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ON THE SPECTRAL THEORY OF OPERATORS ASSOCIATED WITH PERTURBED KLEIN-GORDON AND WAVE TYPE EQUATIONS

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INTRODUCTION

The equation

$$\begin{aligned}\Big\{-\Big(-\mathrm{i}\frac{\partial}{\partial t}-\overline{\mathcal{V}(x)}\Big)\Big(-\mathrm{i}\frac{\partial}{\partial t}-\mathcal{V}(x)\Big)+\sum_{j=1}^{n}\Big(-\mathrm{i}\frac{\partial}{\partial x_j}-\overline{\mathcal{U}_j(x)}\Big)\Big(-\mathrm{i}\frac{\partial}{\partial x_j}-\mathcal{U}_j(x)\Big)+\\ +\mathcal{U}_0(x)+m^2\Big\}u(x,t)=0,\end{aligned} \tag{0.1}$$

where $m \geqslant 0$, $\mathcal{V},\mathcal{U}_j \in L_\infty(\mathbb{R}^n)$, $j=0,1,\ldots,n$, $\mathcal{U}_0=\overline{\mathcal{U}_0}$, $u\in C^2(\mathbb{R}^n\times\mathbb{R})$, which is a perturbed Klein-Gordon ($m>0$) or wave ($m=0$) equation, can be written as the following first order equation in t:

$$\begin{aligned}&\frac{\partial}{\partial t}\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix}=\\ &=\mathrm{i}\begin{pmatrix}\mathcal{V}(x) & 1\\ \sum_{j=1}^{n}\Big(-\mathrm{i}\frac{\partial}{\partial x_j}-\overline{\mathcal{U}_j(x)}\Big)\Big(-\mathrm{i}\frac{\partial}{\partial x_j}-\mathcal{U}_j(x)\Big)+\mathcal{U}_0(x)+m^2 & \overline{\mathcal{V}(x)}\end{pmatrix}\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix}.\end{aligned} \tag{0.2}$$

Here the derivatives are understood in the distribution sense.

We define

$$\left[\begin{pmatrix}u_1\\ v_1\end{pmatrix},\begin{pmatrix}u_2\\ v_2\end{pmatrix}\right]:=(v_1,u_2)_{L_2(\mathbb{R}^n)}+(u_1,v_2)_{L_2(\mathbb{R}^n)},\qquad u_1,u_2,v_1,v_2\in L_2(\mathbb{R}^n). \tag{0.3}$$

For a 2-vector function $\begin{pmatrix} u \\ v \end{pmatrix}$ satisfying the Klein-Gordon equation (0.2) ($m > 0$) the quadratic form $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right]$ represents the charge of the particle described by (0.2). The space $\mathcal{H} := L_2(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)$ provided with the indefinite scalar product $[\cdot,\cdot]$ is a Krein space. It is easy to see that the operator in the Krein space $\mathcal{H}$ corresponding to the matrix in (0.2) in the unperturbed case (i.e. $\mathcal{V} = 0$, $\mathcal{U}_j = 0$, $j = 0, 1, \ldots, n$),

$$\begin{pmatrix} 0 & 1 \\ -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + m^2 & 0 \end{pmatrix},$$

is selfadjoint and nonnegative.

For nonnegative selfadjoint operators A in a Krein space there was studied in [6] a certain class of perturbations of A containing the relatively compact perturbations.

In the present note we apply the perturbation results of [6] to an abstract class of operators which contains the operators in $\mathcal{H}$ corresponding to matrices as in (0.2) (under appropriate assumptions on the potentials $\mathcal{V}, \mathcal{U}_j$, $j = 0, 1, \ldots, n$). Our objective is to study the spectra and the spectral functions of these operators.

This approach to the Klein-Gordon equation employing the charge form $[\cdot,\cdot]$ as the basic inner product was used in a paper of K. Veselić ([20]) and in the unpublished paper [12] of H. Langer and B. Najman. The latter work was the stimulus for the present note. In Section 1 we recall some definitions and results of the spectral theory of selfadjoint operators in Krein spaces connected with the notion of definitizability. Section 2 contains some preliminaries on the unperturbed operator which, essentially, are taken from [12]. In Sections 3 and 4 the perturbed operator is considered. Theorem 3.1 and Corollary 3.2, which give simple sufficient conditions for the (local) definitizability of the perturbed operator, and Proposition 3.3, which is devoted to the localization of the nonreal spectrum and the spectral singularities (in the sense of [14]), are slightly strengthened versions of results from [12]. For the case $m = 0$, definitizability properties and spectral singularities, apparently, have not been studied before. For this case, in Theorem 3.4 and Corollary 3.5 we give conditions for the perturbed operator to be definitizable over some neighbourhood of ∞. In Section 4 we consider the case $m = 0$ under more restrictive assumptions, which imply definitizability over $\overline{\mathbb{R}} \setminus \{0\}$ ($\overline{\mathbb{R}}$: closure of $\mathbb{R}$ in the complex sphere) and, in particular, that the only possible accumulation point of nonreal eigenvalues and spectral singularities is the point 0. In Section 5 results of Section 3 are applied to a Cauchy problem.

In most of the papers dealing with the spectral theory of the Klein-Gordon equation ($m > 0$) instead of the charge from the energy form is used as the basic inner product (e.g. [13], [18], [8], [15], [16], [17], [4]). Since for compact perturbations the energy form has no more than a finite number of negative squares, this leads to problems of the spectral theory in Pontrjagin spaces. It was pointed out in [12] that the charge form and the energy form approaches are closely connected.

I am indebted to Prof. H. Langer and Prof. B. Najman for providing me with a copy of [12].

1. NOTATION AND PRELIMINARIES

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a selfadjoint operator in $\mathcal{H}$. The operator A is called *nonnegative* if the resolvent set $\rho(A)$ is non-empty and $[Ax, x] \geqslant 0$ for every $x \in \mathcal{D}(A)$. The operator A is called *definitizable* if $\rho(A) \neq \emptyset$ and there exists a polynomial p such that $[p(A)x, x] \geqslant 0$ for every $x \in \mathcal{D}(p(A))$.

Let $\sigma_0(A)$ denote the nonreal spectrum $\sigma(A) \setminus \mathbb{R}$ of the selfadjoint operator A in $\mathcal{H}$. Assume that no more than a finite number of accumulation points of $\sigma_0(A)$ are nonreal. An open subset Δ of $\overline{\mathbb{R}}$ is said to be of *positive type (negative type)* with respect to A if the following conditions (i), (ii), (iii) are fulfilled:

(i) No point of Δ is an accumulation point of $\sigma_0(A)$.

(ii) For every closed subset δ of Δ there exist $m \geqslant 1$ and $M > 0$ such that

$$\|R(z; A)\| \leqslant M(1 + |z|)^{2m-2} |\operatorname{Im} z|^{-m}$$

for all z in a neighbourhood of δ (in $\overline{\mathbb{C}}$) with $z \neq \infty$ and $\operatorname{Im} z \neq 0$.

(iii) For every nonnegative (resp. nonpositive) $f \in C^\infty(\overline{\mathbb{R}})$ with $\operatorname{supp} f \subset \Delta$ the operator $f(A)$ (defined, in view of (ii), by extension of the Riesz-Dunford-Taylor functional calculus, see [5, Proposition 1.3]) is nonnegative.

