J. OPERATOR THEORY
29(1993), 323–343

SEMINORMAL COMPOSITION OPERATORS

JAMES T. CAMPBELL and WILLIAM E. HORNOR

1. INTRODUCTION AND MOTIVATION

A *weighted composition operator (w.c.o.)* on an L^2 space is an operator induced by composition with a reasonable transformation of the underlying measure space, followed by a multiplication. (See Section 2 for precise definitions). A bounded operator on Hilbert space is *seminormal* if either the operator or its adjoint is hyponormal. Within the hyponormal class are the nested subclasses of subnormal, quasinormal and normal operators. We say that an operator is *coprefix-normal* if its *adjoint* is prefix-normal.

The last decade has been much succes in characterizing in simple, direct measure-theoretic terms, exactly when w.c.o. may lie in several of these subclasses of the seminormal class. Because these results are scattered through the literature, and because the tools developed do not appear to be so widely known, we present in Section 2 a comprehensive introduction to the basic measure-theoretic techniques and in Section 3 a survey of the known seminormal characterizations. Sections 2 and 3 provide an introduction for the uninitiated reader and serve as a convenient reference.

In Section 4 we characterize the cohyponormal and coquasinormal w.c.o.'s. Most of the prior work on seminormal w.c.o.'s examined the relationship between the transformation and the induced operator using an associated Radon-Nikodym derivative and conditional expectation. The methods we use in Section 4 continue in this vein. Our results also give simplifications of the known characterizations of normal w.c.o.'s, and several examples are described.

We then extend some seminormal results in two ways. In Section 5 we study the seminormality of w.c.o.'s with the assumption that the underlying transformation is conservative. This assumption yields a nicer characterization of the normal w.c.o.'s, as well as the result that a w.c.o. induced by a conservative transformation, whose adjoint is hyponormal, is actually normal. Section 5 is concluded with a discussion of a conjecture on the relationship between conservative transformations and hyponormal composition operators.

Our second extension is to show how the characterizations of the hyponormal and cohyponormal classes for bounded w.c.o.'s carry over to the *unbounded* case. As a consequence of our results, one may argue that the unbounded case is the natural context in which to study w.c.o.'s. For, on one hand, the condition which insures *boundedness* of a w.c.o. (see the paragraphs prior to Lemma 2.1) is violated by many naturally occuring innocuous examples. On the other hand, the condition that simply guarantes a *dense domain* for the operator is also equivalent to the existence of the associated conditional expectation, and to the operator being *closed* (i.e. having a closed graph; see Section 6). These conditions free us to work with adjoints in the unbounded case. For such a w.c.o., we show that the conditions characterizing semi-normality are *the same* as those in the bounded cases (Theorem 6.6).

We thank James Jamison for useful discussion, support, and good coffee.

2. PRELIMINARIES. FACTS ABOUT THE ASSOCIATED CONDITIONAL EXPECTATION.

Throughout this paper (X, Σ, μ) will denote a complete, σ-finite Lebesgue space. When we consider any sub-σ-algebras of Σ, we assume they are completed. All functional equations and set relations are taken modulo sets of measure 0. When we speak of a *measurable function* f we mean that we have chosen a representative f from the equivalence class $[f]$ of a.e. defined functions. We define the *support of* f as $\sigma(f) \equiv \{f \neq 0\}$.

A *transformation* T will be a measurable point transformation mapping X into X with the property that the measure $\mu \circ T^{-1}$ on Σ given by $\mu \circ T^{-1}(A) = \mu(T^{-1}A)$ is absolutely continuous with respect to μ. The function $h(x)$ will always denote $\mathrm{d}\mu \circ T^{-1}/\mathrm{d}\mu$, and we assume (or will have to show for concrete examples) that $h(x)$ is a.e. finite valued. That this last condition is equivalent to the σ-finiteness off $(X, T^{-1}\Sigma, \mu)$ is an exercise. The σ-finiteness of $T^{-1}\Sigma$ plays an important role in obtaining our results; it is implicit in most of the definitions and used explicitly in many of the calculations.

Composition with T defines a linear transformation C_T on the set of measurable functions on X. We have the change of variables formula:

$$\int_X f \circ T \mathrm{d}\mu = \int_X f \cdot h \mathrm{d}\mu, \quad \text{for each } f \in L^1.$$

Consequently C_T induces a bounded operator on L^p $(1 \leqslant p < \infty)$ if and only if h is in L^∞, and in this case the operator norm of C_T is $\|h\|_\infty^{\frac{1}{p}}$. The change of variables also shows that *in any case, $h \circ T$ is always strictly positive*. Indeed, if $A = \{h \circ T = 0\} = T^{-1}\{h = 0\} = T^{-1}B$ we have $\mu(T^{-1}B) = \int \chi_B \circ T \mathrm{d}\mu = \int \chi_B h \mathrm{d}\mu = 0$ [6].

Operators of the form C_T we denote as *composition operators (c.o.'s)*. Throughout Sections 2 - 5 we restrict our discussion to bounded operators. Section 6 contains new results in the unbounded case.

We will find the following lemma useful.

LEMMA 2.1. a) *Suppose $h < \infty$. Then any non-negative function or any function in $L^p(\mu)$, which is also $T^{-1}\Sigma$ measurable, is of the form $g \circ T$ for some Σ-measurable function g.*

b) *If $h \in L^\infty$ then $\overline{(\operatorname{Ran} C_T)} = \{f \in L^p(\mu) : f$ is $T^{-1}\Sigma$ measurable $\}$.*

The proof of part a) we leave as an enlightening exercise. A proof of b) is found in [9]. It turns out that b) is also true under the weaker hypothesis of a), and we give a proof in Section 6.

A useful consequence of Lemma 2.1 is that if f is non-negative or in L^p, and $T^{-1}\Sigma$-measurable, one may define, in a unique fashion, a Σ-measurable function $g = f \circ T^{-1}$, even if T is not invertible. For a non-negative integrable function φ, the change of variables implies that $\varphi \circ T = 0$ if and only if $\mu(\sigma(\varphi) \cap \sigma(h)) = 0$. Now for f as above there exists a Σ-measurable g satisfying $g \circ T = f$. g will be unique if we require $\sigma(g) \subseteq \sigma(h)$ ([2], [6]) hence $f \circ T^{-1}$ is defined as the unique Σ-measurable function g whose support is contained within the support of h, which satisfies $g \circ T = f$.

Associated with each transformation T is a so-called *conditional expectation operator* $E(\cdot|T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non-negative measurable function f, or for each $f \in L^p$ $(1 \leqslant p)$, and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}\Sigma$-measurable, and

(ii) If B is any $T^{-1}\Sigma$-measurable set for which $\int_B f \mathrm{d}\mu$ converges we have $\int_B f \mathrm{d}\mu = \int_B E(f) \mathrm{d}\mu$.

This expectation is at the center of our work, and we list here some useful properties it possesses. The essential nature of E is that while it is defined on a possibly infinite σ-finite measure space it behaves similarly to expectations on standard probability spaces.

E1. If $f \geqslant g$ then $E(f) \geqslant E(g)$.

E2. (Monotone Convergence) If $f_n \uparrow f$ pointwise then $E(f_n) \uparrow E(f)$ pointwise.

E3. If f is $T^{-1}\Sigma$-measurable then $E(fg) = fE(g)$.

E4. $E(1) = 1$.

E5. $|E(fg)|^2 \leqslant E(|f|^2)E(|g|^2)$.

E6. If $f > 0$ then $E(f) > 0$.

E7. If $f \geqslant 0$ then $\sigma(f) \subseteq \sigma(E(f))$.

For proofs and discussions of these facts see [8], [2], [6]. For a deeper study of the properties of E see the interesting paper [17].

Properties E3 and E4 imply that E is an idempotent. As an operator on L^p, E is the projection onto the closure of the range of C_T. E is the identity on L^p if and only if $T^{-1}\Sigma = \Sigma$. On the other hand, if X is non-atomic and E is not the identity, the kernel of E (Ker E) is infinite dimensional ([18], [6]).

