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ALGEBRAS GENERATED BY MUTUALLY ORTHOGONAL IDEMPOTENT OPERATORS

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1. INTRODUCTION

The map $a \mapsto \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ embeds each linear space S of operators acting on a Hilbert space H into a commutative algebra $\mathcal{S}$ of operators acting on $H \oplus H$. This time-honored trick shows that commutative operator algebras can be as badly behaved as arbitrary linear spaces of operators: their ampliations need not be reflexive, the relative weak operator and weak* topologies need not coincide on them and their closures in these topologies may also differ. On the other hand, J. Deddens and P. Fillmore proved that ampliations of operators acting on finite-dimensional spaces are always reflexive. This seemed to suggest that singly generated algebras might be better behaved. W. Wogen dashed these hopes in [5] by showing that every weak*-closed linear space of operators occurs as a direct summand of a singly-generated operator algebra.

In particular, Wogen showed that on infinite-dimensional spaces, (the algebra generated by) an ampliation $a \oplus a$ need not be reflexive and direct sums of reflexive operators need not be reflexive. Further effort produced a reflexive operator whose direct sum with the 0 operator fails to be reflexive. D. Larson and W. Wogen [3] accomplished this by a delicate analysis of certain algebras generated by mutually orthogonal families of idempotent operators. The relative wot and w* topologies agree on these algebras as do their closures in these topologies. In the last paragraph of [3], the authors ask whether these are general properties of algebras generated by commuting families of idempotents.

On finite-dimensional spaces, commuting families of idempotents are diagonalizable and hence they generate reflexive algebras. Despite this encouraging evidence, we show in this paper that every w*-closed linear space of operators is a direct summand of an algebra generated by mutually orthogonal idempotents. This provides negative answers to the questions posed in [3].

We prove our main embedding theorem in Section 2. In order to keep the exposition simple, we stick to the norm topology and postpone applications to later sections. The construction is basically matricial, but we find it helpful to formulate the result in coordinate-free fashion. As a by-product of the proof, we also learn that every linear space of matrices "is" the limit of a sequence of diagonalizable algebras.

Sections 3, 4, and 5 address questions raised in [3]. Let S be a set of bounded operators acting on a common Hilbert space. We write $[S]$ for the linear span of S, $[[S]]$ for the linear span of S and the identity operator, and alg $[S]$ for the linear span of all products of members of S. We will call S *algebraically orthogonal* if each of its members is idempotent and the product of each pair of distinct members of S is the zero operator. We also write nrm(S), w$^*(S)$, and wot(S) for the closures of S in the norm, weak-* and weak operator topologies respectively. Finally, we abreviate $L(\mathbf{C}_n)$ by M_n.

In Section 3, we apply Theorem 2.3 to construct algebraically orthogonal families $\mathcal{F}$ which provide answers to three questions raised in the last paragraph of [3]:

(1) wot$[[\mathcal{F}]]$ need not be reflexive,

(2) w$^*[[\mathcal{F}]]$ need not be closed in the weak operator topology, and

(3) the relative weak operator and weak* topologies need not coincide on wot$[[\mathcal{F}]]$.

In Section 4, we apply Theorem 2.5 to construct a block-diagonal version of (2).

In Section 5, we restrict attention to families $\mathcal{F}$ in "staircase form". We show that the phenomena of the preceding paragraph cannot occur in this setting: if $\mathcal{F}$ is in staircase form, wot$[[\mathcal{F}]]$ must be reflexive and 3-elementary. We also address Larson and Wogen's remaining questions in this setting: concrete criteria are presented for deciding when the identity operator belongs to the reflexive and weak operator closures of $[\mathcal{F}]$.

We close this section with several elementary but useful observations.

PROPOSITION 1.1. *Let $\mathcal{F}$ be an algebraically orthogonal family of operators on a separable Hilbert space H. Then*

(a) alg$[\mathcal{F}] = [\mathcal{F}]$,

(b) $\mathcal{F}$ *is countable,*

(c) nrm$[\mathcal{F}]$ *is singly generated.*

Proof. There is no harm in assuming that $0 \notin \mathcal{F}$.

(a) Clear from the definition of algebraically orthogonal.

(b) For each $a \in \mathcal{F}$ choose a unit vector x_a in its range. Then for $a, b \in \mathcal{F}$ with

$a \neq b$, we have $1 = \|a(x_a)\| = \|a(x_a - x_b)\| \leqslant \|a\|\,\|x_a - x_b\|$. Thus each point in the set $\{x_a \mid a \in \mathcal{S}\}$ is isolated. The conclusion follows since this set has the same cardinality as $\mathcal{S}$.

(c) Suppose $\mathcal{S} = \{T_n\}_{n=1}^{\infty}$. Define a sequence $\{b_n\}_{n=1}^{\infty}$ of positive real numbers satisfying $\frac{b_n}{b_k}\|T_n\| \leqslant 2^{-n}$ for each $1 \leqslant k < n$. Note that $\sum_{n=1}^{\infty} b_n T_n$ converges in norm to an operator T. Since $\mathcal{S}$ is algebraically orthogonal, we have for each $k \geqslant 1$,

$$(b_1^{-1}T)^k - T_1 = \sum_{n=2}^{\infty}(b_1^{-1}b_nT_n)^k.$$

The norm of the right hand member of this equation is majorized by $\sum_{n=2}^{\infty} 2^{-nk}$ so T_1 belongs to the norm closed algebra generated by T. Similarly, the equation

$$[b_2^{-1}(b_1^{-1}T - T_1)]^k T_2 = \sum_{n=3}^{\infty}(b_2^{-1}b_nT_n)^k$$

shows that $T_2 \in \text{nrm}(\text{alg}[T])$ and $T_n \in \text{nrm}(\text{alg}[T])$ for all $n \geqslant 1$ by an easy induction argument. ■

2. EMBEDDING

We begin by making our notion of embedding precice. Recall that a *partial isometry* is an operator u, possibly acting between different Hilbert spaces, for which u^*u is a projection. As multiplication of $u^*uu^* - u^*$ by its adjoint then yields zero, we see that $u^*uu^* = u^*$ whence u^* is also a partial isometry. The ranges of u^* and u are closed and have the same Hilbert space dimension; they are called the *initial* and *final* spaces of u respectively. Conversely, every pair of Hilbert subspaces of the same dimension occur as the initial and final spaces of some partial isometry. When u^*u is the identity operator, u is called an *isometry*.

DEFINITION 2.1. Let u and v be isometries acting between Hilbert spaces H and $\mathcal{H}$ and suppose A is a subset of $L(H)$. Then uAv^* is a subset of $L(\mathcal{H})$ called a *copy* of A. If B is a subset of $L(\mathcal{H})$ satisfying $uAv^* \subseteq B$ and $u^*Bv \subseteq A$, then we say that A is a *(spatial) direct summand* of B.

