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SPECTRAL ANALYSIS FOR SIMPLY CHARACTERISTIC OPERATORS BY MOURRE'S METHOD. II

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1. INTRODUCTION

In [7] the authors developed an abstract theory of multiple commutator estimates for a self-adjoint operator H and a suitable conjugate operator A.

The purpose of this paper is mainly to show how this abstract theory and its consequences can be used in the context of simply characteristic operators.

The plan of the paper is as follows. In Section 2 we present a short-range scattering theory for simply characteristic operators. The results of Section 2 are extended in Section 3 to operators with long-range potentials. In Section 4 we obtain resolvent estimates in Besov spaces in the context of Mourre's commutator methods. Finally, the paper has an Appendix which contains results concerning L^2-boundedness of some multi-commutators of pseudodifferential operators and the quasi-divergence of some functions.

2. THE SHORT-RANGE CASE

The results of [7] have as a particular consequnce an abstract scattering theory. In this section we shall apply this theory to the simply characteristic operators with short-range perturbations.

We shall work under the following hypotheses.

HYPOTHESES

I. The free Hamiltonian H_0 is a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^n)$, with the domain $\mathcal{D}(H_0) = \{u \in \mathcal{H}; p_0\hat{u} \in \mathcal{H}\}$, $H_0 = \mathcal{F}^{-1}p_0\hat{u}$, where $\hat{u}$ is the Fourier transform of u and p_0 is a real valued function which satisfies:

(i) $p_0 : \mathbf{R}^n \to \mathbf{R}$ is a continuous function.

(ii) Let S_p be the set $\{\xi \in \mathbf{R}^n;\ p_0 \text{ is not } C^\infty \text{ in any neighborhood of } \xi\}$, let C_p be the set $\{\xi \in \mathbf{R}^n \setminus S_p;\ \nabla p_0(\xi) = 0\}$ and let $S = S_p \cup C_p$. Then $\overline{p_0(S)}$ is a countable subset of $\mathbf{R}$.

(iii) For any compact interval $I \subset \mathbf{R} \setminus \overline{p_0(S)}$, with $p_0^{-1}(I) \neq \emptyset$, we have

$$\inf\{|\nabla p_0(\xi)|;\ \xi \in p_0^{-1}(I)\} > 0, \tag{2.1}$$

$$\operatorname{dist}(p_0^{-1}(I), S_p) > 0 \tag{2.2}$$

(iv) $\sup\left\{\dfrac{|D^\alpha p_0(\xi)|}{1 + |p_0(\xi)| + |\nabla p_0(\xi)|};\ \xi \in \mathbf{R}^n \setminus S_p\right\} < \infty$ for each multi-index α with $|\alpha| \geqslant 2$.

(v) (local compactness). For any compact interval $I \subset \mathbf{R} \setminus \overline{p_0(S)}$ and for each $r > 0$, the operator

$$F(|x| < r)E_0(I)$$

is compact. Here $F(M)$ denotes the indicator function of the set M and $E_0(I)$ denotes the spectral projection for H_0 onto the interval I.

II. Let $V : \hat{\mathcal{D}} \to \mathcal{H}$ be a symmetric operator such that

(vi) The operator $H_0 + V$ with the domain $\hat{\mathcal{D}}$ has a self-adjoint extension H on $\mathcal{H}$.

(vii) For some $\varepsilon > 0$ the operator $g(H)Vg(H_0)\langle X\rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each g in $C_0^\infty(\mathbf{R})$.

We used the notations: $\hat{\mathcal{D}}$ for the image of $\mathcal{D}$ (the space of test functions defined on $\mathbf{R}^n$) by the Fourier transform and $\langle x\rangle = (1 + |x|^2)^{1/2}$, $x \in \mathbf{R}^n$.

(viii) For any g in $C_0^\infty(\mathbf{R})$ the operator $g(H) - g(H_0)$ is compact.

The main result of this section is the following theorem.

THEOREM 2.1. *Assume that the hypotheses* (i)–(viii) *are satisfied. Then*

(a) *The wave operators* $W_\pm = s - \lim\limits_{t\to\pm\infty} e^{iHt}e^{-iH_0t}E_{ac}(H_0)$ *exist;*

(b) *Range* $W_\pm = \mathcal{H}_c(H)$, *the contiuous subspace of* H;

(c) $\sigma_{sc}(H) = \emptyset$;

(d) *Any eigenvalue of* H *not in* $\overline{p_0(S)}$ *is of finite multiplicity. The eigenvalues of* H *can accumulate only at the points of* $\overline{p_0(S)}$.

Before proving the theorem we wish to make a few remarks about the hypotheses we made.

REMARK 2.2. a) The condition (2.1) can be read as follows:

(2.1)′ If the free energy lies in a compact interval disjoint from thresholds, then the velocity is bounded from below by a positive constant.

b) If we replace the condition (2.1) by the stronger condition

(2.1)″ $\lim_{|\xi|\to\infty,\ \xi\notin S_p}(|p_0(\xi)| + |\nabla p_0(\xi)|) = \infty$,

then the local compactness property of H_0 (i.e. condition (v)) is fulfilled (see the Appendix).

In the same way one can prove a similar theorem with the condition (vii) replaced by the condition

(vii)′ For some $\varepsilon > 0$ the operator $g(H)V\langle X\rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each g in $C_0^\infty(R)$.

This condition is always true when V is a symmetric H_0-compact operator and there is an $\varepsilon > 0$ such that the operator

$$(H_0 + \mathrm{i})^{-1}\langle X\rangle^{1+\varepsilon}$$

has a bounded extension.

The idea of the proof of the theorem is to construct for any interval $I \subset\subset \mathbf{R}\backslash\overline{p_0(S)}$ an operator $A = A_I$ conjugate to H_0 on the interval I, such that H_0 is ∞-smooth with respect to A in the sense of the Definition 2.1 given in [7].

Since we shall use the same technique in Section 3, we shall recall this definition.

Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(H)$. Let E_H denote the spectral measure for H. Denote by $\mathcal{H}_s$ the completition of the vectors ψ satisfying

$$\|\psi\|_s^2 = \int (1+\lambda^2)^{s/2}\mathrm{d}\|E_H(\lambda)\psi\|^2 < \infty.$$

Then $\mathcal{H}_{+2}$ is the domain $\mathcal{D}(H)$ with the graph norm, and $\mathcal{H}_{-2}$ is the dual of $\mathcal{H}_{+2}$ obtained via the inner product on $\mathcal{H}$.