In this definition (iii) can be replaced by a condition on the resolvent of A. We give this condition under the additional assumption that $\infty \notin \Delta$: If (i) and (ii) holds and $\Delta \subset \mathbb{R}$, then (iii) is equivalent to the following ([5, Remark 2.5]*)):

(iii)′ For every $x \in \mathcal{H}$ we have

$$-\mathrm{i} \lim_{\varepsilon \downarrow 0} [\{R(t + \mathrm{i}\varepsilon; A) - R(t - \mathrm{i}\varepsilon; A)\}x, x] \geqslant 0 \quad (\text{resp. } \leqslant 0)$$

for almost every $t \in \Delta$, and for every $x \in \mathcal{H}$, every compact subinterval $\delta \subset \Delta$, and sufficiently small $\varepsilon_0 < 0$ there exists an $M > 0$ such that

$$-\mathrm{i}\,[\{R(t + \mathrm{i}\varepsilon; A) - R(t - \mathrm{i}\varepsilon; A)\}x, x] \geqslant -M \quad (\text{resp. } \leqslant M)$$

*) In the two inequalities of [5, Remark 2.5] the factors $-\mathrm{i}$ are missing.

for every $t \in \delta$ and $\varepsilon \in (0, \varepsilon_0]$.

We say that an open set $\Delta \subset \mathbf{R}$ is of *definite type* if it is of positive or of negative type.

The operator A is called *definitizable over* the open set $\Delta \subset \overline{\mathbf{R}}$ if the above-mentioned conditions (i) and (ii) are fulfilled and for every $t \in \Delta$ all sufficiently small one-sided open neighbourhoods of t are of definite type. A definitizable operator is definitizable over $\overline{\mathbf{R}}$ ([5, Section 2.1]).

By $\Delta(A)$ we denote the union of all open subsets Δ of $\overline{\mathbf{R}}$ such that A is definitizable over Δ. Let Δ_1 be an open subset of $\Delta(A)$. We denote by $\sigma_+(A; \Delta_1)$ ($\sigma_-(A; \Delta_1)$) the set of all points $t \in \sigma(A) \cap \Delta_1$ such that there is no open subset $\Delta \ni t$ of $\Delta(A)$ of negative (resp. positive) type. We remark that, for a definitizable operator A, $\sigma_+(A; \overline{\mathbf{R}})$ and $\sigma_-(A; \overline{\mathbf{R}})$ do not coincide with the sets of spectral points of positive and negative type, respectively, introduced in [12]. The elements of the set

$$c(A) := \sigma_+(A; \Delta(A)) \cap \sigma_-(A; \Delta(A))$$

are called *critical points*. A point $t \in \sigma(A) \cap \Delta(A)$ belongs to $c(A)$ if and only if there is no open neighbourhood of t of definite type.

If $\Delta(A) \neq \emptyset$, the operator A possesses a spectral function $E(\cdot; A)$. For the definition of the spectral function and a construction of it by extension of the functional calculus of A (cf. the above-mentioned condition (iii)) we refer to [5; Section 2.2]. Here we mention only some properties of the spectral function and give some definitions connected with it which will be used in the following. For every connected subset δ of $\Delta(A)$ whose endpoints belong to $\Delta(A) \setminus c(A)$, $E(\delta; A)$ is defined and is a selfadjoint projection in $\mathcal{H}$. An open subset $\Delta_0 \subset \Delta(A)$ is of positive type (negative type) with respect to A if and only if for every closed connected subset δ of Δ_0, $E(\delta; A)$ is defined and nonnegative (resp. nonpositive).

A critical point t is called *regular* if there exists a neighbourhood $\mathfrak{V}_t$ (in $\overline{\mathbf{R}}$) of t such that $\sup \|E(\delta; A)\| < \infty$, where the supremum is taken over all intervals $\delta \subset \mathfrak{V}_t$ such that $E(\delta; A)$ is defined. A non-regular critical point of A is called *singular*. $c_s(A)$ denotes the set of singular critical points of A. As a consequence of these definitions the set of spectral singularities of A (in the sense of [14]) is contained in $c_s(A) \cup (\sigma(A) \setminus \Delta(A))$.

An open set $\Delta_0 \subset \Delta(A)$ is said to be of type π_+ (type π_-) with respect to A if for every compact connected subset δ of Δ_0 such that $E(\delta; A)$ is defined, $(E(\delta; A)\mathcal{H}, [\cdot, \cdot])$ is a Pontrjagin space with a finite rank of negativity (resp. $(E(\delta; A)\mathcal{H}, [\cdot, \cdot])$ is a Pontrjagin space with finite rank of positivity).

Let Δ_0 be a connected open subset of $\overline{\mathbf{R}}$ of type π_+ (type π_-). Then, making use of the spectral function, we easily see that $\sigma_-(A; \Delta_0)$ (resp. $\sigma_+(A; \Delta_0)$) has no accumulation points in Δ_0. From the well-known Pontrjagin invariant subspace theorem it follows that every point $t \in \sigma_-(A;\Delta_0)$ (resp. $t \in \sigma_+(A;\Delta_0)$) is an eigenvalue of A and there exists an eigenvector x of A corresponding to t with $[x,x] \leqslant 0$ (resp. $[x,x] \geqslant 0$).

2. THE OPERATORS A_m AND A'_m

Let $(\mathcal{G}_0,(\cdot,\cdot)_0)$ be a Hilbert space and let H_0 be a nonnegative selfadjont operator in $\mathcal{G}_0$ such that $0 \in \sigma(H_0)$. We set $H_m := H_0 + m^2 I$, $m \geqslant 0$. Define a scalar product $(\cdot,\cdot)_\alpha$ on $\mathcal{D}(H_1^\alpha)$, $\alpha \in \mathbb{R}$, by $(x,y)_\alpha := (H_1^\alpha x, H_1^\alpha y)_0$, $x,y \in \mathcal{D}(H_1^\alpha)$. By $\mathcal{G}_\alpha$, $\alpha \in \mathbb{R}$, we denote the completion of $\mathcal{D}(H_1^\alpha)$ with respect to the norm $\|\cdot\|_1$, $\|x\|_\alpha := (x,x)_\alpha^{\frac{1}{2}}$, $x \in \mathcal{D}(H_1^\alpha)$. The form $(\cdot,\cdot)_0$ can be extended by continuity to $\mathcal{G}_\alpha \times \mathcal{G}_{-\alpha}$ for every $\alpha \in \mathbb{R}$. Extensions of forms by continuity will be denoted, in the following, in the same way as the forms theirselves.

We provide the linear space $\mathcal{H} = \mathcal{G}_0 \oplus \mathcal{G}_0$ with the Krein space inner product $[\cdot,\cdot]$ defined by

$$\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = (u_2,v_1)_0 + (u_1,v_2)_0 \quad u_1,u_2,v_1,v_2 \in \mathcal{G}_0.$$

Set

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{G}_0 \oplus \mathcal{G}_0$. J is a fundamental symmetry of $(\mathcal{H},[\cdot,\cdot])$ and we have

$$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) := \left[J\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right] = (u_1,v_1)_0 + (u_2,v_2)_0, \quad u_1,u_2,v_1,v_2 \in \mathcal{G}_0.$$

Set $\|x\| := (x,x)^{\frac{1}{2}}$, $x \in \mathcal{H}$.

In what follows the operator A_m, $m \geqslant 0$, in $\mathcal{H}$ defined by $\mathcal{D}(A_m) := \mathcal{G}_1 \oplus \mathcal{G}_0$ and

$$A_m \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} 0 & I \\ H_m & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{D}(A_m),$$

will play the role of the unperturbed operator in our abstract setting.

If $\mathcal{G}_0 = L_2(\mathbb{R}^n)$ and H_0 is the Laplace operator with its natural domain, then A_m is the operator corresponding to the matrix in (0.2) in the unperturbed case.

