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COMPLETELY REDUCIBLE ALGEBRAS CONTAINING COMPACT OPERATORS

SHLOMO ROSENOER

1. INTRODUCTION

There is now an extensive theory of those algebras of operators on Hilbert space that are determined by their invariant subspace lattices. Most of the problems considered have meaning and interest in the more general context of operators on Banach spaces, but there are relatively few results in this setting. The present paper concerns certain algebras of Banach space operators which contain compact operators.

Let X be a Banach space and $\mathcal{A}$ an algebra of bounded linear operators on X. Lat$\mathcal{A}$ is the family of all (closed) subspaces of X invariant under each operator in $\mathcal{A}$. We say that $\mathcal{A}$ is *completely reducible* if for every M invariant in lat$\mathcal{A}$ there is N in lat$\mathcal{A}$ such that $M \dot{+} N = X$ (i.e., $M \cap N = 0$, and X is the algebraic sum of M and N). E. Azoff asked whether a unital strongly closed completely reducible algebra must be *reflexive*; that is, must contain all operators leaving invariant its invariant subspaces. Azoff's question was answered positively in some special cases by Fong [6] and Rosenthal and Sourour [15]. Also, in the author's paper [13] it has been shown that every unital strongly closed commutative completely reducible algebra which commutes with a family of compact operators with spanning ranges and zero intesection of kernels is a reflexive algebra of scalar type spectral operators.

It is known that Azoff's question has a negative answer in general. For there are operators on Banach spaces with no non-trivial invariant subspaces [10]. If $\mathcal{A}$ is the unital strongly closed algebra generated by such operator, then $\mathcal{A}$ is completely reducible but not reflexive.

A special class of completely reducible algebras is formed by *reductive* algebras. A unital, strongly closed algebra $\mathcal{A}$ on a Hilbert space is said to be *reductive* if every invariant subspace is reducing (i.e., $M \in \text{lat}\mathcal{A}$ implies $M^\perp \in \text{lat}\mathcal{A}$). It was proved by Rosenthal [14], and, independently, by Loginov and Shul'man [7], that every reductive algebra containing a set of compact operators with spanning ranges is a von Neumann algebra. Note that a reductive algebra is reflexive if and only if it is a von Neumann algebra.

In this paper we will generalize this result and also give a partial answer to the Azoff's question. We will show that every unital, strongly closed completely reducible algebra containing a set of compact operators with spanning ranges is reflexive. Moreover, we will show that every such algebra is a "direct sum" of what we call "factor algebras", which admit a very simple description. Our results seem to be new even in the case of a Hilbert space.

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If X and Y are Banach spaces, we write $\mathcal{L}(X,Y)$ for the set of all bounded linear transformations from X into Y and $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. A transformation $T \in \mathcal{L}(X,Y)$ is called a *quasiaffinity* provided $\text{cl}T(X) = Y$ and $\ker T = 0$. By a *projection* in $\mathcal{L}(X)$ we mean an operator E satisfying $E^2 = E$. If $x \in X$ and $\varphi \in X^*$ we wil write $x \otimes \varphi$ for the rank-one operator on X defined by $(x \otimes \varphi)(y) = \varphi(y)x$. If $\mathcal{A}$ is a subalgebra of $\mathcal{L}(X)$, we write $\mathcal{A}'$ for its commutant and $\mathcal{A}'' = (\mathcal{A}')'$ for its double commutant. An operator algebra $\mathcal{A}$ on X is said to be *spatially isomorphic* to another operator algebra $\mathcal{B}$ on Y if there is a one-to-one bicontinuous linear map S of X onto Y such that $S\mathcal{A}S^{-1} = \mathcal{B}$.

Obviously, an operator algebra $\mathcal{A}$ on X is completely reducible if and only if for every $M \in \text{lat}\mathcal{A}$ there is a projection $E \in \mathcal{A}'$ such that $E(X) = M$. If $\mathcal{A}$ is completely reducible, and $\mathcal{B}$ is spatially isomorphic to $\mathcal{A}$, then $\mathcal{B}$ is completely reducible. Also, if $M \in \text{lat}\mathcal{A}$, the restriction of $\mathcal{A}$ to M, denoted by $\mathcal{A}|M$, is again completely reducible. Note that if $\mathcal{A}$ is completely reducible and reflexive, one has $\mathcal{A}'' = \mathcal{A}$.

Let X be a Banach space, and let $X_1, X_2, \ldots, X_n$ be subspaces of X such that $X = X_1 \dot{+} X_2 \dot{+} \cdots \dot{+} X_n$. Then X is spatially isomorphic to the exterior direct sum of $X_1, X_2, \ldots, X_n$ (denoted by $X_1 \oplus X_2 \oplus \cdots \oplus X_n$) with the norm given by $\|x_1 \oplus x_2 \oplus \cdots \oplus x_n\| = (\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2)^{\frac{1}{2}}$. If $\mathcal{A}$ is the algebra of all operators on X which leave each X_j invariant, then $\mathcal{A}$ is spatially isomorphic to the algebra $\mathcal{L}(X_1) \oplus \mathcal{L}(X_2) \oplus \cdots \oplus \mathcal{L}(X_n)$ on $X_1 \oplus X_2 \oplus \cdots \oplus X_n$. We will often identify these two algebras.

2. COMPLETE REDUCIBILITY OF THE ALGEBRA GENERATED BY $\mathcal{A} \cup \mathcal{A}'$

In order to approach our main result, some preliminary work is needed. In this section, we show that if $\mathcal{A}$ is a completely reducible algebra and if the algebra $\mathcal{B}$ generated by $\mathcal{A}$ and $\mathcal{A}'$ has enough invariant subspaces, then $\mathcal{B}$ is completely reducible.

Let $\mathcal{L}$ be a lattice of subspaces in X. We say that $M \in \mathcal{L}$ is a *cover* for $N \in \mathcal{L}$ if $M \supset N$, $M \neq N$ and there is no $L \in \mathcal{L}$ different from M and N such that $M \subset L \subset N$. $M \in \mathcal{L}$ is called *an atom* in $\mathcal{L}$ if it is a cover for 0.

The following lemma due to Fong is of great importance for our goal.

LEMMA 2.1. *If $\mathcal{A}$ is a completely reducible algebra in $\mathcal{L}(X)$, $M \in (\mathrm{lat}\mathcal{A}) \cap$ $\cap(\mathrm{lat}\mathcal{A}')$ and $N \in \mathrm{lat}\mathcal{A}$, then $M + N$ is closed.*

For the proof see [6], Proposition 19.

THEOREM 2.2. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra. Let $\mathcal{B}$ denote the algebra generated by $\mathcal{A}$ and $\mathcal{A}'$. If $\mathrm{lat}\mathcal{B}$ has no covers, then $\mathcal{B}$ is completely reducible.*

Proof. Since the proof is very close to the proof of the same assertion about commutative algebras in [13], we give merely its outline here. Suppose the converse. Then there exist X_1 and X_2 in $\mathrm{lat}\mathcal{A}$ such that $X_1 \dotplus X_2 = X$, and X_1 is in $\mathrm{lat}\mathcal{A}'$ while X_2 is not. Then one can find infinite sequences $\{E_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$, such that, for each $n \geqslant 1$, E_n is a non-zero projection in $(\mathcal{A}|X_2)'$, F_n is a non-zero projection in $(\mathcal{A}|X_1)'$, $E_iE_j = E_jE_i = 0$, $F_iF_j = F_jF_i = 0$ whenever $i \neq j$, $T_n \in \mathcal{L}(X_2, X_1)$ satisfying

$$(A|X_1)T_n = T_n(A|X_2) \quad \text{for each} \quad A \in \mathcal{A}$$

and $F_nT_nE_n \neq 0$.

