J. OPERATOR THEORY
29(1993), 225–236

QUASICENTRAL APPROXIMATE UNITS FOR THE DISCRETE HEISENBERG GROUP

DAVID BERNIER

Let G be a group on generators $g_1, \ldots, g_n$ and consider the n-tuple of left translation operators $\lambda(g_1), \ldots, \lambda(g_n)$ in $\ell^2(G)$. It is a general problem ([6], [7]) to determine for which normed ideals of operators $\mathcal{I}$ (see [4]) there is a quasicentral approximate unit for $\lambda(g_1), \ldots, \lambda(g_n)$ relative to $\mathcal{I}$.

The only groups for which this question was settled were the groups $\mathbf{Z}^n$ (see [6]) and the non-commutative groups G containing a free semigroup on 2 generators (see [8]). Here we will deal with this question for the discrete Heisenberg group. Our approach makes an essential use of the Kohn laplacian and its fundamental solution on the continuous Heisenberg group.

The right scale of ideals for this question is that of the C_p^- ideals, which are a bit smaller than the C_p ideals of Schatten and von Neumann. We also have that $C_p^- = C(p, 1)$ on the so-called Lorentz scale of ideals.

It is harder to show the non-existence of approximate units than to show their existence. We will show that there is no quasicentral approximate unit for the $\lambda(g_i)$ of the discrete Heisenberg group (a subgroup of the order 3 continuous Heisenberg group) relative to the ideal C_4^-. The index 4 corresponds to the polynomial order of growth of the group, as was found to be the case for the groups studied previously. The detailed statement of this result is contained in Theorem 1 at the end of the paper. Our solution involves a discretization of a continuous potential function. Though random walks on discrete groups do not appear explicitly below, much of what we are doing seems related to random walks.

This work is part of my Ph.D. dissertation written at the University of California, Berkeley under Dan Voiculescu.

1. PRELIMINARIES

H_3 is $\mathbf{R}^3$ equipped with the composition law

$$(u_1, v_1, w_1)(u_2, v_2, w_2) = (u_1 + u_2, v_1 + v_2, w_1 + w_2 + u_1 v_2) \tag{1}$$

The discrete Heisenberg group, which we will denote by Γ, is then the subgroup of H_3 consisting of elements of H_3 with integer components.

We will introduce some symmetrically normed ideals, and refer the reader to [3, 5] for more details. For $1 < p < \infty$, C_p^- is the ideal given by the norm

$$|T|_p^- = \sum_{j=1}^{\infty} \lambda_j j^{-1+\frac{1}{p}}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$ are the eigenvalues of $(T^*T)^{\frac{1}{2}}$, arranged in decreasing order and taking into account the multiplicities. C_p^- is the closure of the set of finite rank operators with respect to the $|\,.\,|_p^-$ norm. C_p^+ is the ideal given by the norm

$$|T|_p^+ = \sup_n \frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{n} j^{-\frac{1}{p}}}$$

where the λ_j, $j \geqslant 1$, are defined as above. C_p^+ is the set of compact operators for which this norm is finite. C_p^+ is not the closure of the finite rank operators with respect to the $|\,.\,|_p^+$ norm (see [4]). For $1 \leqslant p_1 < p_2 < p_3 < \infty$, we have

$$C_{p_1} \subset C_{p_2}^- \subset C_{p_2} \subset C_{p_2}^+ \subset C_{p_3}$$

(C_p is the ideal given by the norm $|T|_p = \left(\sum_{j=1}^{\infty} \lambda_j^p\right)^{\frac{1}{p}}$.) If $\frac{1}{p} + \frac{1}{q} = 1$, then $C_p^* \cong C_q$ and $C_p^{-*} \cong C_q^+$ $(1 < p < \infty)$ (see [4] for more details).

