J. OPERATOR THEORY
29(1993), 307–321

THE INNER DERIVATIONS AND THE PRIMITIVE IDEAL SPACE OF A C^*-ALGEBRA

DOUGLAS W. B. SOMERSET

1. INTRODUCTION

A *derivation* on an algebra A is a linear map from A to A satisfying

$$D(ab) = D(a)b + aD(b) \quad (\forall)\ a, b \in A.$$

Each element $a \in A$ induces an *inner derivation* $D(a, A)$, given by

$$D(a, A) = ab - ba \ \ (b \in A).$$

When A is a Banach algebra it is clear that each inner derivation $D(a, A)$ is a bounded map on A. In fact a simple application of the triangle inequality shows that

$$\|D(a, A)\| \leqslant 2\mathrm{d}(a, Z(A)) \tag{1}$$

where $\mathrm{d}(a, Z(A))$ denotes the distance from a to $Z(A)$, the centre of A. In the case when A is a C^*-algebra it is known that every derivation on A is bounded [17].

The inequality (1) has received considerable attention, mainly devoted to showing that equality holds in various cases. For instance Kadison, Lance and Ringrose [13] showed that equality holds when A is a von Neumann algebra and a is self-adjoint. Stampfli [19] showed that equality holds when A is a primitive C^*-algebra with an identity, and in particular when A is the algebra of bounded operators on Hilbert space. When A is the algebra of bounded operators on a Banach space Johnson [12] and Kyle [15] showed that equality sometimes holds, and sometimes not. Zsido [22] showed that equality holds for all elements in a von Neumannn algebra A. Special cases of this had been obtained by Gajendragadkar [8] and Hall [11]. Apostol and Zsido [1] showed that equality holds when A is a quotient of a von Neumann algebra, and Halpern when A is an AW*-algebra, see [7]. For quotients of AW*-algebras the problem remained open until recently, when it was shown that equality holds here also [18].

On the other hand there are C^*-algebras containing elements for which the inequality (1) is strict [13; 6.2]. To examine the possible behaviour in more detail Archbold [2] introduced two constants, $K(A)$ and $K_s(A)$, defined to be the smallest numbers in $[0, \infty]$ such that

$$\mathrm{d}(a, Z(A)) \leqslant K(A)\|D(a, A)\| \quad (\forall)\ a \in A$$

and

$$\mathrm{d}(a, Z(A)) \leqslant K_s(A)\|D(a, A)\| \quad (\forall)\ a = a^* \in A.$$

Clearly $K(A) = K_s(A) = 0$ when A is commutative. When A is non-commutative it follows from (1) that $K(A) \geqslant \frac{1}{2}$ and $K_s(A) \geqslant \frac{1}{2}$, with equality (1) for all elements, or for all self-adjoint elements, exactly when $K(A) = \frac{1}{2}$, or when $K_s(A) = \frac{1}{2}$. It is elementary to check that $K_s(A) \leqslant K(A) \leqslant 2K_s(A)$. The Closed Graph Theorem implies that $K(A) < \infty$ if and only if the set of inner derivations of A is closed, in the operator norm, in the set of all derivations on A [13; Theorem 5.3]. Examples 6.2 of [13] and 3.1 of [15] are therefore C^*-algebras with $K(A) = K_s(A) = \infty$. The main results of [2] were to show that $K_s(A) \leqslant 1$ when A is a weakly central C^*-algebra, and to give an example of a weakly central C^*-algebra A with $K_s(A) = 1$. Mention should also be made of the inequalities relating $K(A \otimes_\beta B)$ to $K(A)$ and $K(B)$ [4], [2] (where $A \otimes_\beta B$ is any C^*-tensor product of the C^*-algebras A and B).

In this paper we continue the study of $K_s(A)$. For reasons which will emerge in a moment this constant is considerably more amenable than the constant $K(A)$.

The structure of this paper is as follows. In the next section we define a constant $\mathrm{Orc}(A)$ in terms of the hull-kernel topology on the space of primitive ideals of A. We give examples to show that all possible values of $\mathrm{Orc}(A)$, namely every positive integer and ∞, can occur. In Section 3 we give a formula for the distance of a self--adjoint element from the centre in a C^*-algebra with an identity, and show that the self-adjoint part of the centre is a proximinal subspace of the self-adjoint part of the algebra. In Section 4 we prove the main theorem which is that $K_s(A) = \frac{1}{2}\mathrm{Orc}(A)$ when A is a non-commutative C^*-algebra with an identity.

We now give some further notation and conventions. Let A be a C^*-algebra and let J be an ideal in A (which will always mean a closed, two-sided ideal). For $a \in A$ either a_J or $a + J$ will denote the image of a in the quotient C^*-algebra A/J. Let A_{sa} denote the set of self-adjoint elements of A, and A^+ the set of positive elements of A.

It turns out that for $a \in A$ both $\mathrm{d}(a, Z(A))$ and $\|D(a, A)\|$ can be described in terms of the supremum of the distance of a_I from the scalars in the quotient C^*--algebra A/I, as I ranges over appropriate sets of ideals of A. The reason why $K_s(A)$ is easier to study than $K(A)$ is that the distance of a_I from the scalars is more easily described when a is self-adjoint:

Let $a \in A_{\mathrm{sa}}$. Let $\alpha(a)$ and $\beta(a)$ denote the largest and smallest numbers in the spectrum of a. If A has an identity then the nearest scalar to a is $\frac{1}{2}(\alpha(a) + \beta(a))$ and the distance from a to the scalars is $\frac{1}{2}(\alpha(a) - \beta(a))$.

2. DEFINITION OF Orc(A)

In this section we associate to each topological space X a natural number $\mathrm{Orc}(X)$, called the connecting order of X. This number arises as the supremum of the diameters of various subgraphs, so we start by recalling the appropriate definitions from graph theory.