Since JA_m is selfadjoint in the Hilbert space $(\mathcal{H},(\cdot,\cdot))$, A_m is selfadjoint in the Krein space $(\mathcal{H},[\cdot,\cdot])$. We have $[A_m x, x] \geqslant 0$ for every $x \in \mathcal{D}(A_m)$. Evidently, $0 \in \rho(A_m)$ if $m > 0$. Thus for $m > 0$, A_m is a non-negative operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. It is easy to see that for any $m \geqslant 0$ and any $z \in \mathbf{C}\setminus\mathbf{R}$ the operator $A_m - zI$ maps $\mathcal{G}_1 \oplus \mathcal{G}_0$ one-to-one onto $\mathcal{G}_0 \oplus \mathcal{G}_0$ and we have

$$(2.1) \qquad (A_m - zI)^{-1} = \begin{pmatrix} zR_m(z^2) & R_m(z^2) \\ I + z^2 R_m(z^2) & zR_m(z^2) \end{pmatrix}, \quad z \in \mathbf{C}\setminus\mathbf{R},$$

where $R_m(z) := (H_m - zI)^{-1}$, $z \in \rho(H_m)$. Hence A_0 is a nonnegative operator in the Krein space $\mathcal{H}$.

Let $m \geqslant 0$ be arbitrary and assume that the real numbers a and b, $0 < a < b < \infty$, are no eigenvalues of $H_m^{\frac{1}{2}}$. From (2.1), applying [11; proof of Theorem 3.1], we obtain:

$$E((a,b);A_m) = \frac{1}{2}\begin{pmatrix} E((a,b);H_m^{\frac{1}{2}}) & H_m^{-\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}}) \\ H_m^{\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}}) & E((a,b);H_m^{\frac{1}{2}}) \end{pmatrix}$$

and

$$E((-b,-a);A_m) = \frac{1}{2}\begin{pmatrix} E((a,b);H_m^{\frac{1}{2}}) & -H_m^{-\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}}) \\ -H_m^{\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}}) & E((a,b);H_m^{\frac{1}{2}}) \end{pmatrix}$$

(see [12]). Hence

$$\left\| E((a,b);A_m)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\|^2 = \frac{1}{4}\left\| E((a,b);H_m^{\frac{1}{2}})u_1 + H_m^{-\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}})u_2\right\|_0^2 +$$

$$+\frac{1}{4}\left\| H_m^{\frac{1}{2}}E((a,b);H_m^{\frac{1}{2}})u_1 + E((a,b);H_m^{\frac{1}{2}})u_2\right\|_0^2 .$$

It follows that for a fixed a we have

$$\sup\{\|E((a,b);A_m)\| : b \in (a,\infty)\} = \infty$$

if and only if H_0 is unbounded. The same is true for (a,b) replaced by $(-b,-a)$. Therefore, if H_0 is unbounded, ∞ is a singular critical point of A_m, $\infty \in c_s(A_m)$.

Let $m = 0$. If 0 is an isolated point of $\sigma(H_0)$, then, evidently, $0 \notin c_s(A_0)$. If 0 is an accumulation point of $\sigma(H_0)$, then for any fixed b, $0 < b < \infty$,

$$\sup\{\|E((a,b);A_0)\| : a \in (0,b)\} = \infty,$$

i.e. $0 \in c_s(A_0)$.

Set

$$(2.2) \qquad \mathcal{H}_{+\frac{1}{2}}^{(*)} := \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0, \quad \mathcal{H}_{-\frac{1}{2}} := \mathcal{G}_0 \oplus \mathcal{G}_{-\frac{1}{2}}.$$

The form $[\cdot,\cdot]$ restricted to $\mathcal{H}_{+\frac{1}{2}}^{(*)} \times \mathcal{H}$ $(\mathcal{H} \times \mathcal{H}_{+\frac{1}{2}}^{(*)})$ can be extended by continuity to $\mathcal{H}_{+\frac{1}{2}}^{(*)} \times \mathcal{H}_{-\frac{1}{2}}$ (resp. $\mathcal{H}_{-\frac{1}{2}} \times \mathcal{H}_{+\frac{1}{2}}^{(*)}$). Every continuous linear functional on $\mathcal{H}_{+\frac{1}{2}}^{(*)}$ has the form $[\cdot,y]$ for some $y \in \mathcal{H}_{-\frac{1}{2}}$. The spaces $\mathcal{H}_{+\frac{1}{2}}^{(*)}$ and $\mathcal{H}_{-\frac{1}{2}}$ coincide (up to equivalence of norms) for every $m \geqslant 0$ with the spaces $\mathcal{H}^{(*)}_{+\frac{1}{2}}(A_m)$ and $\mathcal{H}_{-\frac{1}{2}}(A_m)$, respectively, introduced in [7; Section 1.1, Section 1.4] (see also [6; Section 2.1]).

If $\tilde{H}_m$ is the extension by continuity of H_m to an operator of $\mathcal{G}_{\frac{1}{2}}$ in $\mathcal{G}_{-\frac{1}{2}}$, then

$$\tilde{A}_m : \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0 \ni \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ \tilde{H}_m u_1 \end{pmatrix} \in \mathcal{G}_0 \oplus \mathcal{G}_{-\frac{1}{2}}$$

is the extension by continuity of A_m to an operator belonging to $\mathcal{L}(\mathcal{H}^{(*)}_{+\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}}) =:$ $=: \mathcal{L}^{(A)}$.

In [6; Section 2.4] (cf. also [2]) with every nonnegative operator C in $\mathcal{H}$ there was associated a certain Krein space $\mathcal{H}_C$. In our case we find by an easy computation that $\mathcal{H}_{A_m} = \mathcal{G}_{\frac{1}{4}} \oplus \mathcal{G}_{-\frac{1}{4}}$ for every $m \geqslant 0$. We denote this space simply by $\mathcal{H}_A$. The Krein space inner product on $\mathcal{H}_A$ is defined by

$$\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right] = (u_2, v_1)_0 + (u_1, v_2)_0, \quad u_1, v_1 \in \mathcal{G}_{\frac{1}{4}},\ u_2, v_2 \in \mathcal{G}_{-\frac{1}{4}}.$$

The operator A'_m in $\mathcal{H}_A$ introduced in [6; Section 2.4] which is defined by

$$\mathcal{D}(A'_m) := \{x \in \mathcal{H}^{(*)}_{+\frac{1}{2}} : \tilde{A}_m x \in \mathcal{H}_A\}, \quad A'_m x := \tilde{A}_m x,\ x \in \mathcal{D}(A'_m),$$

is a nonnegative operator in $\mathcal{H}_A$ and $\infty \notin c_s(A'_m)$ (see [6; Lemma 2.1]). The operators A_m and A'_m coincide on every spectral subspace of A_m (A'_m) corresponding to the complement of a neighbourhood of ∞. By definition $\mathcal{D}(A'_m) = \mathcal{G}_{\frac{3}{4}} \oplus \mathcal{G}_{\frac{1}{4}}$. We have

$$\mathcal{H}^{(*)}_{+\frac{1}{2}}(A'_m) = \mathcal{H}^{(*)}_{+\frac{1}{2}}(A_m) = \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0, \quad \mathcal{H}_{-\frac{1}{2}}(A'_m) = \mathcal{H}_{-\frac{1}{2}}(A_m) = \mathcal{G}_0 \oplus \mathcal{G}_{-\frac{1}{2}}$$

for every $m \geqslant 0$ ([6; Lemma 2.1]). For further properties of A'_m see [6; Lemma 2.1].

3. A CLASS OF PERTURBATIONS OF A_m

3.1. In what follows let Z be an operator belonging to $\mathcal{L}^{(A)}$ of the form

$$Z = \begin{pmatrix} V & 0 \\ U & V^* \end{pmatrix} \tag{3.1}$$

(with respect to the decompositions (2.2)) such that

$$V \in \mathfrak{S}_\infty(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_0), \quad U \in \mathfrak{S}_\infty(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}}) \tag{3.2}$$

and

$$U = U^*. \tag{3.3}$$

Here * denotes the adjoint with respect to $(\cdot,\cdot)_0$. We have $[Zx,y]=[x,Zy]$, $x,y\in \mathcal{H}^{(*)}_{+\frac{1}{2}}$.

