factorizations. Let $A \in \mathcal{L}(\mathcal{K}_1)$, $A = A^\#$ and $B \in \mathcal{L}(\mathcal{K}_2)$, $B = B^\#$ be given. We are interested in factorizations of the type

$$A = C^\# BC, \tag{2.27}$$

where $C \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. Under certain conditions, this kind of factorizations produce unitary operators acting between the Krein spaces induced by A and B.

LEMMA 2.7. *Let $A \in \mathcal{L}(\mathcal{K}_1)$, $A = A^\#$, $B \in \mathcal{L}(\mathcal{K}_2)$, $B = B^\#$, and $C \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ be such that (2.27) holds and, in addition, assume that there exist a regular subspace $\mathcal{L}$ of $\mathcal{K}_2$ such that*

$$\mathcal{R}(B) \subseteq \mathcal{L} \subseteq \mathcal{R}(C). \tag{2.28}$$

If J_1 is a f.s. of $\mathcal{K}_1$ and J_2 is a f.s. of $\mathcal{K}_2$ such that $J_2\mathcal{L} \subseteq \mathcal{L}$ (always exists such a J_2) then:

(i) *C induces a unitary operator in $\mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$.*

(ii) *There exists a uniquely determined unitary operator $V \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$ such that*

$$V|J_1A|^{\frac{1}{2}} = |J_2B|C. \tag{2.29}$$

Proof. Let $C^{-1} : \mathcal{R}(C) \to \mathcal{K}_1$ be an operator such that $CC^{-1}x = x$, $x \in \mathcal{R}(C)$. C^{-1} is closed hence, from (2.28), it follows that $C^{-1}|\mathcal{L}$ is bounded. With respect to the decomposition

$$\mathcal{K}_2 = \mathcal{L}[+]\mathcal{L}^\perp$$

consider the operator $X \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ defined by

$$X = [C^{-1}|\mathcal{L} \quad 0]$$

Then $CX = P \in \mathcal{L}(\mathcal{K}_2)$, where P has the properties $P^\# = P = P^2$ and $P\mathcal{K}_2 = \mathcal{L}$. Multiplying on left with $X^\#$ we have

$$X^\# A = X^\# C^\# BC = PBC = BC.$$

Using Lemma 2.6 this shows that C induces an operator $\tilde{C} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$. From (2.27) it follows that C is isometric. Also, we have

$$\mathcal{R}(J_2B) = J_2\mathcal{R}(B) \subseteq J_2\mathcal{L} \subseteq \mathcal{L} \subseteq \mathcal{R}(C),$$

and, since $\mathcal{R}(J_2B)$ is dense in $\mathcal{K}_B$, it follows that $\mathcal{R}(\tilde{C})$ is also dense in $\mathcal{K}_B$, hence $\tilde{C}$ is unitary.

(ii) Using the identifications of the Krein spaces $\mathcal{H}_A$ with $\mathcal{K}_A$ and of $\mathcal{H}_B$ with $\mathcal{K}_B$ (see proposition 2.5) and also using the unitary operator $\tilde{C} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$, it follows that the linear operator $V : \mathcal{R}(|J_1 A|^{\frac{1}{2}})(\subseteq \mathcal{H}_A) \to \mathcal{H}_B$ defined by (2.29), extends uniquely to a unitary operator in $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$. ■

COROLLARY. *For any operator $A \in \mathcal{L}(\mathcal{K})$, $A = A^{\#}$, where $\mathcal{K}$ is a Krein space, the induced Krein spaces $\mathcal{K}_A$ and $\mathcal{H}_A$ are unique, modulo unitary equivalence, with respect to the f.s. J used for their definitions.*

Proof. Let J be a f.s. of $\mathcal{K}$. We first remark that $\mathcal{K}_A$ can be viewed also as the Krein space $\mathcal{K}_{JA}$, induced by the selfadjoint operator JA acting in the Hilbert space $(\mathcal{K}, (\cdot,\cdot)_J)$. It follows that we have to prove that, if G is another f.s. of $\mathcal{K}$, then the Krein spaces $\mathcal{K}_{JA}$ and $\mathcal{K}_{GA}$ are unitary equivalent.

To this end, notice that the following relation holds

$$JA = (JG)(GA), \tag{2.30}$$

and that the operator JG is the adjoint of the identity operator acting $(\mathcal{K}, (\cdot,\cdot)_J) \to \to (\mathcal{K}, (\cdot,\cdot)_G)$. Applying Lemma 2.7 to the factorization (2.30) it follows that the identity operator induces a unitary operator in $\mathcal{L}(\mathcal{K}_{GA}, \mathcal{K}_{JA})$.

The uniqueness of the definition of the Krein space $\mathcal{H}_A$ follows now from the Proposition 2.5. ■

We can introduce now the *signatures* of the seladjoint operator $A \in \mathcal{L}(\mathcal{K})$ by

$$\kappa^{\pm}[A] = \kappa^{\pm}[\mathcal{K}_A], \quad \kappa^{\circ}[A] = \dim\ker A. \tag{2.31}$$

Using Corollary 2.9 it follows that these definitions are correct, i.e. they do not depend on the f.s. J used in the construction of $\mathcal{K}_A$. Also, as a consequence of Proposition 2.5 we have

$$\kappa^{\pm}[A] = \kappa^{\pm}[\mathcal{H}_A] = \dim\ker(I \mp S_{JA}). \tag{2.32}$$

3. ELEMENTARY ROTATIONS

3.1. Existence of elementary rotations. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Krein spaces and $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. An *elementary rotation* of T is a triple $(U; \mathcal{K}_1', \mathcal{K}_2')$ where $\mathcal{K}_1'$ and $\mathcal{K}_2'$ are Krein spaces, the operator $U \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{K}_2')$ is unitary and extends T, i.e.

$$P_{\mathcal{K}_2} U | \mathcal{K}_1 = T, \tag{3.1}$$

and one of the following equivalent minimality condition hold

$$\mathcal{K}_2 \vee U\mathcal{K}_1 = \mathcal{K}_2[+]\mathcal{K}_2', \quad \mathcal{K}_1 \vee U^{\#}\mathcal{K}_2 = \mathcal{K}_1[+]\mathcal{K}_1'. \tag{3.2}$$

We need now some more notation. Fix J_1 and J_2 f.s. on $\mathcal{K}_1$ and $\mathcal{K}_2$. Then we can define the *defect operators*

$$D_T = |J_1 - T^* J_2 T|^{\frac{1}{2}}, \quad D_{T^*} = |J_2 - T J_1 T^*|^{\frac{1}{2}}, \tag{3.3.}$$

and the *sign operators*

$$J_T = \operatorname{sgn}(J_1 - T^* J_2 T), \quad J_{T^*} = \operatorname{sgn}(J_2 - T J_1 T^*). \tag{3.4}$$

Using these, one defines the defect spaces $\mathcal{D}_T = \overline{\mathcal{R}(D_T)}$ and $\mathcal{D}_{T^*} = \overline{\mathcal{R}(D_{T^*})}$, considered as Krein spaces with indefinite inner products determined by the symmetries $J_T \in \mathcal{L}(\mathcal{D}_T)$ and, respectively, $J_{T^*} \in \mathcal{L}(\mathcal{D}_{T^*})$.

Notice that, with respect to the definition of the Krein space $\mathcal{H}_A$ from Section 1, we have $\mathcal{D}_T = \mathcal{H}_{I-T^{\#}T}$ and $\mathcal{D}_{T^*} = \mathcal{H}_{I-TT^{\#}}$, when J_1 and J_2 are f.s. used in the construction of the induced Krein spaces. In the following an important role will be played by the selfadjoint operator $A \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_2)$

$$A = \begin{bmatrix} I_1 & T^{\#} \\ T & I_2 \end{bmatrix}. \tag{3.5}$$

The Krein space $\mathcal{H}_A$ is constructed using the f.s. J on $\mathcal{K}_1[+]\mathcal{K}_2$

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}. \tag{3.6}$$

LEMMA 3.1. *There exists a unitary operator $\Omega_- \in \mathcal{L}(\mathcal{H}_A, \mathcal{K}_1[+]\mathcal{D}_{T^*})$, uniquely determined such that*

$$\Omega_- |JA|^{\frac{1}{2}} = \begin{bmatrix} I_1 & T^{\#} \\ 0 & \mathcal{D}_{T^*} \end{bmatrix}. \tag{3.7}$$

Proof. Consider the factorization

$$A = \begin{bmatrix} I_1 & 0 \\ T & I_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ 0 & I_2 - TT^{\#} \end{bmatrix} \begin{bmatrix} I_1 & T^{\#} \\ 0 & I_2 \end{bmatrix} \tag{3.8}$$

and notice that the operator $\begin{bmatrix} I_1 & T^{\#} \\ 0 & I_2 \end{bmatrix}$ is invertible. Then apply Lemma 2.7 and obtain that the relation (3.7) determines uniquely a unitary operator

$$\Omega_- \in \mathcal{L}(\mathcal{H}_A, \mathcal{K}_1[+]\mathcal{D}_{T^*}).$$

■

LEMMA 3.2. *There exits a unitary operator $\Omega_+ \in \mathcal{L}(\mathcal{H}_A, \mathcal{D}_T[+]\mathcal{K}_2)$, uniquely determined such that*

$$\Omega_+|JA|^{\frac{1}{2}} = \begin{bmatrix} D_T & 0 \\ T & I_2 \end{bmatrix}. \tag{3.9}$$

Proof. Consider the factorization

$$A = \begin{bmatrix} I_1 & T^{\#} \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} I_1 - T^{\#}T & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ T & I_2 \end{bmatrix} \tag{3.10}$$

and notice that the operator $\begin{bmatrix} I_1 & 0 \\ T & I_2 \end{bmatrix}$ is invertible. Then apply Lemma 2.7 and obtain the unitary operator $\Omega_+ \in \mathcal{L}(\mathcal{H}_A, \mathcal{D}_T[+]\mathcal{K}_2)$, uniquely determined by (3.9). ■

Keeping a certain analogy with the abstract scattering theory, the operators Ω_- and Ω_+ can be considered as *wave operators* associated with the selfadjoint operator A. Then it is natural to introduce the *scattering operator*

$$S(T) \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{D}_{T^*}, \mathcal{K}_2[+]\mathcal{D}_T)$$

defined by

$$S(T) = \Omega_+\Omega_-^{-1}. \tag{3.11}$$

THEOREM 3.3. *The triple $(S(T); \mathcal{D}_{T^*}, \mathcal{D}_T)$ is an elementary rotation of T.*

Proof. Let $S(T)$ be represented by the block-matrix

$$S(T) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

with respect to the decompositions $\mathcal{K}_1[+]\mathcal{D}_{T^*}$ and $\mathcal{K}_2[+]\mathcal{D}_T$. Using this, from (3.7), (3.9), and the definition of $S(T)$ (see (3.11)), we obtain

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} I_1 & T^{\#} \\ 0 & \mathcal{D}_{T^*} \end{bmatrix} = \begin{bmatrix} T & I_2 \\ D_T & 0 \end{bmatrix}, \tag{3.12}$$

and then, performing the product and identifying the corresponding entries in (3.12), we obtain $S_{11} = T$ and $S_{21} = D_T$.