If $w : X \to \mathbb{C}$ is a non-negative finite-valued measurable function we define a linear transformation $W_{T,w}$ of measurable functions on X by the equation

$$W_{T,w}f(x) = w(x)f(Tx). \tag{1}$$

If w is fixed we just write W_T, or (if T is fixed) just W, and denote operators of this form as *weighted composition operators (w.c.o.'s)*. As an application of the properties of the conditional expectation we observe that for each $f \in L^p$ we have

$$||Wf||^p = \int |w|^p |f|^p \circ T\mathrm{d}\mu = \int E(|w|^p)|f|^p \circ T\mathrm{d}\mu = \int hE(|w|^p)\circ T^{-1}|f|^p\mathrm{d}\mu.$$

Setting $J(x) = h(x)E(|w|^p)\circ T^{-1}(x)$ we see that W defines a bounded operator on L^p if and only if $J(x)$ is essentially bounded, and the operator norm of W is given by $||J||_\infty^{1/p}$.

The assumption of non-negativity for w does two things. First it guarantees the existence of $E(w)$, which is important because W could be a bounded operator without w being a bounded function or even an L^p function. Of course this could have been guaranteed by the assumption that w is conditionable. But we also want the non-negativity of w in order to simplify the calculations in the proofs. The astute reader will see how generalizations for complex-valued w may be made.

We need an important notion first used in [9], refined in [2] and [1] and studied deeply in [17].

DEFINITION; If $\mathcal{B} \subseteq \Sigma$ is a σ-algebra and C is any set, we define $\mathcal{B}_C \equiv \{B \cap C : : B \in \mathcal{B}\}$.

If C is an element of $\mathcal{B}$ then every element of $\mathcal{B}_C$ is $\mathcal{B}$-measurable; in fact $\mathcal{B}_C$ in this case consists of the $\mathcal{B}$-measurable subsets of C. If $C \notin \mathcal{B}$ then there will exist elements of $\mathcal{B}_C$ not in $\mathcal{B}$.

With this convention, the following lemma holds:

LEMMA 2.2. a) $\overline{(\operatorname{Ran} W_T)} = \text{c.l.s.}\{w\chi_A \in L^p : A \in (T^{-1}\Sigma)_{\sigma(w)}\}$

b) $f \in \overline{\operatorname{Ran} W_T}$ *if and only if there exists* Σ*-measurable* $g \in L^p(J\mathrm{d}\mu)$ *with* $f = wg \circ T$.

This lemma holds even in case W_T is a densely defined unbounded operator; a proof of this more general fact is given in Section 6.

3. SURVEY OF SEMINORMAL CHARACTERIZATIONS.

This survey is intended to delineate previously known characterizations of subclasses of the bounded seminormal c.o.'s and w.c.o.'s on L^2 (normal, quasinormal, and hyponormal w.c.o.'s, plus subnormal c.o.'s), and to serve as an accesible reference for the reader. In all of the characterizations listed, it is assumed that the weight function w is non-negative and the operators are bounded. Because L^2 is a Hilbert space the adjoint W_T^* acts on L^2, and $W_T^* f = h[E(wf)] \circ T^{-1}$. This is not, in general, a w.c.o.

The article by Nordgren ([19]), appearing in 1978, is seminal. In it (inter alia) he posed the question of determining measure theoretic conditions on T which would characterize the normality of C_T. The question was first answered in 1978 by R. J. Whitley (21]) and independently around the same time by R. K. Singh (who had written his disertation under Nordgren) and A. A. Kumar ([12]). The normality of W_T was characterized in 1990 in ([1]) and may be steated as

THEOREM 3.1. ([1]) *W_T is normal if and only if the following conditions hold:*
a) $wE(w)h \circ T = hE(w^2) \circ T^{-1}$, *and*
b) $(T^{-1}\Sigma)_{\sigma(w)} = \Sigma_{\sigma(w)}$.

COROLLARY 3.2. ([21]; [12]) *C_T is normal if and only if*
a) $h = h \circ T$ *and*
b) $T^{-1}\Sigma = \Sigma$.

In particular, if $\mu(X)$ is finite, condition a) can be shown to imply that $h \equiv 1$, so that C_T is normal if and only if C_T is unitary ([21]).

There are related results in ([2]), although the attempted characterization of normality appearing there is correct only in case T is *conservative* (see section 5 of this paper, Theorem 5.3; [3]). There is a simplification of Theorem 3.1 given as Corollary 4.4.1 of this paper.

Also in the paper of Whitley, the quasinormal C_T were characterized ($h = h \circ T$), and in ([1]) the quasinormal W_T were characterized ($J = J \circ T$ on $\sigma(w)$).

The paper by D. Harrington and R. Whitley ([9]) appearing in 1984 contains many interesting results on seminormal composition operators, and in obtaining those results the authors laid out a nice collection of basic techniques. In that paper they obtained the following characterization of cohyponormality for C_T:

THEOREM 3.3. ([9]) *C_T is cohyponormal if and only if*
a) $\Sigma_{\sigma(h)} \subseteq T^{-1}\Sigma$ *and*
b) $h \leqslant h \circ T$.

This theorem is generalized to the weighted case in our Theorem 4.2. Some other interesting results in the Harrington-Whitley paper are:

1. C_T is coquasinormal if and only if $\Sigma_{\sigma(h)} \subseteq T^{-1}\Sigma$ and $h = h \circ T$ on $\sigma(h)$. This result is generalized to the weighted case below (Thorem 4.4).

2. If $\mu(X)$ is finite then C_T is hyponormal if and only if T is measure-preserving (so that C_T is an isometry and hence quasinormal).

3. If $h \geqslant h \circ T$ then C_T is hyponormal, and if h is $T^{-1}\Sigma$-measurable and C_T is hyponormal then $h \geqslant h \circ T$.

4. Several characterizations of hyponormality that, unfortunately,were norm conditions that needed to be checked on each $f \in L^2$.

In 1986 Alan Lambert was able to give a measure-theoretic characterization of hyponormality for W_T. Here is a statement of his result, slightly sharper than as it appeared in 1986:

THEOREM 3.4. ([13]) *W_T is hyponormal if and only if*
a) $\sigma(w) \subseteq \sigma(J)$ *and*
b) $h \circ T\left(E\left[\frac{w^2}{J}\right]\right) \leqslant \chi_{\sigma(E(w))}$ a.e. *(the fraction is interpreted as 0 of $\sigma(J)$).*

COROLLARY 3.5. ([13]) *C_T is hyponormal if and only if $h>0$ and $h \circ T E\left(\frac{1}{h}\right) \leqslant 1$.*

The idea he used, which is fundamentally different from earlier techniques, is the observation that an arbitrary w.c.o. induces an isometry from $L^2(J\,d\mu)$ to $L^2(\mu)$. This is the starting point for our work in Section 6 on the unbounded case. We do not know of any other proofs of Theorem 3.4 or Corollary 3.5.

From Corollary 3.5 one may easily deduce the hyponormality results mentioned above from ([9]). In ([5]) it was shown that $h \geqslant h \circ T$ implies that C_T is *power hyponormal*, that is, C_T^n is hyponormal for $n \geqslant 0$. However examples are known ([2]) of power hyponormal C_T which do not satisfy $h \geqslant h \circ T$. It is an open problem to characterize the power hyponormal class.

Here is another characterization of normality for W_T given in analogy with the hyponormal result above:

THEOREM 3.6. ([1]) *W_T is normal if and only if the following conditions hold:*
a) $\sigma(w) = \sigma(J)$,
b) $\Sigma_{\sigma(w)} = (T^{-1}\Sigma)_{\sigma(w)}$, and
c) $h \circ TE\left(\left[\frac{w^2}{J}\right]\right) = \chi_{\sigma(E(w))}$ *(the fraction is interpreted as 0 of $\sigma(J)$).*

With the appearance of the three papers ([14], [15], [16]), Lambert gave a measure-theoretic characterization of when C_T is subnormal, and showed that in this case the minimal normal extension of C_T may be described as a composition operator. Let $h_n := \mathrm{d}\mu \circ T^{-n}/\mathrm{d}\mu$.