We regard Z as a perturbation of A_m, and the perturbed operator $A_m \uplus Z$ is defined by

$$\mathcal{D}(A_m \uplus Z) := \{x \in \mathcal{H}^{(*)}_{+\frac{1}{2}} : (\tilde{A}_m + Z)x \in \mathcal{H}\},\ (A_m \uplus Z)x := (\tilde{A}_m + Z)x \quad \text{if } x \in \mathcal{D}(A_m \uplus Z),$$

see [6; Definition 2.2]. Similarly we define the operator $A'_m \uplus Z$ in $\mathcal{H}_A$. The operators $A_m \uplus Z$ and $A'_m \uplus Z$ are closely connected (see [6; Lemma 2.8]). For example, we have $\sigma(A_m \uplus Z) = \sigma(A'_m \uplus Z)$. $A_m \uplus Z$ and $A'_m \uplus Z$ can also be defined by form sums. The operators in $\mathcal{H}$ ($\mathcal{H}_A$) which were associated with a perturbed equation of Klein-Gordon type in [12] are contained in the set of the operators of the form $A_m \uplus Z$ (resp. $A'_m \uplus Z$), $m > 0$.

The following theorem and corollary, which are sharpened versions of some results of [12], follow from Lemma 2.3, Proposition 3.1 and Theorem 3.6 in [6].

THEOREM 3.1. *For every $m \geqslant 0$, $A_m \uplus Z$ is a selfadjoint operator in $\mathcal{H}$ with $\rho(A_m \uplus Z) \neq \emptyset$ and*

$$(A_m \uplus Z - \lambda I)^{-1} - (A_m - \lambda I)^{-1} \in \mathfrak{S}_\infty, \quad \lambda \in \rho(A_m) \cap \rho(A_m \uplus Z).$$

Assume, in addition, that $\rho(H_m) \cap (0,\infty) \neq \emptyset$ and let $t_1 \in (-\infty,0)$ and $t_2 \in (0,\infty)$ such that $t_1^2, t_2^2 \in \rho(H_m)$. Then $A_m \uplus Z$ is definitizable over $(t_2,\infty) \cup \{\infty\} \cup (-\infty,t_1)$ and (t_2,∞) ($(-\infty,t_1)$) is of type π_+ (resp. π_-) with respect to $A_m \uplus Z$. If H_0 is unbounded, then

$$\infty \in c_s(A_m \uplus Z). \tag{3.4}$$

If, in particular, $m > 0$, then $A_m \uplus Z$ is definitizable.

COROLLARY 3.2. *The assertions of Theorem 3.1 with the exception of (3.4) remain true if A_m and $\mathcal{H}$ are replaced by A'_m and $\mathcal{H}_A$, respectively. If $\rho(H_m) \cap (0,\infty) \neq \emptyset$, then $\infty \notin c_s(A'_m \uplus Z)$ and, hence, $\mathrm{i}(A'_m \uplus Z)$ is the infinitesimal generator of a strongly continuous group of unitary operators in $\mathcal{H}_A$.*

If $m > 0$, then by Theorem 3.1, $\sigma_+(A_m \uplus Z;\mathbf{R}) \cap (-\infty,m)$ ($\sigma_-(A_m \uplus Z;\mathbf{R}) \cap (-m,\infty)$) is a bounded set which has no accumulation points in $(-\infty,m)$ (resp. $(-m,\infty)$). The same is true for $A_m \uplus Z$ replaced by $A'_m \uplus Z$.

3.2. The Propositions 2.2 and 2.3 in [12] (see also [16]) dealing with the location of eigenvalues of a special type can be carried over to our situation. We shall formulate the corresponding assertions in the following proposition. First we remark that

$$\gamma'_0 := \sup\{((V^*V - U - H_0)u,u)_0 (u,u)_0^{-1} : u \in \mathcal{G}_1\} < \infty.$$

Indeed, by $V^*V - U \in \mathfrak{S}_\infty(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}})$ there exists a $\gamma < \infty$ such that

$$((V^*V-U)u,u)_0 \leqslant |((V^*V-U)u,u)_0| \leqslant (H_0 u,u)_0 + \gamma(u,u)_0, \quad u \in \mathcal{G}_1.$$

Let us put

$$\gamma_{(m)} := \max\{\gamma_0' - m^2, 0\}.$$

Further, we set

$$\gamma_l := \inf\{\operatorname{Re}(Vu,u)_0(u,u)_0^{-1} : u \in \mathcal{G}_1\} \geqslant -\infty,$$

$$\gamma_r := \sup\{\operatorname{Re}(Vu,u)_0(u,u)_0^{-1} : u \in \mathcal{G}_1\} \leqslant \infty.$$

Then

$$s_{1,m} := \max\{-\gamma_{(m)}^{\frac{1}{2}}, \gamma_l\} \quad\text{and}\quad s_{2,m} := \min\{\gamma_{(m)}^{\frac{1}{2}}, \gamma_r\}$$

are finite real numbers.

PROPOSITION 3.3. *For any $m \geqslant 0$ we have*

$$|\sigma_0(A_m \uplus Z)| \leqslant \gamma_{(m)}^{\frac{1}{2}} \tag{3.5}$$

and

$$\gamma_l \leqslant \operatorname{Re}\sigma_0(A_m \uplus Z) \leqslant \gamma_r. \tag{3.6}$$

If, in addition, the interval $(-\infty, t_1)$ is of type π_- with respect to $A_m \uplus Z$ for some $t_1 \leqslant 0$, then $(-\infty, t_1) \cap (-\infty, s_{1,m})$ is of negative type with respect to $A_m \uplus Z$. Similarly, if (t_2, ∞) is of type π_+ for some $t_2 \geqslant 0$, then $(t_2,\infty) \cap (s_{2,m}, \infty)$ is of positive type.

Proof. Let $x := (u,v)^\tau \in \mathcal{H}$, $\|x\| \neq 0$, be an eigenvector of $A_m \uplus Z$ corresponding to an eigenvalue λ. Then $v = \lambda u - Vu$ and $(\tilde{H}_m + U)u = \lambda v - V^*v$. It follows that

$$\frac{1}{2}[x,x] = \operatorname{Re}\lambda(u,u)_0 - \operatorname{Re}(Vu,u)_0 \tag{3.7}$$

and

$$\lambda^2(u,u)_0 - 2\lambda\operatorname{Re}(Vu,u)_0 + ((V^*V - U - H_m)u,u)_0 = 0. \tag{3.8}$$

Taking the real part of (3.8) and making use of (3.7) we obtain

$$|\lambda|^2(u,u)_0 = ((V^*V - U - H_m)u,u)_0 + \operatorname{Re}\lambda\,[x,x]. \tag{3.9}$$

If $\operatorname{Im}\lambda \neq 0$ we have $[x,x]=0$ and, in view of (3.9) and (3.7),

$$|\lambda|^2(u,u)_0 \leqslant (\gamma_0' - m^2)(u,u)_0 \text{ and } \gamma_l \leqslant \operatorname{Re}\lambda \leqslant \gamma_r.$$

This implies the relations (3.5) and (3.6).

Suppose that under the assumptions of the second part of the proposition $(-\infty,t_1)\cap(-\infty,s_{1,m}) =: \Delta$ is not of negative type. Then there exists a $\lambda \in \sigma_+(A_m \uplus Z;\Delta)$ and an eigenvector $x \in \mathcal{H}$, $x \neq 0$, of $A_m \uplus Z$ corresponding to λ such that $[x,x] \geqslant 0$ (see the remark at the end of Section 1). This contradicts either (3.9) or (3.7). For Δ replaced by $(t_2,\infty)\cap(s_{2,m},\infty)$ a similar reasoning applies. This proves Proposition 3.3.