A *graph* consists of a set of *points* together with a symmetric relation on the points called *adjacency*. The (unordered) pairs of the relation are called the *edges* of the graph. (The edges of our graphs do not have a direction.) A *path of length n* from a point u to a point v is a sequence of points $u = u_0,\ u_1, \ldots, u_n = v$ such that u_i is adjacent to u_{i+1} for each i. Denote by $\mathrm{d}(u, v)$ the minimal length of a path from u to v. If no such path exists set $\mathrm{d}(u, v) = \infty$. The number $\mathrm{d}(u, v)$ is the *distance* from u to v. The *diameter* of a graph is the supremum of the distances between pairs of its points, except that will adopt the non-standard convention that the diameter of a single point is *one* (although the distance from a point to itself is zero).

Now let X be a topological space and for $x, y \in X$ let $x \sim y$ if x and y cannot be separated by disjoint open sets. We will view X as a graph in which two points x and y are adjacent if and only if $x \sim y$. We define $\mathrm{Orc}(X)$, the *connecting order* of X, to be the supremum of the diameters of the connected components of the graph X. Note that $\mathrm{Orc}(X) = 1$ if and only if $\sim$ is an equivalence relation. Because of our non-standard convention on the diameter of a point, $\mathrm{Orc}(X) = 1$ when X is Hausdorff. In Example 2.8 we show that $\mathrm{Orc}(X)$ can be any positive integer, or infinity.

The next step is to give an alternative description of $\mathrm{Orc}(X)$ in the case when X is compact. Define a *chain of length n on X* to be a collection of n closed subsets $X_1, \ldots, X_n$ with the following properties:

(i) $\bigcup_{i=1}^{n} X_i = X$

(ii) X_i and X_j are disjoint if $|i - j| > 1$

(iii) if $n > 1$ then $X_1 \setminus X_2$ and $X_n \setminus X_{n-1}$ are non-empty.

A chain of length n is said to be *admissible* if there exist $x \in X_1 \setminus X_2$ and $y \in X_n \setminus X_{n-1}$ such that $d(x,y) < \infty$. Note that this further condition (on top of (i) and (ii)) implies that $X_i \cap X_{i+1}$ is non-empty for $i = 1, \ldots, n-1$, for otherwise x and y would belong to different clopen subsets of X. In particular, admissibility implies that each X_i is non-empty for $i = 1, \ldots, n$.

If Y and Z are subsets of a topological space X let

$$d(Y,Z) = \inf\{d(y,z) : y \in Y,\ z \in Z\}$$

and, for $n \geqslant 0$, let

$$Y^n = \{x \in X : d(\{x\}, Y) \leqslant n\}.$$

Note that $Y^0 = Y$.

LEMMA 2.1. *Let X be a topological space and let $X_1, \ldots, X_n$ be a chain on X of length $n > 1$. If $x \in X_1 \setminus X_2$ and $y \in X_n \setminus X_{n-1}$ then $d(x,y) \geqslant n$.*

Proof. Since $d(x,y) = \inf\{n \in \mathbb{N} : y \in \{x\}^n\}$ it is sufficient to show that $\{x\}^k \subseteq X_1 \cup \ldots \cup X_k$ for $k = 1, \ldots, n-1$. We prove this by induction on k. If $z \in X \setminus X_1$ then the sets $X_1 \setminus X_2$ and $X \setminus X_1$ are disjoint open neighbourhoods of x and z respectively, so $z \notin \{x\}^1$. Hence $\{x\}^1 \subseteq X_1$ and the induction hypothesis holds for $k = 1$. Suppose now that $m \leqslant n-2$ and $\{x\}^m \subseteq X_1 \cup \ldots \cup X_m$. If $z \notin X_1 \cup \ldots \cup X_{m+1}$ then the sets $(X_1 \cup \ldots \cup X_{m+1}) \setminus X_{m+2}$ and $(X_{m+2} \cup \ldots \cup X_n) \setminus X_{m+1}$ are disjoint open sets containing $\{x\}^m$ and z respectively. Hence $z \notin \{x\}^{m+1}$, so $\{x\}^{m+1} \subseteq X_1 \cup \ldots \cup X_{m+1}$. This completes the induction and shows that $d(x,y) \geqslant n$. ∎

LEMMA 2.2. *Let X be a topological space and let Y and Z be compact subsets of X with $d(Y,Z) \geqslant 2$. Then there are disjoint open sets U and V containing Y and Z respectively.*

Proof. The proof is standard. For $p \in Y$, $q \in Z$, $p \not\sim q$ so there are disjoint open sets $S_{p,q}$ and $T_{p,q}$ containing p and q respectively. For fixed p the sets $T_{p,q}$ ($q \in Z$) form an open cover of Z so by the compactness of Z there is a finite set $\{q_1, \ldots, q_n\} \subseteq Z$ such that $T_{p,q_1} \cup \ldots \cup T_{p,q_n} \supseteq Z$. Set $U_p = S_{p,q_1} \cap \ldots \cap S_{p,q_n}$ and $V_p = T_{p,q_1} \cap \ldots \cap T_{p,q_n}$. Then U_p and V_p are disjoint open sets containing p and Z respectively. By the compactness of Y there is a finite set $\{p_1, \ldots, p_m\} \subseteq Y$ such that $U_{p_1} \cup \ldots \cup U_{p_m} \supseteq Y$. Set $U = U_{p_1} \cup \ldots \cup U_{p_m}$ and $V = V_{p_1} \cap \ldots \cap V_{p_m}$. Then U and V are disjoint open sets containing Y and Z respectively. ∎

COROLLARY 2.3. *Let Y be a compact subset of a topological space X. Then Y^1 is closed.*

Proof. If $z \in X \setminus Y^1$ then $d(\{z\}, Y) \geqslant 2$, so by Lemma 2.2 there are disjoint open sets U containing z and V containing Y. Clearly Y^1 is disjoint from U. This proves that Y^1 is closed. ∎

LEMMA 2.4. *Let X be compact topological space and let $x, y \in X$ with $d(x,y) \geqslant \geqslant n > 1$. Then there exists a chain $X_1, \ldots X_n$ of length n with $x \in X_1 \setminus X_2$ and $y \in X_n \setminus X_{n-1}$.*

