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APPROXIMATION BY PRODUCTS OF POSITIVE OPERATORS

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Many authors have considered the problem of which operators may be factored into products of positive operators. A recent survey of some of the known results is contained in a paper of Pei Yuan Wu [8]. In this paper we consider the problem of which operators may be approximated in norm by products of k positive operators. We prove that quasinilpotent operators, compact and algebraic operators with non-negative spectra, and normal operators whose spectra intersect the non-negative real axis in every component may all be approximated by products of two positive operators. Finally, it is proved that any operator that may be approximated with products of positive operators may be approximated by a product of five such operators.

1. PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space, for $n \geqslant 1$ let $\mathcal{P}_n$ denote the set of operators on $\mathcal{H}$ that admit a factorization into n positive invertible operators, and let $\mathcal{Q}_n$ denote the set of operators on $\mathcal{H}$ that admit a factorization into n positive (possibly non-invertible) operators. Let us agree that $\mathcal{P}_\infty$ (resp. $\mathcal{Q}_\infty$) is the union of all the $\mathcal{P}_n$ (resp. $\mathcal{Q}_n$) over all finite n. We note that $\mathcal{P}_n$ is norm dense in $\mathcal{Q}_n$; this follows from the fact that every positive operator is the limit of positive invertible operators. It follows that $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the same closure, which we denote $\overline{\mathcal{P}}_n$. By definition, we have

$$\overline{\mathcal{P}}_n \subset \overline{\mathcal{P}}_{n+1}$$

for every $n \geqslant 1$. If an operator T is defined on another Hilbert space $\mathcal{K}$, then we will still write $T \in \mathcal{P}_n$ meaning that T may be written as a product of n positive invertible operators acting on $\mathcal{K}$.

If $\mathcal{H}$ is finite dimensional, then there are some observations that immediately come to mind. Ballantine proved (see [1] and the survey [8]) that $\mathcal{P}_4$ (resp. $\mathcal{Q}_4$) is all of $\mathcal{P}_\infty$ (resp. $\mathcal{Q}_\infty$), with the exception of the non-positive scalar operators in $\mathcal{P}_\infty$. Furthermore it was demonstrated that

$$\mathcal{P}_5 = \mathcal{P}_\infty = \{T : \det(T) > 0\}$$

(replace $>$ with $\geqslant$ to obtain the respective statement for the $\mathcal{Q}$'s). We observe that we have

$$\overline{\mathcal{P}}_4 = \mathcal{P}_5 = \mathcal{P}_\infty;$$

this is because if z is a complex number such that $z^n > 0$, then the $n \times n$ diagonal matrix

$$\begin{bmatrix} z + \varepsilon z^{1-n} & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{bmatrix}$$

will have a positive determinant if $\varepsilon > 0$, and by choosing ε small we have succeeded in approximating zI_n with an element of $\mathcal{P}_4$ (where I_n denotes the n-dimensional identity operator).

It has been known for a long time that $\mathcal{P}_2$ is equal to the set of operators similar to positive invertible operators (see Theorem 2 in [1]). We assert that $\overline{\mathcal{P}}_2$ is the set of all operators on $\mathcal{H}$ with non-negative spectrum. This is because, after bringing an operator T with non-negative spectrum to upper triangular form, we may construct an operator S by perturbing the diagonal of T so that S is as close to T in norm as we wish and such that S has distinct positive numbers on its diagonal. We will then have approximated T with an element of $\mathcal{P}_2$ since S is similar to a positive operator.

When $\mathcal{H}$ is infinite dimensional, observations about $\overline{\mathcal{P}}_n$ come more slowly. We proceed to catalogue those observations that we have made, beginning with small n.

2. APPROXIMATION BY PRODUCTS OF TWO POSITIVE OPERATORS

By mimicking the argument used in finite dimensions, we are led to our first infinite dimensional result, which follows.

PROPOSITION 1. *Let A be an algebraic operator. We have $A \in \overline{\mathcal{P}}_2$ if and only if $\sigma(A)$ is contained in the non-negative real axis.*

Proof. Since every element of $\sigma(A)$ is an isolated point, and since, for every $P \in \mathcal{P}_2$, $\sigma(P)$ is contained in the non-negative real axis, the necessity that $\sigma(A)$ is contained in the non-negative real axis is a consequence of what Herrero calls the "l.s.c. of distinct parts of the spectrum" (see Corollary 1.6 of [6] and note that if $P_i \to A$ and G is a neighbourhood of any clopen subset of $\sigma(A)$, then eventually $\sigma(P_i)$ must meet G).