THEOREM 3.7. ([14]) *C_T is subnormal if and only if $\{h_n(x)\}_{n\geqslant 1}$ is a moment sequence for almost every x.*

Here, a *moment sequence* is a sequence of moments for some probability measure on the interval $I = [0, \|h\|_\infty^{1/2}]$. That is, for a.e. x, there exists a probability measure ν_x on I so that $h_n(x) = \int\limits_I t^n \mathrm{d}\nu_x(t)$.

A moment sequence characterization of subnormality involving the Radon-Nikodym derivative is entirely natural. However the relative complexity of this criterion when compared with the other seminormal criteria is intriguing. It would be nice to somehow simplify the subnormal characterization, and a conjecture given at the end of Section 5 addresses this.

Finnaly, in the recent paper [7], Embry-Wardrop and Lambert have provided a characterization of subnormality of the adjoint C_T^*. Let Σ_∞ denote the σ-algebra $\bigcap\limits_{k=1}^{\infty} T^{-k}\Sigma$.

THEOREM 3.8. ([7]) *C_T^* is subnormal if and only if $T^{-1}\Sigma = \Sigma_\infty$, h is $T^{-1}\Sigma$ measurable, and $\{h_n \circ T^n\}$ is a moment sequence for almost every x.*

Another interesting result is Theorem 5 of [7]: If C_T^* is hyponormal, then C_T^* must be power hyponormal. This result should be compared with the weighted shift case, and also Theorem 5.4 below.

Thus the seminormal class for C_T is completely delineated. The coquasi- and cohypo-normality for W_T are determined in this paper, leaving only sub- and cosubnormality for W_T.

4. CHARACTERIZATIONS OF THE COHYPONORMAL AND COQUASINORMAL CLASSES

In this section we assume $W_T : L^2 \to L^2$ is a bounded w.c.o. with non-negative weight function w and adjoint W_T^*. The Hilbert space inner product of f with g will be denoted $\langle f, g\rangle$. We characterize the cohyponormal and coquasinormal w.c.o.'s. For the cohyponormal cases we need the following lemma, inspired by [9].

LEMMA 4.1. *The following are equivalent:*

a) $\operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$.

b) $\Sigma_{\sigma(J)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$.

Proof. First observe that condition a) implies that $\sigma(J) \subseteq \sigma(w)$. For if A is any Σ-set of finite measure disjoint from $\sigma(w)$, we have $\chi_A w = 0$ so that $W_T^*\chi_A = 0$ and hence, by a), $W_T\chi_A = 0$. But this is equivalent to the disjointness of A and $\sigma(J)$.

Suppose a) holds. If b) does not hold then $\exists\ \widehat{A} \in \Sigma$, $\widehat{A} \subseteq \sigma(J)$, with $\widehat{A} \notin (T^{-1}\Sigma)_{\sigma(w)}$. $\widehat{A}$ must contain a set of $\widehat{\widehat{A}}$ of a positive finite measure with the same properties. Indeed if every measurable subset of $\widehat{A}$ with finite measure were in $(T^{-1}\Sigma)_{\sigma(w)}$ then $\widehat{A}$ would be in $(T^{-1}\Sigma)_{\sigma(w)}$ (by σ-finiteness and the fact that $T^{-1}\Sigma$ is a σ-algebra). Finally, since $\widehat{\widehat{A}} \subseteq \sigma(J) \subseteq \sigma(w)$, we have

$$\widehat{\widehat{A}} = \widehat{\widehat{A}} \cap \{\bigcup_{m=1}^{\infty} \{x : 2^{-m} \leqslant w(x) \leqslant 2^m\}\},$$

and the same type of argument gives the existence of a set A of finite measure with the same peoperties, as well as $w\chi_A \in L^2$. But by Lemma 2.2., $w\chi_A \notin \overline{(\operatorname{Ran} W_T)}$. This implies that there is some $f \neq 0$, $f \in (\operatorname{Ran} W_T)^\perp = \operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$ with $0 \neq$ $\neq \langle f, w\chi_A\rangle$. But $0 = W_T^* W_T f = Jf$, so that $\mu(\sigma(f) \cap \sigma(J)) = 0$. Thus $\mu(A \setminus \sigma(J)) >$ 0, a contradiction. Thus a) implies b).

Suppose b) holds and suppose $f \notin \operatorname{Ker} W_T$; then $0 < \|W_T f\|^2 = \int J|f|^2 \mathrm{d}\mu$. Hence there is some measurable subset A of $\sigma(J)$ with positive measure and, say, $\operatorname{Re}(f) > 0$ on A. By b), $\sigma(J) \subseteq \sigma(w)$ so that $w > 0$ on A; and as above we may choose A so that $w\chi_A \in L^2$. Then $\int_A w(\operatorname{Re}(f))\mathrm{d}\mu > 0$. Again by b) we know that $A = T^{-1}B \cap \sigma(w)$ for some Σ-set B. Letting $\{B_n\}$ be a nested sequence of sets of finite measure increasing to B, we apply the Monotone Convergence Theorem and conclude that for some n_0,

$$0 < \int_{T^{-1}B_{n_0}} w\operatorname{Re}(f))\mathrm{d}\mu = \langle W_T\chi_{B_{n_0}}, \operatorname{Re}(f)\rangle = \langle \chi_{B_{n_0}}, W_T^*(\operatorname{Re}(f))\rangle,$$

so that $f \notin \operatorname{Ker} W_T^*$. ■

REMARK 4.1.1. Condition b) implies: $\sigma(J) \subseteq \sigma(w)$ and $(T^{-1}\Sigma)_{\sigma(J)} = \Sigma_{\sigma(J)}$.

THEOREM 4.2. *W_T is cohyponormal if and only if*

a) $\Sigma_{\sigma(J)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$, *and*
b) $J \leqslant J \circ T$.

Proof. Suppose a) and b) hold. Write any $f \in L^2(\mu)$ as $f = f_1 + f_2$ with $f_1 \in \overline{(\operatorname{Ran} W_T)}$ and $f_2 \in (\operatorname{Ran} W_T)^{\perp}$. By Lemma 4.1 we have $(\operatorname{Ran} W_T)^{\perp} = \operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$, hence $\|W_T^* f\| = \|W_T^* f_1\|$ and $\|W_T f\| = \|W_T f_1\|$. Therefore to establish the cohyponormality of W_T it suffices to show that $\|W_T^* f\|^2 \geqslant \|W_T f\|^2$ for $f \in \overline{(\operatorname{Ran} W_T)}$. By Lemma 2.2 any such f is of the form $f = wg \circ T$ for some $g \in L^2(J\mathrm{d}\mu)$. Hence

$$\|W_T^* f\|^2 - \|W_T f\|^2 = \int (wh \circ TE(wf) - Jf)\bar{f}\mathrm{d}\mu = \int (wh \circ TE(w^2) g \circ T - Jf)\bar{f}\mathrm{d}\mu = $$
$$= \int (h \circ TE(w^2) - J)|f|^2 \mathrm{d}\mu = \int (J \circ T - J)|f|^2 \mathrm{d}\mu.$$

By b) W_T is cohyponormal.

Suppose W_T is cohyponormal; then $\operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$ and by Lemma 4.1, a) holds.

Consider the set $A = \{J \circ T < J\} \subseteq \sigma(J)$. By a), $A = T^{-1}B \cap \sigma(w)$ for some $B \in \Sigma$. Write B as the union of an increasing sequence $\{B_n\}$ of Σ-sets each of finite measure and set $A_n = T^{-1}B_n \cap \sigma(w)$. Then $w\chi_{A_n} = W_T \chi_{B_n} \in L^2(\mu)$. By cohyponormality we have

$$0 \leqslant \|W_T^*(w\chi_{A_n})\|^2 - \|W_T(w\chi_{A_n})\|^2 = \int_{A_n} (J \circ T - J) w^2 \mathrm{d}\mu \leqslant 0,$$

hence $\mu(A_n) = 0$ for all n and A is a nullset. ∎

Examples 5.5 and 5.6 (in Section 5) are cohyponormal, non-normal w.c.o.'s.