Define $T_0 \in \mathcal{L}(X_2, X_1)$ by

$$T_0 = \sum_{n=1}^{\infty} 2^{-n} \|F_nT_nE_n\|^{-1} F_nT_nE_n$$

(the series is uniformly convergent).

Let

$$L = \bigcap_{n=1}^{\infty} \ker F_n.$$

Then $L \in \mathrm{lat}(\mathcal{A}|X_1)$ so there is a projection $P \in (\mathcal{A}|X_1)'$ such that $\ker P = L$. Define $S \in (\mathcal{A}|X_2)'$ by

$$S = \sum_{n=1}^{\infty} 2^{-n} \|PF_nT_nE_n\| \, \|F_nT_nE_n\|^{-1} \, \|E_n\|^{-1} E_n.$$

Then the subspace

$$M = \mathrm{cl}\{PT_0x \oplus Sx,\ x \in X_2\}$$

is in lat$\mathcal{A}$. Moreover, M is the graph of a closed unbounded linear transformation defined on a manifold $\mathcal{D} \subset X_2$ and with range in X_1. By the Closed Graph Theorem, $\mathcal{D}$ is not closed. This contradicts Lemma 2.1 and therefore completes the proof. ∎

REMARK 2.3. It is clear from the above that if $\mathcal{A}$ is a completely reducible algebra, $X = X_1 \dotplus X_2$, where $X_1 \in (\mathrm{lat}\mathcal{A}) \cap (\mathrm{lat}\mathcal{A}')$, $X_2 \in \mathrm{lat}\mathcal{A}$, and there exist infinite sequences $\{E_n\}$, $\{F_n\}$, $\{T_n\}$ satisfying the above conditions, then $X_2 \in \mathrm{lat}\mathcal{A}'$.

3. FINITE-RANK OPERATORS IN $\mathcal{A}$

In this section, we will show that a completely reducible algebra containing non--zero compact operators also contains non-zero finite rank operators.

LEMMA 3.1. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra and let $\mathcal{J}$ be a bilateral ideal in $\mathcal{A}$. Let M denote the subspace spanned by the ranges of all operators in $\mathcal{J}$ and N the intersection of their kernels. Then both M and N lie in $(\mathrm{lat}\mathcal{A}) \cap (\mathrm{lat}\mathcal{A}')$ and $M \dotplus N = X$.*

Proof. Obviously, M and N are in $(\mathrm{lat}\mathcal{A}) \cap (\mathrm{lat}\mathcal{A}')$. Hence, there is $N_1 \in \mathrm{lat}\mathcal{A}$ such that $M \dotplus N_1 = X$. Clearly, all operators in $\mathcal{J}$ vanish on N_1, so that $N_1 \subseteq N$. Since the projection onto M along N_1 leaves N invariant, we can write N as $N_1 \dotplus N_2$, where $N_2 \subseteq M$ and $N_2 \in \mathrm{lat}\mathcal{A}$. There exists $M_1 \in \mathrm{lat}\mathcal{A}$ such that $M_1 \dotplus N_2 = M$. Since all operators in $\mathcal{J}$ vanish on N_2, the definition of M yields $M_1 = M$ and $N_2 = 0$, so that $N_1 = N$. ∎

We say that $\mathcal{S} \subseteq \mathcal{L}(X)$ is a *sufficient* set if the ranges of the operators in $\mathcal{S}$ span X. Note that any strongly dense ideal in a unital, completely reducible algebra is a sufficient set.

COROLLARY 3.2. (i) *If $\mathcal{A} \subseteq \mathcal{L}(X)$ is a completely reducible algebra containing a sufficient set of compact (respectively, finite-rank) operators and $M \in \mathrm{lat}\mathcal{A}$, then $\mathcal{A}|M$ contains a sufficient set of compact (respectively, finite-rank) operators.*

(ii) *If $\mathcal{A} \subseteq \mathcal{L}(X)$ is completely reducible and $\mathcal{A}' \cap \mathcal{A}''$ contains no non-trivial projections, then 0 is the only member of $\mathcal{A}$ which vanishes on a non-zero invariant subspace of $\mathcal{A}$.*

Proof. (i) Let $\mathcal{J}$ denote the set of all compact (finite-rank) operators in $\mathcal{A}$. Then $\mathcal{J}$ is a bilateral ideal in $\mathcal{A}$. By Lemma 3.1, the intersection of kernels of all operators in $\mathcal{J}$ is zero. Now apply that lemma to $\mathcal{A}|M$.

(ii) Let $0 \neq N \in \operatorname{lat}\mathcal{A}$. Denote by $\mathcal{J}$ the set of all members of $\mathcal{A}$ vanishing on N. Since N has an $\mathcal{A}$-invariant complement, it is clear that $\mathcal{J}$ is a bilateral ideal in $\mathcal{A}$. An application of Lemma 3.1 finishes the proof. ■

THEOREM 3.3. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra that contains a non-zero compact operator. Then*

(i) $\operatorname{lat}\mathcal{A}$ *has an atom;*

(ii) *if, in addition, $\mathcal{A}$ is unital and uniformly closed and contains a sufficient set of compact operators, then $\mathcal{A}$ contains a sufficient set of finite-rank operators.*

Proof. (i) Let $\mathcal{J}$ denote an ideal of compact operators in $\mathcal{A}$ and M a subspace associated with $\mathcal{J}$ as in Lemma 3.1. Considering the restriction $\mathcal{A}|M$ we can, without any loss of generality, assume that $\mathcal{A}$ contains a sufficient set of compact operators. Let $\mathcal{B}$ be the algebra generated by $\mathcal{A} \cup \mathcal{A}'$. We claim that $\operatorname{lat}\mathcal{B}$ has a cover. Suppose the converse. Then, by Theorem 2.2, $\mathcal{B}$ is completely reducible. Denote by Ω the set of all projections in $\mathcal{B}'$. Since $\mathcal{B}'$ is clearly commutative, so is Ω. Hence, Ω is a commutative Boolean algebra of projections. But, since $\mathcal{B}$ is completely reducible, and $\operatorname{lat}\mathcal{B}$ is a complete lattice, Ω must be a complete Boolean algebra. By a theorem due to Bade [5], Lemma XVII.3.3, Ω is bounded in the sense that there exists $C > 0$ such that $\|E\| < C$ for each $E \in \Omega$. Now, just as in the proof of Theorem 9 in [12], we conclude that Ω is a totally atomic Boolean algebra. Suppose E_0 is an atom in Ω. Then $E_0(X)$ is a cover for 0 in $\operatorname{lat}\mathcal{B}$, a contradiction.