To prove the converse, we may write

$$A = \begin{bmatrix} s_1 I_1 & A_{12} & \dots & A_{1k} \\ 0 & s_2 I_2 & \ddots & \vdots \\ \vdots & \dots & \ddots & A_{k-1\,k} \\ 0 & \dots & 0 & s_k I_k \end{bmatrix}$$

relative to a decomposition of $\mathcal{H}$ as

$$\mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_k$$

(where I_j denotes the identity map on $\mathcal{M}_j$). Given $\varepsilon > 0$, choose distinct positive numbers r_j such that $|r_j - s_j| < \varepsilon$ $(1 \leqslant j \leqslant k)$, and let

$$B = \begin{bmatrix} r_1 I_1 & A_{12} & \dots & A_{1k} \\ 0 & r_2 I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{k-1\,k} \\ 0 & \dots & 0 & r_k I_k \end{bmatrix}$$

It follows that $\|A - B\| < \varepsilon$ and $B \in \mathcal{P}_2$, since B is similar (by Rosenblum's theorem, see [7], p.8) to the positive operator

$$r_1 I_1 \oplus \dots \oplus r_k I_k.$$

■

PROPOSITION 2. *If A is a quasinilpotent operator, then $A \in \overline{\mathcal{P}}_2$.*

Proof. If A is nilpotent, then $A \in \overline{\mathcal{P}}_2$ by Proposition 1. Since every quasinilpotent is a limit of nilpotents (see Remark 5.2 of [6]), it follows that $\overline{\mathcal{P}}_2$ contains all quasinilpotents. ■

Recall that an operator T on $\mathcal{H}$ is called a *Riesz operator* if every non-zero member of $\sigma(T)$ is an isolated point and the corresponding Riesz projection is of finite rank. The only accumulation point of the spectrum of such an operator is 0. Every compact operator is a Riesz operator. For any operator A, we will say that an isolated pole of the resolvent of A is "a pole of finite rank" in case the corresponding Riesz idempotent is finite rank.

PROPOSITION 3. *If A is a Riesz operator, then $A \in \overline{\mathcal{P}}_2$ if and only if $\sigma(A)$ is contained in the non-negative real axis.*

Proof. The necessity that $\sigma(A)$ is contained in the non-negative real axis is a consequence of Corollary 1.6 of [6], as in the proof of Proposition 1.

Assume that A is a Riesz operator and $\sigma(A) \subset [0, \infty)$. Given $\varepsilon > 0$, there are a finite number of points in $\sigma(A)$ whose moduli exceed ε. Using the Riesz decomposition theorem ([7], p. 31) we can express A as

$$A = \begin{bmatrix} F & Z \\ 0 & R \end{bmatrix}$$

where F is a finite rank operator with $\sigma(F) \subset (\varepsilon, \infty)$ and R is a Riesz operator with $\sigma(R) \subset [0, \varepsilon]$. The West decomposition ([2], p. 73) yields $R = K + Q$, where K is normal and compact and Q is quasinilpotent; furthermore $\sigma(K) = \sigma(R)$. Thus $\sigma(R) \subset [0, \varepsilon]$ implies $\|K\| \leqslant \varepsilon$; i.e., R can be approximated with Q. Now using Proposition 2 and the diagonal perturbation technique of Proposition 1. we may approximate F and Q with products of positive operators. ■

PROPOSITION 4. *If $A \in \mathcal{B}(\mathcal{H})$ and $p(A)$ is a Riesz operator for some polynomial p, then $A \in \overline{\mathcal{P}}_2$ if and only if $\sigma(A)$ is contained in the non-negative real axis. (In particular the assertion is true if A is polynomially compact.)*