Since Ω_+ and Ω_- are unitary operators, the same is $S(T)$. Also, $S(T)$ is an extension of T since

$$P_{\mathcal{K}_2} S(T)|\mathcal{K}_1 = S_{11} = T,$$

and the first minimality condition in (3.2) holds

$$\mathcal{K}_2 \vee S(T)\mathcal{K}_1 = \mathcal{K}_2 \vee S_{21}\mathcal{K}_1 = \mathcal{K}_2[+]\overline{D_T\mathcal{K}_1} = \mathcal{K}_2[+]\mathcal{D}_T,$$

while the latter minimality condition in (3.2) is a consequence of the first, using the fact that $S(T)$ is a unitary operator. ■

We can obtain now, as a first consequence of the existence of the unitary operator $S(T)$, an important relation concerning the defect signatures of T.

COROLLARY 3.4. *For any operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$, the following relations hold*

$$\kappa^{\pm}[\mathcal{K}_1] + \kappa^{\pm}[I_2 - TT^{\#}] = \kappa^{\pm}[I_1 - T^{\#}T] + \kappa^{\pm}[\mathcal{K}_2] \tag{3.13}$$

and

$$\kappa^{\circ}[I_2 - TT^{\#}] = \kappa^{\circ}[I_1 - T^{\#}T] \tag{3.14}$$

Proof. Since $S(T) \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{D}_{T^*}, \mathcal{K}_2[+]\mathcal{D}_T)$ is a unitary operator, we have

$$\kappa^{\pm}[\mathcal{K}_1] + \kappa^{\pm}[\mathcal{D}_{T^*}] = \kappa^{\pm}[\mathcal{K}_2] + \kappa^{\pm}[\mathcal{D}_T]$$

and then, using the fact that

$$\kappa^{\pm}[I_1 - T^{\#}T] = \kappa^{\pm}[\mathcal{D}_T], \quad \kappa^{\pm}[I_2 - TT^{\#}] = \kappa^{\pm}[\mathcal{D}_{T^*}],$$

(see (2.32)) we obtain (3.13). As for (3.14), this follows directly form the factorizations (3.8) and (3.10), since

$$\kappa^{\circ}[I_2 - TT^{\#}] = \kappa^{\circ}[A] = \kappa^{\circ}[I_1 - T^{\#}T]$$

■

3.2. Link operators. The elementary rotation $R(T)$. In [2], the existence of elementary rotations was obtained using the existence of the so-called link operators, which, roughly speaking, are the substitutes of the classical defect relations $TD_T = D_{T^*}T$, which are no longer true in Krein space. We show now that the existence and the properties of the link operators can be obtained using the same pattern which produced the elementary rotation $S(T)$. As a consequence, another elementary rotation denoted $R(T)$ is obtained.

We continue to consider Krein spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ with fixed f.s. J_1 and J_2, and an operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$.

PROPOSITION 3.5. *There exists uniquely determined operator $L_T \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$ such that*

$$L_T D_T = D_{T^*} J_2 T, \tag{3.15}$$

and, similarly, there exists an operator $L_{T^*} \in \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T)$, uniquely determined such that

$$L_{T^*} D_{T^*} = D_T J_1 T^*. \tag{3.16}$$

Proof. We consider the Hilbert spaces $(\mathcal{K}_i, (\cdot,\cdot)_{J_i})$ and the selfadjoint operator $H \in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{K}_2)$,

$$H = \begin{bmatrix} J_1 & T^* \\ T & J_2 \end{bmatrix}$$

which has two dual indefinite factorizations

$$H = \begin{bmatrix} I_1 & 0 \\ TJ_1 & I_2 \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 - TJ_1T^* \end{bmatrix} \begin{bmatrix} I_1 & J_1T^* \\ 0 & I_2 \end{bmatrix}, \tag{3.17}$$

and

$$H = \begin{bmatrix} I_1 & T^*J_2 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} J_1 - T^*J_2T & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ J_2T & I_2 \end{bmatrix}. \tag{3.18}$$

Using Lemma 2.7, from (3.17) we obtain a unitary operator (of Krein spaces) $\omega_- \in$ $\in \mathcal{L}(\mathcal{H}_H, \mathcal{K}_1[+]\mathcal{D}_{T^*})$, uniquely determined such that

$$\omega_-|H|^{\frac{1}{2}} = \begin{bmatrix} I_1 & J_1T^* \\ 0 & D_{T^*} \end{bmatrix}, \tag{3.19}$$

and, from (3.18) we obtain a unitary operator (of Krein spaces) $\omega_+ \in \mathcal{L}(\mathcal{H}_H, \mathcal{K}_2[+]\mathcal{D}_T)$, uniquely determined such that

$$\omega_+|H|^{\frac{1}{2}} = \begin{bmatrix} D_T & 0 \\ J_2T & I_2 \end{bmatrix}. \tag{3.20}$$

Then consider the unitary operator $U \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{D}_{T^*}, \mathcal{K}_2[+]\mathcal{D}_T)$

$$U = \omega_+\omega_-^{-1}.$$

Representing U as a block-matrix with respect to the decompositions $\mathcal{K}_1[+]\mathcal{D}_{T^*}$ and $\mathcal{K}_2[+]\mathcal{D}_T$

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

from (3.19) and (3.20) we obtain

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} I_1 & J_1T^* \\ 0 & D_{T^*} \end{bmatrix} = \begin{bmatrix} J_2T & I_2 \\ D_T & 0 \end{bmatrix},$$

and then, performing the product and identifying the corresponding entries, this imply $U_{11} = J_2T$, $U_{21} = D_T$, $U_{12} = J_2D_{T^*}J_{T^*}$, and

$$-U_{22}D_{T^*} = D_TJ_1T^*.$$

Denoting $L_{T^*} = -U_{22} \in \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{D}_T)$, this proves the existence of the operator L_{T^*} such that (3.16) holds, and, since $D_{T^*}\mathcal{K}_2 = \mathcal{D}_{T^*}$, we obtain also its uniqueness.

The statement concerning L_T follows in a similar way, using the unitary operator $U^\# = \omega_-\omega_+^{-1}$. ■

The operators L_T and L_{T^*} are called *link operators* associated to T, J_1 and J_2.

COROLLARY 3.6. *The link operators L_T and L_{T^*} have also the following properties*

$$(J_T - D_TJ_1D_T)|\mathcal{D}_T = L_T^*J_{T^*}L_T, \tag{3.21}$$

and, respectively

$$(J_{T^*} - D_{T^*}J_2D_{T^*})|\mathcal{D}_{T^*} = L_{T^*}^*J_TL_{T^*}. \tag{3.22}$$

Moreover, L_T and L_{T^*} are related by the following equality

$$L_{T^*} = J_TL_T^*J_{T^*} \tag{3.23}$$

Proof. As an outgrowth of the proof of Proposition 3.5, we have the unitary operator $U \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{D}_{T^*}, \mathcal{K}_2[+]\mathcal{D}_T)$

$$U = \begin{bmatrix} J_2T & J_2D_{T^*}J_{T^*} \\ D_T & -L_{T^*} \end{bmatrix} \tag{3.24}$$

and, since $U^\# = U^{-1} = \omega_-\omega_+^{-1}$, the following equality also holds

$$U^\# = \begin{bmatrix} J_1T^* & J_1D_TJ_T \\ D_{T^*} & -L_T \end{bmatrix}. \tag{3.25}$$

Since U is isometry we have

$$\begin{bmatrix} T^*J_2 & D_T \\ J_{T^*}D_{T^*}J_2 & -L_{T^*}^* \end{bmatrix} \begin{bmatrix} J_2 & 0 \\ 0 & J_T \end{bmatrix} \begin{bmatrix} J_2T & J_2D_{T^*}J_{T^*} \\ D_T & -L_{T^*} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_{T^*} \end{bmatrix},$$

and performing the products and identifying the lower-right entries we obtain (3.22). Similarly, by writing that $U^\#$ is isometric and using the representation in (3.25) one obtains (3.21).

From (3.24) and (3.25) we also have

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_{T^*} \end{bmatrix} \begin{bmatrix} T^*J_2 & D_T \\ J_{T^*}D_{T^*}J_2 & -L_{T^*}^* \end{bmatrix} \begin{bmatrix} J_2 & 0 \\ 0 & J_T \end{bmatrix} = \begin{bmatrix} J_1T^* & J_1D_TJ_T \\ D_{T^*} & -L_T \end{bmatrix}$$

and from here, performing the products and identifying the lower-right handed entries, we obtain the relation (3.32). ∎

REMARK 3.7. The relation (3.23) is equivalent with

$$L_{T^*} = L_T^{\#},$$

which emphasises better the duality between L_T and L_{T^*}.