For an n-tuple $(T_1, \ldots, T_n)$ of operators and an ideal $\mathcal{I}$ with norm $|\,.\,|_{\mathcal{I}}$, we define (see [6])

$$k_{\mathcal{I}}((T_1, \ldots, T_n)) = \liminf_{Y \in R_1^+} \Big(\sup_{1 \leqslant j \leqslant n} |[Y, T_j]|_{\mathcal{I}} \Big) \tag{2}$$

where R_1^+ is the set of finite rank positive operators in the unit ball of $\mathcal{L}(\mathcal{H})$. For $1 \leqslant p < \infty$, we denote k_{C_p} and $k_{C_p^-}$ by k_p and k_p^-, respectively.

The properties of the invariant $k_{\mathcal{I}}$ and its relation to the existence of quasicentral approximate units are discussed in [8]. If $\Omega \subset \mathcal{L}(\mathcal{H})$ and $\mathcal{I}$ is an ideal with norm $|\,.\,|_{\mathcal{I}}$, we will say that there is a quasicentral approximate unit for Ω relative to $\mathcal{I}^{(0)}$ if there is a sequence $X_j \in R_1^+$, $X_j \uparrow I$ such that $\lim_{j\to\infty} |[X_j, T]|_{\mathcal{I}} = 0$ for every T in Ω (from [8]) where $\mathcal{I}^{(0)}$ is the closure of the set of finite rank operators in the $|\,.\,|_{\mathcal{I}}$ norm. Then the existence of a quasicentral approximate unit for Ω is equivalent to the requirement that $k_{\mathcal{I}}(\tau) = 0$ for all n-tuples of operators τ with components in Ω (for any natural number n).

Suppose G is an infinite discrete group with operators $g_1, \ldots, g_n$. Let $\lambda(g_i)$ for $1 \leqslant i \leqslant n$, be the unitary operator on $\ell^2(G)$ defined by

$$\lambda(g_i)\xi_g = \xi_{g_i g}$$

(where $\{\xi_g : g \in G\}$ forms a Hilbert basis on $\ell^2(G)$). We wish to give estimates of $k_{\mathcal{I}}(\lambda(g_1), \ldots, \lambda(g_n))$ for various ideals $\mathcal{I}$ with norm $|\,.\,|_{\mathcal{I}}$. The following lemma, whose proof is similar to that of lemma 3.2 in [7], will be used to prove Theorem 1.

LEMMA 1. *$k_{\mathcal{I}}(\lambda(g_1), \ldots, \lambda(g_n)) > 0$ if there exist functions G_j on G (for $1 \leqslant j \leqslant$ $\leqslant n$) such that*

$$\text{(i)} \qquad \sum_{j=1}^{n} (G_j(g_j g) - G_j(g)) = \delta^e(g)$$

(where δ^e is the unit mass measure at the identity e of G)

$$\text{(ii)} \qquad G_j \in \ell_{\mathcal{I}}^{(0)}(G)^*$$

where the duality is given by $\langle \xi, G_j \rangle = \sum_{g\in G} \xi_g G_j(g)$. ($\ell_{\mathcal{I}}^{(0)}(G)$ is the Banach space obtained by completing $C_K(G)$ in the $|\,.\,|_{\mathcal{I}}$ norm. For $\xi \in C_K(G)$, and $j : N^* \to G$ a bijection, we define $|\xi|_{\mathcal{I}}$ to be $\left|\sum_i (\xi \circ j)(i) e_i\right|_{\mathcal{I}}$ where the e_i, $1 \leqslant i < \infty$ are rank one orthogonal projections such that $\sum_{i=1}^{\infty} e_i = I$. This amounts to defining $|\xi|_{\mathcal{I}}$ to be the $|\,.\,|_{\mathcal{I}}$ norm of a diagonal operator whose non-zero diagonal entries (in standard matrix form) are all the values ξ_g for all $g \in G$, repeated according to multiplicity.)