DEFINITION 2.3. Let $I \subset \mathbf{R}$ be an interval and let $m \geqslant 1$ be an integer. A self-adjoint operator A on $\mathcal{H}$ is said to be conjugate to H on the interval I, and H is said to be m-smooth with respect to A, if the following conditions are satisfied:

a) $\mathcal{D}(A) \cap \mathcal{D}(H)$ is a core for H.

b) $\mathrm{e}^{\mathrm{i}A\alpha}$ maps $\mathcal{D}(H)$ into $\mathcal{D}(H)$, and for each $\psi \in \mathcal{D}(H)$

$$\sup_{|\alpha|\leqslant 1} \|H\mathrm{e}^{\mathrm{i}A\alpha}\psi\| < \infty$$

$\mathrm{c_m}$) The form $\mathrm{i}[H, A]$ defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$, is bounded from below and closable. The self-adjoint operator associated with its closure is denoted $\mathrm{i}B_1$. Assume $\mathcal{D}(H) \subset \mathcal{D}(B_1)$. If $m > 1$, assume for $j = 2, \dots, m$ that the form $\mathrm{i}[\mathrm{i}B_{j-1}, A]$, defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$, is bounded from below and closable. The associated self-adjoint operator is denoted iB_j, and it is assumed that $\mathcal{D}(H) \subset \mathcal{D}(B_j)$.

$\mathrm{d_m}$) The form $[B_m, A]$, defined on $\mathcal{D}(H) \cap \mathcal{D}(A)$, extends to a bounded operator from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$.

e) There exist $a > 0$ and a compact operator K on $\mathcal{H}$ such that

$$E_H(I)\mathrm{i}B_1E_H(I) \geqslant aE_H(I) + E_H(I)KE_H(I).$$

If H is m-smooth with respect to A for every integer $m \geqslant 1$, H is said to be ∞-smooth with respect to A.

We pass now to define the operators which we mentioned.

Let $I \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ be an interval. If $p_0^{-1}(I) = \emptyset$ we take $A = A_I = 0$.

If $p_0^{-1}(I) \neq \emptyset$ we proceed as follows. The condition (iii) (2.2) implies the existence of a function χ in $C_b^\infty(\mathbf{R}^n)$ with the following properties

$$\chi(\xi) = 1 \ \text{ if } \ \xi \in p_0^{-1}(I), \quad \operatorname{supp}\chi \subset \mathbf{R}^n \setminus S_p.$$

Here $C_b^\infty(\mathbf{R}^n)$ denotes the space of all smooth functions bounded with all their derivatives.

Next we define the smooth vector field v in phase space by

$$v(\xi) = \frac{\chi(\xi)\nabla p_0(\xi)}{1 + |p_0(\xi)|^2 + |\nabla p_0(\xi)|^2}. \tag{2.3}$$

Let us note that the condition (iv) and the properties of the function χ imply that the components of the vector field v belong to the space $C_b^\infty(\mathbf{R}^n)$.

Since the vector field v is bounded it follows that the Cauchy problem

$$\begin{cases} \left(\dfrac{\mathrm{d}}{\mathrm{d}\alpha}\right) \Gamma(\alpha, \xi) = v(\Gamma(\alpha, \xi)) \\ \Gamma(0, \xi) = \xi \end{cases}$$

defines a group of smooth diffeomorphisms of $\mathbf{R}^n$, $\{\Gamma(\alpha, \cdot)\}_{\alpha \in \mathbf{R}}$.

To this group of diffeomorphisms $\{\Gamma(\alpha, \cdot)\}_{\alpha \in \mathbf{R}}$ we associate a group of unitary operators $\{V(\alpha)\}_{\alpha \in \mathbf{R}}$ on $L^2(\mathbf{R}^n, \mathrm{d}\xi)$ by

$$(V(\alpha)\psi)(\xi) = \left|\det \frac{\partial \Gamma(\alpha, \xi)}{\partial \xi}\right|^{1/2} \psi(\Gamma(\alpha, \xi)), \ \ \psi \in L^2(\mathbf{R}^n, \mathrm{d}\xi). \tag{2.4}$$

If we denote by $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R}^n)$, then we obtain another group of unitary operators on $L^2(\mathbb{R}^n, \mathrm{d}x)$ defined by

$$U(\alpha) = \mathcal{F}^{-1} V(\alpha) \mathcal{F} \quad \text{on} \quad L^2(\mathbb{R}^n, \mathrm{d}x). \tag{2.5}$$

Let now $A = A_I$ be the self-adjoint operator on $L^2(\mathbb{R}^n, \mathrm{d}x)$ such that

$$U(\alpha) = \mathrm{e}^{-\mathrm{i}A\alpha}. \tag{2.6}$$

LEMMA 2.4. a) *$\hat{\mathcal{D}}$ is a core of A, A maps $\hat{\mathcal{D}}$ into $\hat{\mathcal{D}}$ and*

$$A = \sum_1^n \frac{X_j v_j(D) + v_j(D) X_j}{2} \quad \text{on} \quad \hat{\mathcal{D}}. \tag{2.7}$$

b) *For any $m \in \mathbb{N}$, $\hat{\mathcal{D}}$ is a core of A^m;*
c) *The statements* a) *and* b) *are true with $\hat{\mathcal{D}}$ replaced with $\mathcal{S}(\mathbb{R}^n)$.*

Proof. a) Let $\{V(\alpha)\}_{\alpha \in \mathbb{R}}$ be the group of unitary operators defined by (2.4). Concerning this group we make two quasievident remarks:

1) For any $\alpha \in \mathbb{R}$, $V(\alpha)$ maps $\mathcal{D}$ into $\mathcal{D}$;
2) For any $\psi \in \mathcal{D}$ we have

$$\lim_{\alpha \to 0} \frac{V(\alpha)\psi - \psi}{\mathrm{i}\alpha} = 2^{-1} \sum_1^n (D_{\xi_j} v_j(\cdot) + v_j(\cdot) D_{\xi_j})\psi \quad \text{in} \quad \mathcal{D}.$$

The last assertion can be proved by using a Taylor expansion of order two. Now this part of the lemma follows from the definition and the Theorem VIII.11 of [12].

b) Let B be the self-adjoint operator on $\mathcal{H}$ such that $V(\alpha) = \mathrm{e}^{\mathrm{i}B\alpha}$. Then the second part of this lemma follows if we show for every $m \in \mathbb{N}$ $\mathcal{D}$ is a core for B^m.