REMARK 3.8. Since D_T and D_{T^*} are one-to one in the spaces $\mathcal{D}_T$ and, respectively, $\mathcal{D}_{T^*}$, we have

$$L_T \supseteq D_{T^*}^{-1}TJ_1D_T, \quad L_{T^*} \supseteq D_T^{-1}TJ_2D_{T^*}.$$

Also, it is easy to see that $D_{T^*}^{-1}TJ_1D_T$ is densely defined in $\mathcal{D}_T$ and $D_T^{-1}T^*J_2D_{T^*}$ is densely defined in $\mathcal{D}_{T^*}$, hence, the existence of the link operators means that the operators $D_{T^*}^{-1}TJ_1D_T$ and $D_T^{-1}T^*J_2D_{T^*}$ are bounded.

On the other hand, if the interwining relation $TJ_1 = J_2T$ holds, then $L_T = J_2T|\mathcal{D}_T$ and $L_{T^*} = J_1T^*|\mathcal{D}_{T^*}$. In this case, the relations (3.15) and (3.16) read simply $D_{T^*}T = TD_T$ and $D_TT^* = T^*D_{T^*}$, the classical "defect relations".

As a consequence of Proposition 3.5 and its Corollary 3.6, the operator $R(T) \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{D}_{T^*}, \mathcal{K}_2[+]\mathcal{D}_T)$ defined by

$$R(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -L_{T^*}J_{T^*} \end{bmatrix}, \tag{3.26}$$

is an elementary rotation of T. This elementary rotation is explicitly computed in terms of T and it plys a role in the dilation theory.

3.3. The spectral conditions $(\alpha)_+$ and $(\alpha)_-$. In connection with the problem of uniqueness of elementary rotation of a given operator, we introduce now a spectral property. We need first to fix some terminology.

Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$, $A = A^*$, and let $\sigma(A)$ denote its spectrum. A real number t is *isolated on the left (on the right)* with respect to $\sigma(A)$ if there exists $\varepsilon > 0$ such that $(t-\varepsilon, t)\cap\sigma(A) = \emptyset$ (respectively, $(t, t+\varepsilon)\cap\sigma(A) = \emptyset$).

Also, in the following $A = A^+ - A^-$ will always denote the Jordan decomposition of A.

Consider again $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ and fix J_1 and J_2 f.s. on $\mathcal{K}_1$ and respectively $\mathcal{K}_2$. The spectral properties $(\alpha)_+$ and $(\alpha)_-$ are introduced thus:

$(\alpha)_+$ 0 is isolated on the right with respect to $\sigma(J_1 - T^* J_2 T)$.

$(\alpha)_-$ 0 is isolated on the left with respect to $\sigma(J_1 - T^* J_2 T)$.

LEMMA 3.9. *The properties $(\alpha)_+$ and $(\alpha)_-$ do not depend on the f.s. J_1 and J_2.*

Proof. Let G_1 and G_2 be two others f.s. on $\mathcal{K}_1$ and $\mathcal{K}_2$. Then

$$G - 1J_1(J_1 - T^* J_2 T) = G_1 - T^\circ G_2 T, \tag{3.27}$$

where T° denotes the adjoint of T with respect to G_1 and G_2. Denote $X = G_1 J_1 :$ $: (\mathcal{K}_1, (\cdot,\cdot)_{J_1}) \to (\mathcal{K}_1, (\cdot,\cdot)_{G_1})$ and notice that X is the adjoint of identity operator. Then (3.27) means that $J_1 - T^* J_2 T$ is congruent with $G_1 - T^\circ G_2 T$, via an invertible operator. The rest of the proof now is a simple exercise in spectral theory of selfadjoint operators in Hilbert spaces. ∎

In view of Lemma 3.9, the properties $(\alpha)_+$ and $(\alpha)_-$ are associated only with the operator T. These properties are also selfdual. More precisely, let us consider the dual properties

$(\alpha)_+^*$ 0 is isolated on the right with respect to $\sigma(J_2 - T J_1 T^*)$.

$(\alpha)_-^*$ 0 is isolated on the left with respect to $\sigma(J_2 - T J_1 T^*)$.

LEMMA 3.10. *T has the property $(\alpha)_-^*$ (the property $(\alpha)_+^*$) if and only if it has the property $(\alpha)_-$ (respectively, the property $(\alpha)_+$).*

Proof. Consider the Hilbert spaces $(\mathcal{K}_i, (\cdot,\cdot)_{J_i})$ and the selfadjoint operator $H \in$ $\in \mathcal{L}(\mathcal{K}_1 \oplus \mathcal{K}_2)$ as in the proof of Proposition 3.5. From (3.17) it follows that T has the property $(\alpha)_-^*$ (the property $(\alpha)_+^*$) if and only if 0 is isolated on the left (on the right) with respect to $\sigma(H)$. Using now (3.18), the latter holds if and only if T has the property $(\alpha)_-$ (the property $(\alpha)_+$). ∎

In the following we will need other equivalent characterizations of the properties $(\alpha)_+$ and $(\alpha)_-$ which are consequences of spectral theory.

LEMMA 3.11. *For any operator $T \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ the following assertions are equivalent:*

(i) *T has the property $(\alpha)_-$ (the property $(\alpha)_+$).*

(ii) *$(J_1 - T^*J_2T)^-$ (respectively $(J_1 - T^*J_2T)^+$) has closed range.*

(iii) *$J_T^- D_T$ (respectively $J_T^+ D_T$) has closed range.*

(iv) *0 is isolated with respect to $\sigma(J_T^- D_T)$ (respectively $\sigma(J_T^+ D_T)$).*

3.4. Uniqueness of elementary rotations. Two elementary rotations $(U;\mathcal{K}_1',\mathcal{K}_2')$ and $(V;\mathcal{K}_1',\mathcal{K}_2')$ of the same operator $T \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ are called *unitary equivalent* if there exist unitary operators $\varphi_1 : \mathcal{K}_1' \to \mathcal{H}_1'$, $\varphi_2 : \mathcal{K}_2' \to \mathcal{H}_2'$ such that

$$\begin{bmatrix} I_2 & 0 \\ 0 & \varphi_2 \end{bmatrix} U = V \begin{bmatrix} I_1 & 0 \\ 0 & \varphi_1 \end{bmatrix}.$$

THEOREM 3.12. *An operator $T \in \mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$ has unique elementary rotation, modulo unitary equivalence, if and only if T has either property $(\alpha)_-$ or property $(\alpha)_+$.*

Proof. Let $(U;\mathcal{K}_1',\mathcal{K}_2')$ be an elementary rotation of T and assume that T has either the property $(\alpha)_-$ or the property $(\alpha)_+$. Let U be represented by the block-matrix

$$U = \begin{bmatrix} T & A \\ B & C \end{bmatrix} \tag{3.28}$$

with respect to the decomposition $\mathcal{K}_1[+]\mathcal{K}_1'$ and $\mathcal{K}_2[+]\mathcal{K}_2'$. Fix f.s. J_1, J_2, J_1' and J_2' on $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_1'$, and respectively $\mathcal{K}_2'$. We consider the elementary rotation $R(T)$ (see (3.26)) and we will prove that U is unitary equivalent with $R(T)$.

To this end, notice first that since U is isometric we have

$$J_1 - T^*J_2T = B^*J_2'B. \tag{3.29}$$

Then notice that, by the first minimality condition in (3.2), it follows that B has dense range hence, from (3.29) it follows that

$$B = \varphi_2 D_T, \tag{3.20}$$

where $\varphi_2 : \mathcal{R}(D_T)(\subseteq \mathcal{D}_T) \to \mathcal{K}_2'$ is isometric and has dense range. Since $J_T\mathcal{R}(D_T) \subseteq \subseteq \mathcal{R}(D_T)$ and according to Lemma 3.11 either $J_T^-\mathcal{R}(D)$ or $J_T^+\mathcal{R}(D)$ is closed, from Lemma 2.3 it follows that φ_2 extends to a unitary operator in $\mathcal{L}(\mathcal{D}_T,\mathcal{K}_2')$, also denoted by φ_2.

We use now Lemma 3.10 to conclude that T has either the property $(\alpha)_-^*$ or the property $(\alpha)_+^*$ and proceeding similarly as before we obtain

$$A = D_{T^*}\varphi_1, \tag{3.31}$$

where $\varphi_1 \in \mathcal{L}(\mathcal{K}_1', \mathcal{D}_{t^*})$ is unitary.

Since U is isometric we also have

$$T^* J_2 A + B^* J_2' C = 0,$$

whence, taking into account of (3.30), (3.31), and $\varphi_2^* J_2' = J_T \varphi_2$ we obtain

$$T^* J_2 D_{T^*} \varphi_1 + D_T J_T \varphi_2^* C = 0.$$

From here and Proposition 3.5 we infer

$$C = -\varphi_2 L_{T^*} J_{T^*} \varphi_1. \tag{3.32}$$

Putting together (3.30), (3.31), and (3.32) we conclude that $(U; \mathcal{K}_1', \mathcal{K}_2')$ is unitary equivalent with $(R(T); \mathcal{D}_{T^*}, \mathcal{D}_T)$.

Conversely, let us now assume that the operator T has neither the property $(\alpha)_-$ nor the property $(\alpha)_+$. First we prove that there exists an isometric operator $V : \mathcal{R}(D_T) \to \mathcal{D}_T$ with dense range, such that V is unbounded but $V D_T$ is bounded.