2. USING THE FUNDAMENTAL SOLUTION

We will now introduce the Kohn laplacian for H_3, denoted Δ_K, and its fundamental solution U. We have

$$\Delta_K = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 2u\frac{\partial^2}{\partial v \partial w} + u^2\frac{\partial^2}{\partial w^2} \tag{3}$$

By changing coordinates in the fundamental solution for Δ_K given by Folland in [2], we get that U, as defined below, is, up to a scalar multiple, a fundamental solution of Δ_K (in our coordinates)

$$U(u,v,w) = ((u^2+v^2)^2 + (2uv-4w)^2)^{-\frac{1}{2}} \tag{4}$$

Since

$$\Delta_K = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial}{\partial v} + u\frac{\partial}{\partial w}\right) + \frac{\partial}{\partial w}\left(u\frac{\partial}{\partial v} + u^2\frac{\partial}{\partial w}\right) \tag{5}$$

it follows that the vector field $\mathbf{F}$, where

$$\mathbf{F}(u,v,w) = \left(\frac{\partial U}{\partial u}, \frac{\partial U}{\partial v} + u\frac{\partial U}{\partial w}, u\frac{\partial U}{\partial v} + u^2\frac{\partial U}{\partial w}\right) \tag{6}$$

satisfies $\operatorname{div}\mathbf{F} = 0$ in $H_3 \setminus \{(0,0,0)\}$.

If we let $\mathbf{F} = (F_1, F_2, F_3)$, then one can compute that

$$\begin{aligned} F_1(u,v,w) =& (-2u^3 - 6uv^2 + 8vw)((u^2+v^2)^2 + (2uv-4w)^2)^{-\frac{3}{2}} \\ F_2(u,v,w) =& (2u^2v - 8uw - 2v^3)((u^2+v^2)^2 + (2uv-4w)^2)^{-\frac{3}{2}} \\ F_3(u,v,w) =& (2u^3v - 8u^2w - 2uv^3)((u^2+v^2)^2 + (2uv-4w)^2)^{-\frac{3}{2}} \end{aligned}$$

It is then easy to show that

$$F_1(\alpha u, \alpha v, \alpha^2 w) = \alpha^{-3} F_1(u,v,w) \tag{7}$$

and that

$$F_2(\alpha u, \alpha v, \alpha^2 w) = \alpha^{-3} F_2(u,v,w) \tag{8}$$

for any non-zero real number α. We also see that $F_3 = uF_2$.

We introduce the following function N on $\mathbf{R}^6$

$$N(\mathrm{d}u, \mathrm{d}v, \mathrm{d}w, u, v, w) = (\mathrm{d}u^2 + \mathrm{d}v^2 + (\mathrm{d}w - u\mathrm{d}v)^2)^{\frac{1}{2}} \tag{9}$$

(see [5]). Then it is easy to see that

$$N(F_1(u,v,w), F_2(u,v,w), F_3(u,v,w), u, v, w) = (F_1(u,v,w)^2 + F_2(u,v,w)^2)^{\frac{1}{2}}. \tag{10}$$

It follows that

$$\begin{aligned} &N(F_1(\alpha u, \alpha v, \alpha^2 w), F_2(\alpha u, \alpha v, \alpha^2 w), F_3(\alpha u, \alpha v, \alpha^2 w), \alpha u, \alpha v, \alpha^2 w) = \\ &= \alpha^{-3} N(F_1(u,v,w), F_2(u,v,w), F_3(u,v,w), u, v, w) \end{aligned} \tag{11}$$

for any non-zero real number α.

3. METRIC ESTIMATES

We now introduce the Carnot metric on H_3

$$ds^2 = du^2 + dv^2 + (dw - udv)^2 \tag{12}$$

(see [5]). We will now show that, if d is the geodesic distance on H_3 given by (12), then

$$d((0,0,0),(u,v,w)) \leqslant |u| + |v| + 4|w|^{\frac{1}{2}}. \tag{13}$$

Indeed, from (12) one sees that

$$d((0,0,0),(u,0,0)) \leqslant |u|$$

and

$$d((0,0,0),(0,v,0)) \leqslant |v|$$

(using a straight line path). We now assume $w_0 \geqslant 0$ and give an estimate for $d((0,0,0),(0,0,w_0))$ using the path (parametrized by t, $0 \leqslant t \leqslant 2\pi$)