If an operator is both hyponormal and cohyponormal then it is normal. Although the conditions for hyponormality and cohyponormality of W_T are not symmetric, we still may quickly check that together they imply the conditions of Theorem 3.6. It is clear that conditions a) and b) of 3.6 hold. As for condition c) we have

$$\chi_{\sigma(E(w))} \geqslant h \circ TE\left(\left[\frac{w^2}{J}\right]\right) \geqslant h \circ TE\left(\left[\frac{w^2}{h \circ TE(w^2)}\right]\right) = \chi_{\sigma(E(w^2))} = \chi_{\sigma(E(w))},$$

where the first inequality follows from b) of 3.4, the second is from b) of 4.2, and the last equality holds because $w \geqslant 0$ so that $\sigma(E(w)) = \sigma(E(w^2)) = \sigma(E(\chi_{\sigma(w)}) =$ smallest $T^{-1}\Sigma$-measurable set containing $\sigma(w)$ ([17]).

In order to characterize the coquasinormal w.c.o.'s we need the following lemma.

LEMMA 4.3. *Let A be a measurable set of positive measure and suppose $\Sigma_A = (T^{-1}\Sigma)_A$. If f is any non-negative Σ-measurable function supported in A, then* *there exists an increasing sequence* $\{g_n\}$ *of non-negative* $T^{-1}\Sigma$*-measurable functions so that* $g_n\chi_A$ *converges pointwise to* f. *If* $f \in L^p$, *we can choose the* $\{g_n\}$ *so that* $g_n\chi_A \in L^p$ *and the convergence is also in* L^p.

Proof. By the hypotheses we may choose an increasing sequence of non-negative simple functions of the form

$$f_n = \sum_{k_n=1}^{m_n} c_{k_n}\chi_{T^{-1}B_{k_n}\cap A}$$

which converge to f a.e. (and if $f \in L^p$, the f_n may chosen in L^p, and to converge to f in L^p). Let $g_n = \sum_{k_n=1}^{m_n} c_{k_n}\chi_{T^{-1}B_{k_n}}$. ■

THEOREM 4.4. *W_T^* is quasinormal if and only if*

a) $\Sigma_{\sigma(J)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$ *and*

b) $hE(w^2)\circ T^{-1} = wE(w)h\circ T$ *on* $\sigma(J)$.

Proof. Suppose a) and b) hold. By Lemma 4.1, it suffices to verify the quasinormal condition for W_T^* applied to each $f \in \overline{\operatorname{Ran} W_T}$. Hence $f = wg\circ T$ for some $g \in L^2(J\,d\mu)$. By considering the positive and negative parts of real and imaginary parts of f, we may assume that f (and hence g) is nonnegative. We have

$$W_T W_T^* W_T^*(f) = W_T W_T^*(Jg) = wh\circ TE(wJg).$$

Using Lemma 4.3 find an increasing sequence of non-negative $T^{-1}\Sigma$-measurable functions $\{\psi_n\}$ so that $\psi_n\chi_{\sigma(J)} \to Jg$ a.e. and in L^2. There is some Σ-measurable set B so that $\sigma(J) = T^{-1}B\cap\sigma(w)$, and applying condition E2 from Section 2 we have

$$wh\circ TE(wJg) = \lim_{n\to\infty} wh\circ TE(w\psi_n\chi_{\sigma(J)}) = \lim_{n\to\infty} wh\circ TE(w\psi_n\chi_B\circ T\chi_{\sigma(w)}) =$$

$$= \lim_{n\to\infty} wh\circ TE(w\chi_{\sigma(w)})\psi_n\chi_B\circ T\chi_{\sigma(w)} = wh\circ TE(w)Jg.$$

On the other hand, $(W_T^* W_T W_T^*)(f) = J^2g$. By condition b), W_T^* is quasinormal.

Conversely suppose that W_T^* is quasinormal. Then W_T^* is hyponormal and a) holds. Thus Lemma 4.3 applies and we observe that

$$J^2g = Jwh\circ TE(w)g \quad \text{for each } g \in L^2(J\,d\mu). \tag{4.4.1}$$

Taking an increasing sequence of measurable sets each of finite measure whose union is all of $\sigma(J)$, and letting $\{g_n\}$ be the corresponding sequence of characteristic functions of those sets, we apply (4.4.1) to each g_n and see that b) holds. ■

See Example 4.6 below.

At this point it is convenient to apply the technique of Lemma 4.3 and Theorem 4.4 to give simpler reformulations of Theorems 3.1 and 4.4:

COROLLARY (THEOREM 3.1 SIMPLIFIED). *W_T is normal if and only if*

a′) $J = \chi_{\sigma(w)} J \circ T$ *and*

b′) $\Sigma_{\sigma(w)} = (T^{-1}\Sigma)_{\sigma(w)}$.

Proof. Suppose a) and b) of Theorem 3.1 hold. Then by Lemma 4.3 we may find an increasing sequence $\{g_n\}$ of non-negative $T^{-1}\Sigma$-measurable functions so that $g_n \chi_{\sigma(w)}$ converges a.e. to w. Proceeding as in the proof of Theorem 4.4, we see that

$$wE(w) = \lim_{n\to\infty} \chi_{\sigma(w)} g_n E(w) = \lim_{n\to\infty} \chi_{\sigma(w)} E(g_n w) = \chi_{\sigma(w)} E(w^2),$$

by property E2 of Section 2. Hence a′) holds. Conversely, if a′) and b′) hold, the reverse chain of equalities shows that $\chi_{\sigma(w)} E(w^2) = wE(w)$, and a) of Theorem 3.1 holds. ∎

COROLLARY (THEOREM 4.4 SIMPLIFIED). *W_T^* is quasinormal if and only if*

a′) $J = \chi_{\sigma(J)} J \circ T$ *and*

b′) $\Sigma_{\sigma(J)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$.

Proof. Apply Remark 4.1.1 and the above technique. ∎

COROLLARY 4.5. *W_T is normal if and only if W_T^* is quasinormal and $\sigma(J) = \sigma(w)$.*

The proof is left to the reader.

EXAMPLE 4.6. W_T^* quasinormal, not normal. Let $X = \mathbb{Z}$, $\Sigma = 2^{\mathbb{Z}}$, $\mu =$ counting measure. Let $Tx = x+2$ if $x \leqslant -2$, $Tx = x+1$ otherwise, and set $w(n) = 1$ for $n \geqslant 0$, $w(n) = 0$ otherwise. Then $\sigma(J) = \mathbb{N}$ and $\sigma(w) = \mathbb{N} \cup \{0\}$, $(T^{-1}\Sigma)_{\sigma(w)} = \Sigma_{\sigma(J)}$, $J = wE(w)h \circ T$ on $\sigma(J)$, but $J(0) = 0$ and $(wE(w)h \circ T)(0) = 1$.

5. CONSERVATIVE TRANSFORMATIONS

In this section we show how the assumption of conservativity on T yields tighter results in the classifications for the subclasses of the semi-normal class of w.c.o.'s. There are many notions of conservativity for both transformations and operators; we use a simple condition on the transformation T. Recall that we assume $\mu \circ T^{-1}$ is absolutely continuous with respect to μ and that all set and functional relations are defined to within a.e. equivalence. With these conventions, the following theorem holds (see [11] especially pages 13–19):

THEOREM 5.1. *The following are equivalent:*

a) *The only $W \in \Sigma$ with $\{T^{-k}W\}_{k\geqslant 0}$ a pairwise disjoint collection is the empty set.*

b) *For all $A \in \Sigma$, $A = \{x \in A : \exists\, n \geqslant 1$ so that $T^n x \in A\}$.*

c) *For all $A \in \Sigma$, $A = \{x \in A : T^n x \in A$ for infinitely many $n \geqslant 1\}$.*

d) *If $A \in \Sigma$ ($B \in \Sigma$) satisfies $T^{-1}A \subseteq A$ ($B \subseteq T^{-1}B$) then $T^{-1}A = A$ ($B = T^{-1}B$).*

e) *If $f(g)$ is measurable and satisfies $f \geqslant f{\circ}T$ ($g \leqslant g{\circ}T$) then $f = f{\circ}T$ ($g = g{\circ}T$).*

A transformation T satisfying any one (hence all) of the above conditions is called *conservative.* The intuition behind conservativity is that generic points cannot wander throughout the space without returning to the initial event they were observed in. Conservative transformations are studied extensively in the ergodic theory literature (see [11]) and many examples are known. For example, any measure-preserving transformation on a finite measure space is conservative; this is known as Poincare's Recurrence Theorem, and is one of the oldest results in ergodic theory. W.c.o.'s on atomic measure spaces are often used to model weighted shifts; we may deduce that in this case the underlying transformation cannot be conservative. Indeed, suppose that a conservative transformation T acts on a *purely atomic* infinite measure space. Then the space may be decomposed into a pairwise disjoint infinite sequence $\{A_i\}$ of non-empty sets, each containing *finitely many* atoms, so that T acts on each A_i as a cyclic permutation of its atoms.