Proof. The necessity of the condition is clear. For sufficiency, first note that A has a finite number of distinct non-negative points $r_1, \ldots, r_n$ in its essential spectrum and the remaining points of the spectrum are all isolated poles of finite rank. Thus using the Riesz decomposition theorem we can assume

$$A = \begin{bmatrix} r_1 I_1 + R_1 & A_{12} & \ldots & A_{1n} \\ 0. & r_2 I_2 + R_2 & & A_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & r_n I_n + R_n \end{bmatrix},$$

where $R_1, \ldots, R_n$ are Riesz operators and where

$$\sigma(r_i I_i + R_i) \cap \sigma(r_j I_j + R_j) = \emptyset$$

for $i \neq j$. Now, by Proposition 3, each R_i can be approximated with an element of $\mathcal{P}_2$. Since $\mathcal{P}_2$ is invariant under addition of non-negative scalars (i.e., $rI + PQ = = (rP + PQP)P^{-1}$ for positive P, Q and $r \geqslant 0$), we can approximate each $r_i I_i + R_i$ with a product $P_i Q_i$ in $\mathcal{P}_2$ such that, for $i \neq j$,

$$\sigma(P_i Q_i) \cap \sigma(P_j Q_j) = \emptyset.$$

Replacing each $r_i I_i + R_i$ with $P_i Q_i$ in the matrix of A we obtain an operator B that approximates A. But B is similar to the product

$$(P_1 \oplus \ldots \oplus P_n)(Q_1 \oplus \ldots \oplus Q_n)$$

(by Rosenblum's theorem, see [7], p. 8) and is thus in $\mathcal{P}_2$. So

$$A \in \overline{\mathcal{P}}_2. \qquad \blacksquare$$

THEOREM 1. *Let A be a normal operator. We have $A \in \overline{\mathcal{P}}_2$ if and only if every component of $\sigma(A)$ intersects the non-negative axis.*

Proof. The necessity that every component of $\sigma(A)$ intersects the non-negative axis is (as in Proposition 1) a consequence of Corollary 1.6 of [6], which may be seen as follows. If some component of $\sigma(A)$ did not intersect the non-negative axis, then it must be true that a clopen subset σ of $\sigma(A)$ does not intersect the non-negative axis, since the components in a compact space are obtained as the intersection of all the clopen sets containing a fixed point in the space (see [4] p. 246). It is then possible to find a Cauchy domain Ω that is a neighbourhood of σ that satisfies the hypothesis of Corollary 1.6 of [6] and such that the closure of Ω does not intersect the non-negative axis. Thus if A is a norm limit of operators B_i, eventually one has $\sigma(B_i) \cap \Omega \neq \emptyset$ by Corollary 1.6 of [6], which implies that A is not in $\overline{\mathcal{P}}_2$.

To prove the converse, assume first that $\sigma(A)$ has only one component. Choose a non-negative number r such that

$$0 \in \sigma(A - rI).$$

It follows that $A - rI$ is a limit of quasinilpotents (see Proposition 5.6 of [6]), and hence

$$A - rI \in \overline{\mathcal{P}}_2.$$

Note that $\mathcal{P}_2$, and hence $\overline{\mathcal{P}}_2$, is invariant under addition of non-negative scalars (see the proof of Proposition 4). Thus we have

$$A = (A - rI) + rI \in \overline{\mathcal{P}}_2.$$

If $\sigma(A)$ has finitely many components, then $A \in \overline{\mathcal{P}}_2$ since it is a direct sum of normal operators whose spectra have one component intersecting the non-negative axis (and a direct sum of elements in $\overline{\mathcal{P}}_2$ is again in $\overline{\mathcal{P}}_2$).

Finally, assume that $\sigma(A)$ has infinitely many components, each intersecting the non-negative real axis. By the previous paragraph, we will be done if we can prove that, given $\varepsilon > 0$, we can find a normal operator A_ε such that

$$\|A - A_\varepsilon\| < \varepsilon$$

and $\sigma(A_\varepsilon)$ has finitely many components, each meeting the non-negative axis. The remainder of the proof is devoted to constructing this operator A_ε.

Suppose $\varepsilon > 0$ is given and let

$$\{R_{ij} \mid i, j = 1, \ldots, k\}$$

be a grid of squares that covers $\sigma(A)$ such that the diameter of each square R_{ij} is smaller than $\varepsilon/2$. Technically speaking, choose

$$\delta < \sqrt{2}\varepsilon/4,$$

choose an even integer k such that

$$2(\|A\| + 1) < \delta k,$$

and for $i, j = 1, \ldots, k$ let

$$R_{ij} \equiv \left\{ z \;\middle|\; \begin{array}{l} \left(-\frac{k}{2} + j - 1\right)\delta < \operatorname{Re}(z) \leqslant \left(-\frac{k}{2} + j\right)\delta \\ \left(-\frac{k}{2} + i - 1\right)\delta < \operatorname{Im}(z) \leqslant \left(-\frac{k}{2} + i\right)\delta \end{array} \right\}.$$

Thus any two squares of the grid are disjoint and

$$\sigma(A) \subset \bigcup_{i,j=1}^{k} R_{ij}.$$

Let

$$K_{ij} \equiv \sigma(A) \cap R_{ij},$$

suppose that E is the spectral measure for A, and let $\mathcal{I}$ be the set of pairs (i, j) such that $E(K_{ij}) \neq 0$.