$$\begin{aligned} u(t) =& a\sin(t) \\ v(t) =& a - a\cos(t) \\ w(t) =& \frac{a^2}{4}(2t - \sin 2t) \end{aligned}$$

where we fix

$$a = \left(\frac{w_0}{\pi}\right)^{\frac{1}{2}}$$

Then one can compute that the length of this path in the metric given by (12) is

$$2\pi a = 2\pi^{\frac{1}{2}} w_0^{\frac{1}{2}} \leqslant 4 w_0^{\frac{1}{2}}$$

Left invariance of the metric implies $d(g^{-1}, e) = d(e, g)$ for any $g \in H_3$, and it follows that

$$d((0,0,0),(0,0,w)) \leqslant 4|w|^{\frac{1}{2}}$$

for any w. We have $(u,v,w) = (0,0,w)(0,v,0)(u,0,0)$, and the triangle inequality gives

$$\begin{aligned} d((0,0,0),(u,v,w)) \leqslant d((0,0,0),(0,0,w)) + d((0,0,w),(0,0,w)(0,v,0)) + \\ + d((0,0,w)(0,v,0),(u,v,w)). \end{aligned}$$

By the left invariance of the metric and the previous estimates, (13) follows.

Now we have

$$d((0,0,0),(u,v,w)) \leqslant |u|+|v|+4|w|^{\frac{1}{2}} \leqslant 6\max(|u|,|v|,|w|^{\frac{1}{2}}) \leqslant \leqslant 6((u^2+v^2)^2+w^2)^{\frac{1}{4}} \tag{14}$$

If we let S denote the surface given by

$$(u^2+v^2)^2+w^2=1,$$

then F_1 and F_2 are bounded on S, and it follows from (7) and (8) that there is a $K>0$ such that

$$\begin{gathered} |F_1(u,v,w)| \leqslant K((u^2+v^2)^2+w^2)^{-\frac{3}{4}} \\ |F_2(u,v,w)| \leqslant K((u^2+v^2)^2+w^2)^{-\frac{3}{4}} \\ \text{for any } (u,v,w)\in H_3\setminus\{(0,0,0)\}. \end{gathered} \tag{15}$$

It now follows from (10) and (15) that

$$N(F_1(u,v,w),F_2(u,v,w),F_3(u,v,w),u,v,w) \leqslant 432Kd((0,0,0),(u,v,w))^{-3}. \tag{16}$$

In Γ, we let $g_1=(1,0,0)$ and $g_2=(0,1,0)$. Then g_1 and g_2 generate Γ, and for any $g\in\Gamma$, we define the algebraic norm of g, denoted $|g|$, to be the least integer $k\geqslant 0$ such that there exist $i_1,\ldots,i_k\in\{1,2\}$ and $\varepsilon_1,\ldots,\varepsilon_k\in\{-1,1\}$ such that $g_{i_1}^{\varepsilon_1}\cdots g_{i_k}^{\varepsilon_k}=g$, and set $|e|=0$. (see [5])

Then $g_1^{-n}g_2^{-n}g_1^n g_2^n=(0,0,n^2)$, and we conclude that $|(0,0,n^2)|\leqslant 4n$ for any $n\geqslant 0$. It is now easy to show that $|(0,0,k)|\leqslant 12|k|^{\frac{1}{2}}$ for any integer k. For a general element g of Γ, say $g=(m,n,p)$, we have

$$g=(0,0,p)(0,n,0)(m,0,0),$$

and we conclude that

$$|g|\leqslant 12|p|^{\frac{1}{2}}+|n|+|m|. \tag{17}$$

We now want to give a lower bound for $d(e,g)$ for a general element $g\in H_3$. (see (12)) It is trivial that, for $g=(u,v,w)\in H_3$, we have $d(e,g)\geqslant|u|$ and $d(e,g)\geqslant|v|$. We will now show that also

$$d(e,g)\geqslant\min\left(\frac{|w|^{\frac{1}{2}}}{2},\frac{3}{4}|w|\right).$$

Suppose we have C^1 functions α,β, and γ defined on $[0,1]$ such that

$$\alpha(0)=\beta(0)=\gamma(0)=0 \ \text{ and } \ \alpha(1)=u,\ \beta(1)=v,\ \gamma(1)=w.$$