Now we prove that $\operatorname{lat}\mathcal{A}$ has an atom. By the preceding paragraph, there exist M and N such that N is a cover for M in $\operatorname{lat}\mathcal{B}$. There is $L \in \operatorname{lat}\mathcal{A}$ such that $M \dot{+} L = N$. It is easy to verify that

$$(\operatorname{lat}(\mathcal{A}|L)) \cap (\operatorname{lat}(\mathcal{A}|L)') = \{0, L\}.$$

Let us assume that $\operatorname{lat}(\mathcal{A}|L)$ has no covers. Let $\mathcal{L}$ be a maximal chain of subspaces in $\operatorname{lat}(\mathcal{A}|L)$. Then $\mathcal{L}$ is a maximal chain in L. Furthermore, $\mathcal{L}$ is continuous in the sense that each $M \in \mathcal{L}$ has neither an immediate predecessor nor an immediate succesor in $\mathcal{L}$. By Corollary 3.2(i), $\mathcal{A}|L$ contains a non-zero compact operator K. Then, by Ringrose [11], K lies in the radical of $\mathcal{A}|L$. However, Shul'man [16] has shown that every operator algebra on a Banach space whose radical contains a non-zero compact operator must have a common non-trivial invariant subspace with its commutant. Applying that to $\mathcal{A}|L$ we obtain a contradiction. Hence, $\operatorname{lat}(\mathcal{A}|L)$ has a cover, that is, there are M_1 and N_1 in $\operatorname{lat}(\mathcal{A}|L)$ such that $M_1 \neq N_1$ and there is no non-trivial subspace of $\operatorname{lat}\mathcal{A}$ between M_1 and N_1. Let $L_1 \in \operatorname{lat}\mathcal{A}$, $L_1 + M_1 = N_1$. Then L_1 is an atom in $\operatorname{lat}\mathcal{A}$. This proves (i).

(ii) Denote by X_1 the subspace of X spanned by the ranges of all finite-rank operators in $\mathcal{A}$. Then $X_1 \in \operatorname{lat}\mathcal{A}$. We must show that $X_1 = X$. Suppose not. Then there exists a non-zero $X_2 \in \operatorname{lat}\mathcal{A}$ such that $X_1 \dot{+} X_2 = X$. Now $\mathcal{A}|X_2$ is a completely reducible algebra containing, by Corollary 3.2(i), a sufficient set of compact operators. By (i), $\operatorname{lat}(\mathcal{A}|X_2)$ has an atom M. That means that $M \subseteq X_2$, $M \in \operatorname{lat}\mathcal{A}$ and $\mathcal{A}|M$ is transitive. Again by Corollary 3.2(i), there is a compact $K \in \mathcal{A}$ such that $K|M \neq 0$. By Lomonosov's Theorem [8], there exist $A \in \mathcal{A}$ and a non-zero $x \in M$ such that $AKx = x$. Since $\mathcal{A}$ is unital and uniformly closed, the finite-rank Riesz projection E of AK onto the root space corresponding to the eigenvalue 1 lies in $\mathcal{A}$. Then,

$$x = Ex \in X_2 \cap X_1 = 0,$$

an apparent contradiction. This proves (ii). ∎

REMARK 3.4. If $\mathcal{A}$ is a (not necessarily completely reducible) operator algebra containing a sufficient set of finite-rank operators, then $\operatorname{lat}\mathcal{A}$ may have not an atom. To see this, it suffices to consider a nest algebra with a continuous invariant subspace lattice.

4. FACTOR ALGEBRAS

In this section, we will study completely reducible algebras $\mathcal{A}$ such that $\mathcal{A}$ contains a sufficient set of finite-rank operators and $\mathcal{A}' \cap \mathcal{A}''$ contains no non-trivial projections. We will characterize all such unital, strongly closed algebras modulo a spatial isomorphism. First let us fix some notation.

Let X and Y be two Banach spaces. If $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, we will write

$$A \xrightarrow{T} B \text{ if } T \in \mathcal{L}(Y, X) \text{ and } AT = TB.$$

If n is a positive integer, we write $X^{(n)}$ for the (exterior) direct sum of n copies of X. If $T \in \mathcal{L}(X, Y)$, then we write $T^{(n)}$ for $T \oplus T \oplus \cdots \oplus T$ (n copies). That is, $T^{(n)}$ is a transformation from $X^{(n)}$ into $Y^{(n)}$ such that

$$T^{(n)}(x_1 \oplus x_2 \oplus \cdots \oplus x_n) = Tx_1 \oplus Tx_2 \oplus \cdots \oplus Tx_n.$$

For any subset $\mathcal{S} \subset \mathcal{L}(X, Y)$, we write $\mathcal{S}^{(n)}$ for $\{T^{(n)} \in \mathcal{L}(X^{(n)}, Y^{(n)}),\ T \in \mathcal{S}\}$.

DEFINITION 4.1. We say that an operator algebra $\mathcal{A}$ is a *factor algebra* if there exist Banach spaces $X_1, X_2, \ldots, X_n$ and quasiaffinities $T_1, T_2, \ldots, T_{n-1}$, where $T_i \in$ $\in \mathcal{L}(X_{i+1}, X_i)$ such that $\mathcal{A}$ is spatially isomorphic to the algebra

$$\{A_1 \oplus A_2 \oplus \cdots \oplus A_n : A_1 \xrightarrow{T_1} A_2 \xrightarrow{T_2} \ldots A_{n-1} \xrightarrow{T_{n-1}} A_n\}.$$

Furthermore, if each T_i is invertible, or if $\mathcal{A} = \mathcal{L}(X_1)$ (which will mean that $\mathcal{A}$ is spatially isomorphic to $\mathcal{L}(X_1)^{(n)}$ for $n \geqslant 1$), $\mathcal{A}$ is said to be a factor algebra of type I. Otherwise, $\mathcal{A}$ is called a factor algebra of type II.

REMARK 4.2. Every factor algebra is reflexive. This can be easily deduced from the fact that for $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, $T \in \mathcal{L}(Y, X)$, $AT = TB$ if and only if the operator $A \oplus B$ leaves the subspace $\{Ty \oplus y,\ y \in T\}$ invariant.

The following simple but important fact is due to Douglas [4]:

PROPOSITION 4.3. *Suppose $T \in \mathcal{L}(X, Y)$ is a quasiaffinity and $A \in \mathcal{L}(X)$. Then the following are equivalent:*

(a) *$A = TB$ for some $B \in \mathcal{L}(Y)$;*

(b) *The range of A is contained in the range of B.*

Proof. The implication (a)$\Rightarrow$(b) is obvious. Assuming (b), consider the (well--defined) map $B = T^{-1}A$. Clearly $A = TB$. On the other hand, an application of the Closed Graph Theorem shows that B is bounded. ∎

Note. If T is a quasiaffinity and A has finite rank, then AT and TA have the same rank as A. In particular, A leaves the range of T invariant if and only if $R(A) \subseteq R(T)$.

The following lemma clarifies the structure of a factor algebra.

LEMMA 4.4. *Suppose $T_i \in \mathcal{L}(X_{i+1}, X_i)$, $i = 1, \dots, n-1$, are quasiaffinities and set*

$$\mathcal{A} = \{A_1 \oplus \cdots \oplus A_n : A_1 \xrightarrow{T_1} A_2 \dots A_{n-1} \xrightarrow{T_{n-1}} A_n\}.$$

Set $S_1 = I$ and $S_i = T_1 \cdots T_{i-1}$ for $i = 2, \dots, n$. Then

(a) *The general member of $\mathcal{A}$ takes the form $A_1 \oplus \cdots \oplus A_n$, where A_1 leaves the ranges of $S_2, \dots, S_n$ invariant and $A_i = S_i^{-1} A_1 S_i$ for each i.*

(b) *The finite-rank members of $\mathcal{A}$ are strongly dense in $\mathcal{A}$.*

(c) *Each finite-rank member of $\mathcal{A}$ is the sum of members of $\mathcal{A}$ whose direct summands all have rank one.*

(d) *The compressed algebra $\mathcal{B} = \{A_1 \oplus \cdots \oplus A_{n-1} : A_1 \oplus \cdots \oplus A_n \in \mathcal{A}\}$ is strongly dense in*

$$\mathcal{C} = \{A_1 \oplus \cdots \oplus A_{n-1} : A_1 \xrightarrow{T_1} A_2 \dots A_{n-2} \xrightarrow{T_{n-2}} A_{n-1}\}.$$

(e) *$\mathcal{A}$ has no non-trivial strongly closed bilateral ideals.*

Proof. (a) follows from Proposition 4.3.