Proof. First note that by repeated application of Corollary 2.3 the sets $\{y\}^i$ are compact for $i \in \mathbb{N}$. Since $d(x,y) \geqslant n$, $d(\{x\}, \{y\}^{n-2}) \geqslant 2$ so by Lemma 2.2 there are disjoint open sets $U_1 \supseteq \{x\}$ and $V_1 \supseteq \{y\}^{n-2}$. Set $X_1 = X \setminus V_1$ and $Y_2 = X \setminus U_1$. If $n = 2$ then X_1 and $X_2 = Y_2$ have the required properties. Otherwise if $n > 2$ then $d(X_1, \{y\}^{n-3}) \geqslant 2$ since X_1 is disjoint from $\{y\}^{n-2}$. For $n > 2$ we define inductively, for $i = 2, \ldots n-1$, $X_i = (X \setminus V_i) \cap Y_i$ and $Y_{i+1} = (X \setminus U_i)$, where U_i and V_i are disjoint open sets containing $X_1 \cup \ldots \cup X_{i-1}$ and $\{y\}^{n-(i+1)}$ respectively. Note that for $1 \leqslant i \leqslant n-2$, $d((X_1 \cup \ldots \cup X_i), \{y\}^{n-(i+2)}) \geqslant 2$ so the induction can proceed. Finally set $X_n = Y_n$. Then it easily to check that $X_1, \ldots, X_n$ is a chain of length n. ∎

COROLLARY 2.5. *Let X be a compact topological space. Then $\mathrm{Orc}(X)$ is equal to the supremum of the lengths of admissible chains on X.*

Proof. Let $\mathrm{CL}(X)$ denote the supremum of the lengths of admissible chains on X. Lemma 2.1 shows that $\mathrm{Orc}(X) \geqslant \mathrm{CL}(X)$. Conversely, if $x, y \in X$ with $d(x,y) = k < \infty$ then, by Lemma 2.4, there is an admissible chain X of length k. Hence $\mathrm{CL}(X) \geqslant \mathrm{Orc}(X)$. ∎

We will call a subset Y *$\sim$-saturated* if $Y = Y^1$.

PROPOSITION 2.6. *Let X be a compact toplogical space with $\mathrm{Orc}(X) < \infty$. If Y and Z are disjoint, compact, $\sim$-saturated subsets of X then there are disjoint, open $\sim$-saturated sets U and V containing Y and Z respectively.*

Proof. It is only necessary to show that if $p \in Y$ and $q \in Z$ then there are disjoint, open $\sim$-saturated sets S and T containing p and q. The rest of the proof is then exactly as in Lemma 2.2.

So let $p \in Y$ and $q \in Z$. Then $d(p,q) = \infty$. Let $k = \mathrm{Orc}(X)$. By Lemma 2.4 there is a chain $X_1, \ldots X_{2k+2}$ of length $2k+2$ with $p \in X_1 \setminus X_2$ and $q \in X_{2k+2} \setminus X_{2k+1}$. The sets $A = (X_1 \cup \ldots \cup X_{k+1}) \setminus X_{k+2}$ and $B = (X_{k+2} \cup \ldots \cup X_{2k+2}) \setminus X_{k+1}$ are disjoint open sets containing p and q respectively. By Corollary 2.3 (repeatedly applied) the sets $C = (X \setminus A)^k$ and $D = (X \setminus B)^k$ are closed, and they are also $\sim$-saturated since $\mathrm{Orc}(X) = k$. If $S = X \setminus C$ and $T = X \setminus D$ then $S \subseteq A$ and $T \subseteq B$ so S

and T are disjoint, open and $\sim$-saturated. The method of Lemma 2.1 shows that $d(\{p\}, X \setminus A) > k$ and $d(\{q\}, X \setminus B) > k$, so that $p \notin C$ and $q \notin D$. Hence $p \in S$ and $q \in T$, as required. ∎

Now let $C^b(X)$ denote the algebra of bounded, continuous functions on X, and for $x, y \in X$ let $x \approx y$ if $f(x) = f(y)$ for all $f \in C^b(X)$. The relation $\approx$ is an equivalence relation, and the equivalence classes are closed subsets of X. Clearly for any $f \in C^b(X)$ and $r \in \mathbb{R}$ the set $Z = \{z \in X : f(x) = r\}$ is $\sim$-saturated, so if $x, y \in X$ and $x \not\approx y$ then $d(x, y) = \infty$. For compact spaces with $\mathrm{Orc}(X) < \infty$ the converse is true.

COROLLARY 2.7. *Let X be a compact topological space. If* $\mathrm{Orc}(X) < \infty$ *then for $x, y \in X$*

$$x \approx y \Leftrightarrow d(x, y) < \infty.$$

Proof. We have already noted that $x \not\approx y \Rightarrow d(x, y) = \infty$ for any topological space X. Conversely, suppose that X is compact with $\mathrm{Orc}(X) < \infty$. Define an equivalence relation $*$ on X by $x * y$ if $d(x, y) < \infty$. Let $X/*$ denote the topological space of $*$-equivalence classes of X with the quotient topology, and let $X \to X/*$ denote the quotient map. Since $\mathrm{Orc}(X) < \infty$, Corollary 2.3 implies that the equivalence classes are closed, and hence compact. Thus if Y and Z are two different equivalence classes Proposition 2.6 shows that there are disjoint, open, $\sim$-saturated sets U and V containing Y and Z respectively. Then $q(U)$ and $q(V)$ are disjoint, open sets containing $q(Y)$ and $q(Z)$ respectively. This shows that $X/*$ is a compact, Hausdorff space (compact because X is compact and q is continuous). Hence if $x, y \in X$ with $d(x, y) = \infty$ then $q(x) \neq q(y)$, so there is an $f \in C^b(X/*)$ such that $f \cdot q(x) \neq f \cdot q(y)$. But $f \cdot q \in C^b(X)$, so $x \not\approx y$.

∎

One consequence of Corollary 2.7 is that $\sim$ is an equivalence relation on a compact space if and only if $\sim$ coincides with $\approx$. This follows since $\sim$ is an equivalence relation if and only if $\mathrm{Orc}(X) = 1$.

The property of Corollary 2.7 does not characterise spaces with $\mathrm{Orc}(X) < \infty$. For example, for each positive integer i let X_i be a compact space with $\mathrm{Orc}(X) = i$ (see Example 2.8), and let Y be the finest one-point compactification of $\bigcup_{i=1}^{\infty} X_i$. Then it is easy to see that $\mathrm{Orc}(Y) = \infty$ but that, for $x, y \in Y$, $x \approx y \Leftrightarrow d(x, y) < \infty$.