Let s be the length of this path in the metric given by (12), so

$$s = \int_0^1 (\alpha'(t)^2 + \beta'(t)^2 + (\gamma'(t) - \alpha\beta'(t))^2)^{\frac{1}{2}} dt.$$

Then we have $s \geqslant \int_0^1 |\alpha'(t)| dt$ and $s \geqslant \int_0^1 |\beta'(t)| dt$. Suppose

$$\int_0^1 |\alpha'(t)| dt \leqslant \frac{|w|^{\frac{1}{2}}}{2} \quad \text{and} \quad \int_0^1 |\beta'(t)| dt \leqslant \frac{|w|^{\frac{1}{2}}}{2}.$$

Then

$$\begin{aligned} s &\geqslant \int_0^1 |\gamma'(t) - \alpha\beta'(t)| dt \geqslant \left| \int_0^1 (\gamma'(t) - \alpha\beta'(t)) dt \right| \geqslant \\ &\geqslant \left| w - \int_0^1 \alpha(t)\beta'(t) dt \right| \geqslant |w| - \left| \int_0^1 \alpha(t)\beta'(t) dt \right| \geqslant \\ &\geqslant |w| - \int_0^1 |\alpha(t)||\beta'(t)| dt \geqslant |w| - \int_0^1 \frac{|w|^{\frac{1}{2}}}{2} |\beta'(t)| dt \geqslant \\ &\geqslant |w| - \frac{|w|^{\frac{1}{2}}}{2} \frac{|w|^{\frac{1}{2}}}{2} = \frac{3}{4}|w| \end{aligned}$$

So we have

$$s \geqslant \min\left(\frac{|w|^{\frac{1}{2}}}{2}, \frac{3}{4}|w| \right), \text{ for any } \alpha,\ \beta \text{ and } \gamma.$$

This shows that

$$d(e,g) \geqslant \min\left(\frac{|w|^{\frac{1}{2}}}{2}, \frac{3}{4}|w| \right).$$

So

$$d(e,g) \geqslant \max\left(|u|, |v|, \min\left(\frac{|w|^{\frac{1}{2}}}{2}, \frac{3}{4}|w| \right) \right). \tag{18}$$

If $g \in \Gamma$, say $g = (m,n,p)$, then we get

$$d(e,g) \geqslant \max\left(|m|, |n|, \frac{|p|^{\frac{1}{2}}}{2} \right).$$

Combining (17) and (19), we see that

$$d(e,g) \geqslant \frac{|g|}{26} \text{ for any } g \in \Gamma. \tag{20}$$

4. GEOMETRY OF THE FUNDAMENTAL DOMAIN

We let

$$A = \left\{(u,v,w) \in H_3 \text{ such that } -\frac{1}{2} \leqslant u < \frac{1}{2}, -\frac{1}{2} \leqslant v < \frac{1}{2}, -\frac{1}{2} \leqslant w < \frac{1}{2}\right\}.$$

Then $\{gA,\ g \in \Gamma\}$ is a partition of H_3. Our fundamental domain A is closely related to the one discussed by Folland on page 68 in [3]. Given an element $z = (u,v,w) \in H_3$, we want to find the unique $g \in \Gamma$ such that there is an $a \in A$ with $z = ga$. If we write $g = (m,p,n)$, where m, n, and p are integers, then one can show that

$$\begin{aligned} m &= \operatorname{int}(u) \\ p &= \operatorname{int}(v) \\ n &= m \operatorname{int}(v) + \operatorname{int}(w - v \operatorname{int}(u)) \end{aligned} \tag{21}$$

We define int : $\mathbf{R} \to \mathbf{Z}$ as follows

For any real number x, $\operatorname{int}(x)$ is the integer closest to x, and we agree that $\operatorname{int}\left(k + \frac{1}{2}\right) = k + 1$ for any integer k.