Intuitively, one constructs a copy of A by bordering its members with rows and columns of zeros. If $\mathcal{A}$ contains all these zero-bordered matrices and all other members of $\mathcal{A}$ differ from these only in their borders, then A will be a direct summand of $\mathcal{A}$.

EXAMPLE 2.2. Let $\mathcal{H}$ be the direct sum of three copies of H. We use 3 by 3 block matrices to represent operators on $\mathcal{H}$. Define u and $v : H \to \mathcal{H}$ by $u(x) \equiv x \oplus 0 \oplus 0$ and $v(x) \equiv 0 \oplus 0 \oplus x$. Then $u^*(x \oplus y \oplus z) = x$ and $v^*(x \oplus y \oplus z) = z$.

For $A \subseteq L(H)$, we have $uAv^* = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : a \in A \right\}$; calling this a copy of A is certainly justified. On the other hand, given a linear space $\mathcal{A} \subseteq L(\mathcal{H})$, we see that $u^*\mathcal{A}v$ is the set of (1,3) block entries of the members of $\mathcal{A}$. The direct sum terminology is appropriate since the containment $uu^*\mathcal{A}vv^* \subseteq \mathcal{A}$ guarantees that the sum of

$$\{b \in \mathcal{A} \mid \text{the (1,3) block of } b \text{ is zero }\} \text{ and}$$

$$\{b \in \mathcal{A} \mid \text{all blocks other than the (1,3) block of } b \text{ are zero}\}$$

exhausts $\mathcal{A}$. ■

THEOREM 2.3. *Given a subset S of $L(H)$ there exists an algebraically orthogonal subset $\mathcal{S}$ of some $L(\mathcal{H})$ such that the norm-closed span of S is a direct summand of the norm-closed span of $\mathcal{S}$. If S is countable and H is separable then $\mathcal{H}$ can also be taken to be separable.*

Proof. We first concentrate on the case when S is countable and H is infinite--dimensional.

Let $S = \{a_i | i \in \mathbb{N}\}$. Choose a family $\{p_{im} | i, m \in \mathbb{N}\}$ of mutually orthogonal self-adjoint projections on H with $\text{rank}(p_{im}) = \text{rank}(a_i)$ for each i and m. Write q_i for the self-adjoint projection onto the closure of the range of a_i and for each pair (i, m) choose a partial isometry w_{im} having initial space equal to the range of p_{im} and final space equal to the range of q_i. This means $w_{im}^* w_{im} = p_{im}$ and $w_{im} w_{im}^* = q_i$ so that in particular $w_{im} = w_{im} p_{im}$ and $w_{im}^* = p_{im} w_{im}^*$.

For each i and m define an operator on $\mathcal{H} \equiv H \oplus H \oplus H$ by

$$b_{im} \equiv \begin{bmatrix} 0 & mw_{im} & m^2 a_i \\ 0 & p_{im} & mw_{im}^* a_i \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $b_{im} b_{jn} = \begin{bmatrix} 0 & mw_{im}p_{jn} & mnw_{im}w_{jn}^* a_j \\ 0 & p_{im}p_{jn} & np_{im}w_{jn}^* a_j \\ 0 & 0 & 0 \end{bmatrix}$. If $i \neq j$ or $m \neq n$, then all entries in this matrix vanish. ($p_{im}p_{jn} = 0$ since they are mutually orthogonal projections while for example $mw_{im}p_{jn} = mw_{im}p_{im}p_{jn} = 0$.) Similar computations show that $(b_{im})^2 = b_{im}$, so the set $\mathcal{S} \equiv \{b_{im} | i, m \in \mathbb{N}\}$ is a countable algebraically orthogonal family of idempotents.

Take u and v as in Example 2.2. (Note for use in Corollary 3.3 that the ranges of u and v are orthogonal so $u^*v = 0$.) By construction, the (1,3) block of each member of $\mathcal{S}$ belongs to S. Since multiplication by u and v^* are linear and norm-continuous, we see that $u^*(\mathrm{nrm}[\mathcal{S}])v \subseteq \mathrm{nrm}[S]$. On the other hand for each fixed i, we have

$$\lim_{m\to\infty} m^{-2}b_{im} = \begin{bmatrix} 0 & 0 & a_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{nrm}[\mathcal{S}] \quad \text{whence } u(\mathrm{nrm}[S])v^* \subseteq \mathrm{nrm}[\mathcal{S}].$$

We have shown $\mathrm{nrm}[S]$ is a direct summand of $\mathrm{nrm}[\mathcal{S}]$ as desired.

In the general case (H possibly finite-dimensional, S possibly uncountable), first construct a copy of S acting on an infinite-dimensional space whose dimension exceeds the cardinality of S. Since a copy of a copy is a copy, we may as well assume this was true of H. From here on the only change necessary in the original proof is the possible use of a larger index set for i. ∎

EXAMPLE 2.4. No algebraically orthogonal subset of M_n can contain a copy of M_2 in its linear span.

Proof. Let $\mathcal{S}$ be an algebraically orthogonal subset of M_n. Consideration of Jordan Canonical Forms shows that every singly-generated subalgebra of M_n has a separating vector. Thus every subspace of $\mathrm{alg}\,\mathcal{S}$ has a separating vector. Since M_2 does not have a separating vector, no copy of it can be found in any such algebra. ∎

Thus even when the set S of Theorem 2.3 acts on $\mathbb{C}_2$, it may be necessary to take $\mathcal{H}$ to be infinite-dimensional. Reexamining the proof of Theorem 2.3 does, however, lead to the following approximation result.

THEOREM 2.5. *Let S be a subset of M_k. Then there is a sequence $\{\mathcal{S}_m\}$ of algebraically orthogonal subsets of some M_n whose linear spans converge to a copy of $[S]$ in the gap metric.*

We need some definitions to make this statement precise. Given a subset A of a Banach space X, we write $\mathrm{Ball}(A)$ for the intersection of A with the closed unit ball in X.

DEFINITION 2.6. Let X be a finite dimensional Banach space and write $\mathcal{F}(X)$ for the collection of its linear subspaces. The *gap metric* on $\mathcal{F}$ is defined by $\rho(A, B) \equiv$ $\equiv \inf\{r > 0 \mid$ for each $x \in \mathrm{Ball}(A)$, there is an $y \in \mathrm{Ball}(B)$ satisfying $\|x - y\| <$ $< r$ and vice versa$\}$.
(This is the Hausdorff distance between the unit balls of A and B.)