Proposition 3.3 shows that if the intervals $(-\infty,t_1)$ and (t_2,∞) are of type π_- and π_+, respectively, with respect to $A_m \uplus Z$, then in $((-\infty,t_1)\cup(t_2,\infty))\cap((-\infty,s_{1,m})\cup \cup(s_{2,m},\infty))$ there are no spectral singularities of $A_m \uplus Z$. The assumption of this result is fulfilled in the situation of the second part of Theorem 3.1.

3.3. Let $\sigma(H_0) = [0,\infty)$. Then we cannot conclude from Theorem 3.1 that $A_0 \uplus Z$ is definitizable over some open interval. But if, in addition, the following condition $\mathfrak{A}$ is fulfilled, then it follows by Proposition 3.3 and [6, Theorem 3.10] that $A_0 \uplus Z$ is definitizable over a neighbourhood of ∞.
$\mathfrak{A}$: There exists a $p \in [1,\infty)$ such that $V \in \mathfrak{S}_p(\mathcal{G}_{\frac{1}{2}},\mathcal{G}_0)$ and $U = U^* \in \mathfrak{S}_p(\mathcal{G}_{\frac{1}{2}},\mathcal{G}_{-\frac{1}{2}})$.

THEOREM 3.4. *Let $\mathfrak{A}$ be fulfilled. Then $A_0 \uplus Z$ is definitizable over $\Delta_+ \cup \cup\{\infty\} \cup \Delta_-$, where $\Delta_+ := (0,\infty)\cap(\min\{\gamma_{(0)}^{\frac{1}{2}},\gamma_r\},\infty)$ and $\Delta_- := (-\infty,0)\cap \cap(-\infty,\max\{-\gamma_{(0)}^{\frac{1}{2}},\gamma_l\})$. Δ_+ (Δ_-) is of positive (resp. negative) type with respect to $A_0 \uplus Z$. If H_0 is unbounded, then*

$$\infty \in c_s(A_0 \uplus Z). \tag{3.10}$$

The following corollary is a consequence of [6; Theorem 3.10].

COROLLARY 3.5. *The assertions of Theorem 3.4 with the exception of (3.10) remain true if A_0 is replaced by A_0'. In this case we have $\infty \notin c_s(A_0' \uplus Z)$. Hence $\mathrm{i}(A_0' \uplus Z)$ is the infinitesimal generator of a strongly continuous group of unitary operators in $\mathcal{H}_A$.*

3.4. Now we apply the abstract results to the perturbed Klein-Gordon and wave equations. Let $\mathcal{G}_0 = L_2(\mathbf{R}^n)$ and let H_0 be the Laplace operator $-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}$ with its natural domain in $L_2(\mathbf{R}^n)$. Then $\mathcal{G}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}}$ coincide with the Sobolev spaces

$H^1(\mathbf{R}^n)$ and $H^{-1}(\mathbf{R}^n)$, respectively. Making use of a well-known theorem of F. Rellich and of [19; Theorem 4.1] we obtain the following.

PROPOSITION 3.6. *Assume that the following holds:*

(i) *V is the operator of multiplication by a bounded measurable function $\mathcal{V}$ such that $\lim_{|x|\to\infty} \mathcal{V}(x) = 0$. V is regarded as an operator from $H^1(\mathbf{R}^n)$ in $L_2(\mathbf{R}^n)$.*

(ii) *Let $\mathcal{U}_j$, $j=0,1,\dots,n$, be bounded measurable functions such that $\lim_{|x|\to\infty} \mathcal{U}_j(x) = 0$ and $\mathcal{U}_0$ is a real function. U is the bounded operator from $H^1(\mathbf{R}^n)$ in $H^{-1}(\mathbf{R}^n)$ defined by the differential expression*

$$\sum_{j=1}^{n} \left(\mathrm{i}\frac{\partial}{\partial x_j}\mathcal{U}_j(x) + \mathrm{i}\overline{\mathcal{U}_j(x)}\frac{\partial}{\partial x_j} + \overline{\mathcal{U}_j(x)}\mathcal{U}_j(x) \right) + \mathcal{U}_0(x).$$

Then the conditions (3.2) and (3.3) are fulfilled. If, in addition, there exists a constant c such that

$$|\mathcal{V}(x)| \leqslant c(1+|x|)^{-\delta}, \; |\mathcal{U}_j(x)| \leqslant c(1+|x|)^{-\delta}, \quad x \in \mathbf{R}^n, \; j=0,1,\dots,n,$$

for some $\delta > 0$, then the condition $\mathfrak{A}$ is fulfilled.

Under the assumptions of Proposition 3.6 we have

$$[(\tilde{H}_m + U)\varphi](x) = \sum_{j=1}^{n} \left(-\mathrm{i}\frac{\partial}{\partial x_j} - \overline{\mathcal{U}_j(x)} \right) \left(-\mathrm{i}\frac{\partial}{\partial x_j} - \mathcal{U}_j(x) \right) \varphi(x) +$$

$$+ \mathcal{U}_0(x)\varphi(x) + m^2\varphi(x), \quad \varphi \in H^1(\mathbf{R}^n),$$

in the distribution sense.

PROPOSITION 3.7. *Let $m > 0$ and let V and U be given as in Proposition 3.6. Assume, further, that there exists a constant c such that*

$$(3.11) \quad |\mathcal{V}(x)| \leqslant c(1+|x|)^{-1-\varepsilon}, \; |\mathcal{U}_j(x)| \leqslant c(1+|x|)^{-1-\varepsilon}, \quad x \in \mathbf{R}^n, \; j=0,1,\dots,n,$$

and

$$(3.12) \qquad \sum_{j=1}^{n} \frac{\partial \mathcal{U}_j}{\partial x_j} \in L_\infty(\mathbf{R}^n), \; \left| \sum_{j=1}^{n} \frac{\partial \mathcal{U}_j}{\partial x_j}(x) \right| \leqslant c(1+|x|)^{-1-\varepsilon}, \quad x \in \mathbf{R}^n,$$

for some $\varepsilon > 0$. Then the interval (m,∞) $((-\infty,-m))$ is of positive (resp. negative) type with respect to $A_m \dotplus Z$ and to $A'_m \dotplus Z$. We have

$$c(A'_m \dotplus Z) = c(A_m \dotplus Z) \subset [-m,m] \cup \{\infty\}$$

and

$$c_s(A_m \uplus Z) \subset \{m, -m, \infty\},\ c_s(A'_m \uplus Z) \subset \{m, -m\}.$$

Proof. It is sufficient to prove that $A_m \uplus Z$ has no eigenvalues in $(-\infty, -m) \cup$ $\cup (m, \infty)$. Suppose that $\lambda \in \sigma_p(A_m \uplus Z) \cap ((-\infty, -m) \cup (m, \infty))$ and let $\begin{pmatrix} u \\ v \end{pmatrix} \neq 0$ be an eigenvector of $A_m \uplus Z$ corresponding to λ. Then $0 \neq u \in H^1(\mathbf{R}^n)$ and

$$\tilde{H}_0 u + (U - V^*V + \lambda V + \lambda V^*)u = (\lambda^2 - m^2)u$$

(cf. the proof of Proposition 3.3). In view of (3.12) we have $\Delta u \in L_2(\mathbf{R}^n)$. Then according to [3; Corollary 14.5.6] we have

$$\int (1 + |x|^2)^N \left(|u|^2 + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx < \infty$$

for all positive integers N. Then by [3; Theorem 17.2.8] we get $u = 0$, a contradiction.