Corollary 2.7 shows that if X is a compact space with $\mathrm{Orc}(X) < \infty$ then $\approx$ is the finest closed equivalence relation containing $\sim$. It would be interesting to know if this is also true for compact spaces with $\mathrm{Orc}(X) = \infty$.

Suppose now that A is a C^*-algebra, and let $\mathrm{Prim}(A)$ denote the set of primitive ideals of A, with the hull-kernel topology. If A has an identity then $\mathrm{Prim}(A)$ is compact [16; 4.4.4]. We will denote $\mathrm{Orc}(\mathrm{Prim}(A))$ by $\mathrm{Orc}(A)$.

We close this section by showing that $\mathrm{Orc}(A)$ can take all possible values, that is, can be any positive integer or infinity.

EXAMPLE 2.8. We construct the following family of C^*-algebras.

Let A be the C^*-algebra consisting of all continuous function from the interval [0,1] into the 2×2 complex matrices. Let $A(1)$ be the C^*-subalgebra of A consisting of those functions $f \in A$ satisfying

$$f\left(\frac{1}{2^n}\right) = \begin{pmatrix} \lambda_{2n-1}(f) & 0 \\ 0 & \lambda_{2n}(f) \end{pmatrix} \quad (n \geqslant 1),$$

and

$$f(0) = \begin{pmatrix} \lambda(f) & 0 \\ 0 & \lambda(f) \end{pmatrix},$$

for some complex numbers $\lambda(f)$, $\lambda_n(f)$ $(n \geqslant 1)$. For $m \geqslant 2$ let $A(m)$ be the C^*--subalgebra of $A(1)$ defined by

$$A(m) = \{f \in A(1) : \lambda_{2n}(f) = \lambda_{2n+1}(f) \ (1 \leqslant n < m)\}.$$

Let $A(\infty) = \{f \in A(1) : \lambda_{2n}(f) = \lambda_{2n+1}(f) \ (1 \leqslant n < \infty)\}$.

We now describe the primitive ideal spaces of these algebras. It is well-known that $\mathrm{Prim}(A)$ is a Hausdorff space homeomorphic to the interval [0,1]. Hence $\mathrm{Orc}(A) = 1$. Now set $X = \left\{\frac{1}{2^n} : n \geqslant 1\right\} \cup \{0\}$ and set $Y = [0,1] \setminus X$. For $1 \leqslant m \leqslant \infty$ let

$$P_y(m) = \{f \in A(m) : f(y) = (0)\} \quad (y \in Y),$$

let

$$Q(m) = \{f \in A(m) : \lambda(f) = 0\},$$

and let

$$R_i(m) = \{f \in A(m) : \lambda_i(f) = 0\} \quad (1 \leqslant i < \infty).$$

Then

$$\mathrm{Prim}(A(m)) = \bigcup_{y \in Y} \{P_y(m)\} \cup \{Q(m)\} \cup \bigcup_{i \geqslant 1} \{R_i(m)\}.$$

The points $P_y(m)$ $(y \in Y)$ and $Q(m)$ are separated points, while $R_{2i-1}(m) \sim R_{2i}(m)$ in $\mathrm{Prim}(A(m))$. However, when $1 \leqslant i < m$, $R_{2i}(m) = R_{2i+1}(m)$ so it follows that $\mathrm{d}(R_1(m), R_{2m}(m)) = m$. It is now easy to see that $\mathrm{Orc}(A(m)) = m$, when m is finite, and that $\mathrm{Orc}(A(\infty)) = \infty$.

Note that $\mathrm{Orc}(A(1)) = 1$ although $\mathrm{Prim}(A(1))$ is not Hausdorff. Other examples of C^*-algebras with $\mathrm{Orc}(A) = \infty$ have been given in [13; 6.2] and [15; 3.1].

3. THE DISTANCE TO THE CENTRE

In this section we obtain formula for the distance from the centre of a self--adjoint element in a C^*-algebra. At the same time we show that this distance is actually attained.

We start by recalling some facts about the complete regularization of $\mathrm{Prim}(A)$ (see [3] and [5] for further details). As in Section 2, for $P, Q \in \mathrm{Prim}(A)$ let $P \approx Q$ if $f(P) = f(Q)$ for all $f \in C^b(\mathrm{Prim}(A))$. Since $\approx$ is an equivalence relation and the equivalence clasess are closed subsets of $\mathrm{Prim}(A)$ there is a one-to-one correspondence between $\mathrm{Prim}(A)/\approx$ and a set of ideals of A given by

$$[P] \leftrightarrow \cap[P] \quad (P \in \mathrm{Prim}(A)),$$

where $[P]$ denotes the $\approx$-equivalence class of P. The set of ideals obtained in this way is called $\mathrm{Glimm}(A)$, and we identify this set with $\mathrm{Prim}(A)/\approx$ by the correspondence above. The quotient map $\varphi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is known as the *complete regularization map*. Note that if $P, Q \in \mathrm{Prim}(A)$, $G \in \mathrm{Glimm}(A)$ and $P \supseteq G = \cap[Q]$ then, since $[Q]$ is closed, $P \in [Q]$ and so $\varphi_A(P) = \varphi_A(Q) = G$. It follows that if $P \in \mathrm{Prim}(A)$, $G \in \mathrm{Glimm}(A)$ and $P \supseteq G$ then $\varphi_A(P) = G$.

We will consider $\mathrm{Glimm}(A)$ as a topological space with the quotient topology. There is a second topology on $\mathrm{Glimm}(A)$ which is also important [3], but it coincides with the quotient topology when A has an identity, which is the case that we are interested in, so we will ignore it.

When A has an identity it follows from the Dauns-Hofmann Theorem [16; 4.4.8] that for $P, Q \in \mathrm{Prim}(A)$:

$$P \approx Q \Leftrightarrow P \cap Z(A) = Q \cap Z(A).$$

From this it follows that the map $G \to G \cap Z(A)$ $(G \in \mathrm{Glimm}(A))$ is a homeomorphism from $\mathrm{Glimm}(A)$ to $\mathrm{Prim}(Z(A))$ [3; p. 351]. In particular $\mathrm{Glimm}(A)$ is a compact, Hausdorff space.