LEMMA 2.7. *If v and w are vectors in a normed linear space with $\|v\| = 1$, then*

$$\left\| v - \frac{w}{\|w\|} \right\| \leqslant 2\|v - w\|.$$

Proof.

$$\left\| \frac{w}{\|w\|} - w \right\| = \|w\| \left\| \frac{1}{\|w\|} - 1 \right\| = |\,\|w\| - 1| = |\,\|w\| - \|v\|\,| \leqslant \|v - w\|.$$

Therefore, $\left\| v - \frac{w}{\|w\|} \right\| \leqslant \|v - w\| + \left\| \frac{w}{\|w\|} - w \right\| \leqslant 2\|v - w\|.$ ■

LEMMA 2.8. *Let $x_1, x_2, \ldots, x_k$ be independent members of X. Given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x_i - y_i\| < \delta$ for each $i = 1, 2, \ldots, k$ implies*

$$\rho([x_1, \ldots, x_k], [y_1, \ldots, y_k]) < \varepsilon.$$

Proof. Extend $x_1, x_2, \ldots, x_k$ to a basis $x_1, x_2, \ldots, x_n$. Since all norms on the finite dimensional space $L(X)$ are equivalent, there is a constant r satisfying $\|a\| \leqslant \leqslant r \max_{1 \leqslant i \leqslant n} \|ax_i\|$ for each $a \in L(X)$. Assume that $\varepsilon < 1$ and take $\delta = \frac{\varepsilon}{4r}$.

Suppose now that $y_1, \ldots, y_k$ are given. Set $y_i = x_i$ for $i > k$ so that $\|x_i - y_i\| < \delta$ for $i = 1, 2, \ldots, n$. Write I for the identity operator on X and b for the linear transformation which sends x_i to the corresponding y_i. We have $\|I - b\| < \frac{\varepsilon}{4}$. Let x be a unit vector in $[x_1, \ldots, x_k]$. Then $bx \in [y_1, \ldots, y_k]$ satisfies $\|x - bx\| < \frac{\varepsilon}{4}$. In view of the preceding lemma, $\frac{bx}{\|bx\|}$ is a unit vector in $[y_1, \ldots, y_k]$ within distance $\frac{\varepsilon}{2}$ of x.

Moreover, $\|I - b^{-1}\| < \frac{\varepsilon}{2}$, so given a unit vector $y \in [y_1, \ldots, y_k]$, the preceding argument tells us that $\frac{b^{-1}y}{\|b^{-1}y\|}$ is a unit vector in $[x_1, \ldots, x_k]$ within distance ε of y. ■

Proof of Theorem 2.5. There is no loss of generality in assuming that S is independent say $S = \{a_i | i \leqslant r\}$. Also, since a copy of a copy is a copy, there is no harm in assuming that k exceeds the sum of the ranks of the $\{a_i\}$. Choose a family $\{p_i\}$ of mutually orthogonal self-adjoint projections in M_k with $\operatorname{rank}(p_i) = \operatorname{rank}(a_i)$. Write q_i for self-adjoint projection onto the range of a_i and choose a partial isometry w_i having initial space equal to the range of p_i and final space equal to the range of q_i. For each $m \in \mathbb{N}$, set $\mathcal{S}_m \equiv \{b_{im} | i \leqslant r\}$ where

$$b_{im} = \begin{bmatrix} 0 & mw_i & m^2 a_i \\ 0 & p_i & mw_i^* a_i \\ 0 & 0 & 0 \end{bmatrix}.$$

Then each family $\mathcal{S}_m$ is algebraically orthogonal, and for $i = 1, \ldots, r$ we have

$\lim_{m\to\infty} m^{-2}b_{im} = \begin{bmatrix} 0 & 0 & a_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Applying Lemma 2.8, we see that the sequence $\{[\mathcal{S}_m]\}$ converges to a copy of $[S]$ in the gap metric on M_{3k}. ■

3. GENERAL COUNTEREXAMPLES

In this section, we apply Theorem 2.3 to construct counterexamples to three questions raised in [3]. We begin with a brief review of the relevant notation and terminology from [1].

The full algebra of bounded linear operators on a Hilbert space H is denoted by $L(H)$ and we write $T(H)$ for its trace class ideal. When the underlying Hilbert space is clear, reference to it will be suppressed. We use the standard notations for distinguished subsets of T: F for the subideal of finite rank operators and F_n for the collection of operators of rank at most n. We adopt the "inner product" notation for the duality between L and T so that the value of $\langle a,t\rangle$ is the trace of the product of $a \in L$ with $t \in T$. We also employ the corresponding preannihilator and annihilator notations: $S_\perp \equiv \{t \in T | \langle a,t\rangle = 0 \text{ for all } a \in S\}$ when S is a subset of L while $S^\perp \equiv \{a \in L(H) | \langle a,t\rangle = 0 \text{ for all } t \in S\}$ when S is a subset of T.

F and T induce weak topologies on L called the *weak operator* (wot) and *weak** (w*) topologies respectively. The three topologies nrm, wot, and w* will play a role in the sequel; we write $\tau(S)$ for the closure of S in the τ topology. By contrast, the only topology we will consider on T is the one arising from the trace norm.

From topological vector space theory we know that $(S_\perp)^\perp = \text{w}^*[S]$, the weak* closure of the linear span of S, while $(S_\perp \cap F)^\perp = \text{wot}[S]$; we call $(S_\perp \cap F_1)^\perp$ the *reflexive closure* of S, denoted ref(S). In particular, a linear manifold $M \subset L$ is wot-closed iff $(M_\perp \cap F)^\perp = M$. We say M is *n-reflexive* if $(M_\perp \cap F_n)^\perp = M$.

The equation $M_\perp + F = T$ characterizes those linear manifolds on which the relative wot and w* topologies coincide; we say that M is *n-elementary* if $M_\perp +$ $+F_n = T$. "Reflexive" and "elementary" stand for "1-reflexive" and "1-elementary" respectively.

Let $\mathcal{S}$ be an algebraically orthogonal set of operators. The authors of [3] constructed an example satisfying ref$(\mathcal{S}) = \text{wot}[[\mathcal{S}]]$ but id $\notin \text{wot}[\mathcal{S}]$ and they raised the following questions:

(1) Is wot$[[\mathcal{S}]]$ always reflexive?

(2) Is w*$[[\mathcal{S}]]$ always weakly closed?

(3) Must the relative w* and wot topologies agree on wot$[[\mathcal{S}]]$?

(4) When does id belong to wot$[\mathcal{S}]$?

(5) When does id belong to ref([$\mathcal{S}$])?
A counterexample to the first question was added to [3] in proof.

The reader familiar with block matrices will have little trouble applying Theorem 2.3 to construct counterexamples to Questions 1–3 for himself. The basic idea is that when $\mathcal{A}$ is an algebra of 3×3 block matrices with a copy of a linear space M of operators in its northeast corner, then $\mathcal{A}_\perp$ will contain a copy of $M_\perp$ in its southwest corner so $\mathcal{A}$ reflects any pathology S may exhibit. We have chosen a slightly longer treatment below in order to complete our discussion of the coordinate-free approach.