The following result was first proved in [4]:

THEOREM 5.2. *Suppose T is conservative.*

a) *If h is $T^{-1}\Sigma$-measurable then C_T is quasinormal if and only if C_T is hyponormal.*

b) *If $T^{-1}\Sigma = \Sigma$ then C_T is normal if and only if C_T is hyponormal.*

The value in this theorem is that it reduces the calculation involved for determining the hyponormality and subnormality of C_T (see Theorems 3.5 and 3.6). The following theorems continue in this spirit.

THEOREM 5.3. *Suppose T is conservative. Then W_T is normal if and only if*

a) *$T^{-1}\sigma(w) = \sigma(w)$, and*

b) *$T^{-1}(\Sigma_{\sigma(w)}) = \Sigma_{\sigma(w)}$, and*

c) *w is $T^{-1}\sigma$-measurable and $hw^2 \circ T^{-1} = h \circ Tw^2$.*

Proof. Conditions a), b) and c) are stronger than those determining the normality of W_T (Theorem 3.1). Suppose W_T is normal. It was shown in the proof of Theorem 2 of [1] that this implies $\sigma(w) \subseteq T^{-1}\sigma(w)$. By conservativity, a) holds. Hence $T^{-1}(X \setminus \sigma(w)) = X \setminus \sigma(w)$. Since W_T is normal we know that for each $A \in \Sigma_{\sigma(w)}$ there exists a $B \in \Sigma$ so that $A = T^{-1}B \cap \sigma(w)$. But by the invariance of $\sigma(w)$ and $X \setminus \sigma(w)$ we may suppose that $B \subseteq \sigma(w)$. Thus b) holds. Note that a) and b) imply that w is $T^{-1}\Sigma$-measurable. The functional equation in c) now follows from this fact and condition a) of Theorem 3.1. ∎

Example 1 in [1] satisfies W_T is normal, T is not conservative, neither a) nor b) hold, and w is not $T^{-1}\Sigma$-measurable.

THEOREM 5.4. *Suppose T is conservative. If W_T is cohyponormal then W_T is normal.*

Proof. By Theorem 4.2 we have $J \leqslant J \circ T$ hence $J = J \circ T$, so that $\sigma(J) = \sigma(J \circ \circ T) = \sigma(w)$ is invariant under T. Thus a) of Theorem 5.3 holds. By part a) of Theorem 4.2 we have $\Sigma_{\sigma(w)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$. Because the reverse inclusion always holds these algebras are equal. But by the invariance of $\sigma(w)$ we have $(T^{-1}\Sigma)_{\sigma(w)} = T^{-1}(\Sigma_{\sigma(w)})$ so that b) of Theorem 5.3 holds. Thus w is $T^{-1}\Sigma$-measurable, and since $J = J \circ T$, condition c) of Theorem 5.3 holds and W_T is normal. ∎

EXAMPLE 5.5. In this example T is invertible but not conservative and W_T is cohyponormal but normal. Let $X = \mathbb{Z}$ equipped with counting measure on the algebra $2^{\mathbb{Z}}$. Define $T(x) = x + 1$. $w(x)$ will be $1/2$ for all $x \leqslant 0$ and $w(x) = 1$ for all $x \geqslant 1$. Then $h \equiv 1$, w is $T^{-1}\Sigma$-measurable, $J \leqslant J \circ T$ but equality does not hold when $x = 0$.

EXAMPLE 5.6. In the previous example, X contained no subset on which T was conservative (and W_T was a "backwards" weighted shift). We may obtain examples where X contains a subset on which T is conservative, W_T is not equivalent to a shift, yet T is not conservative and W_T is cohyponormal but not normal. Let $X = \mathbb{N} \cup \{0\}$ with counting measure on 2^X and define $T(x) = x - 1$ for nonzero x, $T(0) = 0$. Direct (mildly tedious) calculations show that if $w(x)$ is defined by $w(x) = 1$ if $x \in \{0, 1, 2\}$ and $w(x) = 2^{-n+2}$ for $n \geqslant 3$ then W_T is cohyponormal but not normal.

We conclude this section with a discussion of conservativity and subnormality for composition operators C_T. The conditions for normality, quasinormality, and hyponormality each involve functional equalities or inequalities involving h and $h \circ T$. On the other hand the condition for subnormality involves conditions on the sequence $\{h_n\}$ of Radon-Nikodym derivatives $\mathrm{d}\mu \circ T^{-n}/\mathrm{d}\mu$ (Theorem 3.7). While these conditions arise naturally enough from the definition of subnormality, they do not seem to fit easily with the other seminormal criteria, as well as being several degrees of magnitude more difficult to verify.

The condition for hyponormality implies the condition $E(h) \geqslant h \circ T$, while the condition for quasinormality is simply $h = h \circ T$. The subnormal class lies strictly between these two classes. For every hyponormal example we know, if one applies the Hopf decomposition (see [3]), the relation $h = h \circ T$ holds on the conservative part. We conjecture that if T is conservative and C_T is hyponormal then $h = h \circ T$. We point out that this conjecture is true in finite measure (because C_T hyponormal implies $h \equiv 1$, see result 2 listed after Theorem 3.2), and in case (X, Σ, μ) is purely atomic. To see this, apply the description of a conservative transformation acting on a purely atomic space given in the beginning of this section. For each i, $L^2(A_i)$ is reducing for C_T. But A_i has finite measure so we are in the previous case. If the conjecture holds in general then combined with the quasinormal criterion this would greatly simplify the determination of subnormality in the conservative case.

6. EXTENSIONS TO THE UNBOUNDED CASE

An unbounded operator acting on Hilbert space has a uniquely determined adjoint when the operator is densely defined. For w.c.o.'s acting on $L^2(\mu)$ this is equivalent to the function J attaining only finite values (a.e.). This last condition is sufficient for the w.c.o. to have a closed graph. We establish these results, and characterize the hypo- and cohypo-normal unbounded w.c.o.'s.

We drop the assumption of uniform boundedness on h but retain the assumption that h is finite valued. Thus $(X, T^{-1}\Sigma, \mu)$ is σ-finite, $E(\cdot)$ is defined and behaves as in Section 2, and in particular $E(w)$ exists (we still assume that w is non-negative). Recall that $J(x) = h(x)E(|w|^p) \circ T^{-1}(x)$. J exists and is well-defined by Lemma 2.1 a) and the discussion in the following two paragraphs. All set relations and functional equations are taken μ-mod0. We set $\mathrm{d}\nu = J\mathrm{d}\mu$ and $\mathcal{D} = L^p(\mu) \cap L^p(\nu)$.