Indeed, using Lemma 3.11 it follows that there exists a decreasing sequence of values $\{\mu_n\}_{n\geqslant 1} \subset \sigma(J_T^+ D_T)$, $0 < \mu_n < 1$, such that $\mu_n \to 0$ $(n \to \infty)$, and also there exists a decreasing sequence of values $\{\nu_n\}_{n\geqslant 1} \subset \sigma(J_T^- D_T)$, $0 < \nu_n < 1$, such that $\nu_n \to 0$ $(n \to \infty)$. Let $\{e_n\}_{n\geqslant 1}$ and $\{f_n\}_{n\geqslant 1}$ be orthonormal systems of vectors in $\mathcal{D}_T$ such that

$$e_n \in E((\mu_{n+1}^2, \mu_n^2])\mathcal{K}_1, \quad f_n \in E([-\nu_n^2, -\nu_{n+1}^2))\mathcal{K}_1, \quad n \geqslant 1,$$

where E is the spectral measure of $J_1 - T^* J_2 T$.

We remark that there exists a sequence $\{\lambda_n\}_{n\geqslant 1} \subset \mathbb{C}$, $0 < \lambda_n < 1$, such that $\lambda_n \to 1$ $(n \to \infty)$ and

$$\sup_{n\geqslant 1} \frac{\max\{\mu_n, \nu_n\}}{\sqrt{1 - |\lambda_n|^2}} < \infty. \tag{3.33}$$

(Indeed, $|\lambda_n| = \max\{\sqrt{1-\mu_n^2}, \sqrt{1-\nu_n^2}\}$ works).

Consider now the regular subspaces of the Krein space $\mathcal{D}_T$

$$\mathcal{S}_n = (\mathbb{C}e_n \oplus \mathbb{C}f_n) \subset \mathcal{R}(D_T), \quad n \geqslant 1,$$

and the isometric operators $X_n \in \mathcal{L}(\mathcal{S}_n)$

$$X_n = \frac{1}{\sqrt{1 - |\lambda_n|^2}} \begin{bmatrix} 1 & -\lambda_n \\ \lambda_n & 1 \end{bmatrix}, \quad n \geqslant 1.$$

Using these define the increasing chain of regular subspaces of $\mathcal{D}_T$

$$\mathcal{L}_n = \mathop{[+]}_{k=1}^{n} \mathcal{S}_k \subset \mathcal{R}(\mathcal{D}_T), \quad n \geqslant 1,$$

and the sequence of isometric operators $V_n \in \mathcal{L}(\mathcal{L}_n)$,

$$V_n = \mathop{[+]}_{k=1}^{n} X_k, \quad n \geqslant 1.$$

Then consider $\mathcal{D}_0 = \bigcup_{k \geqslant 1} \mathcal{L}_k \subseteq \mathcal{R}(D_T)$ and define $V : \mathcal{D}_0 \to \mathcal{D}_T$ by

$$V|\mathcal{L}_n = V_n, \quad n \geqslant 1.$$

Since $V\mathcal{D}_0 \subseteq \mathcal{D}_0$ and $J_T\mathcal{D}_0 \subseteq \mathcal{D}_0$ hold, we have

$$\mathcal{R}(D_T) = \mathcal{D}_0 \dot{+} (\mathcal{D}(D_T) \cap \mathcal{D}_0^{\perp})$$

and let $V : \mathcal{R}(D_T)(\subseteq \mathcal{D}_T) \to \mathcal{D}_T$ be the extension such that

$$V|\mathcal{R}(D_T) \cap \mathcal{D}_0^{\perp} = I|\mathcal{R}(D_T) \cap \mathcal{D}_0^{\perp}.$$

Then V is isometric, it has dense range, it is unbounded (since $\sigma_p(V) \supseteq \bigcup_{n \geqslant 1} \sigma(X_n)$ is unbounded), and using (3.33) it is easy to see that

$$B = VD_T \in \mathcal{L}(\mathcal{K}_1, \mathcal{D}_T).$$

Further, since V is isometric it follows

$$J_1 - T^* J_2 T = B^* J_T B,$$

and then considering the operator $T_c = [T \quad B]^t \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2[+]\mathcal{D}_T)$ from here it follows that T_c is isometric , in particular $\mathcal{R}(T_c)$ is a regular subspace of $\mathcal{K}_2[+]\mathcal{D}_T$. Let $\mathcal{K}_1'$ denote the orthogonal complement of $\mathcal{R}(T_c)$

$$\mathcal{K}_2[+]\mathcal{D}_T = \mathcal{R}(T_c)[+]\mathcal{K}_1'.$$

If R denotes the inclusion $\mathcal{K}_1' \hookrightarrow \mathcal{K}_2[+]\mathcal{D}_T$ then

$$\begin{bmatrix} J_2 & 0 \\ 0 & J_T \end{bmatrix} - T_c J_1 T_c^* = R J_1' R^*, \tag{3.34}$$

where J_1' is a fixed f.s. on $\mathcal{K}_1'$. Defining $U \in \mathcal{L}(\mathcal{K}_1[+]\mathcal{K}_1', \mathcal{K}_2[+]\mathcal{D}_T)$

$$U = [T_c \quad R],$$

we claim that $(U;\mathcal{K}_1',\mathcal{D}_T)$ is an elementary rotation of T.

Indeed, U is an extension of T and it is isometric since

$$R^*\begin{bmatrix} J_2 & 0 \\ 0 & J_T \end{bmatrix} R = J_1', \quad R^*\begin{bmatrix} J_2 & 0 \\ 0 & J_T \end{bmatrix} T_c = 0,$$

hold. Using (3.34) it folows that $U^{\#}$ is also isometric. Now the analog of the minimality conditions (3.2) hold since B has dense range and U is unitary.

Finally, the elementary rotations $(U;\mathcal{K}_1',\mathcal{D}_T)$ and $(R(T);\mathcal{D}_{T^*},\mathcal{D}_T)$ are not unitary equivalent since V is not bounded. ∎

4. THE ROLE OF ELEMENTARY ROTATIONS IN DILATION THEORY

4.1. Minimal unitary dilations. Let $\mathcal{H}$ be a Krein space and $T \in \mathcal{L}(\mathcal{H})$. A *unitary dilation* of T is, by definition, a pair $(U;\mathcal{K})$, where $\mathcal{K} \supseteq \mathcal{H}$ is a Krein space extension of $\mathcal{H}$, $U \in \mathcal{L}(\mathcal{K})$ is a unitary operator, such that

(4.1)
$$P_{\mathcal{H}}^{\mathcal{K}} U^n | \mathcal{H} = T^n, \quad n \geqslant 1.$$

Notice that if (4.1) holds then

(4.2)
$$P_{\mathcal{H}}^{\mathcal{K}} U^{\#n} | \mathcal{H} = T^{\#n}, \quad n \geqslant 0.$$

A unitary dilation $(U;\mathcal{K})$ of $T \in \mathcal{L}(\mathcal{H})$ is a *minimal unitary dilation* (in brief, m.u.d.) if

(4.3)
$$\mathcal{K} = \bigvee_{n\in\mathbb{Z}} U^n\mathcal{H}.$$

Begining with the theorem of B. Sz.-Nagy [20], the existence of a minimal unitary dilation has been proved for more and more general classes of operators by C. Davis [9], P. Sorjonen [19], and T. Ya. Azizov [3] (see also [4]). In [7] we generalized the Schäffer form of a minimal unitary dilation of an arbitrary bounded operator acting in a Krein space.

In the following we fix a Krein space $\mathcal{H}$, an operator $T \in \mathcal{L}(\mathcal{H})$, and a f.s. J on $\mathcal{H}$. With respect to these, the defect operators D_T and D_{T^*}, the sign operators J_T and J_{T^*}, and the Krein spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ will be considered (see Section 3). Also, recall the definition of the Krein spaces of the type $l^2(\mathcal{H})$ (see 2.3). We begin by realizing a minimal unitary dilation as an elementary rotation.

PROPOSTION 4.1. *We consider the trivial extension of T (with 0), denoted $\tilde{T} : \mathcal{H}[+]l^2(\mathcal{D}_T) \to \mathcal{H}[+]l^2(\mathcal{D}_{T^*})$. Then, identifying naturally the Krein spaces*

$\mathcal{D}_T[+]l^2(\mathcal{D}_T)$ with $l^2(\mathcal{D}_T)$ and $\mathcal{D}_{T^*}[+]l^2(\mathcal{D}_{T^*})$ with $l^2(\mathcal{D}_{T^*})$, the pair $(R(\tilde{T});\mathcal{K})$ is a minimal unitary dilation of T, where

$$\mathcal{K} = l^2(\mathcal{D}_{T^*})[+]\mathcal{H}[+]l^2(\mathcal{D}_T). \tag{4.4}$$

Proof. With respect to the decompositions $\mathcal{H}[+]l^2(\mathcal{D}_T)$ and $\mathcal{H}[+]l^2(\mathcal{D}_{T^*})$, T has the representation

$$\tilde{T} = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \tag{4.5}$$

hence

$$D_{\tilde{T}} = \begin{bmatrix} D_T & 0 \\ 0 & I|l^2(\mathcal{D}_T) \end{bmatrix}, \quad D_{T^*} = \begin{bmatrix} D_{T^*} & 0 \\ 0 & I|l^2(\mathcal{D}_{T^*}) \end{bmatrix} \tag{4.6}$$

and then obtain $\mathcal{D}_{\tilde{T}} = \mathcal{D}_T[+]l^2(\mathcal{D}_T)$ and $\mathcal{D}_{\tilde{T}^*} = \mathcal{D}_{T^*}[+]l^2(\mathcal{D}_{T^*})$. From (4.5), (4.6), and the uniqueness of link operators it follows that $L_{\tilde{T}}$ is the trivial extension of L_{T^*} and then, identifying naturally $\mathcal{D}_{\tilde{T}}$ with $l^2(\mathcal{D}_T)$ and $\mathcal{D}_{\tilde{T}^*}$ with $l^2(\mathcal{D}_{T^*})$, the elementary rotation $R(\tilde{T}) \in \mathcal{L}(\mathcal{K})$ has the representation

$$R(\tilde{T}) = \left[\begin{array}{cccc|cccc} & & & \vdots & \vdots & & & \\ & I & 0 & 0 & 0 & 0 & 0 & \\ & 0 & I & 0 & 0 & 0 & 0 & \\ \cdots & 0 & 0 & D_{T^*} & T & 0 & 0 & \cdots \\ \hline \cdots & 0 & 0 & -L_{T^*}J_{T^*} & D_T & 0 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & I & 0 & \\ & 0 & 0 & 0 & 0 & 0 & I & \\ & & & \vdots & \vdots & 0 & 0 & \end{array}\right] \tag{4.7}$$

with respect to the decomposition (4.4). $R(\tilde{T})$ is unitary (since it is an elementary rotation) and the axioms (4.1) and (4.3) can be readily verified. ∎

We shall refer to the minimal unitary dilation constructed in Proposition 4.1. as the *canonical minimal unitary dilation* of T.