We assert that we may assume, with no loss of generality, that the range of $E(K_{ij})$ is infinite dimensional for every $(i, j) \in \mathcal{I}$. The reason for this is that we may decompose A into a direct sum

$$A = A_1 \oplus A_2,$$

where A_1 is the restriction of A to the range of the sum of those $E(K_{ij})$ that are finite dimensional. It follows that A_1 is a normal operator on a finite dimensional space with $\sigma(A_1)$ contained in the non-negative real axis, and thus $A_1 \in \overline{\mathcal{P}}_2$. We are then left with the task of showing that $A_2 \in \overline{\mathcal{P}}_2$, and A_2 is an operator that satisfies the hypothesis of this theorem (since $\sigma(A) \setminus \sigma(A_2)$ is finite) with the additional property that, if F is the spectral measure of A_2, then $F(K_{ij})$ is either 0 or infinite rank.

For each $(i,j) \in \mathcal{I}$, let J_{ij} be a normal operator defined on the range of $E(K_{ij})$ with $\sigma(J_{ij}) = \overline{R_{ij}}$. If

$$A_\varepsilon \equiv \bigoplus_{(i,j)\in\mathcal{I}} J_{ij},$$

then

$$\|A_\varepsilon - A\| < \varepsilon$$

and

$$\sigma(A_\varepsilon) = \bigcup_{(i,j)\in\mathcal{I}} \overline{R_{ij}}.$$

It is obvious that $\sigma(A_\varepsilon)$ has finitely many components. If C is one such component, then there exists

$$z \in C \cap \sigma(A).$$

Since C must contain the component of $\sigma(A)$ containing z, which intersects the non-negative axis by hypothesis, we have that C intersects the non-negative axis, which completes the proof of the theorem. ■

COROLLARY 1. *If U is a unitary operator, then $U \in \overline{\mathcal{P}}_2$ if only if $\sigma(U)$ is a connected set containing 1. In particular, the bilateral shift is in $\overline{\mathcal{P}}_2$.*

3. APPROXIMATION BY PRODUCTS OF FOUR POSITIVE OPERATORS

The following result will be proved later in this section once we have shown that biquasitriangular operators are in $\overline{\mathcal{P}}_4$, but we cannot resist the temptation of pluking it as a corollary of Proposition 2 and a nice result of Fong and Surour [3].

COROLLARY 2. *If A is a compact operator, then $A \in \overline{\mathcal{P}}_4$.*

Proof. By Theorem 6 of [3], every compact operator is a product of two quasinilpotent operators. It follows from Proposition 2 that $A \in \overline{\mathcal{P}}_4$. ■

THEOREM 2. *If A is algebraic, then $A \in \overline{\mathcal{P}}_4$.*

Proof. We may write

$$A = \begin{bmatrix} s_1 I_1 & A_{12} & \dots & A_{1k} \\ 0 & s_2 I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{k-1\,k} \\ 0 & \dots & 0 & s_k I_k \end{bmatrix},$$

relative to some decomposition $\mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_k$ of $\mathcal{H}$. We may approximate A as closely as we wish with an operator

$$B = \begin{bmatrix} r_1 I_1 & A_{12} & \ldots & A_{1k} \\ 0 & r_2 I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{k-1\,k} \\ 0 & \ldots & 0 & r_k I_k \end{bmatrix}$$

such that the r_i are distinct non-zero complex numbers. Thus, we are done if we prove that $B \in \overline{\mathcal{P}}_4$. Since $\mathcal{P}_2$ is similarity invariant (recall that $\mathcal{P}_2$ is equal to the set of operators similar to positive invertible operators [1], [8]), it follows that $\mathcal{P}_4$, and hence $\overline{\mathcal{P}}_4$ is similarity invariant. Thus we have that $B \in \overline{\mathcal{P}}_4$ if and only if