LEMMA 6.1. *Suppose* $1 \leqslant p < \infty$.

a) *If* f *is* Σ*-measurable, then* f *is in* $L^p(\nu)$ *if and only if* $wf \circ T$ *is in* $L^p(\mu)$.

b) $W_T : L^p(\mu) \to L^p(\mu)$ *has dense domain* $\mathcal{D}$ *if and only if* J *is finite valued.*

Proof. a) Define $D : L^p(\nu) \to L^p(\mu)$ by $Df = wf \circ T$. Then for each $f \in L^p(\nu)$,

$$\|Df\|_\mu^p = \int w^p |f|^p \circ T \mathrm{d}\mu = \int E(w^p)|f|^p \circ T \mathrm{d}\mu = \int |f|^p \mathrm{d}\nu = \|f\|_\nu^p.$$

b) By a), the domain of W_T is $\mathcal{D}$. But it is an easy exercise to show that $\mathcal{D}$ is dense in $L^p(\mu)$ if and only if J (which is non-negative) is finite valued. ■

If E and F are Banach spaces, $E \oplus_p F$ will denote their L^p direct sum.

LEMMA 6.2. *Suppose W_T has dense domain in $L^p(\mu)$.*

a) *W_T is closed.*

b) $\overline{\operatorname{Ran} W_T} = \{f \in L^p(\mu) : f = wg \circ T$ *for some* $g \in L^p(\nu).\}$

REMARK 6.2.1. Under this hypothesis, $\mathcal{D}$ is also dense in $L^p(\nu)$.

Proof. a) Define $\Phi : L^p(\mu+\nu) \to L^p(\mu) \oplus_p L^p(\mu)$ by $\Phi(f) = f \oplus_p W_T f$. Φ is an isometry, so the range of Φ is closed; but range of Φ is the graph of W_T.

b) For $f \in \overline{\operatorname{Ran} W_T}$, find a sequence $\{f_n\} \subseteq \operatorname{Ran} W_T$ converging to f in $L^p(\mu)$. By Lemma 6.1 there exists a sequence $\{g_n\} \subseteq \mathcal{D}$ with $W_T g_n = f_n$. Because $D : L^p(\nu) \to \to L^p(\mu)$ (as in the proof of Lemma 6.1) is an isometry, $\{g_n\}$ is a Cauchy sequence in $L^p(\nu)$, and converges to say, g, in $L^p(\nu)$. Again because D is an isometry we have $Dg = f$.

Conversely, suppose $f = Dg$ for some $g \in L^p(\nu)$. Since $\mathcal{D}$ is dense in $L^p(\nu)$ we may find a sequence $\{g_n\} \subseteq \mathcal{D}$ with $g_n \to g$ in $L^p(\nu)$. Because W_T and D agree on $\mathcal{D}$, and D is an isometry, the result follosw. ∎

REMARK 6.2.2. These results establish lemma 2.2 and show Lemma 2.1 holds even in the unbounded case.

COROLLARY 6.3. *Suppose W_T acts on L^2. With the conditions of Lemma 6.1, W_T^* is closed, the domain of W_T^* is dense, $W_T^{**} = W_T$, and $(\operatorname{Ran} W_T)^{\perp} = \operatorname{Ker} W_T^*$.*

Proof. The two previous lemmas showed that W_T is densely defined and closed. The result follows immediately from Theorems 13.9 and 13.12 of [20]. ∎

Henceforth our standing assumption is that J is finite valued. We denote the domain of W_T^* by $\mathcal{D}^*$.

LEMMA 6.4. *If W_T has dense domain then for all $g \in \mathcal{D}^*$, $W_T^* g = hE(wg) \circ T^{-1}$.*

Proof. If $g \in \mathcal{D}^*$ we have

$$\langle W_T f, g\rangle = \int wf \circ T\overline{g} \mathrm{d}\mu = \int f \circ TE(w\overline{g}) \mathrm{d}\mu < \infty, \quad \text{for all } f \in \mathcal{D}.$$

Thus $f \circ TE(w\overline{g})$ is a $T^{-1}\Sigma$-measurable function in $L^1(\mu)$, and by Lemma 2.1 there exists a unique Σ-measurable function F, supported in $\sigma(h)$, so that $F \circ T = = f \circ TE(w\overline{g})$; i.e., $F = fE(w\overline{g}) \circ T^{-1}$. By Lemma 6.2 a), $F \in L^1(h\mathrm{d}\mu)$ and we conclude that

$$\langle W_T f, g\rangle = \int f \cdot hE(w\overline{g}) \circ T^{-1} \mathrm{d}\mu, \quad \text{for all } f \in \mathcal{D} \text{ and all } g \in \mathcal{D}^*.$$

The conclusion of the lemma follows because $\mathcal{D}$ is dense in $L^2(\mu)$. ∎

By setting $w = 1$ in results 6.1–6.4 we obtain the corresponding results for C_T. Thus the condition that $h < \infty$ a.e., necessary for the existence of E, is actually sufficient for T to induce a densely defined, closed operator C_T with a computable, densely defined adjoint.

We turn now to the problem of seminormality in the unbounded case. In [10], a definition of hyponormality for unbounded operators was given as:

DEFINITION 6.5. Let $S : \mathcal{H} \to \mathcal{H}$ be a densely defined operator on the Hilbert space $\mathcal{H}$. S is *hyponormal* if

(i) Domain $S \subseteq$ Domain S^*, and

(ii) $\|Sh\| \geqslant \|S^*h\|$ for each $h \in$ Domain S.

Also in [10], a characterization was given for C_T to be cohyponormal in the special case when T is invertible. The preliminary lemmas in this section, combined with this definition of hyponormality, allow us to show that the characterizations of hyponormality and cohyponormality for w.c.o.'s carry over in full generality from the bounded case to the unbounded case:

THEOREM 6.6. *Let W_T be a densely defined w.c.o. Then*

(i) *W_T is hyponormal if and only if the conditions of Theorem 3.4 hold.*

(ii) *W_T is cohyponormal if and only if the conditions of Theorem 4.2 hold.*

Proof. The idea to modify the proofs from the bounded cases by restricting calculations to the correct domains instead of all of $L^2(\mu)$. In the hyponormal case the proof goes through practically verbatim, underscoring the elegance of the original proof due to Lambert. The cohyponormal case requires more care.

Proof of (i): Suppose W_T is hyponormal. Then by Lemma 4.4, $W_T^* f = h[E(wf)] \circ$ $\circ T^{-1}$ for each $f \in \mathcal{D}$. Now we may proceed as in the prof of Theorem 3.4 given in [13]. If $B \subseteq X \setminus \sigma(J)$ has finite measure then $\chi_B \in \mathcal{D}$ and $W_T \chi_B \equiv 0$. By hyponormality, $W_T^* \chi_B \equiv 0$. But $0 = \int hE(w\chi_B) \circ T^{-1} \mathrm{d}\mu = \int w\chi_B \mathrm{d}\mu = \int_B w \mathrm{d}\mu$, so that w vanishes on B and hence on all of $X \setminus \sigma(J)$. This shows $\sigma(w) \subseteq \sigma(J)$.

Still proceeding as in [13], one establishes that

$$\|W_T^* f\|_\mu = \int \frac{h \cdot |E(wf)|^2 \circ T^{-1}}{E(w^2) \circ T^{-1}} \mathrm{d}\nu \text{ for each } f \in \mathcal{D}.$$

Define $G : \mathcal{D} \to L^2(\nu)$ by

$$G(f) = \left[\frac{h}{E(w^2) \circ T^{-1}}\right]^{\frac{1}{2}} \cdot h[E(wf)] \circ T^{-1}, \quad f \in \mathcal{D}.$$

As in [13], calculations verify that

$$\|GF\|_\nu \leqslant \|f\|_\nu, \quad f \in \mathcal{D}.$$

Hence G extends to a contraction on all of $L^2(\nu)$. Then G^* is also a contraction on $L^2(\nu)$ and

$$G^*g = \left[\frac{w\sqrt{J\circ T}}{J}\right] g\circ T, \quad g \in \mathcal{D}.$$

But

$$\|G^*g\|_\nu^2 = \int \left[hE\left[\frac{w^2}{J}\right]\circ T^{-1}\right]\cdot |g|^2 \mathrm{d}\nu, \quad g\in\mathcal{D}.$$

Because $\mathrm{d}\nu = J\mathrm{d}\mu$ and $\mathcal{D}$ is dense in $L^2(\nu)$, we see that $hE\left[\frac{w^2}{J}\right]\circ T^{-1} \leqslant 1$. Because $hE\left[\frac{w^2}{J}\right]\circ T^{-1}$ is non-negative we have $h\circ TE\left[\frac{w^2}{J}\right] \leqslant 1$. Moreover the conditions $\sigma\left(\frac{w^2}{J}\right) = \sigma(w)$, $\frac{w^2}{J} \geqslant 0$, $w \geqslant 0$ imply that

$$\sigma\left(E\left[\frac{w^2}{J}\right]\right) = \sigma(E(w)),$$

and since $h\circ T$ we have

$$h\circ TE\left[\frac{w^2}{J}\right] \leqslant \chi_{\sigma(E(w))}.$$

Thus conditions (a) and (b) of Theorem 3.4 hold. (This last inequality is not explicitly established in [13]. The inequality given there has 1 in place of $\chi_{\sigma(E(w))}$.)