For any $g \in \Gamma$, gA is a parallelipiped centered at g. Using (21), one can show that A has 10 nearest neighboring cells, i.e. there are 10 elements of Γ (other than e) such that $(\partial A) \cap (\partial gA)$ has a positive surface area. One way to see this is to look at the plane sections of $\bigcup_{g\in\Gamma} \partial A_g$ with u constant, lying in a v, w plane, with v being the independent variable. Then one finds that such a section of $\bigcup_{g\in\Gamma} \partial A_g$ with $u = u_0$ forms a tessalation of parallelograms with sides vertical or at a slope of $\operatorname{int}(u_0)$. One can also show that $(u_0, 0, 0)$ is the center of $\overline{A_{(\operatorname{int}(u_0),0,0)}}$.

These values of g are given below

$$\begin{aligned} h_1 &= (1,0,0) \\ h_2 &= (0,1,0) \\ h_3 &= (0,0,1) \\ h_4 &= (1,0,1) \\ h_5 &= (1,0,-1) \end{aligned} \tag{22}$$

and their inverses.

It follows (by left translation by any $g \in \Gamma$) that the nearest neighbors of the parallelepiped gA are the $(gh_j)A$ and $(gh_j^{-1})A$ for $1 \leqslant j \leqslant 5$.

We let

$$C = \int_{\partial A} \mathbf{F} \cdot \mathrm{d}\vec{S} \tag{23}$$

($\mathrm{d}\vec{S}$ being the outward normal to A.) A numerical computation gives

$$C \simeq -6.28,$$

(this shows that C is non-zero) and we define the vector field $\mathbf{H}$ to be a scalar multiple of $\mathbf{F}$ as follows

$$\mathbf{H}(u, v, w) = \frac{1}{C}\mathbf{F}(u, v, w). \tag{24}$$

So we have that

$$\int_{\partial A} \mathbf{H} \cdot \mathrm{d}\vec{S} = 1$$

$$\mathrm{d}\vec{S} \text{ outward normal to } A.$$

We let

$$A_g = gA \text{ for any } g \in \Gamma. \tag{25}$$

For the rest of this paper, we will agree that $\partial\vec{A}_g$ indicates that $\mathrm{d}\vec{S}$ is an outward normal to A_g, where we are doing a surface integral over some part of ∂A_g.

We now define real-valued functions G_j on Γ $(1 \leqslant j \leqslant 5)$.

$$G_j(g) = \int_{(\overrightarrow{\partial A_{g^{-1}h_j}}) \cap (\partial A_{g^{-1}})} \mathbf{H} \cdot \mathrm{d}\vec{S} \tag{26}$$

Then

$$\sum_{j=1}^{5}(G_j(h_j g) - G_j(g)) = \sum_{j=1}^{5}\left(\int_{(\overrightarrow{\partial A_{g^{-1}}}) \cap (\partial A_{g^{-1}h_j^{-1}})} \mathbf{H} \cdot \mathrm{d}\vec{S} - \int_{(\partial A_{g^{-1}}) \cap (\overrightarrow{\partial A_{g^{-1}h_j}})} \mathbf{H} \cdot \mathrm{d}\vec{S}\right) =$$

$$= \sum_{j=1}^{5}\left(\int_{(\overrightarrow{\partial A_{g^{-1}}}) \cap (\partial A_{g^{-1}h_j^{-1}})} \mathbf{H} \cdot \mathrm{d}\vec{S} + \int_{(\overrightarrow{\partial A_{g^{-1}}}) \cap (\partial A_{g^{-1}h_j})} \mathbf{H} \cdot \mathrm{d}\vec{S}\right) = \int_{\overrightarrow{\partial A_{g^{-1}}}} \mathbf{H} \cdot \mathrm{d}\vec{S} =$$

$$= \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{if } g \neq e \end{cases}$$

(since $\operatorname{div} \mathbf{H} = 0$ inside A_g if $g \neq e$). So

$$\sum_{j=1}^{5}(G_j(h_j g) - G_j(g)) = \delta^e(g) \tag{27}$$

for any $g \in \Gamma$, where δ^e is the function on Γ with value 1 at e and 0 elsewhere. This last property is an assumption in Lemma 1 of section 1, and will be used in the next section.