Let $\mathcal{G} = \{g \in \mathcal{S}(\mathbb{R});\ \hat{g} \in C_0^\infty(\mathbb{R})\}$. To prove that $\mathcal{D}$ is a core for B^m it suffice to show that for $g \in \mathcal{G}$ we have $g(B)\mathcal{D} \subset \mathcal{D}$. Indeed, from this assertion it follows that the space $\mathcal{M} = \bigvee_{g \in \mathcal{G}} g(B)\mathcal{D} \subset \mathcal{D} \subset \mathcal{D}(B^m)$ and it is obvious to see that $\mathcal{M}$ is a core for B^m.

Let $I \subset \mathbb{R}$ be a symmetric compact interval and let $K \subset \mathbb{R}^n$ be a compact set. Let $g \in \mathcal{G}$ with $\operatorname{supp} \hat{g} \subset I$ and $\varphi \in \mathcal{D}_K$. Then using the representation

$$g(B) = (2\pi)^{-1/2} \int \hat{g}(t) V(t) \mathrm{d}t$$

we obtain that $g(B)\varphi \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} g(B)\varphi \subset K_I$, where $K_I = \Gamma(I \times K)$.

c) Since the vector field has its components in $C_b^\infty(\mathbb{R}^n)$ it follows that A is a bounded operator on $\mathcal{S}(\mathbb{R}^n)$ and this concludes the proof of the lemma. ∎

LEMMA 2.5. *Let $q \in C^1(\mathbf{R}^n \setminus S_p) \cap C(\mathbf{R}^n)$ such that $\nabla q \cdot v$ is a bounded function. Then*

a) *For any $\alpha \in \mathbf{R}$, $\mathcal{D}(U(\alpha)q(D)U(-\alpha)) = \mathcal{D}(q(D))$ and*

$$U(\alpha)q(D)U(-\alpha) - q(D) = b(\alpha, D), \text{ where}$$

$$b(\alpha,\xi) = \int_0^\alpha \nabla q(\Gamma(\tau,\xi)) \cdot v(\Gamma(\tau,\xi))\mathrm{d}\tau. \tag{2.8}$$

b) *For any $\alpha \in \mathbf{R}$, $U(\alpha)$ maps $\mathcal{D}(q(D))$ into $\mathcal{D}(q(D))$ and for each $\varphi \in \mathcal{D}(q(D))$*

$$\sup_{|\alpha|\leqslant 1} \|q(D)U(\alpha)\varphi\| \leqslant \|q(D)\varphi\| + \sup|\nabla q \cdot v|\,\|\varphi\| \tag{2.9}$$

c) *The form $\mathrm{i}[q(D), A]$ defined on $\hat{\mathcal{D}}$ has a bounded extension and*

$$\mathrm{i}[q(D), A] = (\nabla q \cdot v)(D). \tag{2.10}$$

Proof. a) For a borelian function $g : \mathbf{R}^n \to \mathbf{C}$, we denote by M_g the operator $\varphi \to g\varphi$ on $L^2(\mathbf{R}^n)$.

Since $U(\alpha)q(D)U(-\alpha) = \mathcal{F}^{-1}M_{q\circ\Gamma(\alpha,\ )}\mathcal{F}$ for any $\alpha \in \mathbf{R}$, it follows that $\hat{\mathcal{D}}$ is a common core for $U(\alpha)q(D)U(-\alpha)$ and $q(D)$.

On the other hand we have

$$U(\alpha)q(D)U(-\alpha) - q(D) = b(\alpha, D) \text{ on } \hat{\mathcal{D}},$$

with b given in (2.8).

Now a) follows from previous relation by observing that $b(\alpha, D)$ is a bounded operator.

b) is an easy consequence of a)

c) follows from (2.8) by derivation.

∎

REMARK 2.6. a) Let $m_I = \sup\{|\tau|;\ \tau \in I\}$ and let $c_I = \inf\{|\nabla p_0(\xi)|;\ \xi \in p_0^{-1}(I)\}$. Then the part c) of Lemma 2.5 implies that

$$E_0(I)\mathrm{i}[H_0, A]E_0(I) \geqslant c_I^2(1 + m_I^2 + c_I^2)^{-1}E_0(I)$$

b) If we denote as usual

$$\mathrm{ad}_A(H_0) = [H_0, A]$$
$$\mathrm{ad}_A^{k+1}(H_0) = \mathrm{ad}_A(\mathrm{ad}_A^k(H_0)),\ k \in \mathbf{N},\ k \geqslant 1,$$

with the commutators understood as in Definition 2.3, then we obtain from Lemma 2.5 that for every $m \in \mathbb{N},\ m \geqslant 1$

$$\mathrm{ad}_A^m(H_0) \in \mathcal{B}(\mathcal{H}).$$

c) Lemma 2.5 b) together with the previous remarks imply that A is a conjugate operator for H_0 on the interval I, and H_0 is ∞-smooth with respect to A.

Remark 2.6. b) has the following consequence.

LEMMA 2.7. *Let $m \in \mathbb{N}$ and let $g \in C_0^\infty(\mathbb{R})$. Then*

a) *$\mathrm{ad}_A^m(g(H_0))$ is bounded.*

b) *The following formula holds*

$$g(H_0)(A+\mathrm{i})^{-m} = (A+\mathrm{i})^{-m}\left\{\sum_0^m \binom{m}{k}(-1)^k \mathrm{ad}_A^k(g(H_0))(A+\mathrm{i})^{-k}\right\}$$

Here $\mathrm{ad}_A^0(g(H_0)) = g(H_0)$.

Proof. a) Using formula

$$\mathrm{ad}_A(\mathrm{e}^{\mathrm{i}H_0t}) = \mathrm{e}^{\mathrm{i}H_0t}A - A\mathrm{e}^{\mathrm{i}H_0t} = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}H_0s}\mathrm{ad}_A(H_0)\mathrm{e}^{\mathrm{i}H_0(t-s)}\mathrm{d}s$$

it can be shown by induction that for every $k \in \mathbb{N},\ k \geqslant 1$

$$\mathrm{ad}_A^k(\mathrm{e}^{\mathrm{i}H_0t}) \in \mathcal{B}(\mathcal{H}),$$

$$\mathrm{ad}_A^k(\mathrm{e}^{\mathrm{i}H_0t}) = \mathrm{i}\cdot \sum_{k_1+k_2+k_3=k-1} \frac{(k-1)!}{k_1!k_2!k_3!}\cdot\int_0^t \mathrm{ad}_A^{k_1}(\mathrm{e}^{\mathrm{i}H_0s})\mathrm{ad}^{k_2+1}(H_0)\mathrm{ad}^{k_3}(\mathrm{e}^{\mathrm{i}H_0(t-s)})\mathrm{d}s,$$

$$\|\mathrm{ad}^k\mathrm{e}^{\mathrm{i}H_0t})\|_{\mathcal{B}(\mathcal{H})} \leqslant C|t|^k.$$