PROPOSITION 3.1. *Let A be a C^*-algebra and let $a \in A$.*

(i) *The function $P \to \|a + P\|$ $(P \in \mathrm{Prim}(A))$ is lower semi-continuous on* $\mathrm{Prim}(A)$.

(ii) *The function $G \to \|a + G\|$ $(G \in \mathrm{Glimm}(A))$ is upper semi-continuous on* $\mathrm{Glimm}(A)$.

Proof. (i) is a standard fact about C^*-algebras, see for example [16; 4.4.4]. (ii) can be found in [20; proof of Theorem 3.1] for example. ■

COROLLARY 3.2. *Let A be a C^*-algebra with an identity and let $a \in A_{sa}$.*

(i) *The functions $P \to \alpha(a_P)$ and $P \to \beta(a_P)$ ($P \in \mathrm{Prim}(A)$) are lower semi--continuous and upper semi-continuous respectively on* $\mathrm{Prim}(A)$.

(ii) *The functions $G \to \alpha(a_G)$ and $G \to \beta(a_G)$ ($G \in \mathrm{Glimm}(A)$) are upper semi-continuous and lower semi-continuous respectively on* $\mathrm{Glimm}(A)$.

Proof. (i) and (ii) follow immediately from Proposition 3.1(i) and (ii), together with the fact that since A has an identity $\alpha(a_P) = \|(\|a\| + a_P)\| - \|a\|$ and $\beta(a_P) = \|a\| - \|(\|a\| - a_P)\|$. ■

For the next theorem we need the following **Fact**: if X is a compact Hausdorff space and $f, g : X \to \mathbf{R}$ are functions which are respectively upper semi-continuous and lower semi-continuous and satisfy $f \leqslant g$ then there is a continuous function $h : X \to \mathbf{R}$ such that $f \leqslant h \leqslant g$. This property characterises normal topological spaces [21], [14]; it was proved for paracompact spaces in [6] and for metric spaces in [10].

THEOREM 3.3. *Let A be a C^*-algebra with an identity and let a be a self-adjoint element of A. Then the distance $\mathrm{d}(a, Z(A))$ of a from $Z(A)$ is attained, that is, there is a $z \in Z(A)$ such that*

$$\|a - z\| = \mathrm{d}(a, Z(A)),$$

and this distance is given by

$$\mathrm{d}(a, Z(A)) = \sup\left\{\frac{1}{2}(\alpha(a_G) - \beta(a_G)) : G \in \mathrm{Glimm}(A)\right\}.$$

Proof. Let $a \in A$. Set $\gamma = \sup\left\{\frac{1}{2}(\alpha(a_G) - \beta(a_G)) : G \in \mathrm{Glimm}(A)\right\}$. For $z \in Z(A)$ and $G \in \mathrm{Glimm}(A)$ it is clear that $\|a_G - z_G\| \geqslant \frac{1}{2}(\alpha(a_G) - \beta(a_G))$, from which it follows that $\mathrm{d}(a, Z(A)) \geqslant \gamma$. The real-valued functions f and g, defined on $\mathrm{Glimm}(A)$ by

$$f(G) = \alpha(a_G) - \gamma$$

and

$$g(G) = \beta(a_G) + \gamma \quad (G \in \mathrm{Glimm}(A))$$

are upper semi-continuous and lower semi-continuous respectively, Corollary 3.2(ii). Since $\mathrm{Glimm}(A)$ is a compact, Hausdorff space and $f \leqslant g$ it follows from the Fact (above) that there is a continuous function h on $\mathrm{Glimm}(A)$ with $f \leqslant h \leqslant g$. Since A

has an identity Glimm(A) is homeomorphic to Prim($Z(A)$) so there is a $z \in Z(A)_{sa}$ with $z_G = h(G)$ for $G \in$ Glimm(A). Hence for all $G \in$ Glimm(A)

$$f(G) = \alpha(a_G) - \gamma \leqslant z_G \leqslant \beta(a_G) + \gamma = g(G)$$

so $\alpha(a_G) - z_G \leqslant \gamma$ and $z_G - \beta(a_G) \leqslant \gamma$, that is, $\|a - z\| \leqslant \gamma$. Hence $\mathrm{d}(a, Z(A)) = \|a - z\| = \gamma$. ■

COROLLARY 3.4. *Let A be a C^*-algebra with an identity and let $a \in A_{sa}$. Then there is a $G \in$ Glimm(A) such that* $\mathrm{d}(a, Z(A)) = \frac{1}{2}(\alpha(a_G) - \beta(a_G))$.

Proof. The function $G \to \frac{1}{2}(\alpha(a_G) - \beta(a_G))$ ($G \in$ Glimm(A)) is upper semi-continuous, by Corollary 3.2(ii). It therefore obtains its supremum on the compact set Glimm(A). The result now follows from Theorem 3.3. ■

If X is a Banach space and Y is a closed subspace such that each $x \in X$ attains its distance from Y then Y is said to be *proximinal*. Theorem 3.3 shows therefore that when A has an identity $Z(A)_{sa}$ is a proximinal subspace of the real Banach space A_{sa}. It would be interesting to know whether $Z(A)$ is a proximinal subspace of A.

4. THE MAIN THEOREM

In this section we prove the main result of the paper (Theorem 4.4), which is that if A is a non-commutative C^*-algebra with an identity then $K_s(A) = \frac{1}{2}\mathrm{Orc}(A)$. A number of applications are then given.

The proof of the first lemma is elementary, and is left to the reader.

LEMMA 4.1. *Let A be a C^*-algebra and let $a \in A$. Then*

$$\|D(a, A)\| = \sup\{\|D(a_P, A/P)\| : P \in \mathrm{Prim}(A)\}.$$

The next theorem is due to Stampfli [19], and is crucial for the most of the work in this area. It allows one to forget about the derivation and concentrate on the distance to the scalars in the primitive quotients.