5. BOUNDING THE G_j BY THE ALGEBRAIC NORM OF g

We now give some more estimates relating to the geodesic distance d on H_3 introduced in (12). For $g \in \Gamma$, let

$$d(e, A_g) = \inf_{y \in A_g} d(e, y) \tag{28}$$

and let

$$d_0 = \sup_{x \in A} d(e, x). \tag{29}$$

Then by the left invariance of d, the triangle inequality and (20), we get

$$d(e, A_g) \geqslant d(e, g) - d_0 \geqslant \frac{|g|}{26} - d_0. \tag{30}$$

From (16) and (30), we get

$$\sup_{(u,v,w) \in A_g} N(F_1(u,v,w), F_2(u,v,w), F_3(u,v,w), u, v, w) \leqslant 432K \left(\frac{|g|}{26} - d_0\right)^{-3} \tag{31}$$

(assuming that $\frac{|g|}{26} - d_0 > 0$).

So if $|g| > 52 d_0$, we have

$$\sup_{(u,v,w) \in A_g} N(F_1(u,v,w), F_2(u,v,w), F_3(u,v,w), u, v, w) \leqslant 432 \cdot 52^3 \cdot K \cdot |g|^{-3}. \tag{32}$$

We now want to give a bound on $|G_j(g^{-1})|$ in terms of $|g|$, for an arbitrary element g of Γ.

We make a change of coordinates in (26), left-translating by g. If $g = (m, p, n) \in \Gamma$ and (u, v, w) is in the domain of integration of (26), we change coordinates as follows

$$\begin{aligned} u' &= u - m \\ v' &= v - p \\ w' &= w - mv + mp - n \end{aligned} \tag{33}$$

$$\mathbf{H}'(u', v', w') = (H_1(u,v,w), H_2(u,v,w), H_3(u,v,w) - mH_2(u,v,w))$$

Then one can check that

$$G_j(g^{-1}) = \int_{(\overrightarrow{\partial A_{gh_j}})\cap(\partial A_g)} \mathbf{H}\cdot \mathrm{d}\vec{S} = \int_{(\overrightarrow{\partial A_{h_j}})\cap(\partial A)} \mathbf{H}'\cdot \mathrm{d}\vec{S} \tag{34}$$

One can also check that

$$\begin{aligned} &N(H_1'(u',v',w'), H_2'(u',v',w'), H_3'(u',v',w'), u',v',w') = \\ &= N(H_1(u,v,w), H_2(u,v,w), H_3(u,v,w), u,v,w). \end{aligned} \tag{35}$$

It is easy to show that for $(u',v',w') \in \overline{A}$ and any $(\mathrm{d}u',\mathrm{d}v',\mathrm{d}w') \in \mathbf{R}^3$,

$$(\mathrm{d}u'^2 + \mathrm{d}v'^2 + \mathrm{d}w'^2)^{\frac{1}{2}} \leqslant 2N(\mathrm{d}u',\mathrm{d}v',\mathrm{d}w',u',v',w'). \tag{36}$$

Using (36), (35), (34), (32), and (24), we get the following bound on $G_j(g^{-1})$, if $|g| > 52d_0$,

$$|G_j(g^{-1})| \leqslant 864\cdot 52^3 \cdot \frac{K}{|C|}\cdot |g|^{-3} \quad (1 \leqslant j \leqslant 5). \tag{37}$$

THEOREM 1. *For the generators h_1, h_2, h_3, h_4 and h_5 of Γ given in* (22), *we have*

$$0 < k_4^-(\lambda(h_1), \lambda(h_2), \lambda(h_3), \lambda(h_4), \lambda(h_5)) < \infty$$

(see section 1 for the definition of the operators $\lambda(h_i) \in \ell^2(\Gamma)$, $i = 1$ to 5).