Now the part a) of the lemma follows from the representation

$$g(H_0) = (2\pi)^{-1/2}\int \hat{g}(s)\mathrm{e}^{\mathrm{i}H_0s}\mathrm{d}s$$

where $\hat{g}$ denotes the Fourier transform of g.

b) The proof of the second part of the lemma is elementary and is made by induction. ■

COROLLARY 2.8. *Let $m \in \mathbb{N}$ and let $g \in C_0^\infty(\mathbb{R})$. Then $g(H_0)$ maps $\mathcal{D}(A^m)$ into $\mathcal{D}(A^m)$.*

Now we need to make some notations. We denote by χ^+ (resp. χ^-) the indicator function of $(0,+\infty)$ (resp. $(-\infty,0)$). For a self-adjoint operator A, P_A^+ (resp. P_A^-)

denotes the spectral projection corresponding to $(0,+\infty)$ (resp. $(-\infty,0)$) and $\langle A\rangle$ denotes the operator $(1+A^2)^{1/2}$.

Lemma 2.4 has the following consequence.

LEMMA 2.9. *Let* $s \geqslant 0$. *Then*

$$\langle A\rangle^s \langle X\rangle^{-s} \equiv J_s$$

is a bounded operator on $\mathcal{H}$.

Proof. We need only to prove the case $s = k \in \mathbf{N}$ and then use the complex interpolation. Thus we must show that the operators

$$A^j \langle X\rangle^{-k}, \quad j \leqslant k,$$

are bounded. But this follows from Lemma 2.4 and the estimate

$$|(\langle X\rangle^{-k}\varphi, A^j\psi)| \leqslant C\|\varphi\|\,\|\psi\|, \qquad \varphi,\psi \in \hat{\mathcal{D}}$$

■

Remark 2.6 c), Corollary 2.8 and Theorem 4.2 of [7] have the following consequnce which can be interpreted as a propagation property.

THEOREM 2.10. *Let* $0 \leqslant s' < s$ *and let* $g \in C_0^\infty(I)$. *Then there is a constant* $c = c(g,s,s')$ *such that*

$$\|\langle A\rangle^{-s} \mathrm{e}^{-\mathrm{i}H_0 t} g(H_0) \langle A\rangle^{-s}\| \leqslant c\langle t\rangle^{-s'}, \quad t \in \mathbf{R}, \tag{2.11}$$

$$\|\chi^{\pm}(t)\langle A\rangle^{-s} \mathrm{e}^{-\mathrm{i}H_0 t} g(H_0) P_A^{\pm}\| \leqslant c\langle t\rangle^{-s'}, \quad t \in \mathbf{R}. \tag{2.12}$$

From now on the proof of Theorem 2.1 follows the same way as the proof of Theorem 4.3 of [7] or the proof of Theorem 1.1 of [2].

3. THE LONG-RANGE CASE

Some of the results of Section 2 concerning to the unperturbed Hamiltonian H_0, such as ∞-smootheness, can be extended to a perturbed Hamiltonian $H_L = H_0 + V_L$, where V_L is a long-range potential.

The purpose of this section is to prove that under certain conditions, the perturbed Hamiltonian $H_L = H_0 + V_L$ is ∞-smooth with respect to certain conjugate operators on any finite interval $I \subset \mathbf{R} \setminus \overline{p_0(S)}$.

We recall now some definitions and notations which we shall use.

Let $F(M)$ denotes the indicator function of the set M and assume that $\mathbb{R}^n$ is divided into unit "cubes" C_k, $k \in \mathbb{N}$ so that

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} \overline{C}_k \quad \text{and} \quad C_k \cap C_j = \emptyset,\ k \neq j.$$

We say that $f \in c_0(L^p)$, $p \geqslant 1$, if

$$\|f\|_{0,p} = \sup_{k\in\mathbb{N}} \|F(C_k)f\|_p < \infty \quad \text{and} \quad \lim_{k\to\infty} \|F(C_k)f\|_p = 0.$$

Also we say that a function f is quasi-divergent if

$$\lim_{k\to\infty} |C_k \cap B_m| = 0.$$

for all $m \in \mathbb{N}$, where $B_m = \{x \in \mathbb{R}^n;\ |f(x)| \leqslant m\}$ and $|M|$ denotes the Lebesgue measure of the measurable set M.

Finally we denote by $C^\infty_\infty(\mathbb{R}^n)$ the space of all smooth functions φ such that

$$\lim_{x\to\infty} D^\alpha \varphi(x) = 0$$

We shall work under the following hypotheses.

HYPOTHESES

I. The free Hamiltonian H_0 is a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, with the domain $\mathcal{D}(H_0) = \{u \in \mathcal{H};\ p_0\hat{u} \in \mathcal{H}\}$, $H_0 = \mathcal{F}^{-1}p_0\hat{u}$, where $\hat{u}$ is the Fourier transform of u and p_0 is a real valued function which satisfies:

(i) $p_0 : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

(ii) Let S_p be the set $\{\xi \in \mathbb{R}^n;\ p_0$ is not C^∞ in any neighborhood of $\xi\}$, let C_p be the set $\{\xi \in \mathbb{R}^n \setminus S_p;\ \nabla p_0(\xi) = 0\}$ and let $S = S_p \cup C_p$. Then $\overline{p_0(S)}$ is a countable subset of $\mathbb{R}$.

(iii) For any compact interval $I \subset \mathbb{R} \setminus \overline{p_0(S)}$, with $p_0^{-1}(I) \neq \emptyset$, we have

$$\operatorname{dist}(p_0^{-1}(I), S_p) > 0.$$

(iv) $F(S_p) \in c_0(L^1)$.

(v) $\lim_{\xi\to\infty,\ \xi\notin S_p} (|p_0(\xi)| + |\nabla p_0(\xi)|) = \infty$.

(vi) $\sup \left\{ \dfrac{|D^\alpha p_0(\xi)|}{1 + |p_0(\xi)| + |\nabla p_0(\xi)|},\ \xi \in \mathbb{R}^n \setminus S_p \right\} < \infty$ for each multi-index with $|\alpha| \geqslant 2$.