THEOREM 4.2. *Let A be primitive C^*-algebra with an identity and let $a \in A$. Let $\lambda(a)$ denote the scalar nearest to a. Then* $\|D(a, A)\| = 2\|a - \lambda(a)\|$.

We have already remarked that when a is self-adjoint $\lambda(a) = \frac{1}{2}(\alpha(a) + \beta(a))$ and $\|a - \lambda(a)\| = \frac{1}{2}(\alpha(a) - \beta(a))$. Combining this with Lemma 4.1 and Theorem 4.2 we get the following:

COROLLARY 4.3. *Let A be a C^*-algebra with an identity and let $a \in A_{sa}$. Then*

$$\|D(a,A)\| = \sup\{\alpha(a_P) - \beta(a_P) : P \in \mathrm{Prim}(A)\}.$$

We now prove the main theorem of the paper.

THEOREM 4.4. *Let A be a non-commutative C^*-algebra with an identity. Then*

$$K_s(A) = \tfrac{1}{2}\mathrm{Orc}(A).$$

Proof. First we show that $K_s(A) \leqslant \frac{1}{2}\mathrm{Orc}(A)$. Since $\frac{1}{2} \leqslant K_s(A) \leqslant \infty$ and $1 \leqslant \mathrm{Orc}(A) \leqslant \infty$ this inequality is immediate if either $K_s(A) = \frac{1}{2}$ or $\mathrm{Orc}(A) = \infty$. So suppose that $\mathrm{Orc}(A) < \infty$ and that there is a positive integer n such that$K_s(A) > n/2$. We show that $\mathrm{Orc}(A) \geqslant n+1$. By Corollary 2.5 it is sufficient to produce an admissible chain on $\mathrm{Prim}(A)$ of length $n+1$. Since $K_s > n/2$ there is an $a \in A_{sa}$ such that $(n/2)\|D(a,A)\| < \mathrm{d}(a, Z(A))$. By substracting a suitable element of $Z(A)$ (Theorem 3.3) we may assume that $\mathrm{d}(a, Z(A)) = \|a\|$, and by multiplying by a suitable real number we may assume that $\|a\| = n/2$. Let $\varepsilon \in \mathbf{R}^+$ with $\varepsilon < \frac{1}{2}$ such that $\|D(a,A)\| < 1-\varepsilon$. Corollary 4.3 implies that $\alpha(a_P) - \beta(a_P) <$ $< 1-\varepsilon$ for all $P \in \mathrm{Prim}(A)$. Consider the following collection of sets:

$$X_1 = \left\{P \in \mathrm{Prim}(A) : \beta(a_P) \geqslant n/2 - 1 + \frac{\varepsilon}{2}\right\}$$
$$X_i = \left\{P \in \mathrm{Prim}(A) : \alpha(a_P) \leqslant n/2 - i + 2 - \frac{\varepsilon}{2},\ \beta(a_P) \geqslant n/2 - i + \frac{\varepsilon}{2}\right\} \quad 2 \leqslant i \leqslant n$$
$$X_{n+1} = \left\{P \in \mathrm{Prim}(A) : \alpha(a_P) \leqslant -n/2 + 1 - \frac{\varepsilon}{2}\right\}.$$

We claim that $X_1, \ldots, X_{n+1}$ is an admissible chain on $\mathrm{Prim}(A)$ of length $n+1$. That each X_i is closed follows from Corollary 3.2(i). Since $\alpha(a_P) - \beta(a_P) < 1-\varepsilon$ for all $P \in \mathrm{Prim}(A)$, $\bigcup_{i=1}^{n+1} X_i = \mathrm{Prim}(A)$. It is trivial that each X_i is disjoint from X_j if $|i-j| > 1$. Let G be a Glimm ideal such that $\mathrm{d}(a, Z(A)) = \frac{1}{2}(\alpha(a_G) - \beta(a_G))$ (Corollary 3.4). Then there are $P, Q \in \mathrm{Prim}(A)$ with $P, Q \supseteq G$ such that $\alpha(a_P) =$ $= \alpha(a_G) = n/2$ and $\beta(a_Q) = \beta(A_G) = -n/2$. Hence $Q \in X_1 \setminus X_2$ and $P \in X_{n+1} \setminus X_n$ and $P \approx Q$. Since $\mathrm{Orc}(A) < \infty$ it follows from Corollary 2.7 that $\mathrm{d}(P,Q) < \infty$. This shows that $X_1, \ldots, X_{n+1}$ is an admissible chain on $\mathrm{Prim}(A)$ of length $n+1$, and hence that $\frac{1}{2}\mathrm{Orc}(A) \geqslant K_s(A)$.

We now show that $K_s(A) \geqslant \frac{1}{2}\mathrm{Orc}(A)$. This is true when $\mathrm{Orc}(A) = 1$ because $K_s(A) \geqslant \frac{1}{2}$ automatically. So suppose that n is a positive integer greater than 1 and $\mathrm{Orc}(A) \geqslant n$. By Lemma 2.4 there is an admissible chain $X_1, \ldots, X_n$ on $\mathrm{Prim}(A)$ of length n. We will construct an element $a \in A_{sa}$ such that $\mathrm{d}(a, Z(A)) = n/2$ and $\|D(a,A)\| = 1$.

For $i = 1, \ldots, n$ set $R_i = \bigcap\{P \in \text{Prim}(A) : P \in X_i\}$. We will arrange that $\alpha(a_{R_i}) = i$ and $\beta(a_{R_i}) = i - 1$. Since each primitive ideal contains an R_i it will follow that $\alpha(a_P) - \beta(a_P) \leqslant 1$ for all $P \in \text{Prim}(A)$, and hence that $\|D(a, A)\| \leqslant 1$, by Corollary 4.3. In fact it will turn out that $\|D(a, A)\| = 1$. Since $X_1, \ldots, X_n$ is an admissible chain there exist $P_1 \in X_1 \backslash X_2$ and $P_n \in X_n \backslash X_{n-1}$ with $\mathrm{d}(P_1, P_n) < \infty$. Let G be the Glimm ideal contained in P_1 and P_n. We will arrange that $\alpha(a_{P_n}) = n$ and $\beta(a_{P_1}) = 0$. By Theorem 3.3 $\mathrm{d}(a, Z(A)) \geqslant \frac{1}{2}(\alpha(a_G) - \beta(a_G)) \geqslant \frac{1}{2}(\alpha(a_{P_n}) - \beta(a_{P_1})) = n/2$. But since $\alpha(a) = n$ and $\beta(a) = 0$ it will follow that $\mathrm{d}(a, Z(A)) = n/2$.