4. A MORE SPECIAL CLASS OF PERTURBATIONS OF A_0

4.1. In this section we consider the case $m = 0$. First we introduce two conditions $\mathfrak{B}$ and $\mathfrak{C}$ which together with $\mathfrak{A}$ imply the assumptions of [6; Theorem 3.10]. These conditions are suitable, in particular, for applications to differential operators in the case when the unperturbed operator has constant coefficients (see [10; II] and Section 4.2).

$\mathfrak{B}$: There exists a Hilbert space $\mathcal{K}$ and operators $V_1 \in \mathfrak{S}_\infty(\mathcal{G}_{\frac{1}{2}}, \mathcal{K})$, $V_2 \in \mathcal{L}(\mathcal{K}, \mathcal{G}_0)$, $U_1 \in \mathcal{L}(\mathcal{G}_{\frac{1}{2}}, \mathcal{K})$ and $\tilde{U} \in \mathcal{L}(\mathcal{K})$ with $\iota^* V_2 \in \mathfrak{S}_\infty(\mathcal{K}, \mathcal{G}_{-\frac{1}{2}})$ and $U_1 \iota' \in \mathfrak{S}_\infty(\mathcal{G}_1, \mathcal{K})$ such that $V = V_2 V_1$ and $U = U_1^* \tilde{U} U_1$. Here ι and ι' denote the natural embeddings of $\mathcal{G}_{\frac{1}{2}}$ in $\mathcal{G}_0$ and of $\mathcal{G}_1$ in $\mathcal{G}_{\frac{1}{2}}$, respectively.

$\mathfrak{C}$: For a sufficiently small $\varepsilon > 0$ there exist a Hilbert space $\mathcal{C}$, a unitary operator F from $E((0,\varepsilon); H_0)\mathcal{G}_0$ onto $L_2((0,\varepsilon); \mathcal{C})$ and locally Hölder continuous functions $T(\cdot\,; Y)$, $Y = V_1^*,\ \iota^* V_2,\ U_1^*$, on $(0, \varepsilon)$ with values in $\mathcal{L}(\mathcal{K}, \mathcal{C})$ such that the following holds:

(i) For every Borel set $b \subset (0, \varepsilon)$ the operator $FE(b; H_0)F^{-1}$ is the operator of multiplication by the characteristic function of b.

(ii) There exist dense subsets $\mathcal{D}_Y$, $Y = V_1^*,\ \iota^* V_2,\ U_1^*$, of $\mathcal{K}$ such that for any $u \in \mathcal{D}_Y$ one has

$$T(\lambda; Y)u = (F\tilde{E}((0,\varepsilon); H_0)Yu)(\lambda) \ \text{ a.e. in } (0, \varepsilon).$$

Here $\tilde{E}((0,\varepsilon);H_0)$ denotes the extension by continuity of $E((0,\varepsilon);H_0)$ to a bounded operator of $\mathcal{G}_{-\frac{1}{2}}$ into $\mathcal{G}_{\frac{1}{2}}$.

LEMMA 4.1. *If the conditions $\mathfrak{B}$ and $\mathfrak{C}$ are fulfilled, then every open one-sided neighbourhood of 0 contains points which are no accumulation points of $\sigma_0(A_o \uplus Z)$.*

Proof. We write Z in the form $Z = W_2\tilde{W}W_1$ where

$$W_1 := \begin{pmatrix} V_1 & 0 \\ 0 & V_2^* \\ U_1 & 0 \end{pmatrix} : \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0 \to \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K},$$

$$\tilde{W} := \begin{pmatrix} I_{\mathcal{K}} & 0 & 0 \\ 0 & I_{\mathcal{K}} & 0 \\ 0 & 0 & \tilde{U} \end{pmatrix} : \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \to \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K},$$

$$W_2 := \begin{pmatrix} V_2 & 0 & 0 \\ 0 & V_1^* & U_1^* \end{pmatrix} : \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K} \to \mathcal{G}_0 \oplus \mathcal{G}_{-\frac{1}{2}}.$$

For every $z \in \rho(A_0)$ the operator $(A_0 - zI)^\curlyvee + Z$ is matricially coupled with

$$G(z) := I + \tilde{W}W_1(A_0 - zI)^{\curlyvee -1}W_2$$

(see e.g. [1]). Hence for every such z we have $z \in \rho(A_0 \uplus Z)$ if and only if $G(z)$ has a bounded inverse.

Fix some $z_0 \in \rho(A_0) \cap \rho(A_o \uplus Z)$ and let $z \in \rho(A_0)$. Then

$$G(z) = G(z_0)(I + (z - z_0)G(z_0)^{-1}\tilde{W}W_1(A_0 - z_0I)^{-1}(A_0 - zI)^{\curlyvee -1}W_2).$$

First we show that

$$(4.1) \qquad (z - z_0)G(z_0)^{-1}\tilde{W}W_1(A_0 - z_0I)^{-1}(A_0 - zI)^{\curlyvee -1}W_2 \in \mathfrak{S}_\infty$$

for every $z \in \rho(A_0)$. By (2.1)

$$(A_0 - z_0I)^{-1}(A_0 - zI)^{\curlyvee -1} = (z_0 + z)\begin{pmatrix} z & 1 \\ z_0 z & z_0 \end{pmatrix}\begin{pmatrix} R_0(z_0^2)R_0(z^2) & 0 \\ 0 & R_0(z_0^2)\tilde{R}_0(z^2) \end{pmatrix} +$$

$$+ \begin{pmatrix} R_0(z_0^2) & 0 \\ zR_0(z^2) + z_0R_0(z_0^2) & \tilde{R}_0(z^2) \end{pmatrix},$$

where $\tilde{R}_0(\zeta)$ denotes the extension by continuity of $R_0(\zeta)$ to a bounded operator of $\mathcal{G}_{-\frac{1}{2}}$ in $\mathcal{G}_{\frac{1}{2}}$. The elements of the 3×3 operator matrix

$$W_1\begin{pmatrix} z & 1 \\ z_0 z & z_0 \end{pmatrix}\begin{pmatrix} R_0(z_0^2)R_0(z^2) & 0 \\ 0 & R_0(z_0^2)\tilde{R}_0(z^2) \end{pmatrix}W_2$$

have the form

$$XR_0(z_0^2)\tilde{R}_0(z^2)Y \tag{4.2}$$

up to scalar factors where X is equal to V_1, $V_2^*\iota$ or U_1 and Y is equal to V_1^*, $\iota^* V_2$ or U_1^*. It follows from Condition $\mathfrak{B}$ that the operators (4.2) are compact.

In the same way one verifies that the elements of the matrix

$$W_1 \begin{pmatrix} R_0(z_0^2) & 0 \\ zR_0(z^2) + z_0 R_0(z_0^2) & \tilde{R}_0(z^2) \end{pmatrix} W_2 =$$

$$= \begin{pmatrix} V_1 R_0(z_0^2) V_2 & 0 & 0 \\ V_2^*\iota(z\tilde{R}_0(z^2) + z_0\tilde{R}_0(z_0^2))\iota^* V_2 & V_2^*\iota\tilde{R}_0(z^2)V_1^* & V_2^*\iota\tilde{R}_0(z^2)U_1^* \\ U_1 R_0(z_0^2)\iota^* V_2 & 0 & 0 \end{pmatrix}$$

are compact. Therefore, (4.1) holds for every $z \in \rho(A_0)$.