II. (vii) V_L is C^∞ real valued function which satisfies

$$|D^\alpha V_L(x)| \leqslant C_\alpha \langle x\rangle^{-\varepsilon-|\alpha|}, \quad x \in \mathbf{R}^n,$$

for some $\varepsilon > 0$ and all $\alpha \in \mathbf{N}^n$.

From the hypotheses (iv) and (v) it follows that p_0 is a quasi-divergent function (see the Appendix). Now from Theorem 9 of [5] we obtain that V_L is a symmetric H_0-compact operator. We denote by H_L the operator $H_0 + V_L$ with the domain $\mathcal{D}(H_L) = \mathcal{D}(H_0)$.

III. Let $V : \hat{\mathcal{D}} \to \mathcal{H}$ be a symmetric operator such that

(viii) The operator $H_0 + V_L + V$ with the domain $\hat{\mathcal{D}}$ has a self-adjoint extension H.

(ix) For some $\varepsilon > 0$ the operator $g(H)Vg(H_L)\langle X\rangle^{1+\varepsilon}$has a bounded extension to the whole of $\mathcal{H}$ for each g in $C_0^\infty(\mathbf{R})$.

(x) For any g in $C_0^\infty(\mathbf{R})$ the operator $g(H) - g(H_0)$ is compact.

The main results of this section are the following theorems.

THEOREM 3.1. *For any interval $I \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ there is a self-adjoint operator A_I such that A_I is conjugate to H_L on the interval I and H_L is ∞-smooth with respect to A_I.*

As a consequence we have that the eigenvalues of H_L which are not is $\overline{p_0(S)}$ are of finite multiplicity and they can accumulate only at the points of $\overline{p_0(S)}$.

THEOREM 3.2. *Assume that the hypotheses* (i)–(x) *are satisfied. Then*

(a) *The wave operators $W_\pm = s - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_Lt}E_{ac}(H_L)$ exist;*

(b) *Range $W_\pm = \mathcal{H}_c(H)$, the continuous subspace of H;*

(c) $\sigma_{cs}(H) \neq \emptyset$;

(d) *Any eigenvalue of H not in $\overline{p_0(S)} \cup \sigma_p(H_L)$ is of finite multiplicity. The eigenvalues of H can accumulate only at the points of $\overline{p_0(S)} \cup \sigma_p(H_L)$.*

REMARK 3.3. a) In the proof of Theorem 3.2 we shall need the following local compactness property: for any compact interval $I \subset \mathbf{R} \setminus \overline{p_0(S)}$ and for each $r > 0$, the operator

$$F(|x| < r)E_L(I)$$

is compact. Here $E_L(I)$ denotes the spectral projection for H_L onto the interval I.

This property is a consequence of the quasi-divergence of p_0.

b) The quasi-divergence of p_0 implies that the condition (x) is equivalent to the following one:

(x)′ For any φ in $C_0^\infty(\mathbf{R})$ the operator $\varphi(H) - \varphi(H_L)$ is compact.

c) In the same way one can prove Theorem 3.2 with the condition (ix) replaced by the condition:

(ix)′ For some $\varepsilon > 0$ the operator $g(H)V\langle X\rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each g in $C_0^\infty(\mathbf{R})$.

This condition is always true when V is a symmetric H_0-compact operator and there is an $\varepsilon > 0$ such that the operator

$$(H_0 + \mathrm{i})^{-1}V\langle X\rangle^{1+\varepsilon}$$

has a bounded extension.

d) The condition (v) implies that for any compact interval $I \subset \mathbf{R} \setminus \overline{p_0(S)}$, with $p_0^{-1}(I) \neq \emptyset$, we have

$$\inf\{|\nabla p_0(\xi)|;\ \xi \in p_0^{-1}(I)\} = c_I > 0.$$

This remark implies that all the constructions in Section 2 can be made in the context of the hypotheses of this section.

Let $I \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ be an interval and let $J \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ be another interval such that $I \subset\subset J$. Let $A = A_I$ be the self-adjoint operator associated to the interval J defined in Section 2.

Then the Remark 2.6. a) and the quasi-divergence of the function p_0 have the following consequence.

There is a compact operator K on $\mathcal{H}$ such that

$$E_L(I)\mathrm{i}[H_0, A]E_L(I) \geqslant \frac{c_J^2}{1 + m_J^2 + c_J^2}E_L(I) + E_L(I)KE_L(I) \tag{3.1}$$

where $m_J = \sup\{|t|;\ t \in J\}$.

Since the components of the vector field v, which defines the operator A, belong to the space $C_\infty^\infty(\mathbf{R}^n)$, we can prove the following lemmas.

LEMMA 3.4. *The operator* $\mathrm{ad}_A(V_L)$ *defined on* $\mathcal{S}(\mathbf{R}^n)$ *has a bounded extension to a compact operator on* $\mathcal{H}$.

LEMMA 3.5. *Let* $m \in \mathbf{N}$. *Then the operator* $\mathrm{ad}_A^m(V_L)$ *defined on* $\mathcal{S}(\mathbf{R}^n)$ *has a bounded extension to the whole of* $\mathcal{H}$.

For the proofs of these lemmas we refer to the Appendix.

These lemmas end the proof of Theorem 3.1.

We note, also, that Lemma 3.5 gives, as in Section 2 (cf. the proof of Lemma 2.7), the following corollary.

COROLLARY 3.6. Let $m \in \mathbf{N}$ and let $g \in C_0^\infty(\mathbf{R})$. Then

a) $\mathrm{ad}_A^m(g(H_L))$ is bounded.

b) The following formula holds

$$g(H_L)(A+\mathrm{i})^{-m} = (A+\mathrm{i})^{-m}\left\{\sum_0^m \binom{m}{k}(-1)^k \mathrm{ad}_A^k(g(H_L))(A+\mathrm{i})^{-k}\right\}.$$

Here $\mathrm{ad}_A^0(g(H_L)) = g(H_L)$.

c) $g(H_L)$ maps $\mathcal{D}(A^m)$ into $\mathcal{D}(A^m)$.

Now Theorem 3.1, Corollary 3.6 and Theorem 4.2 of [7] lead to the following theorem.

THEOREM 3.7. *Let* $0 \leqslant s' < s$ *and let* $g \in C_0^\infty\ (I \setminus \sigma_p(H_L))$. *Then there is a constant* $c = c(g, s, s')$ *such that*

$$\|\langle A\rangle^{-s}\mathrm{e}^{-\mathrm{i}H_Lt}g(H_L)\langle A\rangle^{-s}\| \leqslant c\langle t\rangle^{-s'},\quad t \in \mathbf{R}. \tag{3.2}$$

$$\|\chi^{\pm}(t)\langle A\rangle^{-s}\mathrm{e}^{-\mathrm{i}H_Lt}g(H_L)P_A^{\pm}\| \leqslant c\langle t\rangle^{-s'},\quad t \in \mathbf{R}. \tag{3.3}$$

From now on the proof of Theorem 3.2 follows the same way as the proof of Theorem 4.3 of [7] or the proof of Theorem 1.1 of [2].