$$r_1 I_1 \oplus \ldots \oplus r_k I_k \in \overline{\mathcal{P}}_4.$$

To prove the relation above, assume that $\varepsilon > 0$ has been given. We will repeatedly use the fact that direct sum of operators in $\overline{\mathcal{P}}_4$ are in $\overline{\mathcal{P}}_4$. Notice first that if $\mathcal{M}_j$ is infinite dimensional, then $r_j I_j$ is in $\overline{\mathcal{P}}_4$. This is because it is posible to choose a complex number z in the ε disk around r_j and an integer n such that $r_j z^n > 0$. Thus, $r_j I_j$ is ε close to an infinite direct sum of $(n+1) \times (n+1)$ matrices identical to

$$\begin{pmatrix} r_j & 0 & \ldots & 0 \\ 0 & z & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & z \end{pmatrix},$$

which is in $\mathcal{P}_4$ since each summand is a non-scalar matrix with positive determinant (see Theorem 4 of [1]). Thus $r_j I_j$ is in $\overline{\mathcal{P}}_4$ in case $\mathcal{M}_j$ is infinite dimensional.

This reduces the problem to the case where $\mathcal{M}_1$ is infinite dimensional and $\mathcal{M}_j$ is finite dimensional for $j = 2, 3, \ldots, k$. Let

$$d = r_2 r_3 \ldots r_k,$$

and choose a complex number z in the ε disk around r_1 and an integer n such that $dz^n > 0$ and $z \neq r_2$. Then, borrowing n dimensions from $\mathcal{M}_1$, we have that

$$r_1 I_1 \oplus \ldots \oplus r_k I_k$$

is ε close to

$$r_1 I_{11} \oplus z I_{12} \oplus r_2 I_2 \oplus r_3 I_3 \oplus \ldots \oplus r_k I_k,$$

where I_{12} is the identity on an n dimensional subspace of $\mathcal{M}_1$, and

$$I_1 = I_{11} \oplus I_{12}.$$

By our comments above, we know that $r_1 I_{11} \in \overline{\mathcal{P}}_4$, and by the choice of z and n we have that

$$zI_{12} \oplus r_2 I_2 \oplus r_3 I_3 \oplus \ldots \oplus r_k I_k$$

is a non-scalar operator with

$$\det(zI_{12} \oplus r_2 I_2 \oplus r_3 I_3 \oplus \ldots \oplus r_k I_k) > 0.$$

It follows that

$$zI_{12} \oplus r_2 I_2 \oplus r_3 I_3 \oplus \ldots \oplus r_k I_k \in \overline{\mathcal{P}}_4$$

by Theorem 4 of [1]. Thus

$$r_1 I_1 \oplus \ldots \oplus r_k I_k \in \overline{\mathcal{P}}_4$$

and the proof is complete. ■

Recall that an operator T is a *quasitriangular operator* if there exists an increasing sequence of finite rank projections $\{P_n\}$ such that $P_n \to I$ strongly and $\|(I - P_n)TP_n\| \to 0$. We say that T is *biquasitriangular* if both T and T^* are quasitriangular. The class of all biquasitriangular operators on a Hilbert space coincides with the uniform closure of the set of algebraic operators on the Hilbert space. In particular, every normal operator is biquasitriangular. (See [6] for a complete discussion.)

COROLLARY 3. *If A is a biquasitriangular operator, then*

$$A \in \overline{\mathcal{P}}_4.$$

Thus, every normal operator is in $\overline{\mathcal{P}}_4$.

Proof. If A is a biquasitriangular operator, then A is a limit of algebraic operators (see Theorem 6.15 of [6]), and thus in $\overline{\mathcal{P}}_4$. ■

4. APPROXIMATION BY PRODUCTS OF FIVE POSITIVE OPERATORS

We are now in a position to see why membership in $\overline{\mathcal{P}}_n$ implies membership in $\overline{\mathcal{P}}_5$. Given an operator A, let $\nu(A)$ denote the nullity of A.