Let now $\varepsilon > 0$ be as in Condition $\mathfrak{C}$. Then one proves as in [10; proofs of Proposition 4.1 and Theorem 3.9] that $G(z)|\mathbf{C}^+$ and $G(z)|\mathbf{C}^-$, where $\mathbf{C}^\pm := \{z : \operatorname{Im} z \gtrless 0\}$, can be extended to locally Hölder continuous functions on $\mathbf{C}^+ \cup (-\varepsilon^{\frac{1}{2}}, 0) \cup (0, \varepsilon^{\frac{1}{2}})$, and $\mathbf{C}^- \cup (-\varepsilon^{\frac{1}{2}}, 0) \cup (0, \varepsilon^{\frac{1}{2}})$, respectively. From this fact and the relation (4.1) the assertion of Lemma 4.1 follows as in the proof of Lemma 4.20 in [9].

Assuming now that, in addition to the assumption of Theorem 3.4, the conditions $\mathfrak{B}$ and $\mathfrak{C}$ are fulfilled we obtain, as a consequence of Lemma 4.1 and [6; Theorem 3.10 and Lemma 2.8], the following theorem.

THEOREM 4.2. *Let the perturbation Z (see (3.1)–(3.3)) satisfy the conditions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$. Then the operator $A_0 \uplus Z$ is definitizable over $\overline{\mathbf{R}} \setminus \{0\}$. In particular, the only possible accumulation point of $\sigma_0(A_0 \uplus Z)$ in $\overline{\mathbf{C}}$ is the point 0. $(0, \infty)$ $((-\infty, 0))$ is of type π_+ (resp. of type π_-) with respect to $A_0 \uplus Z$. The same is true for $A_0 \uplus Z$ replaced by $A_0' \uplus Z$.*

Under the assumptions of this theorem, $\sigma_+(A_0 \uplus Z; \mathbf{R} \setminus \{0\}) \cap (-\infty, 0)$ $((\sigma_-(A_0 \uplus Z; \mathbf{R} \setminus \{0\}) \cap (0, \infty))$ is a bounded set which has no accumulation points in $(-\infty, 0)$ (resp. $(0, \infty)$). The same is true for $A_0 \uplus Z$ replaced by $A_0' \uplus Z$.

4.2. Now we consider the perturbed wave equation.

PROPOSITION 4.3. *Let, in addition to the assumptions of Proposition 3.6, the relations (3.11) hold for some constant c and some $\varepsilon > 0$. Then the Conditions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are fulfilled and, therefore, the assertions of Theorem 4.2 are true.*

If, additionally, (3.12) holds for some $\varepsilon > 0$, then the interval $(0, \infty)$ $((-\infty, 0))$ is of positive (resp. negative) type with respect to $A_0 \uplus Z$ and to $A_0' \uplus Z$.

Proof. As to Condition $\mathfrak{A}$ see Proposition 3.6. To show that $\mathfrak{B}$ and $\mathfrak{C}$ hold set $\mathcal{K} := \mathcal{K}_0 \oplus \sum_{\alpha,\beta=0}^{n} \oplus \mathcal{K}_{\alpha,\beta}$, where all the spaces $\mathcal{K}_0, \mathcal{K}_{\alpha,\beta}$, $\alpha,\beta = 0,1,\ldots,n$, coincide with $L_2(\mathbb{R}^n)$. $\mathcal{K}$ is equipped with the scalar product $(\cdot,\cdot)_{\mathcal{K}}$ defined by

$$((f_0,\{f_{\alpha,\beta}\}),(g_0,\{g_{\alpha,\beta}\}))_{\mathcal{K}} := (f_0,g_0)_{L_2} + \sum_{\alpha,\beta=0}^{n} (f_{\alpha,\beta}, g_{\beta,\alpha})_{L_2}.$$

Set $\nu(x) := (1+|x|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^n$, $\delta := 1+\varepsilon$ and define the linear mappings V_1, V_2, U_1 and $\tilde{U}$ (see Condition $\mathfrak{B}$) by

$$V_1 : f \mapsto \nu^{-\frac{\delta}{2}} f : \mathcal{G}_{\frac{1}{2}} \to \mathcal{K}_0 \subset \mathcal{K},$$

$$V_2 : f \mapsto \begin{cases} \nu^{\frac{\delta}{2}} \mathcal{V} f & \text{if } f \in \mathcal{K}_0 \\ 0 & \text{if } f \in \mathcal{K} \ominus \mathcal{K}_0 \end{cases} : \mathcal{K} \to \mathcal{G}_0$$

$$U_1 : f \mapsto \begin{pmatrix} \nu^{-\frac{\delta}{2}} f & -\mathrm{i}\nu^{-\frac{\delta}{2}} \frac{\partial}{\partial x_1} f & \cdots & -\mathrm{i}\nu^{-\frac{\delta}{2}} \frac{\partial}{\partial x_n} f \\ \vdots & \vdots & & \vdots \\ \nu^{-\frac{\delta}{2}} f & -\mathrm{i}\nu^{-\frac{\delta}{2}} \frac{\partial}{\partial x_1} f & \cdots & -\mathrm{i}\nu^{-\frac{\delta}{2}} \frac{\partial}{\partial x_n} f \end{pmatrix} : \mathcal{G}_{\frac{1}{2}} \to \mathcal{K} \ominus \mathcal{K}_0 \subset \mathcal{K}$$

$$\tilde{U} | \mathcal{K}_0 = 0$$

$$\tilde{U} | \mathcal{K} \ominus \mathcal{K}_0 : \begin{pmatrix} g_{00} & g_{01} & \cdots & g_{0n} \\ g_{10} & g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & & \vdots \\ g_{n0} & g_{n1} & \cdots & g_{nn} \end{pmatrix} \mapsto$$

$$\mapsto \begin{pmatrix} \nu^{\delta} \left(\sum_{j=1}^{n} \overline{\mathcal{U}}_j \mathcal{U}_j + \mathcal{U}_0 \right) g_{00} & -\nu^{\delta} \overline{\mathcal{U}}_1 g_{01} & \cdots & -\nu^{\delta} \overline{\mathcal{U}}_n g_{0n} \\ -\nu^{\delta} \mathcal{U}_1 g_{10} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\nu^{\delta} \mathcal{U}_n g_{n0} & 0 & \cdots & 0 \end{pmatrix} : \mathcal{K} \ominus \mathcal{K}_0 \to \mathcal{K} \ominus \mathcal{K}_0.$$

It is easy to see that these operators satisfy Condition $\mathfrak{B}$. One verifies as in [10; II, Section 2.3] that Condition $\mathfrak{C}$ is fulfilled.

The last assertion can be proved by the same reasoning as in the proof of Proposition 3.7.

5. ON THE CAUCHY PROBLEM FOR PERTURBED KLEIN-GORDON AND WAVE TYPE EQUATIONS

Throughout this section we assume, in addition to (3.2) and (3.3), that

$$\sup\{\|Vx\|_{-\frac{1}{2}} : x \in \mathcal{G}_{\frac{1}{2}},\ \|x\|_0 \leqslant 1\} < \infty. \tag{5.1}$$

Under the assumptions of Proposition 3.6 this condition is fulfilled. Then one verifies without difficulty that for every function $\mathbf{R} \ni t \mapsto u(t) \in \mathcal{G}_{\frac{1}{2}}$ belonging to $C(\mathbf{R};\mathcal{G}_{\frac{1}{2}}) \cap C^1(\mathbf{R};\mathcal{G}_0)$ the following statements are equivalent:

(i) There exists a $v \in C(\mathbf{R};\mathcal{G}_0) \cap C^1(\mathbf{R};\mathcal{G}_{-\frac{1}{2}})$ such that $x := (u,v)^\tau$ satisfies the relation $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \mathrm{i}(\tilde{A}_m + Z)x(t)$.