(b) Suppose $A = A_1 \oplus \cdots \oplus S_n^{-1} A_1 S_n \in \mathcal{A}$ and $x = x_1 \oplus \cdots \oplus x_n \in X_1 \oplus \cdots \oplus X_n$ are given. Note that the range of $S_k^{-1} S_n = T_k \cdots T_{n-1}$ is dense in X_k for each k. Fix an $\varepsilon > 0$. We will construct a finite sequence $y_n, \dots, y_1$ in the following way. Set

$y_n = S_n^{-1}A_1S_nx_n$. Assume that for some k, $1 \leqslant k \leqslant n-1$, we have already produced the vectors $y_n, \ldots, y_{k+1}$. Now consider two cases. If S_kx_k is not in the linear span of $\{S_ix_i | i > k\}$, choose $y_k \in R(S_k^{-1}S_n)$ with $\|S_k^{-1}A_1S_kx_k - y_k\| < \varepsilon$. On the other hand, if $S_kx_k = \sum_{i>k} c_iS_ix_i$, set $y_k = S_k^{-1}\sum_{i>k} c_iS_iy_i$. In the latter case, we have

$$\|S_k^{-1}A_1S_kx_k - y_k\| \leqslant \sum_{i>k} |c_i| \, \|S_k^{-1}S_i(S_i^{-1}A_1S_ix_i - y_i)\|.$$

Since $S_k^{-1}S_i$ is bounded for $i > k$ and only finitely many c_i appear in these linear combinations, we can arrange, by an appropriate choice of y_k's, that

$$\|S_k^{-1}A_1S_kx_k - y_k\| < \varepsilon \quad \text{for all } 1 \leqslant k \leqslant n. \tag{$*$}$$

Define B_1 to be a linear transformation on X_1 which vanishes on a subspace of X_1 complementary to the span of $\{S_kx_k\}$, and sends S_kx_k to S_ky_k (it is clear from the above that such B_1 exists). Then B_1 is a finite-rank operator whose range is contained in the range of S_n. Thus $B = \sum_{i=1}^{n}{}^{\oplus} S_i^{-1}B_1S_i$ is a finite-rank member of $\mathcal{A}$ and $(*)$ shows that we can make $\|(A-B)x\|$ arbitrarily small.

For a finite subset $\{x_1, \ldots, x_m\} \subset \sum_{i=1}^{n}{}^{\oplus} x_i$, where $x_j = \sum_{i=1}^{n}{}^{\oplus} x_{ji}$, define the factor algebra

$$\tilde{\mathcal{A}} = \left\{A_1^{(m)} \oplus \cdots \oplus A_n^{(m)} : A_1 \xrightarrow{T_1} A_2 \ldots A_{n-1} \xrightarrow{T_{n-1}} A_n\right\}$$

and a vector $\tilde{x} = \sum_{i=1}^{n}{}^{\oplus} \sum_{j=1}^{m}{}^{\oplus} x_{ji}$. To finish the proof, apply the result of the preceding paragraph to $\tilde{A} = \sum^{\oplus} A_i^{(m)}$ and $\tilde{x}$.

(c) Every finite-rank operator B_1 whose range is contained in the range of S_n can be expressed as a sum of rank one operators, with ranges contained in the range of S_n. Now apply (a).

(d) In view of (a), every rank one member of $\mathcal{C}_1 = \mathcal{C}|X_1$ takes the form $C_1 = = S_{n-1}x \otimes \varphi$ for some $x \in X_{n-1}$ ans $\varphi \in X_1^*$. In view of (b) and (c), it suffices to express any $C = \sum_{i=1}^{n-1}{}^{\oplus} S_i^{-1}C_1S_i$ as a strong limit of the members of $\mathcal{B}$. Choose $y \in X_n$ with $\|x - T_ny\| < \varepsilon$ and take $B_1 = S_ny \otimes \varphi$. Then $B = \sum_{i=1}^{n-1}{}^{\oplus} S_i^{-1}B_1S_i$ belongs to $\mathcal{B}$. Moreover, for each $i \leqslant n-1$, we have $S_i^{-1}(C_1 - B_1)S_i = S_i^{-1}S_{n-1}(x - T_ny) \otimes S_i^*\varphi$, so B and C are in fact close in norm whenever ε is sufficiently small.

(e) Let $\mathcal{J}$ be a non-zero bilateral ideal in $\mathcal{A}$. We will show that $\mathcal{J}$ is strongly dense in $\mathcal{A}$. Let $\mathcal{A}_1 = \mathcal{A}|X_1$ and $\mathcal{J}_1 = \mathcal{J}|X_1$. Then $\mathcal{J}_1$ is a non-zero bilateral ideal in $\mathcal{A}_1$. On the other hand, since $\mathcal{A}_1$ is strongly dense in $\mathcal{L}(X_1)$, the strong closure of $\mathcal{J}_1$ is an ideal in $\mathcal{L}(X_1)$, hence $\mathcal{J}_1$ is strongly dense in $\mathcal{L}(X_1)$. Let $\{T_\alpha\}$ be a net in $\mathcal{J}_1$ converging strongly to the identity. Fix $x \in X_n$, $\varphi \in X_1^*$ and $\varepsilon > 0$. Then $\lim_\alpha T_\alpha(S_n x \otimes \varphi) = S_n x \otimes \varphi$. It follows that we can choose $y \in X_n$ such that $\|S_n(y - x)\| < \varepsilon$ and $S_n y \otimes \varphi \in \mathcal{J}_1$. To complete the proof, apply the same argument as in the second part of the proof of (d). ∎

LEMMA 4.5. *Every factor algebra is completely reducible.*

Proof. Adopt the notation of Lemma 4.4. We will proceed by induction on n. If $n = 1$, then $\mathcal{A} = \mathcal{L}(X)$ is completely reducible. Suppose our assertion is true for all integers not exceeding $n - 1$. For the inductive step, suppose $M \in \operatorname{lat}\mathcal{A}$ and set $M_1 = \{x_1 \oplus \cdots \oplus x_{n-1} \in X_1 \oplus \cdots \oplus X_{n-1} : x_1 \oplus \cdots \oplus x_{n-1} \oplus 0 \in M\}$. Then M_1 is invariant under $\mathcal{B}$ whence M_1 is invariant under $\mathcal{C}$ by Lemma 4.4(d). By the induction hypothesis, there exists $N_1 \in \operatorname{lat}\mathcal{C}$ such that $M_1 \dotplus N_1 = X_1 \oplus \cdots \oplus X_{n-1}$. If $M_1 = M$, then $N_1 \oplus X_n$ is the desired complement to M.

Otherwise set $N = N_1 \oplus 0$ and note that $M \cap N = 0$. Fix a vector $x = x_1 \oplus \cdots \oplus x_n \in M$ and $\varphi \in X_1^*$ with $\varphi(S_n x_n) \neq 0$. Given $y \in X_n$, the rank one operator $y \otimes S_n^* \varphi$ belongs to $\mathcal{A}_n = \mathcal{A}|X_n$ and sends x_n to a non-zero multiple of y. This shows that the canonical projection onto X_n along $X_1 \oplus \cdots \oplus X_{n-1}$ sends M onto the whole of X_n, whence $M + N$ exhausts X, and the proof is complete. ∎

The following example answers a question of Fong [6].