THEOREM 3. *The closures of each of the following sets are equal to $\overline{\mathcal{P}}_5$;*

1) *the set of Fredholm operators with index 0, denoted* $\mathcal{F}_0$*;*
2) *the set of operators* A *such that* $\nu(A) = \nu(A^*)$*, denoted* $\mathcal{V}$*.*
3) *the set of invertible operators, denoted* $\mathcal{B}(\mathcal{H})^{-1}$*;*
4) $\mathcal{P}_n$ *for* $n \geqslant 5$*;*
5) $\mathcal{Q}_n$ *for* $n \geqslant 5$*;*
6) $\mathcal{P}_\infty$*.*

Proof. If $n \geqslant 5$, then the following inclusions are transparent;

$$\mathcal{P}_5 \subset \mathcal{P}_n \subset \mathcal{P}_\infty \subset \mathcal{B}(\mathcal{H})^{-1} \subset \mathcal{F}_0 \subset \mathcal{V}.$$

Thus, if we prove that $\mathcal{V} \subset \overline{\mathcal{P}}_5$, we will have established the equality of the closures of the above sets (and the fact that the closures coincide with $\overline{\mathcal{Q}}_n$ for $n \geqslant 5$ follows from the fact that $\mathcal{P}_n$ is dense in $\mathcal{Q}_n$, as mentioned in the preliminaries). If $A \in \mathcal{V}$, and if $A = UP$ is the polar decomposition of A, then it is possible to extend U to a unitary operator V such that $A = VP$ (mimic the proof of Problem 135 in [5]). Since V is biquasitriangular, we have by Corollary 3 that V is an element of $\overline{\mathcal{P}}_4 = \overline{\mathcal{Q}}_4$. Since $P \in \mathcal{Q}_1$, it follows that $A = VP \in \overline{\mathcal{Q}}_5 = \overline{\mathcal{P}}_5$. ■

5. QUESTIONS

There are a number of avenues left for investigation in the area of approximating operators with products of positive operators. Even the sleepiest reader may have noticed no mention of $\overline{\mathcal{P}}_3$. We have not made any progress in deciding which operators are approximable by a product of three positive operators, but we feel there are tractable theorems that will be found (e.g. using Ballantine's characterization [1, Theorem 6] for products of three positive operators on finite dimensional spaces).

Our results clearly show an abundance of new operators when passing from $\overline{\mathcal{P}}_2$ to $\overline{\mathcal{P}}_4$; is it possible that $\overline{\mathcal{P}}_4$ is everything obtainable, that is does $\overline{\mathcal{P}}_4$ equal $\overline{\mathcal{P}}_5$? Another possibility is that $\overline{\mathcal{P}}_4$ is precisely the set of biquasitriangular operators, in which case we would know that $\overline{\mathcal{P}}_4$ is properly included in $\overline{\mathcal{P}}_5$. This follows from the observation that if S is the unilateral shift of multiplicity one, then $S + 2$ is in $\overline{\mathcal{P}}_5$, but $S + 2$ is not biquasitriangular (see [6, Theorem 6.15]).

REFERENCES

1. BALLANTINE, C. S., Products of positive definite matrices. IV, *Linear Algebra Appl.*, 3(1970), 79–114.
2. BARNES, B. A.; MURPHY, G. J.; SMYTH, M. R. F.; WEST, T. T. , *Riesz and Fredholm theory in Banach algebras*, Pitman, Boston, 1982.

3. FONG, C. K.; SUROUR, A. R., Sums and products of quasinilpotent operators, *Proc. Royal Soc. Edinburgh*, Sect. A **99**(1984), 193–209.
4. GILLMAN, L.; JERISON, M., *Rings of continous function*, Springer-Verlag, New York, 1976.
5. HALMOS, P. R., *A Hilbert space problem book*, 2d. ed., Springer-Verlag, New York, 1982.
6. HERRERO, D. A., *Approximation of Hilbert space operators. I*, Pitman Research Notes in Mathematics Series, No. 72, John Wiley and Sons, New York, 1982.
7. RADJAVI, H.; ROSENTHAL, P., *Invariant subspaces*, Springer, Berlin, 1973.
8. WU, P.-Y., The operators factorization problems, *Linear Algebra Appl.*, **117**(1988), 35–63.

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Added in proof. As Pei Yuan Wu pointed out, it follows from results of Apostol and Morrel (*Indiana Univ. Math. J.*, **26**(1977), 424–442) that A belongs to the closure of $\mathcal{P}_2$ if and only if every component of $\sigma(A)$ and $\sigma_e(A)$ intersects the non-negative real axis and the semi-Fredholm index of $A-\lambda$ is zero whenever defined.