(ii) $u \in C^2(\mathbf{R};\mathcal{G}_{-\frac{1}{2}})$ and

$$\left[-\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V^*\right)\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V\right) + \tilde{H}_m + U\right] u(t) = 0. \tag{5.2}$$

A function $u \in C(\mathbf{R};\mathcal{G}_{\frac{1}{2}}) \cap C^1(\mathbf{R};\mathcal{G}_0) \cap C^2(\mathbf{R};\mathcal{G}_{-\frac{1}{2}})$ satisfying (5.2) will be called a solution of (5.2). As a consequence of the Corollaries 3.2 and 3.5 we obtain the following proposition.

PROPOSITION 5.1. *Let the assumptions of Theorem 3.1 with $\rho(H_m) \cap (0,\infty) \neq \emptyset$ or of Theorem 3.5 be fulfilled and assume that (5.1) holds. Then for every $t_0 \in \mathbf{R}$, $u_0 \in \mathcal{G}_{\frac{1}{2}}$, $u_0' \in \mathcal{G}_0$ there exists a uniquely determined solution $t \mapsto u(t;t_0;u_0,u_0')$ of (5.2) such that*

$$u(t_0;t_0;u_0,u_0') = u_0, \quad \dot{u}(t_0;t_0;u_0,u_0') = u_0', \tag{5.3}$$

where $\dot{u}(t;t_0;u_0,u_0') := \frac{\partial}{\partial t}u(t;t_0;u_0,u_0')$. The mappings

$$\mathbf{R} \times \mathbf{R} \times (\mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0) \ni (t,t_0;u_0,u_0') \mapsto u(t,t_0;u_0,u_0') \in \mathcal{G}_{\frac{1}{2}} \tag{5.4}$$

and

$$\mathbf{R} \times \mathbf{R} \times (\mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0) \ni (t,t_0;u_0,u_0') \mapsto \dot{u}(t,t_0;u_0,u_0') \in \mathcal{G}_0 \tag{5.5}$$

are continuous.

Proof. By the Corollaries 3.2 and 3.5 the operator $\mathrm{i}(A_m' \dotplus Z)$ in $\mathcal{H}_A$ generates a strongly continuous group of bounded operators $T(t)$, $t \in \mathbf{R}$, in $\mathcal{H}_A$. It is easy to see that the restriction $S(t)$ of $T(t)$ to $\mathcal{H}^{(*)}_{+\frac{1}{2}}$ is a strongly continuous group of bounded operators in $\mathcal{H}^{(*)}_{+\frac{1}{2}}$. For every $t \in \mathbb{R}$ and every $x \in \mathcal{H}^{(*)}_{+\frac{1}{2}}$ the limit $\lim_{h\to 0} h^{-1}(S(t+h)-S(t))x$ exists in $\mathcal{H}_{-\frac{1}{2}}$ and is equal to $\mathrm{i}(\tilde{A}_m + Z)S(t)x$. If $t_0 \in \mathbb{R}$, $u_0 \in \mathcal{G}_{\frac{1}{2}}$, $u_0' \in \mathcal{G}_0$, then on account of the equivalence of the above conditions (i) and (ii) the first component of $S(t-t_0)(u_0, -Vu_0 - \mathrm{i}u_0')^\tau$ is a solution of (5.2) with the properties (5.3).

Suppose that there exist two different solutions of (5.2) satisfying the conditions (5.3). Then there exists a function $x_0 \in C(\mathbb{R}; \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_0) \cap C^1(\mathbb{R}; \mathcal{G}_0 \oplus \mathcal{G}_{-\frac{1}{2}})$ not identically equal to 0 such that $\frac{\mathrm{d}}{\mathrm{d}t}x_0(t) = \mathrm{i}(\tilde{A}_m + Z)x_0(t)$ and $x_0(t_0) = 0$. Let y_0 be an arbitrary element of $\mathcal{D}(A_m' \dotplus Z)$. Then by the selfadjointness of $A_m' \dotplus Z$ in $\mathcal{H}_A$ we have $\frac{\mathrm{d}}{\mathrm{d}t}[x_0(t), T(t-t_0)y_0] = 0$ for all $t \in \mathbb{R}$ and, hence, $[x_0(t), T(t-t_0)y_0] = 0$ for all $t \in \mathbb{R}$. Since $y_0 \in \mathcal{D}(A_m' \dotplus Z)$ was arbitrary, we have $x_0(t) \equiv 0$, a contradiction. This proves the uniqueness statement. The rest of Proposition 5.1 follows from well-known properties of strongly continuous groups of bounded operators.

REFERENCES

1. BART, H.; GOHBERG, I.; KAASHOEK, M. A., The coupling method for solving integral equations, in *Operator Theory: Advances and Applications*, Vol. **12**, Birkhäuser Verlag, Basel, 1984, pp. 39–73.
2. ĆURGUS, B., On the regularity of the critical point infinity of definitizable operators, *Integral Equations Operator Theory*, **8**(1985), 462–488.
3. HÖRMANDER, L., *The analysis of linear partial differential operators*, Vol. I–IV, Springer Verlag, Berlin – Heidelberg – New York – Tokyo, 1983–1985.
4. JONAS, P., *On the local wave operators for definitizable operators in Krein space and on a paper of T. Kako*, Akad. Wiss. DDR, Zentralinst. Mathematik Mechanik Preprint P-46/79, Berlin 1979.
5. JONAS, P., On a class of selfadjoint operators in Krein space and their compact perturbations, *Integral Equations Operator Theory*, **11**(1988), 351–384.
6. JONAS, P., On a problem of the perturbation theory of selfadjoint operators in Krein space, *J. Operator Theory*, **25**(1991), 183–211.
7. JONAS, P.; LANGER, H., Some questions in the perturbation theory of J-nonnegative operators in Krein space, *Math. Nachr.*, **114**(1983), 205–226.
8. KAKO, T., Spectral and scattering theory for the J-selfadjoint operators associated with the perturbed Klein-Gordon type equation, J. Fac. Sci. Univ. Tokyo, *Sc. I A Math.*, **23**(1976), 199–221.
9. KATO, T.; KURODA, S. T., The abstract theory of scattering, *Rocky Mountain J. of Mathematics*, **1**(1971), 127–171.
10. KURODA, S. T., Scattering theory for differential operators, I, II; I, operator theory: *J. Math. Soc. Japan*, **25**(1973), 75–104; II, selfadjoint elliptic operators: ibid. **25**(1973), 222–234.
11. LANGER, H., Spectral functions of definitizable operators in Krein spaces; in *Functional Analysis*, Proceedings of a Conference held in Dubrovnik, Lecture Notes in Mathematics 948, Springer Verlag, Berlin 1982.

12. LANGER, H.; NAJMAN, B., A Krein space approach to the Klein-Gordon equation (manuscript).
13. LUNDBERG, L. -E., Spectral and scattering theory for the Klein-Gordon equation, *Commun. Math. Phys.*, **31**(1973), 243–257.
14. NAGY, B., Operators with spectral singularities, *J. Operator Theory*, **15**(1986), 307–325.
15. NAJMAN, B., Spectral properties of operators of Klein-Gordon type, *Glasnik Mat. Ser. III*, **15**(1980), 97–112.
16. NAJMAN, B., Localization of the critical points of Klein-Gordon type operators, *Math. Nachr.*, **99**(1980), 33–42.
17. NAJMAN, B., Eigenvealues of the Klein-Gordon equation, *Proc. Roy. Soc. Edinburgh. Sect. A*, **26**(1983), 181–190.
18. SCHECHTER, M., The Klein-Gordon equation and scattering theory, *Ann. Physics*, **101**(1976), 601–609.
19. SIMON, B., *Trace ideals and their applications*, London Math. Soc., Lecture Notes 35, Cambridge Univ. Press, 1979.
20. VESELIĆ, K., A spectral theory for the Klein-Gordon equation with an external electrostatic potential, *Nuclear Physics* A 147 (1970), 215–224.

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