Conversely suppose conditions (a) and (b) of Theorem 3.4 hold. As in [13], define H on $\mathcal{D}$ by

$$Hf = w\left[\frac{\sqrt{J\circ T}}{J}\right] f\circ T, \quad f\in\mathcal{D}.$$

It follows that

$$\|Hf\|_\nu^2 = \|f\|_\nu^2 = \|W_Tf\|_\mu^2, \quad \text{for each } f\in\mathcal{D},$$

so that H extends to a contraction on all of $L^2(\nu)$ and moreover

$$\|H^*f\|_\nu^2 = \|W_T^*f\|_\mu^2 \quad \text{for each } f\in\mathcal{D}.$$

These conditions imply that

$$\|W_T^*f\|_\mu^2 = \|H^*f\|_\nu^2 \leqslant \|f\|_\nu^2 = \|W_Tf\|_\mu^2, \quad \text{for each } f\in\mathcal{D}.$$

Thus inequality (ii) of definition 4.5 holds, and consequently condition (i) of 6.5 holds.

Proof of Theorem 6.6. part (ii): We need the following extension of Lemma 4.1:

LEMMA 6.6.1. *Let W_T be a densely defined w.c.o. The following are equivalent:*
a) $\operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$.

b) $\Sigma_{\sigma(J)} \subseteq (T^{-1}\Sigma)_{\sigma(w)}$.

Proof of Lemma 6.6.1. Conditions a) implies $\sigma(J) \subseteq \sigma(w)$. Indeed, if A is any Σ-set of finite measure disjoint from $\sigma(w)$ then $w\chi_A = 0$ so that $hE(\chi_A w)\circ T^{-1} = 0$. Thus $\chi_A \in \mathcal{D}^*$, and $W_T^*\chi_A = 0$ implies that $W_T\chi_A = 0$, by a). This implies that A is disjoint from $\sigma(J)$.

Because Lemma 2.2. and $\overline{(\operatorname{Ran} W_T)}^{\perp} = \operatorname{Ker} W_T^*$ still hold in the densely defined case, the rest of proof that a) implies b) goes through as in the bounded case.

Suppose b) holds. If W_T is the 0 operator then a) holds trivially. If $W_T \neq 0$, choose $f \in \mathcal{D}$ outside of $\operatorname{Ker} W_T$:

$$0 < \|W_T f\|^2 = \int J|f|^2 \mathrm{d}\mu < \infty.$$

We may find a set $A \subseteq \sigma(J)$ of positive measure with $\chi_A w \in L^2(\mu)$ and

$$0 < \int_A w\mathrm{Re}(f)\mathrm{d}\mu.$$

By b), $A = T^{-1}B \cap \sigma(w)$ for some Σ-set B. But $w\chi_A = w\chi_B \circ T \in L^2(\mu)$ implies $\chi_B \in L^2(\nu)$. Since $\mathcal{D}$ is dense in $L^2(\nu)$ we can find an increasing non-negative sequence $\{f_n\} \subseteq \mathcal{D}$ with $f_n \uparrow \chi_B$ in $L^2(\nu)$ and a.e.. By Monotone Convergence,

$$0 < \int w\chi_A \mathrm{Re}(f)\mathrm{d}\mu = \int w\chi_B \circ T\mathrm{Re}(f)\mathrm{d}\mu = \int W_T\chi_B \mathrm{Re}(f)\mathrm{d}\mu = \lim_{n\to\infty} \int (W_T f_n)\mathrm{Re}(f)\mathrm{d}\mu.$$

Hence there exists n_0 such that

$$0 < \int (W_T f_{n_0})\mathrm{Re}(f)\mathrm{d}\mu = \langle W_T f_{n_0}, \mathrm{Re}(f)\rangle = \langle f_{n_0}, W_T^*(\mathrm{Re}(f))\rangle,$$

so that $f \notin \operatorname{Ker} W_T^*$. ∎

Continuation of the proof of Theorem 6.6. part (ii). We first suppose conditions a) and b) of Theorem 4.2 hold and show that W_T^* is hyponormal. Our task is simplified by observing that it is sufficient to establish the inequality $\|W_T^* f\| \geqslant \|W_T f\|$ for each $f \in \{\overline{\operatorname{Ran} W_T} \cap \mathcal{D}^*\}$. As in the proof of Theorem 4.2 write $f \in L^2(\mu)$ as $f_1 + f_2$, with $f_1 \in \overline{\operatorname{Ran} W_T}$ and $f_2 \in (\operatorname{Ran} W_T)^{\perp}$. By Corollary 4.3, Lemma 4.1, and condition a), $(\operatorname{Ran} W_T)^{\perp} = \operatorname{Ker} W_T^* \subseteq \operatorname{Ker} W_T$ so that $\|W_T^* f\| = \|W_T^* f_1\|$ and $\|W_T f\| = W_T f_1\|$.

Set $\mathcal{K} = \{\overline{\operatorname{Ran} W_T} \cap \mathcal{D}^*\}$; by definition $\mathcal{K} = \{f = wg\circ T : g \in L^2(\nu)$ and $h[E(wf)]\circ$ $\circ T^{-1} \in L^2(\mu)\}$. The fact which allows our calculations to go through in the unbounded case is

CLAIM. $\mathcal{K} = \{f = wg \circ T : g \in L^2((J + J^2)\mathrm{d}\mu)\}$.

Proof of Claim. If $f = wg \circ T \in \mathcal{K}$ then $g \in L^2(J\mathrm{d}\mu)$ and

$$\infty > \int h^2 |E(wf)|^2 \circ T^{-1} \mathrm{d}\mu = \int h \circ T |E(wf)|^2 \mathrm{d}\mu = \int h \circ T |E(w^2)|^2 |g|^2 \circ T \mathrm{d}\mu$$

$$= \int h^2 |E(w^2)|^2 \circ T^{-1} |g|^2 \mathrm{d}\mu = \int |g|^2 J^2 \mathrm{d}\mu.$$

Hence $g \in L^2(J^2\mathrm{d}\mu)$ and $\mathcal{K} \subseteq \{f = wg \circ T : g \in L^2((J + J^2)\mathrm{d}\mu)\}$. Conversely if $g \in L^2((J + J^2)\mathrm{d}\mu)$, the reverese calculation shows that $f = wg \circ T \in \mathcal{K}$. ■

Now the proof that $\|W_T^* f\| \geqslant \|W_T f\|$ for each $f \in \mathcal{K}$ goes through as in the bounded case:

$$\|W_T^* f\|^2 - \|W_T f\|^2 = \int (wh \circ T E(wf) - Jf) \overline{f} \mathrm{d}\mu = \int (wh \circ T E(w^2) g \circ T - Jf) \overline{f} \mathrm{d}\mu =$$

$$= \int (h \circ T E(w^2) - J) |f|^2 \mathrm{d}\mu = \int (J \circ T - J) |f|^2 \mathrm{d}\mu.$$

By condition b) of our hypotheses, W_T^* is hyponormal.

