J. OPERATOR THEORY
29(1993), 363–371

THE ACTION OF A DUAL ALGEBRA ON ITS PREDUAL

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1. INTRODUCTION

For a dual algebra $\mathcal{A}$ on a separable, complex Hilbert space, there are natural left and right actions of the algebra on its predual. These actions give rise to two algebras of operators on the predual, $\mathcal{A}_L$ and $\mathcal{A}_R$. We show that each of these algebras is the commutant of the other. We show that the lattice of invariant subspaces of $\mathcal{A}_R$ is anti-isomorphic to the lattice of weak* closed left ideals of $\mathcal{A}$. We give two sufficient conditions for the reflexivity of $\mathcal{A}_L$ and $\mathcal{A}_R$. Finally, we apply the previous results to describe the invariant subspaces of a von Neumann algebra acting on its predual and the invariant subspaces of H^∞ acting on L^1/H^1_0.

2. THE ALGEBRAS $\mathcal{A}_L$ AND $\mathcal{A}_R$

Let $\mathcal{X}$ be a complex Banach space. Then $\mathcal{X}^*$ will denote the dual space of $\mathcal{X}$, and $\mathcal{L}(\mathcal{X})$ will denote the algebra of bounded operators on $\mathcal{X}$. Let $\mathcal{H}$ be a separable, complex Hilbert space, and let $\mathcal{C}_1(\mathcal{H})$ be the ideal of trace class operators on $\mathcal{H}$. It is well known that $\mathcal{L}(\mathcal{H})$ is (isometrically isomorphic to) the dual space of $\mathcal{C}_1(\mathcal{H})$ via the pairing

$$\langle T, L\rangle = \mathrm{tr}(TL), \quad T \in \mathcal{L}(\mathcal{H}),\ L \in \mathcal{C}_1(\mathcal{H}).$$

A subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ that contains the identity operator I and is closed in the weak* topology on $\mathcal{L}(\mathcal{H})$ is called a *dual algebra*. (For an in-depth development of the theory of dual algebras, see [1].) Let ${}^\perp\mathcal{A}$ denote the set $\{K \in \mathcal{C}_1(\mathcal{H}) : \langle A, K\rangle = 0$ for all $A \in \mathcal{A}\}$. The *predual* of $\mathcal{A}$ is the quotient space $Q_\mathcal{A} = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}$. It follows from general Banach space theory that $\mathcal{A}$ is (isometrically isomorphic to) the dual space of $Q_{\mathcal{A}}$ via the pairing

$$\langle A, [K] \rangle = \mathrm{tr}(AK), \quad A \in \mathcal{A},\ [K] \in Q_{\mathcal{A}}.$$

Consequently, $Q_{\mathcal{A}}$ can be identified with the space of weak* continuous linear functionals on $\mathcal{A}$.

Fix $A \in \mathcal{A}$. Define maps $L_A : Q_{\mathcal{A}} \to Q_{\mathcal{A}}$ and $R_A : Q_{\mathcal{A}} \to Q_{\mathcal{A}}$ by

$$L_A([K]) = [AK] \text{ and } R_A([K]) = [KA], \quad [K] \in Q_{\mathcal{A}}.$$

First we establish some basic properties of L_A and R_A.

THEOREM 1. *Suppose $\mathcal{A}$ is a dual algebra, and $A \in \mathcal{A}$. Then L_A and R_A are elements of $\mathcal{L}(Q_{\mathcal{A}})$, and $\|L_A\| = \|R_A\| = \|A\|$.*

Proof. The linearity of L_A is clear. We will show that L_A is bounded. For each $[K] \in Q_{\mathcal{A}}$, there is a $T \in \mathcal{A}$ with $\|T\| \leqslant 1$ such that $\|L_A([K])\| = \|[AK]\| = \langle T, [AK] \rangle$ by the weak* compactness of the unit ball of $\mathcal{A}$. Now

$$\langle T, [AK] \rangle = \mathrm{tr}(TAK) = \langle TA, [K] \rangle \leqslant \|TA\|\,\|[K]\| \leqslant \|A\|\,\|[K]\|.$$

Thus $\|L_A([K])\| \leqslant \|A\|\,\|[K]\|$ for all $[K] \in Q_{\mathcal{A}}$. So $\|L_A\| \leqslant \|A\|$.

We must now show that $\|L_A\| \geqslant \|A\|$. We have

$$\|L_A([K])\| = \|[AK]\| \geqslant |\langle I, [AK] \rangle| = |\langle A, [K] \rangle| \text{ for all } [K] \in Q_{\mathcal{A}}.$$

Thus $\|L_A\| \geqslant \sup\{|\langle A, [K] \rangle| : [K] \in Q_{\mathcal{A}},\ \|[K]\| \leqslant 1\} = \|A\|$. The proof for R_A is similar, but it uses the fact that $\mathrm{tr}(TKA) = \mathrm{tr}(ATK)$. ■

We can now define maps $\pi_L : \mathcal{A} \to \mathcal{L}(Q_{\mathcal{A}})$ and $\pi_R : \mathcal{A} \to \mathcal{L}(Q_{\mathcal{A}})$ by

$$\pi_L(A) = L_A \text{ and } \pi_R(A) = R_A, \quad A \in \mathcal{A}.$$

COROLLARY 2. *The maps π_L and π_R are isometric algebra homomorphisms.*

Proof. It is easy to show that the maps are algebra homomorphisms, and Theorem 1 shows that they are isometries. ■

Thus $Q_{\mathcal{A}}$ is a Banach bimodule over $\mathcal{A}$. Let $\mathcal{A}_L = \pi_L(\mathcal{A})$ and $\mathcal{A}_R = \pi_R(\mathcal{A})$. We proceed to study the action of $\mathcal{A}$ on $Q_{\mathcal{A}}$ via these algebras. If $\mathcal{S}$ is a subset of $\mathcal{L}(\mathcal{X})$, then $\mathcal{S}' = \{T \in \mathcal{L}(\mathcal{X}) : ST = TS \text{ for all } S \in \mathcal{S}\}$. The *weak topology* on $\mathcal{L}(\mathcal{X})$ is the topology induced by the functions $\varphi(T) = \langle f, Tx \rangle$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. The next result establishes a connection between $\mathcal{A}_L$ and $\mathcal{A}_R$.