The construction used in Proposition 4.1 is actually more general, more precisely to any elementary rotation of T one can associate a minimal unitary dilation of T which is an elementary rotation of a trivial extension of T. In order to characterize the m.u.d. of this type we need some more notation.

Let $U \in \mathcal{L}(\mathcal{K})$ be a unitary operator . A subspace $\mathcal{L} \subset \mathcal{K}$ is called *wandering* for U is $\mathcal{L}$ is nondegenerate and

$$U^p\mathcal{L} \perp U^q\mathcal{L}, \quad p,q \in \mathbf{Z},\ p \neq q. \tag{4.8}$$

Since U is unitary, (4.8) is equivalent with

$$U^n\mathcal{L} \perp \mathcal{L}, \quad n \in \mathbf{Z},\ n \neq 0. \tag{4.9}$$

If $(U;\mathcal{K})$ is a m.u.d. of $T \in \mathcal{L}(\mathcal{H})$, one defines the subspaces $\mathcal{L}$ and $\mathcal{L}_\#$ in $\mathcal{K}$,

$$\mathcal{L} = \overline{(U-T)\mathcal{H}}, \quad \mathcal{L}_\# = \overline{(U^\# - T^\#)\mathcal{H}}, \tag{4.10}$$

and also the subspaces

$$M_+(\mathcal{L}) = \bigvee_{n\geqslant 0} U^n\mathcal{L}, \quad M_-(\mathcal{L}_\#) = \bigvee_{n\leqslant 0} U^n\mathcal{L}_\#. \tag{4.11}$$

LEMMA 4.2. *For any m.u.d. $(U,\mathcal{K})$ of T, the subspaces $\mathcal{L}$ and $\mathcal{L}_\#$ introduceed in (4.10) are wandering for U. In addition, the subspaces $M_+(\mathcal{L})$ and $M_-(\mathcal{L}_\#)$ are nondegenerate, mutually orthogonal and orthogonal to $\mathcal{H}$, and*

$$\mathcal{K} = M_-(\mathcal{L}_\#) \vee \mathcal{H} \vee M_+(\mathcal{L}). \tag{4.12}$$

Proof. Let $h, k \in \mathcal{H}$ be arbitrary and consider n a positive integer. Then, using (4.1) we have

$$[U^n(U-T)h,(U-T)k] = [U^{n+1}h,Uk] - [U^nTh,Uk] - [U^{n+1}h,Tk]+$$

$$+[U^nTh,Tk] = [T^nh,k] - [T^nh,k] - [T^{n+1}h,Tk] + [T^{n+1}h,Tk] = 0.$$

From here we infer that (4.9) holds.

Also, for $h, k \in \mathcal{H}$ and $n \geqslant 0$, we have

$$[U^n(U-T)h,k] = [U^{n+1}h,k] - [U^nTh,k] = [T^{n+1}h,k] - [T^{n+1}h,k] = 0,$$

hence

$$U^n\mathcal{L} \perp \mathcal{H}, \quad n \geqslant 0. \tag{4.13}$$

Similarly it can be proved that the following hold

$$U^n\mathcal{L}_\# \perp \mathcal{L}_\#, \quad n \in \mathbf{Z},\ n \neq 0 \tag{4.14}$$

and

$$U^{\#n}\mathcal{L}_\# \perp \mathcal{H}, \quad n \geqslant 0. \tag{4.15}$$

Moreover, we have

$$U^p\mathcal{L} \perp U^{\#q}\mathcal{L}_\#, \quad p,q \geqslant 0, \tag{4.16}$$

and, using the definition of the spaces $\mathcal{L}$ and $\mathcal{L}_\#$, it is easy to prove that

$$\mathcal{H} \vee U\mathcal{H} \vee \ldots \vee U^n\mathcal{H} = \mathcal{H} \vee \mathcal{L} \vee \ldots \vee U^{n-1}\mathcal{L}, \quad n \geqslant 0$$

and

$$\mathcal{H} \vee U^\#\mathcal{H} \vee \ldots \vee U^{\#n}\mathcal{H} = \mathcal{H} \vee \mathcal{L}_\# \vee \ldots \vee U^{\#(n-1)}\mathcal{L}_\#, \quad n \geqslant 0$$

From (4.17) and (4.18), it follows that the minimality condition (4.3) implyes (4.12). Then, taking into account that (4.13), (4.15) and (4.16) infer that $M_+(\mathcal{L})$, $M_-(\mathcal{L}_\#)$, and $\mathcal{H}$ are mutually ortogonal, using (4.12) we obtain that $M_+(\mathcal{L})$ and $M_-(\mathcal{L}_\#)$ are nondegenerate. At their turn, these imply that $\mathcal{L}$ and $\mathcal{L}_\#$ are nondegenerate (for this we use (4.9) and (4.14)). Finally, $\mathcal{L}$ and $\mathcal{L}_\#$ are wandering spaces of U since (4.9) and (4.14) hold. ∎

COROLLARY 4.3. *For any m.u.d.* $(U,\mathcal{K})$ *of* $T \in \mathcal{L}(\mathcal{H})$, *the following assertions are equivalent:*

(α) $\mathcal{L}$ *is a regular subspace.*

(β) $\mathcal{L}_\#$ *is a regular subspace.*

(γ) $\mathcal{H} \vee U\mathcal{H}$ *is a regular subspace.*

(δ) $\mathcal{H} \vee U^\#\mathcal{H}$ *is a regular subspace.*

(η) $P^{\mathcal{K}}_{\mathcal{H}\vee U\mathcal{H}}U|\mathcal{H} \vee U^\#\mathcal{H}$ *is an elementary rotation of* T.

Proof. Taking into account that $\mathcal{L}$, $\mathcal{L}_\#$, and $\mathcal{H}$ are mutually orthogonal and specializing (4.17) and (4.18) for $n = 1$, we get

$$\mathcal{H} \vee U\mathcal{H} = \mathcal{H}[+]\mathcal{L}$$

and

$$\mathcal{H} \vee U^\#\mathcal{H} = \mathcal{H}[+]\mathcal{L}_\#.$$

Then notice that

$$U(\mathcal{H} \vee U^\#\mathcal{H}) = \mathcal{H} \vee U\mathcal{H}.$$

The equivalence of the five assertions follows now easily. ∎

We are now in a position to characterize those m.u.d. of T which are elementary rotations of trivial extensions of T.

PROPOSTIONS 4.4. *Let* $(U;\mathcal{K})$ *be a m.u.d. of* $T \in \mathcal{L}(\mathcal{H})$. *The following assertions are equivalent:*

(i) *$M_+(\mathcal{L})$ is a regular subspace of $\mathcal{K}$.*

(ii) *$M_-(\mathcal{L}_\#)$ is a regular subspace of $\mathcal{K}$.*

(iii) *Modulo the identification of $M_+(\mathcal{L})$ with $UM_+(\mathcal{L})$ and of $M_-(\mathcal{L}_\#)$ with $U^\# M_-(\mathcal{L}_\#)$, $(U;\mathcal{K})$ is an elementary rotation of a trivial extension of T.*

Proof. (i) $\Rightarrow$ (ii). If either $M_+(\mathcal{L})$ or $M_-(\mathcal{L}_\#)$ is regular, then using Lemma 4.2 we obtain the following decomposition

$$\mathcal{K} = M_-(\mathcal{L}_\#)[+]\mathcal{H}[+]M_+(\mathcal{L}),$$

hence both of $M_+(\mathcal{L})$ and $M_-(\mathcal{L}_\#)$ are regular.