LEMMA 3.1. *Let A and B be subsets of $L(H)$ and $L(K)$ respectively and suppose u and v are isometries mapping H into K. Suppose $uAv^* \subseteq B$ and $u^*Bv \subseteq A$. Finally, let G stand for T, F, or F_k. Then*

(1) $u^*Bv = A$
(2) $v^*B_\perp u \subseteq A_\perp$
(3) $vA_\perp u^* \subseteq B_\perp$
(4) $v^*B_\perp u = A_\perp$
(5) $v^*(B_\perp + G(K))u = A_\perp + G(H)$
(6) $(A_\perp \cap G(H))^\perp = u^*(B_\perp \cap G(K))^\perp v$

Proof. (1) Multiply the first inclusion of the hypothesis on the left by u^* and on the right by v and compare with the second inclusion of the hypothesis.

(2) Let $x = v^*\varphi u$ for some $\varphi \in B_\perp$ and suppose $a \in A$. Then $\langle a, x\rangle = \langle a, v^*\varphi u\rangle = = \langle uav^*, \varphi\rangle = 0$ whence $x \in A_\perp$.

(3) Similar to (2).

(4) Multiply the inclusion of (3) by v^* on the left and by u on the right and compare with the inclusion of (2).

(5) We have $v^*G(K)u \subseteq G(H)$ and $vG(H)u^* \subseteq G(K)$ whence $v^*G(K)u = G(H)$. Thus an appeal to (4) completes the argument.

(6) ($\subseteq$) Let $x \in (A_\perp \cap G(H))^\perp$ and suppose $t \in B_\perp \cap G(K)$. By (2), we have $v^*tu \in A_\perp \cap G(K)$ so $\langle x, v^*tu\rangle = \langle uxv^*, t\rangle = 0$. Thus $uxv^* \in (B_\perp \cap G(K))^\perp$. Multiply on the left by u^* and on the right by v. The proof of the opposite inclusion is similar. ■

PROPOSITION 3.2. *Suppose A is a spatial direct summand of B.*

(1) wot[A] *is a spatial direct summand of* wot[B].
(2) w*[A] *is a spatial direct summand of* w*[B].
(3) *Any of the following properties enjoyed by B will also be enjoyed by A:*
k-reflexive, w-closed,* wot*-closed, k-elementary,*
relative wot *and* w* *topologies agree.*

Proof. (1), (2) Multiplication by a fixed operator is linear and continuous in these topologies so the containments $uAv^* \subseteq B$ and $u^*Bv \subseteq A$ persist when A and B are replaced by the wot and w* closures of their linear spans.

(3) If B is k-reflexive, then $(B_\perp \cap F_k(K))^\perp = B$. By parts (1) and (6) of the preceding lemma, $(A_\perp \cap F_k(H))^\perp = u^*Bv = A$ so A is k-reflexive. Replace F_k by F and T to obtain the desired conclusions concerning wot and w* closure respectively.

If B is k-elementary, apply parts (5) and (4) of the lemma to conclude $A_\perp + + F_k(H) = v^*T(K)u = T(H)$ whence A is k-elementary. Replace F_k by F to establish the assertion concerning relative topologies. ∎

COROLLARY 3.3. *Given a trace class operator $t \in L(H)$ there exists an algebraically orthogonal subset $\mathcal{S}$ of $L(\mathcal{H})$ such that $t^\perp$ is a spatial direct summand of* w*$[[\mathcal{S}]]$.

Proof. Apply Theorem 2.3 to a countable w*-dense subset S of $t^\perp$. This gives us an algebraically orthogonal subset $\mathcal{S}$ of some $L(\mathcal{H})$ and isometries u, v satisfying $u(\mathrm{nrm}[S])v^* \subseteq \mathrm{nrm}[\mathcal{S}]$ and $u^*(\mathrm{nrm}[\mathcal{S}])v \subseteq \mathrm{nrm}[S]$. Moreover, since the ranges of u and v are orthogonal, these inclusions persist when the identity is added to $\mathcal{S}$, i.e. $\mathrm{nrm}[S]$ is a spatial direct summand of $\mathrm{nrm}[[\mathcal{S}]]$. Appeal to part (2) of the preceding result completes the proof. ∎

EXAMPLE 3.4. There is an algebraically orthogonal set $\mathcal{S}$ such that wot$[[\mathcal{S}]]$ is not reflexive.

Proof. Take t to be of rank two. Then $t^\perp$ is wot-closed but non-reflexive. Choosing $\mathcal{S}$ as in Corollary 3.3 and applying Proposition 3.2(1), we get $t^\perp$ as a direct summand of wot$[[\mathcal{S}]]$. Proposition 3.2(3) completes the proof. ∎

EXAMPLE 3.5. There is an algebraically orthogonal set $\mathcal{S}$ such that w*$[[\mathcal{S}]]$ is not wot-closed.

Proof. Take t to be of infinite rank in Corollary 3.3. ∎

EXAMPLE 3.6. There is an algebraically orthogonal set $\mathcal{S}$ such that the relative wot and w* topologies do not agree on wot$[[\mathcal{S}]]$.

Proof. Take $t = 0$ in Corollary 3.3. ∎

REMARK 3.7. By Proposition 1.1, every algebra generated by an algebraically orthogonal set of operators is in fact singly generated, so the construction in this section serves some of the purposes of [5].

4. BLOCK DIAGONAL CONSTRUCTION

In the sequel, M_n will denote $L(\mathbb{C}_n)$ equipped with the operator norm and the subordinate metric. While singly-generated subalgebras of M_n are well-behaved, techniques going back to D. Hadwin and E. Nordgren show that direct sums of such algebras can be quite refractory. In this section we use Theorem 2.5 to show that the answer to Larson and Wogen's third question remains negative in this block diagonal setting. See Section 7 of [1] for further discussion and historical comments on the following quantified version of the k-elementary concept.

DEFINITION 4.1. Let S be a subspace of $L(H)$ and suppose r is a positive number. Then we say S has Property $\mathbb{A}_{1/k}(r)$ if $\operatorname{Ball} T(H) \subseteq S_\perp + r\operatorname{Ball} F_k(H)$.

In this definition, balls are taken relative to the Banach space $T(H)$ equipped with the trace norm. Clearly any S with Property $\mathbb{A}_{1/k}(r)$ must be k-elementary. While the converse even fails on finite-dimensional spaces, Corollary 7.11 of [1] shows that every singly generated subalgebra of M_n has Property $\mathbb{A}_1(r)$ for some r. Proposition 4.3 shows that no r works uniformly however.

PROPOSITION 4.2. *The collection of subspaces of M_n having Property $\mathbb{A}_{1/k}(r)$ is a closed subset of $\mathcal{F}(M_n)$.*

Proof. Write C for the collection and let S be one of its limit points. Choose a basis E for S and a sequence $\{S_i\}_{i=1}^\infty \subseteq C$ converging to S in the gap metric.