EXAMPLE 4.6. *There is completely reducible algebra $\mathcal{A}$ such that the strongly closed algebra generated by $\mathcal{A} \cup \mathcal{A}'$ is not completely reducible.*

Proof. Let X be a Banach space and T a singular quasiaffinity in $\mathcal{L}(X)$. Let $\mathcal{A}$ be the following operator algebra on $X^{(2)}$:

$$\mathcal{A} = \{A \oplus B : AT = TB\}.$$

By the previous lemma $\mathcal{A}$ is completely reducible. Note that for each $A \in \mathcal{L}(X)$, $TA \oplus AT \in \mathcal{A}$. Let us show that $X \oplus 0 \in \operatorname{lat}\mathcal{A}'$. Suppose not. Then, for some non-zero $S \in \mathcal{L}(X)$, an operator

$$\begin{pmatrix} * & * \\ S & * \end{pmatrix}$$

commutes with $\mathcal{A}$. It follows that $STA = ATS$ for each $A \in \mathcal{L}(X)$, so that $ST = TS$ is a multiple of the identity. Since T is a quasiaffinity, $TS \neq 0$, but then T must be invertible, contradicting our hypothesis. Hence, $X \oplus 0 \in (\operatorname{lat}\mathcal{A}) \cap (\operatorname{lat}\mathcal{A}')$. On the other hand, it is easy to see that $X \oplus 0$ has no complement in $\operatorname{lat}\mathcal{A}'$. Therefore, the algebra generated by $\mathcal{A} \cup \mathcal{A}'$ is not completely reducible. ∎

LEMMA 4.7. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra containing a sufficient set of finite-rank operators. Let X_1 and X_2 be non-zero subspaces in* lat$\mathcal{A}$ *such that $X_1 \dot{+} X_2 = X$. Suppose that*

$$\operatorname{lat}\mathcal{A} = \{M \dot{+} N,\ M \in \operatorname{lat}(\mathcal{A}|X_1),\ N \in \operatorname{lat}(\mathcal{A}|X_2)\}.$$

Then $\mathcal{A}|X_1$ contains a subalgebra $\mathcal{J}$ such that the intersection of the kernels of all operators in $\mathcal{J}$ is zero and $B \oplus 0 \in \mathcal{A}$ for each $B \in \mathcal{J}$.

Proof. Put $\mathcal{A}_1 = \mathcal{A}|X_1$ and $\mathcal{J} = \{B \in \mathcal{A}_1 : B \oplus 0 \in \mathcal{A}\}$. Let $M = \bigcap_{B\in\mathcal{J}} \ker B$. Then $\mathcal{J}$ is a bilateral ideal in $\mathcal{A}_1$. Hence $M \in \operatorname{lat}\mathcal{A}_1$. We must show that $M = 0$. Supposing the converse, we obtain, by Corollary 3.2(i) and Theorem 3.3(i), that $\mathcal{A}_1|M$ has an atom. By Lomonosov's Theorem, there exist a finite-rank $A \in \mathcal{A}$ and a non-zero $x \in M$ such that $Ax = x$. Then, for some polynomial p, $E = p(A)$ is a finite-rank projection such that $Ex = x$. Clearly, $E \in \mathcal{A}$. Let $E_1 = E|X_1$. Note that $E_1(M) \neq 0$.

Now consider the algebra $\mathcal{B} = E\mathcal{A}E|E(X)$. It is easy to see that $\mathcal{B}$ is a unital, completely reducible subalgebra of $\mathcal{L}(E(X))$. Since every such algebra on a finite--dimensional space is reflexive ([3], p. 125), so is $\mathcal{B}$. Let P denote the projection onto X_1 along X_2 and $P_0 = EPE|E(X) = PE|E(X)$. By our hypothesis, one has $\operatorname{lat}P \supseteq \operatorname{lat}\mathcal{A}$. It follows that $\operatorname{lat}P_0 \supseteq \operatorname{lat}\mathcal{B}$. Since $\mathcal{B}$ is reflexive, $P_0 \in \mathcal{B}$, so that $PE = E_1 \oplus 0$ is in $\mathcal{A}$. We conclude that $E_1 \in \mathcal{J}$, but the definition of M implies that $E_1(M) = 0$. Contradiction. ∎

LEMMA 4.8. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a unital, strongly closed completely reducible algebra containing a non-zero compact operator. Suppose the following conditions satisfied:*

(i) *$X = X_1 \oplus \cdots \oplus X_n$, where each X_i is an atom in* lat$\mathcal{A}$;

(ii) *0 is the only member of $\mathcal{A}$ which vanishes on a non-zero invariant subspace of $\mathcal{A}$. Then there exist quasiaffinities $T_i \in \mathcal{L}(X_{i+1}, X_i)$, $i = 1, \ldots, n-1$, such that, after a suitable renumbering of X_i's,*

$$\mathcal{A} \subseteq \{A_1 \oplus A_2 \oplus \cdots \oplus A_n : A_1 \xrightarrow{T_1} A_2 \ldots A_{n-1} \xrightarrow{T_{n-1}} A_n\}.$$

Proof. Our proof will be by induction on n. If $n = 1$, then $\mathcal{A}$ is transitive whence $\mathcal{A} = \mathcal{L}(X)$ by Lomonosov's Theorem [9, Theorem 8.23]. Suppose our lemma is true for all $k \leqslant n-1$. For the inductive step, fix k, $1 \leqslant k \leqslant n-1$. In view of condition (ii) and Lemma 4.7 (note that by Lemma 3.1 and Theorem 3.3(ii), $\mathcal{A}$ contains a sufficient set of finite-rank operators), $\mathcal{A}|X_k \oplus X_n$ has an invariant subspace other than the obvious ones. Since $\mathcal{A}$ is completely reducible, there exists a projection E

on $X_k \oplus X_n$ commuting with $\mathcal{A}|X_k \oplus X_n$ and such that the subspaces X_k and X_n are not both invariant under E.

Write E as a matrix

$$E = \begin{pmatrix} * & V \\ S & * \end{pmatrix}, \quad \text{where}$$

$V \in \mathcal{L}(X_n, X_k)$ and $S \in \mathcal{L}(X_k, X_n)$. A routine computation shows that $A_k V = V A_n$ and $A_n S = S A_k$ for every $A \in \mathcal{A}$, where $A_i = A|X_i$. Note that either V or S is non--zero. Then this non-zero transformation must be a quasiaffinity, for if, say $V \neq 0$, then $\operatorname{cl} R(V) \in \operatorname{lat}(\mathcal{A}|X_k)$ and $\ker V \in \operatorname{lat}(A|X_n)$ must equal, respectively, X_k and 0. The case $S \neq 0$ is dealt with similarly.