(iii) $\Rightarrow$ (i). Let $\tilde{T} : \mathcal{H}[+]\mathcal{H}_1 \to \mathcal{H}[+]\mathcal{H}_2$ be a trivial extension of T, such that U is an elementary rotation of $\tilde{T}$. Represent

$$U = \begin{bmatrix} \tilde{T} & A \\ B & C \end{bmatrix},$$

with $A : \mathcal{K}_1 \to \mathcal{H}[+]\mathcal{H}_2$ and $B : \mathcal{H}[+]\mathcal{H}_1 \to \mathcal{K}_2$, $\mathcal{R}(A^\#)$ is dense in $\mathcal{K}_1$ and $\mathcal{R}(B)$ is dense in $\mathcal{K}_2$. Since $U^\#$ is isometric we have

$$I - \tilde{T}\tilde{T}^\# = AA^\#.$$

Then representing $A = [A_1 \;\; A_2]$, $A_1 \in \mathcal{L}(\mathcal{K}_1, \mathcal{H})$ and $A_2 \in \mathcal{L}(\mathcal{K}_1, \mathcal{H}_2)$ we have

$$A_1A_2^\# = 0, \quad A_2A_2^\# = I_2, \quad A_1A_1^\# = I - TT^\#.$$

In particular, $A_2^\#$ is isometry, hence $\mathcal{R}(A_2^\#)$ is a regular subspace of $\mathcal{K}_1$ and, since $\mathcal{R}(A_2^\#) \perp \mathcal{R}(A_1^\#)$ and $\mathcal{R}(A^\#)$ is dense in $\mathcal{K}_1$ it follows that $\overline{\mathcal{R}(A_1^\#)} = \mathcal{R}(A_2^\#)^\perp$ is also a regular subspace of $\mathcal{K}_1$ and

$$\mathcal{K}_1 = \mathcal{R}(A_2^\#)[+]\overline{\mathcal{R}(A_1^\#)}. \tag{4.19}$$

On the other hand, since the m.u.d. $(U, \mathcal{K})$ is obtained from the elementary rotation by identification of two pairs of subspaces, from (4.19) it follows that with respect of the action of U on $\mathcal{K}$ we have

$$\mathcal{K} = \mathcal{K}_1'[+]\mathcal{K}_1[+]\mathcal{H}[+]\mathcal{H}_1, \tag{4.20}$$

where

$$\mathcal{H}_2 = \mathcal{K}_1[+]\mathcal{K}_1'; \tag{4.21}$$

and A is represented by the block-matrix

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

with $A_1 \in \mathcal{L}(\mathcal{K}_1, \mathcal{H})$ such that $A_1 A_1^\# = I - TT^\#$, $\mathcal{R}(A_1^\#)$ is dense in $\mathcal{K}_1'$, and $A_2 \in \mathcal{L}(\mathcal{K}_1;, \mathcal{H}_2)$ is unitary.

Similarly, with respect to the action of $U^\#$ on $\mathcal{K}$ we have

$$\mathcal{K} = \mathcal{H}_2[+]\mathcal{H}[+]\mathcal{K}_2[+]\mathcal{K}_2' \tag{4.22}$$

where

$$\mathcal{H}_1 = \mathcal{K}_2[+]\mathcal{K}_2' \tag{4.23}$$

and B is represented by the block-matrix

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

with $B_1 \in \mathcal{L}(\mathcal{H}, \mathcal{K}_2)$ such that $B_1^\# B_1 = I - T^\# T$, $\mathcal{R}(B_1)$ is dense in $\mathcal{K}_2$, and $B_2 \in \mathcal{L}(\mathcal{H}_1, \mathcal{K}_2')$ is unitary.

Further, with respect to the decompositions (4.20) and (4.22) of $\mathcal{K}$, U is represented by

$$U = \left[\begin{array}{cc|cc} A_2 & 0 & 0 & 0 \\ 0 & A_1 & T & 0 \\ \hline 0 & C & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{array}\right], \tag{4.24}$$

where A_2 and B_2 are unitary and $\begin{bmatrix} T & A_1 \\ B_1 & C \end{bmatrix}$ is an elementary rotation of T. Then, from here we obtain

$$\mathcal{L} = \overline{B_1 \mathcal{H}} = \mathcal{K}_2 \subset \mathcal{H}_1$$

and $U^n \mathcal{L} \subset \mathcal{H}_1$, $n \geqslant 1$, hence

$$M_+(\mathcal{L}) \subseteq \mathcal{H}_1. \tag{4.25}$$

Similarly, from (4.24) we also have

$$\mathcal{L}_\# = \overline{A_1^\# \mathcal{H}} = \mathcal{K}_1 \subset \mathcal{H}_2$$

and then $U^{\# n} \mathcal{L}_\# \subset \mathcal{H}_2$, $n \geqslant 1$, hence

$$M_-(\mathcal{L}_\#) \subseteq \mathcal{H}_2. \tag{4.26}$$

Using Lemma 4.2, from (4.25), (4.26), (4.20) and (4.21) it follows $M_+(\mathcal{L}) = \mathcal{H}_1$ and $M_-(\mathcal{L}_\#) = \mathcal{H}_2$, hence these are regular subspaces of $\mathcal{K}$.

(i) $\Rightarrow$ (iii). Assume that $M_+(\mathcal{L})$ is a regular suspace of $\mathcal{K}$, hence the same is $M_-(\mathcal{L}_\#)$. Using Corollary 4.3 it follows that, with respect to the decomposition of the domain

$$\mathcal{K} = U^\# M_-(\mathcal{L}_\#)[+]\mathcal{L}_\#[+]\mathcal{H}[+]M_+(\mathcal{L}),$$

and the decomposition of the range

$$\mathcal{K} = M_-(\mathcal{L}_\#)[+]\mathcal{H}[+]\mathcal{L}[+]UM_+(\mathcal{L})$$

U is represented by the block-matrix in (4.24), where A_2 and B_2 are unitary operators and $\left(\begin{bmatrix} T & A_1 \\ B_1 & C_1 \end{bmatrix}; \mathcal{L}_\#, \mathcal{L}\right)$ is an elementary rotation of T. Then, modulo the identification of $M_+(\mathcal{L})$ with $UM_+(\mathcal{L})$ and of $M_-(\mathcal{L}_\#)$ with $U^\# M_-(\mathcal{L}_\#)$, U is an elementary rotation of the trivial extension $\tilde{T} : \mathcal{H}[+]M_+(\mathcal{L}) \to \mathcal{H}[+]M_-(\mathcal{L}_\#)$ of T. ∎

As a consequence, we can characterize those operators T having the property that any m.u.d. of T is an elementary rotation of a trivial extension of T.

PROPOSITION 4.5. *If T is an operator on the Krein space $\mathcal{K}$, then the following assertions are equivalent:*

(i) *Any m.u.d. $(U;\mathcal{K})$ of T is, modulo the identification of $M_+(\mathcal{L})$ with $UM_+(\mathcal{L})$ and of $M_-(L_\#)$ with $U^\# M_-(L_\#)$, an elementary rotation of a trivial extension of T.*

(ii) $\min\{\kappa^-[I - T^\# T], \kappa^+[I - TT^\#]\} = \min \kappa^-[I - TT^\#], \kappa^+[I - T^\# T]\} = 0.$

Proof. (ii) $\Rightarrow$ (i) Let $(U;\mathcal{K})$ be a m.u.d. of T. Then for arbitrary $h, k \in \mathcal{H}$ we have

$$[(U - T)h, (U - T)k] = [(I - T^\# T)h, k]$$

and

$$[(U^\# - T^\#)h, (U^\# - T^\#)k] = [(I - TT^\#)h, k].$$

Then we obtain from here

$$\kappa^\pm[\mathcal{L}] = \kappa^\pm[I - T^\# T], \quad \kappa^\pm[\mathcal{L}_\#] = \kappa^\pm[I - TT^\#].$$

Since the hypothesis (ii) holds it follows that since the hypothesis (ii) holds, either $\mathcal{L}$ and $\mathcal{L}_\#$ are definite subspaces of the same sign, or at least one of the subspaces $\mathcal{L}$ and $\mathcal{L}_\#$ is null. Taking into account that, from Lemma 4.2, we obtain

$$M_-(\mathcal{L}_\#) \vee M_+(\mathcal{L}) = \mathcal{K} \cap \mathcal{H}^\perp, \tag{4.24}$$

in the first case $M_-(\mathcal{L}_\#) \vee M_+(\mathcal{L})$ is a uniformly definite subspace of $\mathcal{K}$, hence both of $M_-(\mathcal{L}_\#)$ and $M_+(\mathcal{L})$ are uniformly definite subspaces. In the latter case, either $M_-(\mathcal{L}_\#)$ is null or $M_+(\mathcal{L})$ is null hence, using again Lemma 4.2 we conclude that either $M_+(\mathcal{L}) = \mathcal{K} \cap \mathcal{H}^\perp$ or $M_-(\mathcal{L}_\#) = \mathcal{K} \cap \mathcal{H}^\perp$. We proved that, in any case, the subspaces $M_+(\mathcal{L})$ and $M_-(\mathcal{L}_\#)$ are regular. Applying now Proposition 4.4 it follows that (i) holds.

(i) $\Rightarrow$ (ii). Assume that (ii) does not hold and we will produce a m.u.d. $(U, \mathcal{K})$ of T such that the subspaces $M_+(\mathcal{L})$ and $M_-(\mathcal{L}_\#)$ are not regular.

To this end, let us first note that, as a consequence of (2.32) we have

$$\kappa^\pm[I - T^\# T] = \kappa^\pm[\mathcal{D}_T], \quad \kappa^\pm[I - TT^\#] = \kappa^\pm[\mathcal{D}_{T^*}]$$

Since (ii) does not hold, from here we obtain that there exist two vectors $e \in \mathcal{D}_T$ and $f \in \mathcal{D}_{T^*}$ such that e anf f are definite of opposite sign. Taking into account the definition of the Krein spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, without restricting the generality we can assume that $e \in \mathcal{R}(D_T)$, $f \in \mathcal{R}(D_{T^*})$, e is positive in $\mathcal{D}_{T^*}$, f is negative in $\mathcal{D}_{T^*}$, and $(e, e)_{J_T} = (f, f)_{J_{T^*}} = 1$ (recall that J_T and J_{T^*} are f.s. on $\mathcal{D}_T$ and, respectively, $\mathcal{D}_{T^*}$).