Let $t \in \operatorname{Ball} T$ be given. Since each S_i has property $\mathbb{A}_{1/k}(r)$, for each i there exists $f_i \in F_k$ with $\|f_i\|_1 \leqslant r$ and $(t - f_i) \perp S_i$. Let f_0 be the limit of a subsequence of $\{f_i\}_{i=1}^\infty$, say $\{f_{i_j}\}$. Then $f_0 \in r\operatorname{Ball} F_k$. Given $a \in E$, apply the definition of the gap metric to express it as the limit of a sequence $\{a_i\} \subseteq L(H)$ satisfying $a_i \in S_i$ for each i. We have

$$(t - f_0)(a) = (t - f_{i_j})(a_{i_j}) + (f_{i_j} - f_0)(a_{i_j}) + (t - f_0)(a - a_{i_j}).$$

The first term on the right always vanishes and the remaining terms converge to zero as j increases. This shows $(t - f_0) \perp E$ so $t \in S_\perp + r\operatorname{Ball} F_k$. ∎

PROPOSITION 4.3. *For each integer k there is an algebraically orthogonal family $\mathcal{F}_k$ acting on a finite-dimensional space H_k such that* $\operatorname{alg}[\mathcal{F}_k]$ *fails to have Property* $\mathbb{A}_{1/k}(k)$.

Proof. Apply Theorem 2.5 to express a copy of M_{k+1} as the limit of a sequence of spans of algebraically orthogonal families. Since M_{k+1} is not k-elementary no copy of it can have Property $\mathbb{A}_{1/k}(k)$. The preceding proposition thus allows us to take one of these families for $\mathcal{F}_k$. ∎

We now have the building blocks for the construction of our block-diagonal example. See Sections 7 and 10 of [1] for discussion of the following two results which provide the "glue".

PROPOSITION 4.4. *Let $\{S_n\}$ be a sequence of subspaces of L. Assume that for each n there is an r such that S_n has Property $\mathbb{A}_{1/k}(r)$. If $\sum^{\oplus} S_n$ is k-elementary, then the $\{S_n\}$ share Property $\mathbb{A}_{1/k}(r)$ for some common r.*

Proof. Suppose the $\{S_n\}$ fail to share Property $\mathbb{A}_{1/k}(r)$ for any common r. Rearranging the $\{S_n\}$ if necessary, we may assume that S_{2n} does not have Property $\mathbb{A}_{1/k}(n^3)$ for any n. Take $t_n = 0$ for odd n and choose $t_{2n} \in n^{-2}\mathrm{Ball}\,T$ such that any $f \in F_k$ with $(t_{2n} - f) \in (S_{2n})_\perp$ must satisfy $\|f\| \geqslant n$. But then $\sum^{\oplus} t_n \notin \left(\sum^{\oplus} S_n\right)_\perp + F_k$ so $\sum^{\oplus} S_n$ is not k-elementary. ∎

PROPOSITION 4.5. *If the relative* weak* *and* wot *topologies agree on a linear space S of operators then S has Property $\mathbb{A}_{1/k}(k)$ for some integer k.*

Proof. For each integer n, set $C_n = S_\perp + n\,\mathrm{Ball}\,F_n$. Since $S_\perp + F$ exhausts T by hypothesis, some C_n is of second category. Assuming for the moment that the underlying Hilbert space is finite-dimensional (the case which will be used in the sequel), this C_n is the sum of a closed set with a compact set. Thus C_n has non-empty interior so C_{2n} contains an open ball of some radius ε about the origin in T. Choose k larger than $2n/\varepsilon$.

When the underlying space is infinite-dimensional, C_n need not be closed, but being analytic, it can be expressed as the symmetric difference of an open set and a set of first category. Then C_{2n} will still contain an open ball about the origin in T. ∎

EXAMPLE 4.6. There is a family $\mathcal{F}$ of operators satisfying
(1) $\mathcal{F}$ is algebraically orthogonal,
(2) $\mathcal{F}$ is block diagonal with each block being finite, but
(3) the relative w* and wot topologies do not agree on wot(alg[$\mathcal{F}$]).

Proof. Take H to be direct sum of the Hilbert spaces $\{H_k\}$ of Proposition 4.3 and consider the operators in each $\mathcal{F}_k$ as acting on H. Take $\mathcal{F} = \bigcup \mathcal{F}_k$. Clearly $\mathcal{F}$ is algebraically orthogonal and block diagonal and wot(alg[$\mathcal{F}$]) is the direct sum of the algebras generated by the $\{\mathcal{F}_k\}$. Applications of Propositions 4.4 and 4.5 complete the proof. ∎

It is interesting to speculate on the minimal possible dimension of the space H_k guaranteed by Proposition 4.3. That $\dim(H_k) > k$ follows since (every subspace of) M_k is k-elementary; the proof given in this paper yields $\dim(H_k) = 3(k+1)^2$. We close this section with an alternate proof of Proposition 4.3 (our original argument) which gives $\dim(H_k) = k^2 + 1$.

LEMMA 4.7. *Suppose the minimal polynomial of $w \in M_n$ equals its characteristic polynomial. Then the function* alg *is continuous at w.*

Proof. Note that alg$[w]$ is n-dimensional with basis $\{w^i\}_{i=0}^{n-1}$. If b is close to w, then the powers of b will be close to the powers of w whence Lemma 2.8 shows that alg$[b]$ will be close to alg$[w]$. ■

LEMMA 4.8. (a) *If $b \in M_n$ has n distinct eigenvalues, then* alg$[b]$ *is generated by an algebraically orthogonal subset $\mathcal{P}$ of M_n.*

(b) *The set of such operators is dense in M_n.*

Proof. (a) By the spectral theorem, b is diagonalizable and alg$[b]$ is generated by the spectral idempotents of b.

(b) Assuming as we may that b is upper-triangular, note that small perturbations of its diagonal entries can make them distinct. ■

Example 8.6 of [1] exhibits a weighted shift w_k on $\mathbb{C}_{k^2+1}$ with no zero weights for which alg$[w_k]$ fails to have Property $\mathbb{A}_{1/k}(k)$. Combining Proposition 4.2 with the last two lemmas completes the alternate proof of 4.3.

5. STAIRCASE FORMS

We have seen that algebras generated by mutually orthogonal idempotents can be quite refractory. The specific examples considered in [3] however, are well-behaved. As a first step towards a general theory, we show that this good behavior extends to algebras in staircase form.

Fix an orthonormal basis $\{e_i\}$ for a separable infinite-dimensional Hilbert space H. The *staircase positions* of a matrix consist of its diagonal positions and those positions immediately above or immediately below an odd diagonal position. An operator is in *staircase form* if the non-zero entries of its matrix are confined to the staircase positions. For such an operator to have rank one, its non-zero entries must be confined to a single (even) row or a single (odd) column; it will then be idempotent iff the diagonal entry of that row or column is one. Also if an "i'th row" idempotent is to be orthogonal to a "$i \pm 1$'st column" idempotent, their entries in the $(i, i \pm 1)$ position must be negatives of each other.