4. BESOV SPACE ESTIMATES

In [8], the authors show haw Mourre's commutator methods can be used to prove resolvent estimates in Besov spaces.

In this section we shall use this approach to prove this type of estimates for a regular perturbation of a simply characteristic operator.

DEFINITION 4.1. [8]. Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$.

a) We define the Banach space

$$B_A = \{u \in \mathcal{H};\ \sum_0^\infty R_j^{1/2}\|F(A \in \Omega_j)u\| < \infty\}.$$

where $F(A \in \Omega_j)$ is the spectral projection for A onto the set $\Omega_j = \{t \in \mathbf{R};\ 2^{j-1} \leqslant |t| \leqslant 2^j\}$, $j \geqslant 1$, $\Omega_0 = \{t \in \mathbf{R};\ |t| \leqslant 1\}$, and $R_j = 2^j$. We write $\|\cdot\|_{B_A}$ for the obvious norm on B_A.

b) The dual space B_A^* of B_A with respect to the inner product on $\mathcal{H}$ is the Banach space obtained by completing $\mathcal{H}$ in the norm

$$\|u\|_{B_A^*} = \sup_j R_j^{-1/2} \|F(A \in \Omega_j)u\|.$$

c) The case $A = |X|$, $\mathcal{H} = L^2(\mathbf{R}^n)$ gives the usual spaces $B(\mathbf{R}^n)$ and $B^*(\mathbf{R}^n)$.

DEFINITION 4.2. Let H_0 be the self-adjoint realisation in $L^2(\mathbf{R}^n)$ of the operator of convolution with a real continuous function p_0 defined in $\mathbf{R}^n$ which satisfies hypotheses (i)–(iv) given in Section 2.

The hypotheses on the symbol of the free Hamiltonian allows us to construct a family of self-adjoint operators $\mathcal{A} = \{A_I;\ I \text{ a compact interval contained in } I \backslash \overline{p_0(S)}\}$ as in Section 2.

We say that $H = H_0 + V$ is a regular perturbation of H_0 if V satisfies the following conditions:

a) V is a symmetric H_0-compact operator.

For any operator $A \in \mathcal{A}$ we have:

b) The form $B = \mathrm{i}[V, A]$ defined on $\mathcal{H}_{+2} \cap \mathcal{D}(A)$ extends to a bounded operator from $\mathcal{H}_{+2}$ to $\mathcal{H}$ which is an H_0-compact operator.

c) The form $\mathrm{i}[B, A]$ extends from $\mathcal{H}_{+2} \cap \mathcal{D}(A)$ to a bounded operator from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$.

THEOREM 4.3. *Let $H = H_0 + V$ be a regular perturbation of H_0. Let $R(z) = (H - z)^{-1}$ for $\operatorname{Im} z \neq 0$. Then*

a) *Any eigenvalue of H not in $\overline{p_0(S)}$ is of finite multiplicity. The eigenvalues of H can accumulate only at the points of $\overline{p_0(S)}$.*

b) *For $\lambda \in \mathbf{R} \setminus (\overline{p_0(S)} \cup \sigma_p(H))$, the estimate*

$$\sup_{\delta \neq 0} \|R(\lambda + \mathrm{i}\delta) f\|_{B^*(\mathbf{R}^n)} \leqslant c(\lambda) \|f\|_{B(\mathbf{R}^n)}$$

holds, where $c(\lambda)$ can be choosen uniform in λ running over a fixed compact subset of $\mathbf{R} \setminus (\overline{p_0(S)} \cup \sigma_p(H))$.

Proof. Let $I \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ be an interval and let $J \subset\subset \mathbf{R} \setminus \overline{p_0(S)}$ be another interval such that $I \subset\subset J$. Let $A = A_J$ be the self-adjoint operator associated to the interval J defined in Section 2.

Then Remark 2.6. a) and Definition 4.2 imply that A is conjugate to H on the interval I, and H is 1-smooth with respect to A.

So the first part of Theorem 4.3 follows from the abstract results of Eric Mourre.

Concerning the second part of the theorem we shall use Proposition 2.1 of [8] to obtain that

$$\sup_{\delta\neq 0} \|R(\lambda + i\delta)f\|_{B_A^*} \leqslant c_1(\lambda)\|f\|_{B_A}$$

holds with $c_1(\lambda)$ uniformly bounded in λ, when λ runs in a compact subset of $\mathbf{R} \setminus (\overline{p_0(S)} \cup \sigma_p(H))$.

Next, we show that the abstract spaces B_A and B_A^* look like $B(\mathbf{R}^n)$ and $B^*(\mathbf{R}^n)$.

LEMMA 4.4. *Let $H = H_0 + V$ be a regular perturbation of H_0. Then for any $g \in C_0^\infty(\mathbf{R})$, the operator $g(H)$ is bounded mapping from $B(\mathbf{R}^n)$ to B_A and from B_A^* to $B^*(\mathbf{R}^n)$.*

Proof. We show $g(H) : B(\mathbf{R}^n) \to B_A$ since the other assertion follows by duality. To do this, we use a variant of the interpolation Lemma 2.5 in [1]: let $T : L^2(\mathbf{R}^n) \to$ $\to L^2(\mathbf{R}^n)$ be a linear operator with $T : L_N^2(\mathbf{R}^n) \to \mathcal{D}(|A|^N)$ for some $N > \frac{1}{2}$. Then $T : B(\mathbf{R}^n) \to B_A$. Here $L_N^2(\mathbf{R}^n)$ denotes the space

$$\{u \in L_{\text{loc}}(\mathbf{R}^n);\ \langle X\rangle^N u \in L^2(\mathbf{R}^n)\}.$$

A proof of this interpolation result is obtained by mimicking the proof of Lemma 2.5 in [1].