We consider now the Krein space

$$\mathcal{K} = l^2(\mathcal{D}_{T^*})[+]\mathcal{H}[+]l^2(\mathcal{D}_T) \tag{4.25}$$

and denote

$$l^2_1(\mathcal{D}_T) = Sl^2(\mathcal{D}_T) \subset l^2(\mathcal{D}_T), \quad l^2_1(\mathcal{D}_{T^*}) = Sl^2(\mathcal{D}_{T^*}) \subset l^2(\mathcal{D}_{T^*}), \tag{4.26}$$

where S is the right shift operator (see Section 2.3). We will define now a bounded unitary operator

$$V : l^2_1(\mathcal{D}_{T^*})[+]l^2(\mathcal{D}_T) \to l^2(\mathcal{D}_{T^*})[+]l^2_1(\mathcal{D}_T). \tag{4.27}$$

For any integer number $h \geqslant 1$ we define the vectors

$$e_k = 0[+]\cdots[+]0[+]\underset{k^{\text{th}}\text{ position}}{e}[+]0[+]\cdots \in l^2(\mathcal{D}_T),$$

$$f_k = 0[+]\cdots[+]0[+]\underset{k^{\text{th}}\text{ position}}{f}[+]0[+]\cdots \in l^2(\mathcal{D}_{T^*}),$$

and using these define vectors $x_k, y_k \in l^2(\mathcal{D}_{T^*})[+]l^2(\mathcal{D}_T)$ by

$$x_k = \frac{k+1}{\sqrt{2k^2+2k+1}}\left(\frac{k}{k+1} f_k[+]e_k\right), \quad k \geqslant 2, \tag{4.28}$$

$$y_k = \frac{k+1}{\sqrt{2k^2+2k+1}}\left(f_k[+]\frac{k}{k+1} e_k\right), \quad k \geqslant 2, \tag{4.28}$$

and $x_1 = 0[+]e_1$, $y_1 = f_1[+]0$.

Consider the subspaces

$$l_1^2(\mathbb{C}f)[+]l^2(\mathbb{C}e) \subseteq l_1^2(\mathcal{D}_{T^*})[+]l^2(\mathcal{D}_T)$$

and

$$l^2(\mathbb{C}f)[+]l_1^2(\mathbb{C}e) \subseteq l^2(\mathcal{D}_{T^*})[+]l_1^2(\mathcal{D}_T).$$

Then define

$$\begin{cases} Vy_k = y_{k-1}, \ k \geqslant 2, \\ Vx_k = x_{k+1}, \ k \geqslant 1, \end{cases} \tag{4.30}$$

and extend V by linearity. V is isometry, with domain dense in $l_1^2(\mathbb{C}f)[+]l^2(\mathbb{C}e)$ and range dense in $l^2(\mathbb{C}f)[+]l_1^2(\mathbb{C}e)$. We prove that V is bounded.

Indeed, from (4.28) and (4.29) we have

$$0[+]e_k = \frac{(2k+1)\sqrt{2k^2+2k+1}}{k^2(k+1)}\left(x_k - \frac{k}{k+1}y_k\right), \quad k \geqslant 2$$

$$f_k[+]0 = \frac{(2k+1)\sqrt{2k^2+2k+1}}{k^2(k+1)}\left(-\frac{k}{k+1}x_k + y_k\right), \quad k \geqslant 2$$

and then, using (4.30) we obtain for any $k \geqslant 2$

$$V(0[+]e_k) = \frac{(2k+1)\sqrt{2k^2+2k+1}}{k^2(k+1)}\frac{k+2}{\sqrt{2k^2+6k+3}}\left(\frac{k+1}{k+2}f_{k+1}[+]e_{k+1}\right) - \\ -\frac{k^2}{(k+1)\sqrt{2k^2-2k+3}}\left(f_{k-1}[+]\frac{k-1}{k}e_{k-1}\right)$$

and

$$V(f_k[+]0) = \frac{(2k+1)\sqrt{2k^2+2k+1}}{k^2(k+1)}\frac{k(k+2)}{\sqrt{2k^2+6k+3}}\left(\frac{k+1}{k+2}f_{k+1}[+]e_{k+1}\right) + \\ +\frac{k}{\sqrt{2k^2-2k+3}}\left(f_{k-1}[+]\frac{k-1}{k}e_{k-1}\right).$$

From here it folows that, with respect to the Hilbert space orthonormal basis $\{f_k \oplus \oplus 0, 0 \oplus e_k\}_{k\geqslant 1}$, V has a tridiagonal matrix representation such that each diagonal is uniformly bounded, hence V is bounded and extends uniquely to a unitary operator

$$V : l_1^2(\mathbb{C}f)[+]l^2(\mathbb{C}e) \to l^2(\mathbb{C}f)[+]l_1^2(\mathbb{C}e). \tag{4.31}$$

We remark now that $\mathbb{C}e$ is a regular subspace of $\mathcal{D}_T$ and $\mathbb{C}f$ is a regular subspace of $\mathcal{D}_{T^*}$ so let $\mathcal{F} \subset \mathcal{D}_T$ and $\mathcal{F}_* \subset \mathcal{D}_{T^*}$ be their orthogonal complements, $\mathbb{C}e[+]\mathcal{F} = \mathcal{D}_T$ and $\mathbb{C}f[+]\mathcal{F}_* = \mathcal{D}_{T^*}$. Then we have the natural identifications

$$l^2(\mathcal{D}_T) = l^2(\mathbb{C}e)[+]l^2(\mathcal{F}), \quad l^2(\mathcal{D}_{T^*}) = l^2(\mathbb{C}f)[+]l^2(\mathcal{F}_*).$$

With respect to these, we extend V to a unitary operator as indicated in (4.27) by letting

$$V(y[+]x) = S^{\#}y[+]Sx, \quad x \in l^2(\mathcal{F}),\ y \in l_1^2(\mathcal{F}_*). \tag{4.32}$$

We extend now V to a unitary operator in $\mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ is given in (4.25). To this end, note first that as a consequence of (4.26) we have the decompositions

$$l^2(\mathcal{D}_T) = \mathcal{D}_T[+]l_1^2(\mathcal{D}_T), \quad l^2(\mathcal{D}_{T^*}) = \mathcal{D}_{T^*}[+]l_1^2(\mathcal{D}_{T^*}),$$

and then extend V to the whole $\mathcal{K}$ be letting

$$V|\mathcal{D}_{T^*}[+]\mathcal{H} = R(T). \tag{4.33}$$

From the construction of the unitary operator V it follows easily that $(V,\mathcal{K})$ is a m.u.d. of T. It remains to prove that $M_+(\mathcal{L})$ is a not regular subspace of $\mathcal{K}$. To this end, note first that, using (4.30), (4.32), and (4.33) we obtain

$$M_+(\mathcal{L}) = l^2(\mathcal{F})[+]\bigvee_{k\geqslant 1} \mathbb{C}x_k.$$

Now $M_+(\mathcal{L})$ is not regular since $(x_k, x_k) = 1$ but

$$[x_k, x_k] = \frac{2k+1}{(2k^2+2k+1)} \to 0 \ (k \to \infty).$$

Finally, using Proposition 4.4 it follows that (i) does not hold. ■

As a consequence of the preceding result it is possible to investigate the uniqueness of m.u.d. of a given operator T. In order to do this we need first to recall some definitions.

Two m.u.d $(U_1, \mathcal{K}_1)$ and $(U_2, \mathcal{K}_2)$ of the operator $T \in \mathcal{L}(\mathcal{H})$ are *unitary equivalent* if there exists a unitary operator $W \in \mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ such that W acts as the identity operator on $\mathcal{H}$ and $WU_1 = U_2W$.

COROLLARY 4.6. *The operator $T \in \mathcal{L}(\mathcal{H})$ has unique m.u.d. up to unitary equivalence if and only if T is either doubly contractive or doubly expansive.*

Proof. If the operator T is either doubly contractive or doubly expansive, any m.u.d. of T is unitary equivalent with the canonical m.u.d. of T (see Proposition 4.1) e.g. as indicated in [8, Theorem 1.4].

Conversely, if T is neither doubly contractive nor doubly expansive then one of the following statements holds:

(a) Either $\mathcal{D}_T$ or $\mathcal{D}_{T^*}$ are indefinite.

(b) T satisfies the condition (ii) in Proposition 4.5.

In the first case, we can follow the pattern used by C. Davis [9] to construct a m.u.d. of T which is not unitary equivalent with the canonical m.u.d. of T. In the latter case we use Proposition 4.5 to produce a m.u.d. $(U, \mathcal{K})$ of T such that $M_+(\mathcal{L})$ is not a regular subspace of $\mathcal{K}$. Then this is not unitary equivalent with the canonical m.u.d. of T. ∎

4.2. Characteristic functions. Let $(U, \mathcal{K})$ be a m.u.d. of the operator $T \in \mathcal{L}(\mathcal{H})$, such that the subspaces $\mathcal{L}$ and $\mathcal{L}_{\#}$, introduced in (4.10), are regular. Then, for arbitrary integer $n \geqslant 1$, we can define the regular subspaces

$$M_n(\mathcal{L}) = \bigvee_{k=-n+1}^{n+1} U^{\#k}\mathcal{L}, \quad M^n(\mathcal{L}_{\#}) = \bigvee_{k=-n}^{n} U^k\mathcal{L}_{\#} \tag{4.34}$$