To follow the recipe of the preceding paragraph, suppose $\{a_i\}$ is a sequence of complex numbers. Define $S \equiv \{x_i \otimes y_i | i \in \mathbb{N}\}$ where the vectors $\{x_i\}$ and $\{y_i\}$ are given by the equations

$$\begin{aligned} x_1 &= e_1 + a_2 e_2 & y_1 &= e_1 \\ x_2 &= e_2 & y_2 &= -a_2 e_1 + e_2 + a_3 e_3 \\ &\vdots & &\vdots \\ x_{2n-1} &= -a_{2n-1}e_{2n-2} + e_{2n-1} + a_{2n}e_{2n} & y_{2n-1} &= e_{2n-1} \\ x_{2n} &= e_{2n} & y_{2n} &= -a_{2n}e_{2n-1} + e_{2n} + a_{2n+1}e_{2n+1}. \end{aligned}$$

PROPOSITION 5.1. *Let S be as constructed above. Then the members of S are in staircase form, they are rank one idempotents, and they are mutually orthogonal. Every family of operators with these properties is contained in such an S.*

Proof. It is clear that the members of S are in staircase form and of rank one. The equation $[x_i \otimes y_i][x_j \otimes y_j] = \langle x_j, y_i\rangle[x_i \otimes y_j]$ shows that these operators are also mutually orthogonal idempotents. The last assertion of the proposition is addressed by the opening paragraph of the section. ∎

Such S will be the object of study in the sequel. Multiplying the basis vectors $\{e_n\}$ by appropriate constants of absolute value one shows there is no loss of generality in taking each a_i real and non-negative; to make the exposition smoother, we will also not allow any of the $\{a_i\}$ to vanish. The main example considered in [3] corresponds to taking $a_{2n} = a_{2n+1} = 4^n$ for $n \geqslant 1$; the alternate example of [3] has $a_n = 2^n$ for all n.

THEOREM 5.2. wot[[S]] *is reflexive.*

LEMMA 5.3. *Suppose $\{p_n\}$ is a sequence in L which approaches the identity operator in the strong operator topology. Then we have $\|p_n t p_n - t\| \to 0$ in the trace norm for each $t \in T$.*

Proof. Write $t = \sum_{i=1}^{\infty} \lambda_i u_i \otimes v_i$ where $\{u_i\}$ and $\{v_i\}$ are orthonormal sets. For each i, we have $\|p_n u_i - u_i\| \to 0$ and $\|p_n v_i - v_i\| \to 0$ as $n \to \infty$. Thus $p_n u_i \otimes p_n v_i \to$ $\to u_i \otimes v_i$ as $n \to \infty$. Choose N so large so that

$$\sum_{i=N+1}^{\infty} |\lambda_i| < \frac{\varepsilon}{4} \quad \text{and} \quad \|p_n u_i \otimes p_n v_i - u_i \otimes v_i\| < \frac{\varepsilon}{2N(|\lambda_i| + 1)} \quad \text{for } i \leqslant N \text{ and } n \geqslant N.$$

Then $\|p_n t p_n - t\| < \varepsilon$ for $n \geqslant N$. ∎

LEMMA 5.4. *Set $s_n \equiv x_1 \otimes y_1 + \cdots + x_n \otimes y_n$ and $p_n \equiv e_1 \otimes e_1 + \cdots + e_n \otimes e_n$.*

1) $s_n = \begin{cases} p_n + a_{n+1}(e_{n+1} \otimes e_n) & \text{if } n \text{ is odd} \\ p_n + a_{n+1}(e_n \otimes e_{n+1}) & \text{if } n \text{ is even,} \end{cases}$

2) *if* $m < n$, *then* $p_n s_m = s_m p_n = s_m$,

3) *if* $m > n$, *then* $p_n s_m = s_m p_n = p_n$,

4) *if* n *is even, then* $p_n s_n = s_n$ *while* $s_n p_n = p_n$,

5) *if* n *is odd, then* $p_n s_n = p_n$ *while* $s_n p_n = s_n$.

Proof. For 1) argue by induction. For the first part of 2), apply 1) to get $p_n s_m = p_n p_m + a_{n+1}(p_n e_{m+1} \otimes e_m) = p_n + a_{n+1}(e_{m+1} \otimes e_m) = s_m$. The remaining arguments are similar. ∎

Proof of Theorem 5.2. We must show $(\mathrm{wot}[[S]])_\perp \cap F_1$ is total in $(\mathrm{wot}[[S]])_\perp$. Define the $\{p_n\}$ and the $\{s_n\}$ as in Lemma 5.4 and suppose $t \perp \mathrm{wot}[[S]]$ is given. Set $f_n = p_n t p_n - [\mathrm{tr}(p_n t p_n)](e_n \otimes e_n)$. Since $I \in \mathrm{wot}[[S]]$, t must have trace zero, whence Lemma 5.3 tells us $\|f_n - t\|_1 \to 0$ as $n \to \infty$. We will complete the proof by expressing each f_{2n} as a sum of rank one members of $(\mathrm{wot}[[S]])_\perp$.

From Lemma 5.4(1), we conclude that $\langle s_m, e_n \otimes e_n\rangle = \begin{cases} 1 & \text{if } m \geqslant n \\ 0 & \text{if } m < n \end{cases}$, while (2)–(5) of the same lemma tell us that

$$\langle s_m, p_n t p_n\rangle = \langle p_n s_m p_n, t\rangle = \begin{cases} \mathrm{tr}(p_n t p_n) & \text{if } m \geqslant n \\ \langle s_m, t\rangle & \text{if } m < n. \end{cases}$$

Since $\langle s_m, t\rangle = 0$ for all m (by hypothesis), we see that $\langle s_m, f_n\rangle = 0$ for each m and n. Since $\{s_m\}$ and S have the same linear span, we conclude that $\{f_n\} \subseteq (\mathrm{wot}[[S]])_\perp$.

Lemma 5.4(4) also yields $f_{2n} = p_{2n} f_{2n} = s_{2n} p_{2n} f_{2n} = s_{2n} f_{2n} = \sum_{i=1}^{2n} (x_i \otimes y_i) f_{2n}$. Moreover, $\mathrm{wot}[[S]]$ is an algebra so its preannihilator is closed under multiplication by members of $\mathrm{wot}[[S]]$. Thus each summand $(x_i \otimes y_i) f_{2n}$ in the expression for f_{2n} is a rank one member of $(\mathrm{wot}[[S]])_\perp$ as desired. ∎

THEOREM 5.5. (1) *$\mathrm{w}^*[S]$ is always 3-elementary.*

(2) *If* $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$ *then $\mathrm{w}^*[S]$ is 2-elementary.*

LEMMA 5.6. *Let $t \in T$ and suppose $\{p_n\}$ and $\{q_n\}$ are two given orthogonal families of projections.*

(1) $\sum_n \|p_n t q_n\| \leqslant \|t\|$.