Now apply the induction hypothesis to (the strong closure of) the algebra $\mathcal{A}|X_1 \oplus \oplus \cdots \oplus X_{n-1}$ to see that there exist quasiaffinities $T_1, \ldots, T_{n-2}$ such that (perhaps after renumbering of X_i's, $1 \leqslant i \leqslant n-1$)

$$\mathcal{A}|X_1 \oplus \cdots \oplus X_{n-1} \subseteq \{A_1 \oplus A_2 \oplus \cdots \oplus A_{n-1} : A_1 \xrightarrow{T_1} A_2 \ldots A_{n-2} \xrightarrow{T_{n-2}} A_{n-1}\}.$$

By the above, there exists a quasiaffinity T such that $A_n \xrightarrow{T} A_1$ or $A_1 \xrightarrow{T} A_n$ for all $A \in \mathcal{A}$. In the first case we are done, for

$$A_n \xrightarrow{T} A_1 \xrightarrow{T_1} A_2 \ldots A_{n-2} \xrightarrow{T_{n-2}} A_{n-1} \quad \text{for all } A \in \mathcal{A}.$$

So consider the second case: $A_1 \xrightarrow{T} A_n$. Apply the induction hypothesis to (the strong closure of) $\mathcal{A}|X_2 \oplus \cdots X_n$ to see that after a proper rearrangement of $X_2, \ldots, X_n$ we may find suitable quasiaffinities $V_2, \ldots, V_{n-1}$ such that

$$A_2 \xrightarrow{V_2} A_3 \ldots \xrightarrow{V_{n-1}} A_n \quad \text{for all } A \in \mathcal{A}.$$

Note that for each k, $2 \leqslant k \leqslant n$ there exist a quasiaffinity S_k satisfying $A_1 \xrightarrow{S_k} A_k$. Indeed, set $S_k = T_1 \cdots T_{k-1}$ for $k \leqslant n-1$ and $S_n = T$. Then there is a quasiaffinity (one of S_k's) intertwining A_1 and A_2. Calling it V_1, we have

$$A_1 \xrightarrow{V_1} A_2 \ldots A_{n-1} \xrightarrow{V_{n-1}} A_n \quad \text{for all } A \in \mathcal{A},$$

which completes the proof. ∎

In the folowing theorem which is the main result of this section we generalize one of the results of Azoff [1].

THEOREM 4.9. *Let $\mathcal{A}$ be a unital, strongly closed subalgebra of $\mathcal{L}(X)$. Then the following are equivalent:*

(i) *$\mathcal{A}$ is a completely reducible algebra containing a non-zero compact operator and $\mathcal{A} \cap \mathcal{A}''$ contains no projections other than 0 and I.*

(ii) *$\mathcal{A}$ is a factor algebra.*

Proof. (i)$\Rightarrow$(ii): Apply Theorem 3.3(i) to find an atom in lat$\mathcal{A}$ and use Lomonosov's result to see that $\mathcal{A}$ contains a non-zero finite-rank projection. Denote the minimal rank of such operators by n. Note that since $\mathcal{A}' \cap \mathcal{A}''$ has no non-trivial projections, Corollary 3.2(ii) tells us that 0 is the only member of $\mathcal{A}$ which vanishes on a non-zero invariant subspace of $\mathcal{A}$.

We claim that X may be expressed as a direct sum of exactly n atoms in lat$\mathcal{A}$. Indeed, it is clear from the above that X is a direct sum of at most n such subspaces. Suppose that $X = \sum_{i=1}^{m} {}^{\oplus} X_i$, where each X_i is an atom in lat$\mathcal{A}$ and $m < n$. Let $E = \sum_{i=1}^{m} {}^{\oplus} E_i$ be a projection of rank n in $\mathcal{A}$. Then, for some k, E_k is a projection of rank at least two. Now apply the argument of Barnes [2] to transitive algebra $\mathcal{A} = \mathcal{A}|X_k$. That is, the algebra $E_k\mathcal{A}_k E_k|E_k(X_k)$ is a transitive algebra of operators on a finite-dimensional space and therefore must contain all operators on this space. In particular, it contains a rank-one projection. If so, the algebra $E\mathcal{A}E \subseteq \mathcal{A}$ contains an operator $F = \sum_{i=1}^{m} {}^{\oplus} F_i$ such that F_k is a projection of rank one. Hence the rank of F is strictly smaller that the rank of E. On the other hand, F is a projection, for otherwise $F^2 - F$ would be a non-zero operator in $\mathcal{A}$ vanishing on X_k. This contradiction proves the validity of our claim.

Assume henceforth that $X = \sum_{i=1}^{n} {}^{\oplus} X_i$. By Lemma 4.8, there exist quasiaffinities $T_i \in \mathcal{L}(X_{i+1}, X_i)$, $1 \leqslant i \leqslant n-1$ such that

$$\mathcal{A} \subseteq \{A_1 \oplus \cdots \oplus A_n : A_1 \overset{T_1}{\to} A_2 \ldots A_{n-1} \overset{T_{n-1}}{\to} A_n\}.$$

Obviously each E_i is a rank-one projection; by Lemma 4.4(a), E takes the form $E = \sum_{i=1}^{n} {}^{\oplus} S_i^{-1} S_n x \otimes S_i^* \varphi$ for some $x \in X_n$ and $\varphi \in X_1^*$. Thus for each $A \in \mathcal{A}$, $AE =$ $= \sum_{i=1}^{n} {}^{\oplus} S_i^{-1} S_n A_n x \otimes S_i^* \varphi$ also belongs to $\mathcal{A}$. Since $\mathcal{A}|X_n$ is transitive, for each $y \in X_n$, we see that $\sum_{i=1}^{n} {}^{\oplus} S_i^{-1} S_n y \otimes S_i^* \varphi$ is a norm limit of such operators. Similarly, the transitivity of $\mathcal{A}|X_1$ shows that $\sum_{i=1}^{n} {}^{\oplus} S_i^{-1} S_n y \otimes S_i^* \psi$ belongs to $\mathcal{A}$ for each $\psi \in X_1^*$. Thus $\mathcal{A}$ contains all rank n members of the factor algebra $\{A_1 \oplus \cdots A_n :$ $: A_1 \overset{T_1}{\to} A_2 \ldots A_{n-1} \overset{T_{n-1}}{\to} A_n\}$ and the proof is completed by appealing to Lemma 4.4(b).

(ii)$\Rightarrow$(i): Let $\mathcal{A}$ be the factor algebra as above. We only have to prove that $\mathcal{A}' \cap \mathcal{A}''$ contains no non-trivial projections. Let P be a projection in $\mathcal{A}' \cap \mathcal{A}''$. Since

P commutes with canonical projections onto co-ordinate subspaces, it must take the form $P = \sum_{i=1}^{n}{}^{\oplus} P_i$, where each P_i is a projection on X_i. Furthermore, since each P_i commutes with the strongly dense algebra $\mathcal{A}|X_i$, it is either 0 or I. Suppose that P is neither 0 nor I. Then, for some i and j, $i < j$, P_i and P_j are not both 0 or I at the same time. Let the operator A be given by the matrix (A_{pq}), $1 \leqslant p,\ q \leqslant n$, where $A_{pq} = \delta_{i,p}\delta_{q,j}T_iT_{i+1}\ldots T_{j-1}$. It is easily seen that $AP \neq PA$. On the other hand, A belongs to $\mathcal{A}'$. Thus we conclude that E is 0 or I. ■

5. PROOF OF THE MAIN THEOREM

Throughout the following four lemmas $\mathcal{A}$ is a unital, strongly closed, completely reducible algebra on X which contains a sufficient set of finite-rank operators, and Ω denotes the family of all projections in $\mathcal{A}' \cap \mathcal{A}''$. Recall that Ω is a commutative Boolean algebra.