Before proceeding with the construction of a we prove an algebraic lemma:

LEMMA 4.5. *Let A be a C^*-algebra, let $X_1, \ldots, X_n$ be a chain on* Prim(A) *of length n, and for $i = 1, \ldots, n$ let $R_i = \bigcap\{P \in \text{Prim}(A) : P \in X_i\}$. Suppose that a^i $(i = 1, \ldots, n)$ (i is a superscript, not a power) are elements of A satisfying*

$$a^i_{R_i+R_{i+1}} = a^{i+1}_{R_i+R_{i+1}} \quad (i = 1, \ldots, n-1).$$

Then there exists a unique $a \in A$ such that $a_{R_i} = a^i_{R_i}$ for $i = 1, \ldots, n$.

Proof. The uniqueness of a follows from the fact that $\bigcap_{i=1}^{n} R_i = \{0\}$. We prove the existence of a by induction, as follows. Suppose that $1 \leqslant j \leqslant n - 1$ and there is $b^j \in A$ such that $b^j_{R_i} = a^i_{R_i}$ for $i = 1, \ldots, j$. (Note that this hypothesis holds for $j = 1$.) Set $Q_j = R_1 \cap \ldots \cap R_j$. Note that $Q_j + R_{j+1} = \bigcap\{P \in \text{Prim}(A) : P \in X_j \cap X_{j+1}\} = R_j + R_{j+1}$. Hence $b^j_{Q_j+R_{j+1}} = b^j_{R_j+R_{j+1}} = a^j_{R_j+R_{j+1}} = a^{j+1}_{R_j+R_{j+1}}$, that is $b^j - a^{j+1} \in Q_j + R_{j+1}$. Therefore there exist $c^j \in Q_j$ and $d^{j+1} \in R_{j+1}$ such that $b^j - a^{j+1} = c^j + d^{j+1}$, and hence such that $c^j_{R_{j+1}} = b^j_{R_{j+1}} - a^{j+1}_{R_{j+1}}$. Set $b^{j+1} = b^j - c^j$. Then for $i = 1, \ldots, j$ $b^{j+1}_{R_i} = b^j_{R_i} = a^i_{R_i}$, while $b^{j+1}_{R_{j+1}} = b^j_{R_{j+1}} - c^j_{R_{j+1}} = a^{j+1}_{R_{j+1}}$. Hence b^{j+1} satisfies the induction hypothesis. It follows by induction that $a = b^n$ has the required properties. ∎

Proof of Theorem 4.4 (continued) In view of the preceding lemma it is sufficient, for the construction of a, to find the following elements in A:

(i) an a^1 such that $\alpha(a^1_{R_1}) = \alpha(a^1_{P_1}) = 1$, $\beta(a^1_{R_1}) = \beta(a^1_{P_1}) = 0$ and $a^1_{R_1+R_2} = 1_{R_1+R_2}$,

(ii) for $2 \leqslant i \leqslant n-1$ and a^i such that $\alpha(a^i_{R_i}) = i$, $\beta(a^i_{R_i}) = i - 1$, $a^i_{R_{i-1}+R_i} = (i-1)1_{R_{i-1}+R_i}$ and $a^i_{R_i+R_{i+1}} = i1_{R_i+R_{i+1}}$,

(iii) an a^n such that $\alpha(a^n_{R_n}) = \alpha(a^n_{P_n}) = n$, $\beta(a^n_{R_n}) = \beta(a^n_{P_n}) = n - 1$ and $a^n_{R_{n-1}+R_n} = (n-1)1_{R_{n-1}+R_n}$.

The preceding lemma will then yield an a which has the properties mentioned in the previous part of this proof. This will show that $K_S(A) \geqslant \frac{1}{2}\text{Orc}(A)$.

We produce the a^i's as follows:

(i) Since $P_1 \supseteq R_1$ but $P_1 \not\supseteq R_2$ there exists $r^1 \in R_2^+$ such that $\|r^1_{P_1}\| = \|r^1_{R_1}\| = 1$. Set $a^1 = 1 - r^1$.

(ii) For $2 \leqslant i \leqslant n-1$ note that

$$\frac{A}{R_{i-1}+R_i} = \frac{(R_{i-1}+R_i)+(R_i+R_{i+1})}{R_{i-1}+R_i} \cong \frac{R_i+R_{i+1}}{(R_{i-1}+R_i)\cap(R_i+R_{i+1})}.$$

Hence there exists $r^i \in (R_i + R_{i+1})^+$ with $\|r^i\| = 1$ such that $r^i_{R_{i-1}+R_i} = 1$. For $2 \leqslant i \leqslant n-1$ set $a^i = i - r^i$.

(iii) Since $P_n \supseteq R_n$ but $P_n \not\supseteq R_{n-1}$ there exists $r^n \in R_{n-1}^+$ such that $\|r^n_{P_n}\| = \|r^n_{R_n}\| = 1$. Set $a^n = n - 1 + r^n$.

It is easy to check that these a^i's have the required properties. ■

We now mention some interesting consequences of Theorem 4.4:

(i) Since $\mathrm{Orc}(A)$ is either a positive integer or ∞, it follows from Theorem 4.4 that if A is a non-commutative C^*-algebras with an identity then $K_s(A)$ is either a positive integer multiple of a half, or ∞. The C^*-algebras of Example 2.8 show that all these possibilities do occur.

(ii) Since $\mathrm{Orc}(A) = 1$ if and only if $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$ it follows from Theorem 4.4 that if A is a non-commutative C^*-algebra with an identity then $K_s(A) = \frac{1}{2}$ if and only if $\sim$ is an equivalence relation on $\mathrm{Prim}(A)$.