(2) *If in addition each q_n has rank one, then there is an $f \in F_1$ which is supported on $\bigvee q_n$ and satisfies $p_n f q_n = p_n t q_n$ for each n.*

Proof. (1) Write $t = \sum_i u_i \otimes v_i$ where $\{u_1\}$ and $\{v_i\}$ are orthogonal sets of vectors in the underlying Hilbert space. For each i, Holder's inequality gives

$$\sum_n \|p_n u_i\| \, \|q_n v_i\| \leqslant \left(\sum_n \|p_n u_i\|^2\right)^{\frac{1}{2}} \left(\sum_n \|q_n v_i\|^2\right)^{\frac{1}{2}} \leqslant \|u_i\| \, \|v_i\|.$$

This leads to the desired conclusion since

$$\sum_n \|p_n t q_n\| = \sum_n \left\| p_n \left(\sum_i u_i \otimes v_i\right) q_n \right\| \leqslant \sum_{n,i} \|p_n (u_i \otimes v_i) q_n\| \leqslant \sum_i \|u_i\| \, \|v_i\| = \|t\|.$$

(2) For each n choose a unit vector w_n in the range of q_n. Write $\lambda_n = \sqrt{\|p_n t w_n\|} = \sqrt{\|p_n t q_n\|}$. By (1), the sequence $\{\lambda_n\}$ is square summable and hence the series $\sum\limits_{\lambda_n > 0} \lambda_n^{-1} p_n t w_n$ and $\sum \lambda_n w_n$ converge to vectors x and y respectively. Set $f \equiv x \otimes y$ to complete the proof. ∎

LEMMA 5.7. *Let $t \in T$. In order for t to belong to $S_\perp$ it is necessary and sufficient that the entries of its matrix satisfy the following equations:*

$$t_{11} + t_{22} + \cdots + t_{2n-1,2n-1} + t_{2n-1,2n} a_{2n} = 0, \ n \geqslant 1$$
$$t_{11} + t_{22} + \cdots + t_{2n,2n} + t_{2n+1,2n} a_{2n+1} = 0, \ n \geqslant 0.$$

These equations only involve entries of t in the transpose staircase positions.

Proof. The equations are equivalent to having $\langle s_n, t\rangle = 0$ for the idempotents $\{s_n\}$ of Lemma 5.4. These idempotents have the same linear span as S. ∎

Proof of Theorem 5.5. Define the following projections via their ranges:

$$\operatorname{ran} p_1 \equiv [e_1, e_2], \qquad \operatorname{ran} p_{2n} \equiv [e_{2n}],$$
$$\operatorname{ran} p_{2n+1} \equiv [e_{2n}, e_{2n+1}, e_{2n+2}], \qquad \operatorname{ran} q_n \equiv [e_n].$$

Let $t \in T$. The first part of Lemma 5.6 tells us that the following sums define trace class operators:

$$t_1 \equiv \sum_{n=1}^{\infty} p_{4n+1} t q_{4n+1}, \quad t_2 \equiv \sum_{n=1}^{\infty} p_{4n+3} t q_{4n+3}, \quad t_3 \equiv \sum_{n=1}^{\infty} p_{2n} t q_{2n}.$$

Let f_1, f_2, and f_3 be the rank one operators associated with these operators by the second part of Lemma 5.6. Now Lemma 5.7 tells us that $t - \sum\limits_{i=1}^{3} t_i$ and each $t_i - f_i$ belongs to $S_\perp$. Thus

$$t = \left(t - \sum_{i=1}^{3} t_i\right) + \sum_{i=1}^{3} (t_i - f_i) + \sum_{i=1}^{3} f_i \in S_\perp + F_3$$

and we have established the first part of the theorem.

For the second part, let $t' \in T$. Define t by setting its diagonal entries equal to those of t', using the equations of Lemma 5.7 to solve for the other t_{ij} appearing in those equations and setting all other entries of t equal to those of t'. Then $t - t' \in S_\perp$. Applying the second part of Lemma 5.6 to $t_1 \equiv \sum_{n=1}^{\infty} p_{4n+1} t q_{4n+1}$ and $t_2 \equiv \sum_{n=1}^{\infty} p_{4n+3} t q_{4n+3}$ yields $t = t_1 + t_2$ and $t' = (t'-t) + (t_1 - f_1) + (t_2 - f_2) + (f_1 + f_2) \in S_\perp + F_2$. ■

In particular, $w^*[S] = \text{wot}[S]$. It is conceivable that $w^*[S]$ is always 2-elementary. In the main example of [3], $w^*[[S]]$ is reflexive but $w^*[S]$ is not, so $w^*[[S]]$ is not elementary.

THEOREM 5.8. *$I \in \text{wot}([S])$ if and only if* $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$.

Proof. ($\Leftarrow$) Suppose $I \notin \text{wot}([S])$. Then there exists $t \in (\text{wot}[S])_\perp$ with $\langle I, t\rangle = 1$, i.e., $\text{tr}(t) = 1$. This implies $\sum_{i=1}^{\infty} t_{ii} = 1$ and the equations of Lemma 5.7 are satisfied. Therefore, $\lim_{n\to\infty} t_{2n-1,2n} a_{2n} = -1$ and $\lim_{n\to\infty} t_{2n+1,2n} a_{2n+1} = -1$. Thus, for sufficiently large n, we have $|t_{2n-1,2n}| \geqslant \frac{1}{2a_{2n}}$ and $|t_{2n+1,2n}| \geqslant \frac{1}{2a_{2n+1}}$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{a_n} \leqslant 2\sum_{n=1}^{\infty} (|t_{2n-1,2n}| + |t_{2n+1,2n}|) \leqslant 2\|t\|_1 < \infty$.

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$. By Theorem 5.5, the relative wot and w* topologies agree on $w^*[S]$. It follows from Corollary 10.4 of [1] (due to B. Chevreau – J. Esterle and P. Dixon) that $w^*[S] = \text{wot}[S]$. Set $t_{11} = 1$, $t_{nn} = 0$ for all $n \geqslant 1$, and (by solving the equations of Lemma 5.7) take $t_{2n-1,2n} = \frac{-1}{a_{2n}}$, $t_{2n+1,2n} = \frac{-1}{a_{2n+1}}$; also set the remaining t_{ij} equal to zero. Since $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$, the resulting operator $t \in T$. By construction $t \in S_\perp = (\text{wot}[S])_\perp$ with $\langle I, t\rangle = 1$ so $I \notin \text{wot}[S]$. ■

DEFINITION 5.9. A *pseudovector* is a sequence $z = \{z_n\}_{n=1}^{\infty}$ not necessarily in ℓ_2 satisfying the condition "for each n, either $z \perp x_n$ or $z \perp y_n$". (Inner products are computed in the usual way.)