LEMMA 5.1. *Choose a non-zero $B \in \mathcal{A}$ and let $L = \bigvee\{AB(X),\ A \in \mathcal{A}\}$. Then*

(i) *B does not vanish on every non-zero $L_1 \subseteq L$, $L_1 \in \operatorname{lat}\mathcal{A}$;*

(ii) *L is the range of a projection in Ω;*

(iii) *If, in addition, $E_0 \in \Omega$, $E_0 \neq 0$ and B is an operator of the least rank such that $E_0B \neq 0$, then L is the range of a minimal projection in Ω.*

Proof. (i) Suppose that $L_1 \in \operatorname{lat}\mathcal{A}$, $L_1 \subseteq L$ and $B|L_1 = 0$. Denote by E a projection onto L_1 in $\mathcal{A}'$. Then $EAB(X) = ABE(X) = 0$ for each $A \in \mathcal{A}$, so that $E(L) = 0$. Since $L_1 \subseteq L$, we have $L_1 = E(L) = 0$.

(ii) It is clear that $L \in (\operatorname{lat}\mathcal{A}) \cap (\operatorname{lat}\mathcal{A}')$. Hence there exists $M \in \operatorname{lat}\mathcal{A}$ such that $L \dot{+} M = X$. To prove (ii), it suffices to show that $M \in \operatorname{lat}\mathcal{A}'$. Suppose the converse. Denote, for each $A \in \mathcal{A}$, $A_1 = A|L$ and $A_2 = A|M$. Then $B_2 = 0$, for $B(X) \subseteq L$. Since M is not in $\operatorname{lat}\mathcal{A}'$, there exists a non-zero $T \in \mathcal{L}(M, L)$ satisfying $B_1A_1T = 0$ for each $A \in \mathcal{A}$. Choose $x \in M$ with $Tx \neq 0$ and put $N = \operatorname{cl}\mathcal{A}Tx$. Then $N \subset L$, $N \in \operatorname{lat}\mathcal{A}$ and $B|N = 0$. By (i), we conclude that $N = 0$. Since $\mathcal{A}$ is unital, $Tx \in N$, so that $Tx = 0$. This contradiction shows that $M \in \operatorname{lat}\mathcal{A}'$, as asserted.

(iii) By (ii), one can find $E \in \Omega$ such that $E(X) = L$. It is clear that $E_0E \neq 0$. Suppose that E is not minimal in Ω. Then there exist non-zero E_1 and E_2 in Ω such that $E = E_1 + E_2$ and $F = E_0E_1 \neq 0$. Repeating the same argument as in the proof of Lemma 4.8, we see that for every $M \in \operatorname{lat}\mathcal{A}$, $M = F(M) \dot{+} (I - F)(M)$. By Lemma 4.7, there exists $A \in \mathcal{A}$ such that $FAB \neq 0$ and $(I - F)A = 0$. Then $E_0AB \neq 0$ and, since $E_2B \neq 0$ by (i) and $E_2AB = 0$ ($E_2 \leqslant I - F$), we conclude that $\operatorname{rank}(AB) < \operatorname{rank} B$. This contradicts our assumption about B. Now the proof is complete. ■

LEMMA 5.2. *If E is a minimal projection in Ω, then $E \in \mathcal{A}$.*

Proof. Let B denote an operator of the least rank such that $EB \neq 0$. Denote by F the minimal projection onto the subspace spanned by $AB(X)$, $A \in \mathcal{A}$. (Lemma 5.1 (iii)). Clearly $EF \neq 0$ and, since both E and F are minimal in Ω, $E = F$. It follows that $EB = B$. Let $\mathcal{J}$ denote a set of those $A \in \mathcal{A}$ for which $EA = A$. Then $\mathcal{J}_0 = \mathcal{J}|E(X)$ is a non-zero bilateral ideal in $\mathcal{A}|E(X)$. Since $\mathcal{A}$ is strongly closed, so is $\mathcal{J}_0$. Let $\mathcal{A}_0$ denote the strong closure of $\mathcal{A}|E(X)$. It is easy to see that $\mathcal{A}_0$ satisfies the hypothesis of Theorem 4.9. Hence $\mathcal{A}_0$ is a factor algebra and $\mathcal{J}_0$ is a non-zero bilateral strongly closed ideal in $\mathcal{A}_0$. By Lemma 4.4(e), one has $\mathcal{J}_0 = \mathcal{A}_0$. In particular, the identity operator in $E(X)$ lies in $\mathcal{J}_0$. This means that $E \in \mathcal{A}$, as asserted. ■

LEMMA 5.3. *Every family of minimal projections in Ω has its supremum in Ω.*

Proof. Let $\{E_\alpha\}$ be an arbitrary non-empty family of minimal projections in Ω. The previous lemma makes it clear that $E_\alpha \in \mathcal{A}$ for each α. Put

$$M = \bigvee_\alpha E_\alpha(X), \quad L = \left(\bigcap_\alpha \ker E_\alpha\right) \cap M.$$

Obviously, $M \in (\operatorname{lat}\mathcal{A}) \cap (\operatorname{lat}\mathcal{A}')$. We claim that $L = 0$. Indeed, since $L \in \operatorname{lat}\mathcal{A}$, there is $L_1 \in \operatorname{lat}\mathcal{A}$ such that $L \dotplus L_1 = M$. Then $E_\alpha(X) = E_\alpha(M) \subseteq L_1$ for each α and, by the definition of M, $L_1 = M$ and $L = 0$. This proves our claim.

Let $N \in \operatorname{lat}\mathcal{A}$, $M \dotplus N = X$. It is clear that $E_\alpha(N) = 0$ for each α. Let E denote the projection onto M along N. Then, for every $T \in \mathcal{A}'$,

$$E_\alpha ET(I-E) = ETE_\alpha(I-E) = 0 \quad (\text{all } \alpha).$$

From the preceding paragraph it follows that $ET(I-E) = 0$ for every $T \in \mathcal{A}'$; that is, $N \in \operatorname{lat}\mathcal{A}'$. Hence, $E \in \Omega$ and E is the supremum of $\{E_\alpha\}$. ■

LEMMA 5.4. *Ω is a complete totally atomic Boolean algebra.*

Proof. Let $E \in \Omega$ and let $\{E_\alpha\}$ be the family of all minimal projection in Ω dominated by E. We claim that E is the supremum of $\{E_\alpha\}$. Indeed, denote this supremum by F; then, by the previous lemma, $F \in \Omega$. Evidently, $F \leqslant E$. Suppose that $F \neq E$. Let B be an operator of the least rank in $\mathcal{A}$ such that $(E-F)B \neq 0$, and let L denote the subspace spanned by $AB(X)$ with $A \in \mathcal{A}$. By Lemma 5.1(iii), $L = G(X)$, where G is a minimal projection in Ω. Then $EG \neq 0$, hence $G \leqslant E$. But then F does not dominate G, a contradiction. This proves our claim.

Now let $\{E_\alpha\}$ be an arbitrary non-empty set in Ω. Put

$$L = \bigvee_\alpha E_\alpha(X).$$

Since, by the above, each $E_\alpha(X)$ is spanned by the ranges of minimal projections in Ω, the same can be said of L. By Lemma 5.3, there exists $E \in \Omega$ such that $E(X) = L$. That is, $E = \bigvee_\alpha E_\alpha$ is in Ω. To finish the proof, it suffices to notice that

$$\bigwedge_\alpha E_\alpha = I - \bigvee_\alpha (I - E_\alpha),$$

so that $\bigwedge_\alpha E_\alpha$ is also in Ω. ∎

Let Ω be a complete, bounded Boolean algebra in $\mathcal{L}(X)$, and let $\{E_\alpha\}$ be a family of non-zero mutually disjoint elements of Ω such that $\bigvee_\alpha E_\alpha = I$. Let $X_\alpha = E_\alpha(X)$ and $\mathcal{S}_\alpha \subseteq \mathcal{L}(X_\alpha)$. We will write $\bigoplus_\alpha \mathcal{S}_\alpha$ for the set of all operators $T \in \mathcal{L}(X)$ which commute with each E_α and such that $T|X_\alpha \in \mathcal{S}_\alpha$. Note that $T = \bigoplus_\alpha T_\alpha$ if and only if $T_\alpha = E_\alpha T|E_\alpha(X)$ for each α.