Since $g(H) : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$, we need only to show that $(|A|+1)g(H)\langle X\rangle^{-1}$ is a bounded operator. But, by [4] Lemma 4.12 and Lemma 2.9 of Section 2 we obtain that

$$Ag(H)\langle X\rangle^{-1} = [A, g(H)]\langle X\rangle^{-1} + g(H)A\langle X\rangle^{-1}$$

is a bounded operator. ∎

From now the proof of Theorem 4.3 follows the same way as the proof of Theorem 1.1 of [8]

REMARK 4.5. a) Assume that p_0 satisfies the hypotheses (i)–(vi) of Section 3 and that $V = V_L$ satisfies the hypotheses (vii) of the same section. Then $H = H_0 + V$ is a regular perturbation of H_0.

Moreover, if p_0 and V_L satisfy the above conditions with the conditions (vi) and (vii) replaced with the conditions

(vi)′ $\sup\left\{\dfrac{|D^\alpha p_0(\xi)|}{1+|p_0(\xi)|+|\nabla p_0(\xi)|},\ \xi \in \mathbf{R}^n \setminus S_p\right\} < \infty$ for each multi-index α with $2 \leqslant |\alpha| \leqslant m$, and

(vii)′ V_L is a C^m real valued function which satisfies

$$|D^\alpha V_L(x)| \leqslant C_\alpha\langle x\rangle^{-\epsilon-|\alpha|},\quad x \in \mathbf{R}^n,$$

for some $\varepsilon > 0$ and all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant m$, where $m = 2[n/2] + 3$, then all the conclusions of the Theorem 4.3 hold for $H = H_0 + V_L$.

b) The Theorem 4.3 establishes the existence and the uniqueness of the weak-* limit in $B^*(\mathbb{R}^n)$ for $R(\lambda + i\delta)f$ as $\delta \downarrow 0$, when $f \in B(\mathbb{R}^n)$, and $\lambda \in \mathbb{R} \setminus (\overline{p_0(S)} \cup \sigma_p(H))$. This results follows from the $B - B^*$ estimates, the density of $L^2_s(\mathbb{R}^n)$ in $B(\mathbb{R}^n)$ for $s > 1/2$, and the existence of the boundary values $R(\lambda \pm i0)$ in the $L^2_s - L^2_{-s}$-topology for $s > 1/2$ (cf, Theorem 2.2 of [7] and Lemma 2.9 of Section 2).

APPENDIX

In this appendix we propose to discuss certain results concerning the L^2-boundedness and compactness of multi-commutators of pseudodifferential operators and the quasi-divergence of some functions.

A. We give here the proofs of Lemma 3.4 and Lemma 3.5.

Let $\mathcal{F}$ denote the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$. Let $t \in \mathbb{R}$ and let $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. We define the operator

$$a_t(X, D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

by

$$\langle a_t(X, D)\varphi, \psi\rangle = (2\pi)^{-n/2}\langle((1 \otimes \mathcal{F}^{-1})a) \circ T_t, \psi \otimes \varphi\rangle \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

where $T_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is a linear map defined by

$$T_t(x, y) = (tx + (1 - t)y, x - y)$$

Then we have

$$a_1(X, D) - a_0(X, D) = \sum_{0<|\alpha|<k} \frac{i^{|\alpha|}}{\alpha!}(\partial_x^\alpha \partial_\xi^\alpha a)_0(X, D) +$$

$$\text{(A.1)} \qquad + k \sum_{|\alpha|=k} \frac{i^k}{\alpha!} \int_0^1 (1 - k)^{k-1} (\partial_x^\alpha \partial_\xi^\alpha a)_t(X, D)\mathrm{d}t$$

with the integral converging weakly.

We shall use the theorem of Calderon-Vaillancourt in a variant due to Cordes [3] (see also Kato [9]).

THEOREM A.1. *Let $m = [n/2] + 1$ and $0 \leqslant t \leqslant 1$.*

a) If $D_x^\alpha D_\xi^\alpha a \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for $|\alpha|, |\beta| \leqslant 2m$, then $a_t(X, D)$ is L^2-bounded. Moreover, the one parameter family of operators $a_t(X, D)$, $0 \leqslant t \leqslant 1$, is uniformly bounded and uniformly continuous in operator norm.

b) *If* $a \in C_\infty^{2m}(\mathbf{R}^n \times \mathbf{R}^n)$, *then the operator* $a_t(X, D)$ *is compact in the space* $\mathcal{H} = L^2(\mathbf{R}^n)$.

Let $s \in \mathbf{R}$. We define the space

$$M_s = \{a \in C^\infty(\mathbf{R}^n);\ (\forall)\ \alpha,\ (\exists)\ C_\alpha > 0,\ |\partial^\alpha a(x)| \leqslant C_\alpha \langle x\rangle^{s-|\alpha|},\ x \in \mathbf{R}^n\}.$$

Then for $a \in M_s$ and A defined by (2.6) we have

$$[a(X), A] = \sum_1^n [X_j a(X), v_j(D)] + \mathrm{i}[a(X), (\operatorname{div} v)(D)] \tag{A.2}$$

Now Lemma 3.4 is an easy consequnce of (A.2) and the following result.

LEMMA A.2. *Let* $\varepsilon > 0$ *and let* $a \in M_{1-\varepsilon}$. *Then*

a) *If* $b \in C_b^\infty(\mathbf{R}^n)$, *then the commutator* $[a(X), b(D)]$ *is* L^2*-bounded.*

b) *If* $b \in C_\infty^\infty(\mathbf{R}^n)$, *then the commutator* $[a(X), b(D)]$ *is a compact operator in the space* $L^2(\mathbf{R}^n)$.

Proof. To prove this lemma we observe first that

$$[a(X), b(D)] = (a \otimes b)_1(X, D) - (a \otimes b)_0(X, D)$$

Then, by applying (A.1) with $k = 1$ we obtain

$$[a(X), b(D)] = \mathrm{i} \sum_1^n \int_0^1 (\partial_j a \otimes \partial_j b)_t(X, D)\mathrm{d}t$$

Now this lemma is an easy corollary of the Theorem A.1. ■

Let $a \in M_s$. Then, starting from the equality (A.2) and using the Jacobi's identity, we can prove by induction that for any $m \in \mathbf{N}$, $\mathrm{ad}_A^m(a(X))$ is a finite sum of the terms of the following form

$$[[\ldots[[a_k(X), b_1(D)], b_2(D)], \ldots], b_k(D)]$$

where $a_k \in M_{s+k}$, $b_j \in C_\infty^\infty(\mathbf{R}^n)$, $j = 1, \ldots, k$, $k \leqslant m$.

Now Lemma 3.5 is implied by the following result.