Suppose W_T^* is hyponormal so that $\mathcal{D}^* \subseteq \mathcal{D}$ and $\|W_T^* f\| \geqslant \|W_T f\|$ for all $f \in \mathcal{D}^*$. In particular Ker $W_T^* \subseteq$ Ker W_T so by Lemma 6.6.1 condition a) of Theorem 4.2 holds. Let $A = \{J \circ T < J\} \subseteq \sigma(J)$. We want to show that A is a nullset. By condition a), $A = T^{-1}B \cap \sigma(w)$ for some Σ-set B. Because $L^2((J + J^2)\mathrm{d}\mu) \cap L^2(\mu)$ is dense in $L^2(\mu)$, for each Σ-set C with finite measure contained in B we may find a non-negative increasing sequence $\{f_n\} \subseteq L^2((J + J^2)\mathrm{d}\mu) \cap L^2(\mu)$ with $f_n \uparrow \chi_C$ pointwise and in $L^2(\mu)$. For each n,

$$\|W_T^* W_T f_n\|^2 = \int wh \circ T E(w w f_n \circ T) w \overline{f_n \circ T} \mathrm{d}\mu = \int w^2 h \circ T E(w^2) |f_n|^2 \circ T \mathrm{d}\mu =$$

$$= \int h \circ T E(w^2)^2 |f_n|^2 \circ T \mathrm{d}\mu = \int h^2 E(w^2)^2 \circ T^{-1} |f_n|^2 \mathrm{d}\mu < \infty,$$

since $f_n \in L^2((J + J^2)\mathrm{d}\mu) \subseteq L^2(J^2\mathrm{d}\mu)$. Thus $W_T f_n \in \mathcal{D}^*$ for each n and

$$0 \leqslant \|W_T^* W_T f_n\|^2 - \|W_T W_T f_n\|^2 = \int (J \circ T - J) W_T f_n \mathrm{d}\mu.$$

Because f_n is supported in C for each n, the support of $W_T f_n$ is contained in A so the last integral is non-positive and the entire line reduces to 0. But then

$$0 = \int (J \circ T - J) W_T f_n \mathrm{d}\mu = \int_{T^{-1}C} (J \circ T - J) w f_n \circ T \mathrm{d}\mu.$$

Because $0 \leqslant f_n \leqslant \chi_C$ we have $0 \leqslant f_n \circ T \leqslant \chi_C \circ T$ and hence

$$\int_{T^{-1}C} (J \circ T - J) w f_n \circ T \mathrm{d}\mu \leqslant \int_{T^{-1}C \cap \sigma(w)} (J \circ T - J) w \chi_C \circ T \mathrm{d}\mu = \int_{T^{-1}C \cap \sigma(w)} (J \circ T - J) w \mathrm{d}\mu \leqslant 0.$$

Because $J \circ T - J$ is strictly negative on $T^{-1}C \cap \sigma(w)$ and w is strictly positive there, $\mu(T^{-1}C \cap \sigma(w)) = 0$. By σ-finiteness, we see that A is a nullset and condition b) holds. ■

COROLLARY 6.7. *Let C_T be a densely defined composition operator. Then*
a) *C_T is hyponormal if and only if $h > 0$ and $h \circ TE\left(\frac{1}{h}\right) \leqslant 1$.*
b) *C_T is cohyponormal if and only if* (i) $\Sigma_{\sigma(h)} \subseteq T^{-1}\Sigma$ *and* (ii) $h \leqslant h \circ T$.

EXAMPLE 6.8. This is an example of an unbounded densely defined cohyponormal composition operator, whose adjoint is an unbounded densely defined hyponormal *weighted* composition operator.

Let (X, Σ, μ) be the unit interval and Lebesgue measurable sets equipped with Lebesgue measure, and $T(x) = x^2$. Then $h = \frac{1}{2\sqrt{x}}$, $h \circ T = \frac{1}{2x}$, $T^{-1}\Sigma = \Sigma$ and $h \leqslant h \circ T$, so by Corollary 6.7, C_T is cohyponormal.

But $C_T^* = W_S$, where $S(x) = \sqrt{x}$ and $w(x) = \frac{1}{2\sqrt{x}}$. $S^{-1}\Sigma = \Sigma$, $h_S \equiv \frac{\mathrm{d}\mu \circ S^{-1}}{\mathrm{d}\mu} = 2x$, $J_S = \frac{1}{2x}$, $J_S \circ S = \frac{1}{2\sqrt{x}}$. Directly from Theorem 6.6, W_S is hyponormal. Observe that if $\mathcal{D}$ denotes the domain of C_T and $\mathcal{D}^*$ denotes the domain of W_S then $\mathcal{D}^* \underset{\neq}{\subset} \mathcal{D}$.

REFERENCES

1. CAMPBELL, J.; EMBRY-WARDROP, M; FLEMING, R.; SIVARAMAKRISHNAN, N., Normal and quasinormal weighted composition operators, to appear Glasgow Math. J.
2. CAMPBELL, J.; JAMISON, J., On some classes of weighted composition operators, *Glasgow Math. J.*, **32**(1990). 82–94.
3. CAMPBELL, J.; JAMISON, J., Corrigendum to: On some classes of weighted composition operators, *Glasgow Math. J.*, **32**(1990). 261–263.
4. CAMPBELL, J.; JAMISON, J., The analysis of composition operators on L^p and the Hopf decomposition, *J. Math. Anal. and Appl.*, **159**(1991), 520–531.
5. DIBRELL, P.; CAMPBELL, J., Hyponormal powers of composition operators, *Proc. Amer. Math. Soc.*, **102**(1988), 914–918.
6. EMBRY-WARDROP, M; LAMBERT, A., Measurable transformations and centered composition operators, *Proc. Royal Irish Acad.*, **90A**(1990), 165–172.
7. EMBRY-WARDROP, M; LAMBERT, A., Subnormality for the adjoint of a composition operator on L^2, *J. Operator Theory*, **25**(1991), 309–318.
8. FOGUEL, S. R., *Selected topics in the study of Markov operators*, Carolina Lectures Series no. 9, Dept. Math., UNC-CH, Chapel Hill, N.C. 27514 (1980).
9. HARRINGTON, D.; WHITLEY, R., Seminormal composition operators, *J. Operator Theory*, **11**(1984), 125–135.
10. JANAS, J., On unbounded hyponormal operators, *Scand. Math.*, (1989), 273–281.
11. KRENGEL, U., *Ergodic theorems*, de Gruyter Studies in mathematics 6, Walter de Gruyter, Berlin-New York (1985).

12. KUMAR, A.; SINGH, R., Characterizations of invertible, unitary, and normal composition operators, *Bull. Austral. Math. Soc.*, **19**(1978), 81–95.
13. LAMBERT, A., Hyponormal composition operators, *Bull Lond. Math. Soc.*, **18**(1986), 395–400.
14. LAMBERT, A., Subnormal composition operators, *Proc. Amer. Math. Soc.*, **103**(1988), 750–754.
15. LAMBERT, A., A bi-measurable transformation generated by a non m.p.t., *Rocky Mtn. J. of Math.*, **20**(1990), 467–474.
16. LAMBERT, A., Normal extensions of subnormal composition operators, *Mich. Math. J.*, **35**(1990), 443–450.
17. LAMBERT, A., Localizing sets for sigma algebras and related point transformations, *Proc. Royal Soc. Edinburgh*, **118A**(1991), 111-118.
18. LAMBERT, A.; WEINSTOCK, B., Banach-Lie algebras and related operator algebras induced by conditional expectations, *Mich. Math. J.*, **40**(1993), 359–376.
19. NORDGREN, E. Composition operators on Hilbert spaces, Hilbert space operators proceedings 1977, Lect. Notes in Mathematics no. 693, Springer-Verlag Berlin (1978).
20. RUDIN, W., *Functional analysis*, McGraw-Hill, New York (1973).
21. WHITLEY, R., Normal and quasinormal composition operators, *Proc Amer. Math. Soc.*, **76**(1978), 114–118.

JAMES T. CAMPBELL
Department of Mathematical Sciences,
Memphis State University,
Memphis, TN. 38152,

WILLIAM E. HORNOR
Department of Mathematics,
The University of Southern Mississippi,
P.O. Box 5045,
Hattiesburg, MS 39406-5045.

Received March 1, 1991; revised July 23, 1991.