Denoting by $P^{\mathcal{K}}_{M^n(\mathcal{L}_{\#})}$ the selfadjoint projection onto the regular subspace $M^n(\mathcal{L}_{\#})$, we can introduce the operator $Q_n \in \mathcal{L}(M_n(\mathcal{L}), M^n(\mathcal{L}_{\#}))$

$$Q_n = P^{\mathcal{K}}_{M^n(\mathcal{L}_{\#})}|M_n(\mathcal{L}). \tag{4.35}$$

Considering now the unitary operators $\Omega^{(n)} : \mathcal{L}_{\#}^{2n+1} \to M^n(\mathcal{L}_{\#})$,

$$\Omega^{(n)}(f_k)_{k=1}^{2n+1} = \sum_{k=1}^{2n+1} U^{k-n-1}f_k, \tag{4.36}$$

and $\Omega_{(n)} : \mathcal{L}^{2n+1} \to M_n(\mathcal{L})$,

$$\Omega_{(n)}(f_k)_{k=1}^{2n+1} = \sum_{k=1}^{2n+1} U^{k-n}f_k, \tag{4.37}$$

(recall that $\mathcal{H}^n$ denotes the Krein space direct sum of n copies of the Krein spaces $\mathcal{H}$), we can introduce the operator $\Theta : \mathcal{L}^{2n+1} \to \mathcal{L}_{\#}^{2n+1}$,

$$\Theta_n = \Omega^{(n)\#}Q_n\Omega_{(n)}. \tag{4.38}$$

On the other hand, since $\mathcal{L}$ and $\mathcal{L}_{\#}$ are regular subspaces, according to Corollary 4.3, $(\hat{U}; \mathcal{L}_{\#}, \mathcal{L})$ is elementary rotation of T, where $\hat{U} = P_{\mathcal{H}[+]\mathcal{L}}U|\mathcal{H}[+]\mathcal{L}_{\#}$. Consider the representation

$$U = \begin{bmatrix} T & A \\ B & C \end{bmatrix}. \tag{4.39}$$

PROPOSITION 4.7. *With respect to the canonical decompositions of the Krein spaces $\mathcal{L}^{2n+1}$ and $\mathcal{L}_{\#}^{2n+1}$, Θ_n has a lower triangular Toeplitz block-matrix representation, $\Theta_n = ((\Theta_n)_{ij})_{i,j=1}^{2n+1}$, where*

$$(\Theta_n)_{ij} = \begin{cases} 0, & i-j<0, \\ C^{\#}, & i=j, \\ A^{\#}T^{\#(i-j)}B^{\#}, & i-j>0. \end{cases}$$

Proof. For arbitrary k and p, $-n \leqslant k \leqslant n$ and $-n+1 \leqslant p \leqslant n+1$, as a consequence of the definition of the operator Θ, we have

$$[\Theta_n \Omega_{(n)}^{\#} U^k g, \Omega^{(n)\#} U^p f] = [U^{k-p} g, f], \quad f \in \mathcal{L}_{\#},\ g \in \mathcal{L}.$$

From here, using the definition of A, B and C we obtain the formulae (4.40). ■

From the preceding result it follows that the "symbol" of Θ_n is the $2n+1$-th polynomial approximant of the Taylor expansion, in a neighbourhood of the origin, of the operator valued function

$$\Theta(\lambda) = -C^{\#} + \lambda A^{\#}(I - \lambda T^{\#})^{-1} B^{\#}, \quad g \in \Lambda_T, \tag{4.41}$$

where $\Lambda = \{\lambda \in \mathbb{C} | I - \lambda T^{\#} \text{ is invertible}\}$. This operator function plays the role of a *characteristic function* associated with the m.u.d. $(U;\mathcal{K})$ of T.

Using the fact that U in (4.39) is unitary, it can be proved that

$$I - \Theta(\lambda)^{\#}\Theta(\mu) = (1-\overline{\lambda}\mu) B (I - \overline{\lambda} T)^{-1} (I - \mu T)^{\#} B^{\#}, \quad \lambda, \mu \in \Lambda_T,$$

which is a property usually fulfilled by this kind of functions.

In the ramaining part of this section we will show that the approach used in constructing the elementary rotation $(S(T), \mathcal{D}_{T^*}, \mathcal{D}_T)$ as in Theorem 3.3 leads also to the definition of a characteristic function.

We consider the selfadjoint operator $A_{n+1} \in \mathcal{L}(\mathcal{H}^{n+1})$ defined by the block-matrix

$$A_{n+1} = \begin{bmatrix} I & T^{\#} & T^{\#2} & \cdots & T^{\#n} \\ T & I & T^{\#} & \cdots & T^{\#n-1} \\ T^2 & T & I & & T^{\#n-2} \\ \vdots & & & & \\ T^n & T^{n-1} & T^{n-2} & \cdots & I \end{bmatrix}. \tag{4.42}$$

If J denoted a fixed f.s. of $\mathcal{H}$ then let J_{n+1} be the direct sum of $n+1$ copies of J. J_{n+1} is a f.s. of $\mathcal{H}^{n+1}$.

LEMMA 4.8. *There exist unitary operators* $\Omega_-^{(n+1)} : \mathcal{H}_{A_{n+1}} \to \mathcal{H}[+]\mathcal{D}^n_{T^*}$ *and* $\Omega_+^{(n+1)} : \mathcal{H}_A \to \mathcal{D}^n_T[+]\mathcal{H}$, *uniquely determined such that*

$$\Omega_-^{(n+1)}|J_{n+1}A_{n+1}|^{\frac{1}{2}} = \begin{bmatrix} I & T^\# & \cdots & T^{\#n} \\ 0 & D_{T^*} & \cdots & D_{T^*}T^{\#n-1} \\ & 0 & \ddots & \vdots \\ & & & D_{T^*} \end{bmatrix}. \tag{4.43}$$

and

$$\Omega_+^{(n+1)}|J_{n+1}A_{n+1}|^{\frac{1}{2}} = \begin{bmatrix} D_T & 0 & & & \\ D_T T & D_T & & 0 & \\ \vdots & & \ddots & & \\ D_T T^{n-1} & D_T T^{n-2} & \cdots & D_T & 0 \\ T^n & T^{n-1} & \cdots & & I \end{bmatrix}. \tag{4.44}$$

Proof. We consider the factorizations of A_{n+1}

$$A_{n+1} = \begin{bmatrix} I & 0 & & \\ T & I & 0 & \\ T^2 & T & & \\ \vdots & \vdots & \ddots & \\ T^n & T^{n-1} & & I \end{bmatrix} \begin{bmatrix} I & & & \\ & I-TT^\# & 0 & \\ & 0 & \ddots & \\ & & & I-TT^\# \end{bmatrix} \begin{bmatrix} I & T^\# & \cdots & T^{\#n} \\ 0 & I & & \\ & 0 & \ddots & \\ & & & I \end{bmatrix}$$

and

$$A_{n+1} = \begin{bmatrix} I & T^\# & \cdots & T^{\#n} \\ 0 & I & & \\ & 0 & \ddots & \\ & & & I \end{bmatrix} \begin{bmatrix} I-T^\#T & & 0 & \\ & \ddots & & \\ 0 & & I-T^\#T & \\ & & & I \end{bmatrix} \begin{bmatrix} I & & & \\ T & I & 0 & \\ T^2 & T & & \\ \vdots & \vdots & \ddots & \\ T^n & T^{n-1} & & I \end{bmatrix}$$

Applying Lemma 2.7 to these factorizations we obtain the unitary operators $\Omega_-^{(n+1)}$ and $\Omega_+^{(n+1)}$ as in (4.43) and (4.44). ■

Using the preceding result we can introduce a unitary operator $S_{n+1}(T)$: $: \mathcal{H}[+]\mathcal{D}^n_{T^*} \to \mathcal{D}^n_T[+]\mathcal{H}$ by

$$S_{n+1}(T) = \Omega_+^{(n+1)}\Omega_-^{(n+1)\#}. \tag{4.45}$$

Note that with respect to the definition (3.11) we have $S(T) = S_1(T)$ and consider its representation

$$S(T) = \begin{bmatrix} T & A \\ B & C \end{bmatrix}. \tag{4.46}$$

PROPOSITION 4.9. *With respect to the natural decompositions of $\mathcal{H}[+]\mathcal{D}^n_{T^*}$ and $\mathcal{D}^n_T[+]\mathcal{H}$, $S_{n+1}(T)$ has the following block-matrix representation*

$$S_{n+1}(T) = \left[\begin{array}{c|cccccc} B & C & 0 & \cdots & & 0 & 0 \\ BT & BA & & & & & \\ \vdots & \vdots & & & & & \\ BT^{n-1} & BT^{n-2}A & BT^{n-3}A & \cdots & & BA & C \\ \hline T^n & T^{n-1}A & \cdots & & T^2A & TA & A \end{array}\right] \tag{4.47}$$

Proof. The unitary operator $S_{n+1}(T)$ is uniquely determined with the property

$$S_{n+1}(T)\begin{bmatrix} I & T^\# & \cdots & T^{\#n} \\ 0 & D_{T^*} & \cdots & D_{T^*}T^{\#n-1} \\ & 0 & \ddots & \vdots \\ & & & D_{T^*} \end{bmatrix} = \begin{bmatrix} D_T & 0 & & & \\ D_TT & D_T & & 0 & \\ \vdots & & \ddots & & \\ D_TT^{n-1} & D_TT^{n-2} & \cdots & D_T & 0 \\ T^n & T^{n-1} & \cdots & & I \end{bmatrix}$$

By direct computation it is easy to verify that $S_{n+1}(T)$ has the block-matrix representation as indicated in (4.47). ■

The $n \times n$ lower triangular Toeplitz block-matrix in (4.47) leads now to the definition of an operator valued function of the type considered in (4.41).

REMARK 4.10. If one uses the canonical m.u.d. of T (see Proposition 4.1) then the analog of the definition in (4.41) is the operator valued function Θ_T

$$\Theta_T(\lambda) = -L_TJ_T + \lambda D_{T^*}(I - \lambda T^\#)^{-1}D^\#_T, \quad \lambda \in \Lambda_T, \tag{4.48}$$

(where $D_T \in \mathcal{L}(\mathcal{H}, \mathcal{D}_T)$ and $D_{T^*} \in \mathcal{L}(\mathcal{D}_{T^*}, \mathcal{H})$) which was introduced in [7] as the characteristic function of T. One can obtain this function by generalizing the pattern used in Proposition 3.5 as suggested by Lemma 4.8 and Proposition 4.9.

On the other hand, since elementary rotations are in general not unique, the results from this section show that the same geometric properties (e.g. minimal unitary dilations or scattering theoretical interpretation) lead to possibly non-unitarily equivalent characteristic functions.

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