(iii) A C^*-algebra is *quasi-standard* if $\sim$ is an open equivalence relation on $\mathrm{Prim}(A)$ [3]. It follows from (ii) above that if A is a quasi-standard C^*-algebra with an identity then $K_s(A) \leqslant \frac{1}{2}$.

(iv) Suppose that A is a non-commutative C^*-algebra with an identity and the centre of A is equal to the scalar multiples of the identity. Then $\mathrm{Orc}(A)$ is simply the diameter of $\mathrm{Prim}(A)$ as a graph, so $\mathrm{Orc}(A) = 1$ if and only if A is prime. Thus a non-commutative C^*-algebra A with an identity is prime if and only if the centre of A is trivial and $K_s(A) = \frac{1}{2}$.

(v) Let A be a C^*-algebra with an identity and suppose that for each Glimm ideal $G \subset A$ the set of ideals of A containing G is totally ordered. (This is equivalent to supposing that each ideal containing G is prime). Since for $P, Q \in \mathrm{Prim}(A)$ $P \supset Q \Rightarrow P \sim Q$, it follows that $\mathrm{Orc}(A) = 1$, and hence that $K_s(A) \leqslant \frac{1}{2}$.

When A is an AW*-algebra the proof of [9; Lemma 11] shows that each ideal of A which contains a Glimm ideal is prime. Let I be an ideal in A and let $\pi : A \to A/I$ denote the canonical homomorphism. It easy to check that if P is an ideal of A/I containing a Glimm ideal of A/I then $\pi^{-1}(P)$ contains a Glimm ideal of A. Hence P is prime. It follows from the paragraph above that $K_s(A/I) \leqslant \frac{1}{2}$.

(vi) A C^*-algebra A with an identity is *weakly central* if whenever M and N are distinct maximal ideals of A then $M \cap Z(A) \neq N \cap Z(A)$. It follows easily that A is weakly central if and only if each Glimm ideal of A is contained in a unique maximal ideal. Hence if A is weakly central $\mathrm{Orc}(A) \leqslant 2$, so $K_s(A) \leqslant 1$ [2].

Finally we mentioned in the introduction that the Closed Graph Theorem implies that $K(A) < \infty$ if and only if the set of inner derivations of A is closed, in the operator norm, in the set of all derivations of A [13; Theorem 5.3]. The question of whether this set was closed was one of the motives for the original work in this area [13; Introduction]. Theorem 4.4 implies that $K(A) < \infty$ if and only if $\mathrm{Orc}(A) < \infty$, from which we immediately obtain the final result.

COROLLARY 4.6. *Let A be a C^*-algebra with an identity. Then the set of inner derivations of A is closed, in the operator norm, in the set of all derivations of A if and only if* $\mathrm{Orc}(A) < \infty$.

The author wishes to thank Rob Archbold for his interest in the work, and the United Kingdom Science and Engineering Research Council for its financial support.

REFERENCES

1. APOSTOL, C.; ZSIDO, L., Ideals in W^*-algebras and the function η of A. Brown and C. Pearcy, *Rev. Roum. Math. Pures Appl.*, **18**(1973), 1151–1170.
2. ARCHBOLD, R. J., On the norm of an inner derivation of a C^*-algebra, *Math. Proc. Camb. Phil. Soc.*, **84**(1978), 273–291.
3. ARCHBOLD, R. J.; SOMERSET, D. W. B., Quasi-standard C^*-algebras, *Math. Proc. Camb. Phil. Soc.*, **107**(1990), 349–360.
4. BATTY, C. J. K., Derivations of tensor products of C^*-algebras., *J. London Math. Soc.* (2), **17**(1978), 129–140.
5. DAUNS, J.; HOFMANN, K-H., *Representations of rings by continuous sections*, Memoirs AMS, **83**, American Mathematical Society, 1968.
6. DIEUDONNÉ, J., Une généralization des espaces compacts, *J. Math. Pures Appl.*, **23**(1944), 65–72.
7. ELLIOTT, G. A., On derivations of AW*-algebras, *Tohoku Math. J.*, **30**(1978), 263–276.
8. GAJENDRAGADKAR, P., Norm of a derivation of a von Neumann algebra, *Trans. Amer. Math. Soc.*, **170**(1972), 165–170.
9. GLIMM, J., A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.*, **72**(1960), 216–244.
10. HAHN, H., Uber halbstetige und unstetige Functionen, *Sitzungsber. Akad. Wiss. Wenn Abt. II* a, **126**(1917), 91–110.
11. HALL, A. A., Ph. D. Thesis, (University of Newcastle upon Tyne, 1972).
12. JOHNSON, B. E., Norms of derivations on $L(X)$, *Pacific J. Math.*, **38**(1971), 465–469.
13. KADISON, R. V.; LANCE, E. C.; RINGROSE, J. R., Derivations and automorphisms of operator algebras, II, *J. Functional Analysis*, **1**(1967), 204–221.
14. KATETOV, M., On real-valued functions in topological spaces, *Fund. Math.*, **38**(1951), 85–91, correction **40**(1953), 203–205.
15. KYLE, J., Norms of derivations, *J. London Math. Soc.*, (2) **16**(1977), 297–312.
16. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

17. SAKAI, S., On a conjecture of Kaplanský, *Tohoku Math. J.*, **12**(1960), 31–33.
18. SOMERSET, D. W. B., Inner derivations and primal ideals of C^*-algebras, *J. London Math. Soc.*, to appear.
19. STAMPFLI, J. G., The norm of a derivation, *Pacific J. Math.*, **33**(1970), 737–748.
20. TOMIYAMA, J., Topological representations of C^*-algebras, *Tohoku Math. J.*, **15**(1963), 96–102.
21. TONG, HING, Some characterizations of normal and perfectly normal spaces, *Bull. Amer. Math. Soc.*, **54**(1948), 65; *Duke Math. J.*, **19**(1952), 289–292.
22. ZSIDO, L., The norm of a derivation of a W^*-algebra, *Proc. Amer. Math. Soc.*, **38**(1973), 147–150.

DOUGLAS W.B. SOMERSET
Thackit Eaves, Highclere,
Newbury, Berkshire, RG15 9QU,
United Kingdom.

Received March 1, 1991; revised September 16, 1992.