THEOREM 5.10. *The following are equivalent:*

(1) *the identity operator, I, does not belong to* ref$[S]$,

(2) *the sequence of real numbers defined inductively by*

$$z_1 = 1,\ z_2 = \frac{-1}{a_2},\ z_{n+2} = \frac{a_{n+1} z_n - z_{n+1}}{a_{n+2}}$$

is square summable.

LEMMA 5.11. *Let $z = \{z_n\}_{n=1}^{\infty}$ be a pseudovector.*

(1) *If $z_n = 0$, then $z_k = 0$ for all $k \leqslant n$.*

(2) *The non-zero coordinates of z alternate in sign.*

(3) *The sequence of real numbers defined inductively by*

$$z_1 = 1, \; z_2 = \frac{-1}{a_2}, \; z_{n+2} = \frac{a_{n+1}z_n - z_{n+1}}{a_{n+2}}$$

is the unique pseudovector whose first coordinate is 1.

(4) *If one non-zero pseudovector belongs to ℓ_2, then they all do.*

Proof. Set $w_n = \begin{cases} x_n & \text{if } n \text{ is odd} \\ y_n & \text{if } n \text{ is even} \end{cases}$.
Then $w_1 = e_1 + a_2 e_2$, while $w_n = -a_n e_{n-1} + e_n + a_{n+1}e_{n+1}$ for $n \geqslant 2$.

(1) Set $v_k = \begin{cases} e_k & \text{if } z \perp e_k \\ w_k & \text{otherwise.} \end{cases}$
Every finite set of $\{x_n\}$ is independent as is every finite set of $\{y_n\}$. Since $x_j \perp y_k$ for $j \neq k$, we see that $v_1, \dots, v_n$ are independent. Since $v_n = e_n$, we have $[v_1, \dots, v_n] \subseteq$ $\subseteq [e_1, \dots, e_n]$, and hence these two spaces coincide. Thus $z \perp v_k$ for $k \leqslant n$ implies $z \perp e_k$ for $k \leqslant n$.

(2) Suppose z_n is the first non-zero coordinate of z. By (1), no later coordinates of z can be zero, so $z \perp w_k$ for $k \geqslant n$. This means

$$z_n + a_{n+1}z_{n+1} = 0,$$

$$-a_{n+k-1}z_{n+k-2} + z_{n+k-1} + a_{n+k}z_{n+k} = 0 \text{ for } k \geqslant 2.$$

Solving inductively for z_{n+k} establishes the result.

(3) By (1), $z \perp w_k$ for all k and the coordinates of z can be solved for inductively.

(4) Let z and z' be non-zero pseudovectors. Choose n with z_n and z'_n non-zero. By (1), z_{n+1} and z'_{n+1} are also non-zero. Choose a constant K such that $|z'_n| \leqslant K|z_n|$ and $|z'_{n+1}| \leqslant K|z_{n+1}|$. Using the equations from the proof of (2) and an inductive argument, we conclude that $|z'_k| \leqslant K|z_k|$ for all $k \geqslant n$. ∎

Proof of Theorem 5.10. (1) ⇒ (2) Suppose $I \notin \operatorname{ref}[S]$. Then there exists $f = u \otimes v$ with $f \perp S$ but f not orthogonal to the identity I. Therefore $\langle u, y_n\rangle\langle x_n, v\rangle = 0$ for all n but $\langle u, v\rangle \neq 0$. Let $J \equiv \{n \in \mathbf{N} \mid u \perp y_n\}$ and set $M \equiv \operatorname{span}\{y_n \mid n \in J\}$ and $N \equiv \operatorname{span}\{x_n \mid n \notin J\}$. Then $M \perp N$, $u \in M^{\perp}$, and $v \in N^{\perp}$.

The projection of u on N (denoted by $p_N(u)$) belongs to $M^{\perp}$. Thus $u \notin N$ implies $u - p_N(u)$ is a non-zero vector in $M^{\perp} \cap N^{\perp}$. On the other hand, if $u \in N$ then $v \notin M$ and $v - p_M(v)$ is a non-zero vector in $M^{\perp} \cap N^{\perp}$. In either case, we have a square-summable pseudovector and an appeal to Lemma 5.11 completes the argument.

(2) $\Rightarrow$ (1) Set $x \equiv \sum z_n e_n$. Then $x \otimes x$ is orthogonal to S but not to I. ■

COROLLARY 5.12. (1) *If* $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < 1$ *then* $I \notin \operatorname{ref}[S]$.

(2) *If* $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$ *then* $I \in \operatorname{ref}[S]$.

(3) *If either of the sequences* $\left\{\prod_{k=1}^{n} \frac{a_{2k-1}}{a_{2k}}\right\}_{n=1}^{\infty}$ *or* $\left\{\prod_{k=1}^{n} \frac{a_{2k}}{a_{2k+1}}\right\}_{n=1}^{\infty}$ *fails to be square summable, then* $I \in \operatorname{ref}[S]$.

Proof. (1) Choose a number δ with $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} < \delta^3 < 1$. Choose N such that $n > N$ implies $\frac{a_n}{a_{n+1}} < \delta^3$ and $\frac{1}{a_n} < \delta(1-\delta)$. Assuming n is so large and that $|z_n| < c\delta^n$ and $|z_{n+1}| < c\delta^{n+1}$, we have

$$|z_{n+2}| \leqslant \frac{a_{n+1}}{a_{n+2}}|z_n| + \frac{|z_{n+1}|}{a_{n+2}} \leqslant c\delta^3(\delta^n) + c\delta(1-\delta)\delta^{n+1} = c\delta^{n+2}.$$

Thus an inductive argument shows that the series $\sum |z_n|^2$ is majorized by a convergent geometric series and Theorem 5.10 applies.

(3) Since the z_n alternate in sign, we have $|z_{n+2}| \geqslant |z_n| \frac{a_{n+1}}{a_{n+2}}$ for each n. From here an inductive argument yields $|z_{2n+1}| \geqslant |z_1| \prod_{k=1}^{n} \frac{a_{2k}}{a_{2k+1}}$ and $|z_{2n}| \geqslant |z_2| \prod_{k=1}^{n-1} \frac{a_{2k-1}}{a_{2k}}$. Apply Theorem 5.10 and the comparison test.

(2) The hypothesis implies the sequences in (3) do not appoach zero. ■

EXAMPLE 5.13. (1) If $a_{2n-1} = a_{2n}$ for all n, (as in the main example of [3]), then $I \in \operatorname{ref}[S]$.

(2) If $a_n = 2^n$ for all n, (as in the second example of [3]), then $I \notin \operatorname{ref}[S]$.

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