LEMMA A.3. *Let* $\varepsilon > 0$ *and let* $k \in \mathbf{N}$, $k \geqslant 1$. *If* $a \in M_{k-\varepsilon}$, $b_j \in C_b^\infty(\mathbf{R}^n)$, $j = 1, \ldots, k$, *then the multi-commutator*

$$[[\ldots[[a(X), b_1(D)], b_2(D)], \ldots], b_k(D)],$$

defined on $\mathcal{S}(\mathbf{R}^n)$, *has a bounded extension to the whole space* $L^2(\mathbf{R}^n)$.

Proof. The proof of this lemma is made by induction. The case $k = 1$ was proved in Lemma A.2 a). Assume that the statement is true for k. Let $a \in M_{k+1-\varepsilon}$. Then using (A.1) we obtain as in the proof of Lemma A.2 that

$$[a(X), b_1(D)] = \sum_{0<|\alpha|<k+1} \frac{i^{|\alpha|}}{\alpha!}(\partial^\alpha b_1)(D)(\partial^\alpha a)(X) + B$$

with B a bounded operator and $\partial^\alpha a \in M_{k+1-\varepsilon-|\alpha|} \subset M_{k-\varepsilon}$ for $1 \leqslant |\alpha|$. Now the lemma follows from the induction hypothese. ∎

B: We shall prove the following result.

PROPOSITION B.1. a) *Assume that* $p_0 : \mathbb{R}^n \to \mathbb{R}$ *is a function which satisfies the conditions* (i), (ii) *and* (2.1)″ *of the Section 2. Let* $I \subset \mathbb{R}\setminus\overline{p_0(S)}$ *be a compact interval and let* $r > 0$. *Then*

$$F(|x| \leqslant r)E_0(I)$$

is a compact operator on $L^2(\mathbb{R}^n)$.

b) *Assume that* $p_0 : \mathbb{R}^n \to \mathbb{R}$ *is a function which satisfies the conditions* (i), (ii), (iv) *and* (v) *of the Section 3. Then* p_0 *is quasi-divergent function.*

In order to prove Proposition B.1, it suffices to show (cf. Corollary 3 of [5] for the part a)) that the following lemma is true.

LEMMA B.2. a) *Assume that* p_0 *satisfies the conditions* (i), (ii) *and* (2.1)″ *of Section 2. Let* $I \subset \mathbb{R} \setminus \overline{p_0(S)}$ *be a compact interval. Then*

$$\lim_{k\to\infty} |C_k \cap p_0^{-1}(I)| = 0.$$

b) *Assume that* p_0 *satisfies the conditions* (i), (ii), (iv) *and* (v) *of the Section 3. Let* $I \subset \mathbb{R}$ *be a compact interval. Then*

$$\lim_{k\to\infty} |C_k \cap p_0^{-1}(I)| = 0.$$

Here $|A|$ *denotes the Lebesgue measure of the measurable set* A.

Proof. We shall prove only the part b) of this lemma since the first part can be done in a similar manner.

b) Since $F(S_p) \in c_0(L^1)$ it suffices to show that

$$\lim_{k\to\infty} |C_k \cap (p_0^{-1}(I) \setminus S_p)| = 0.$$

If we denote by

$$b_k = \inf\{|\nabla p_0(\xi)|;\ \xi \in C_k \cap (p_0^{-1}(I) \setminus S_p)\},$$

then the compactness of I and the condition (iv) imply that

$$\lim_{k\to\infty} b_k = \infty.$$

Let $k_0 \in \mathbf{N}$ such that $b_k > 0$ for any $k \in \mathbf{N},\ k \geqslant k_0$.

The proof of the lemma is compleated by the following estimate:

$$\text{(B.1)} \qquad |C_k \cap (p_0^{-1}(I) \setminus S_p)| \leqslant n\sqrt{n}|I|b_k^{-1}, \quad k \in \mathbf{N},\ k \geqslant k_0.$$

Let $B_j = \{\xi \in \mathbf{R}^n \setminus S_p;\ |\nabla p_0(\xi)| \leqslant \sqrt{n}|\partial_j p_0(\xi)|\}$ and let $\Phi_j : \mathbf{R}^n \setminus S_p \to \mathbf{R}^n$ defined by

$$\Phi_j(\xi) = (\xi_1, \ldots, \xi_{j-1}, p_0(\xi), \xi_{j+1}, \ldots, \xi_n)$$

for $j = 1, \ldots, n$. Then Φ_j is a local diffeomorphism at every point $p_0^{-1}(I) \cap B_j$.

Since $C_k \cap p_0^{-1}(I) = \bigcup\limits_1^n C_k \cap p_0^{-1}(I) \cap B_j$, then (B.1) follows from

$$\text{(B.1)}' \qquad |C_k \cap p_0^{-1}(I) \cap B_j| \leqslant \sqrt{n}|I|b_k^{-1}, \quad k \in \mathbf{N},\ k \geqslant k_0,\ j = 1, \ldots, n.$$

This estimate can be obtained by making a change of variable. Let us write $C_k \cap p_0^{-1}(I) \cap B_j$ as a disjoint union

$$\bigcup_l \Phi_j^{-1}(C_{kI}) \cap C_k \cap B_j \cap M_l,$$

where $C_{kI} = \pi_1(C_k) \times \cdots \times \pi_{j-1}(C_k) \times I \times \pi_{j+1}(C_k) \times \cdots \times \pi_n(C_k)$ and M_l, $l \in \mathbf{N}$, are disjoint measurable sets which have neighborhoods on which Φ_j is a diffeomorphism.

Then

$$C_k \cap p_0^{-1}(I) \cap B_j = \bigcup_l \Phi_j^{-1}(A_{kjl}),$$

where A_{kjl}, $l \in \mathbf{N}$, are disjoint measurable subsets of C_{kI} such that $\Phi_j^{-1}(A_{kjl}) = \Phi_j^{-1}(C_{kI}) \cap C_k \cap B_j \cap M_l$, Φ_j is a diffeomorphism in a neighborhood of $\Phi_j^{-1}(A_{kjl})$ and

$$|\det(\Phi_j^{-1})'(\eta)| \leqslant \sqrt{n} b_k^{-1},\ \eta \in A_{kjl}.$$

Hence

$$|C_k \cap p_0^{-1}(I) \cap B_j| \leqslant \sum_l \int_{\Phi_j^{-1}(A_{kjl})} d\xi = \sum_l \int_{A_{kjl}} |\det(\Phi_j^{-1})'(\eta)| d\eta \leqslant$$

$$\leqslant \sqrt{n} b_k^{-1} \int_{C_{kI}} d\eta = \sqrt{n}|I|b_k^{-1}.$$

■

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