Now we prove the main result of this paper.

THEOREM 5.5. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a unital, strongly closed, completely reducible algebra containing a sufficient set of compact operators. Then there exist a complete bounded totally atomic Boolean algebra of projections $\Omega \subset \mathcal{A} \cap \mathcal{A}'$ with the set of atoms $\{E_\alpha\}$ and a collection of algebras $\{\mathcal{A}_\alpha\}$, $\mathcal{A}_\alpha \subseteq \mathcal{L}(X_\alpha)$, where $X_\alpha = E_\alpha(X)$, such that each $\mathcal{A}_\alpha$ is a factor algebra, $\mathcal{A}_\alpha$ is a factor algebra of type* II *only for finitely many α, and $\mathcal{A} = \bigoplus_\alpha \mathcal{A}_\alpha$. Moreover, $\mathcal{A}$ is reflexive and $\mathcal{A}'' = \mathcal{A}$.*

Proof. By Theorem 3.3(ii), $\mathcal{A}$ contains a sufficient set of finite-rank operators. Let Ω denote the collection of all projections in $\mathcal{A}' \cap \mathcal{A}''$. By Lemma 5.4, Ω is a complete totally atomic Boolean algebra, so that by [5], Lemma XVII.3.3, Ω is bounded. Let $\{E_\alpha\}$ be the set of all minimal projections (atoms) in Ω. For each α, $E_\alpha \in \mathcal{A}$ (Lemma 5.1); it follows that $\mathcal{A}_\alpha = \mathcal{A}|X_\alpha$ is strongly closed. Also, $\Omega \subset \mathcal{A}$, for $\mathcal{A}$ is strongly closed. If E is a projection on X_α which commutes with both $\mathcal{A}_\alpha$ and $\mathcal{A}'_\alpha$, then EE_α belongs to Ω. Since E_α is an atom in Ω, it follows that $EE_\alpha = 0$ or $EE_\alpha = E_\alpha$. By Theorem 4.9, we see that $\mathcal{A}_\alpha$ is a factor algebra in $\mathcal{L}(X_\alpha)$ for each α.

Let T be in $\mathcal{L}(X)$ such that $\mathrm{lat}T \supseteq \mathrm{lat}\mathcal{A}$. Then, for each α, T commutes with E_α and $\mathrm{lat}(T|X_\alpha) \supseteq \mathrm{lat}\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ is reflexive (Remark 4.2), $T|X_\alpha \in \mathcal{A}_\alpha$. Hence $TE_\alpha \in \mathcal{A}$. Let Σ denote the set of all finite sums of mutually disjoint E_α's. Then Σ is an increasing net in Ω and, by [5], Lemma XVII.3.4, the identity operator on X

is the strong limit of Σ. Since, for every $E \in \Sigma$, $TE \in \mathcal{A}$ and $\mathcal{A}$ is strongly closed, $T \in \mathcal{A}$. This proves that $\mathcal{A}$ is reflexive and $\mathcal{A} = \bigoplus_{\alpha} \mathcal{A}_\alpha$.

It remains to be shown that the number of those $\mathcal{A}_\alpha$'s which are of type II is finite. Let us suppose the converse. Then there exists an infinite sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of factor algebras of type II such that $\mathcal{A} = \bigoplus_{n=1} \mathcal{A}_n$ is completely reducible. We may choose, for each $n \geqslant 1$, a subspace $M_n \in (\text{lat}\mathcal{A}_n) \cap (\text{lat}\mathcal{A}'_n)$ having no complement in $\text{lat}\mathcal{A}'_n$ (cf. Example 4.6). Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \oplus \cdots$, that is,

$$M = \{x \in X : E_n x \in M_n \text{ for } n \geqslant 1\}.$$

Then $M \in (\text{lat}\mathcal{A}) \cap (\text{lat}\mathcal{A}')$ and, since $\mathcal{A}$ is completely reducible, there is $L \in \text{lat}\mathcal{A}$ such that $M \dot{+} L = X$. It is clear that $L = L_1 \oplus L_2 \cdots \oplus L_n \oplus \cdots$, where $M_n \dot{+} L_n = X_n$, $L_n \in \text{lat}\mathcal{A}_n$. Note that L_n is not in $\text{lat}\mathcal{A}'_n$ for $n \geqslant 1$. Hence there exists a non--zero $T_n \in \mathcal{L}(L_n, M_n)$ such that, for each $A \in \mathcal{A}_n$, $(A|M_n)T_n = T_n(A|L_n)$. Now, using Remark 2.3, it can be shown that $L \in \text{lat}\mathcal{A}'$. Then $L_n \in \text{lat}\mathcal{A}'_n$ for $n \geqslant 1$, a contradiction. This completes the proof. ■

COROLLARY 5.6. [12]. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a commutative, unital, strongly closed completely reducible algebra that contains a sufficient set of compact operators. Then $\mathcal{A}$ is the uniformly closed algebra generated by a complete totally atomic Boolean algebra of projections Ω. Also, each atom in Ω is finite-dimensional, and every operator in $\mathcal{A}$ is a scalar type spectral operator.*

Proof. Since $\mathcal{A}$ is commutative, so is each $\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ is a factor algebra, it must contain only the multiples of the identity. It follows that $\mathcal{A}$ is the strongly closed algebra generated by $\{E_\alpha\}$. By [5], XVII.3.17, $\mathcal{A}$ is the uniformly closed algebra generated by $\{E_\alpha\}$. Furthermore, by [5], XVII.3.25, $\mathcal{A}$ is an algebra of scalar type spectral operators. Since $\mathcal{A}$ contains a sufficient set of compact operators, each E_α must be finite-dimensional. ■

COROLLARY 5.7. *Let $\mathcal{A}$ be a unital, strongly closed completely reducible algebra on a separable Hilbert space that contains a sufficient set of compact operators. Then $\mathcal{A}$ is spatially isomorphic to the orthogonal direct sum of finitely many factor algebras of type* II *and of finitely or countably many factor algebras of type* I.

Proof. Suppose $\mathcal{A} \subseteq \mathcal{L}(H)$, where H is a separable Hilbert space. Let Ω denote, as above, the set of all projections in $\mathcal{A}' \cap \mathcal{A}''$. By [5], Lemma XV.6.2, here exists an invertible $S \in \mathcal{L}(H)$ such that $S^{-1}\Omega S$ is a totally atomic Boolean algebra of self--adjoint projections. Since H is separable, the number of atoms in $S^{-1}\Omega S$ is at most countable. Denote these atoms by $P_1, P_2, \ldots, P_n, \ldots$. Then $\mathcal{B} = S^{-1}\mathcal{A}S = \bigoplus_n \mathcal{B}_n$, where $\mathcal{B}_n = \mathcal{B}|P_n(H)$. ■

Note that the converse to Corollary 5.7 is false, that is, an infinite orthogonal sum of factor algebras of type I is not necessary completely reducible.

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SHLOMO ROSENOER
Department of Mathematics,
Toronto, Ontario, Canada,
M5S 1A1.

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