Proof. It is easy to show by induction on k that for integers m, n and p,

$$|(m,p,n)| \leqslant k \Rightarrow |m| \leqslant k,\ |p| \leqslant k \text{ and } |n| \leqslant k^2.$$

It follows that, for $k \geqslant 1$, there are at most $30k^4$ elements $g \in \Gamma$ satisfying $|g| \leqslant k$.

For $1 < p < \infty$, we will denote by $\ell_p^{-(0)}(\Gamma)$ the Banach space arising (as in lemma 1 of section 1) from the ideal C_p^- introduced in section 1. From (37) and the previous fact on the growth of Γ, we see that there exists an $M > 0$ such that

$$s_k^j \leqslant Mk^{-\frac{3}{4}}$$

where $\{s_k^j\}_{k=1}^{\infty}$ is the decreasing sequence of the values $|G_j(g)|$ for all $g \in \Gamma$ (for $1 \leqslant j \leqslant 5$). It follows, since $k^{-1+\frac{1}{p}} = k^{-\frac{1}{p'}}$ if $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, that

$$G_j \in (\ell_4^{-(0)}(\Gamma))^* \quad \text{for } 1 \leqslant j \leqslant 5, \tag{38}$$

where the duality is as in lemma 1.

So from lemma 1, we get

$$k_4^-(\lambda(h_1), \lambda(h_2), \lambda(h_3), \lambda(h_4), \lambda(h_5)) > 0$$

where the $\lambda(h_j)$ are the unitaries on $\ell^2(\Gamma)$ arising from the elements $h_j \in \Gamma$.

It is not hard to show that

$$k_4^-(\lambda(h_1), \lambda(h_2), \lambda(h_3), \lambda(h_4), \lambda(h_5)) < \infty$$

(see section V of [8]).

REFERENCES

1. DOYLE, P.; SNELL, L., *Random walks and electric networks*, Mathematical Association of America, 1984.
2. FOLLAND, G. B., A fundamental solution for a subelliptic operator, *Bull. of the Amer. Math. Soc.*, **79**(1973), 373–376.
3. FOLLAND, G. B., *Harmonic analysis in phase space*, Annals of Mathematics Studies, Study **122**, Princeton University Press, 1989.
4. GOHBERG, I. C.; KREIN, M. G., *Introduction to the theory of non-selfadjoint operators*, Moscow, 1965.
5. GROMOV, M., *Structures métriques pour les variétés riemanniennes*, Fernand Nathan, 1981.
6. VOICULESCU, D., Some results on norm-ideal perturbations of Hilbert space operators I, *J. Operator Theory*, **2**(1979), 3–37.
7. VOICULESCU, D., Some results on norm-ideal perturbations of Hilbert space operators II, *J. Operator Theory*, **5**(1981), 77–100.
8. VOICULESCU, D., On the existence of quasicentral approximate units relative to normed ideals, Part I, *J. Functional Analysis*, **91**(1990), 1–36.

DAVID BERNIER
Department of Mathematics Faculty of Science,
Prince of Songkla University,
Hat Yai, Songkhla,
P.O. Box 3 Khohong 90112,
Thailand.

Received January 7, 1991.

Added in proof. Quasicentral approximate units were originally introduced by W.B. Arveson in a 1977 paper of his in the *Duke Math. Journal* (vol. 44, pp. 329–355).

In Lemma 1, for $\xi \in C_K(G)$ and $g \in G$, ξ_g is synonymous with $\xi(g)$.

For a general discrete group G , functions G_j satisfying condition (i) of Lemma 1 are equivalent to current flows on the directed Cayley graph of G having a single unit source at the identity and a single sink at infinity. Using symmetric random walks on G, it is simple to define such flows. For our group Γ , an extensive computer simulation seems to indicate that the resulting G_j's have the asymptotics needed to obtain the estimate in Theorem 1 (see [1] and Section 2.6 of my thesis, UC Berkeley, 1991). The same kind of simulation can be done for other groups.

I would like to thank Dan Voiculescu for his helpful guidance and encouragement while I was a student at UC Berkeley.