THEOREM 3. *Let $\mathcal{A}$ be a dual algebra. Let $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$.*

(a) $(\mathcal{A}_L)' = \mathcal{A}_R$, *and* $(\mathcal{A}_R)' = \mathcal{A}_L$.

(b) $(\mathcal{A}_L)'' = \mathcal{A}_L$, and $(\mathcal{A}_R)'' = \mathcal{A}_R$.

(c) *$\mathcal{A}_L$ and $\mathcal{A}_R$ are weakly closed.*

(d) $\pi_L(\mathcal{Z}) = \pi_R(\mathcal{Z}) = \mathcal{A}_L \cap \mathcal{A}_R$.

(e) *$\mathcal{A}$ is abelian if and only if $\mathcal{A}_L = \mathcal{A}_R$.*

Proof. First we prove $(\mathcal{A}_L)' = \mathcal{A}_R$. A simple computation shows that $\mathcal{A}_R \subset \subset (\mathcal{A}_L)'$. So suppose that $G \in (\mathcal{A}_L)'$. Then $\langle I, G([K])\rangle$, $[K] \in Q_{\mathcal{A}}$, defines a bounded linear functional on $Q_{\mathcal{A}}$. Thus there is an operator $S \in \mathcal{A}$ such that $\langle S, [K]\rangle = = \langle I, G([K])\rangle$ for all $[K] \in Q_{\mathcal{A}}$. Fix $[K] \in Q_{\mathcal{A}}$. We have

$$\langle A, R_S([K])\rangle = \langle I, L_A R_S([K])\rangle = \langle I, R_S L_A([K])\rangle = \langle S, L_A([K])\rangle =$$

$$= \langle I, G(L_A([K]))\rangle = \langle I, L_A(G([K]))\rangle = \langle A, G([K])\rangle \quad \text{for all } A \in \mathcal{A}$$

It follows that $G = R_S$, and so $(\mathcal{A}_L)' = \mathcal{A}_R$. The proof that $(\mathcal{A}_R)' = \mathcal{A}_L$ is similar.

Statements (b) and (c) immediately follow from (a). We prove (d). Suppose $Z \in \mathcal{Z}$. Fix $[K] \in Q_{\mathcal{A}}$. Then

$$\langle A, L_Z([K])\rangle = \langle AZ, [K]\rangle = \langle ZA, [K]\rangle = \langle A, R_Z([K])\rangle \quad \text{for all } A \in \mathcal{A}$$

Thus $L_Z = R_Z$. This implies that $\pi_L(\mathcal{Z}) = \pi_R(\mathcal{Z})$, and $\pi_L(\mathcal{Z}) \subset \mathcal{A}_L \cap \mathcal{A}_R$. Now suppose that $G \in \mathcal{A}_L \cap \mathcal{A}_R$. Then there are operators S and T in $\mathcal{A}$ such that $G = L_S = R_T$. So for all $A \in \mathcal{A}$ and $[K] \in Q_{\mathcal{A}}$, we have

$$\langle AS, [K]\rangle = \langle A, L_S([K])\rangle = \langle A, R_T([K])\rangle = \langle TA, [K]\rangle.$$

So $AS = TA$ for all $A \in \mathcal{A}$. In particular, $S = IS = TI = T$, and so $S \in \mathcal{A}'$. Hence $G \in \pi_L(\mathcal{Z})$, and the result follows. Finally, statement (e) is an easy consequence of (d). ∎

3. INVARIANT SUBSPACES AND IDEALS

If $\mathcal{M}$ is a subspace of $\mathcal{X}$, then $\mathcal{M}^\perp$ is the weak* closed subspace $\{f \in \mathcal{X}^* : \langle f, x\rangle = = 0 \text{ for } x \in \mathcal{M}\}$. If $\mathcal{F}$ is a subspace of $\mathcal{X}^*$, then ${}^\perp\mathcal{F}$ is the (norm closed) subspace $\{x \in \mathcal{X} : \langle f, x\rangle = 0 \text{ for } f \in \mathcal{F}\}$. Let $\mathcal{S}$ be a subset of $\mathcal{L}(\mathcal{X})$. Then $\operatorname{Lat}\mathcal{S}$ is the lattice $\{\mathcal{M} \subset \mathcal{X} : \mathcal{M} \text{ is a (norm closed) subspace, and } S\mathcal{M} \subset \mathcal{M} \text{ for all } S \in \mathcal{S}\}$. For $T \in \in \mathcal{L}(\mathcal{X})$, T^* will denote the Banach space adjoint of T. We use $\mathcal{S}^*$ to denote the set $\{S^* : S \in \mathcal{S}\}$, and $\operatorname{Lat}_W \mathcal{S}^*$ is the lattice $\{\mathcal{M} \subset \mathcal{X}^* : \mathcal{M} \text{ is a weak* closed subspace of } \mathcal{X},$

and $S^*\mathcal{M} \subset \mathcal{M}$ for all $S \in \mathcal{S}\}$. The following proposition describes the relationship between $\operatorname{Lat}\mathcal{S}$ and $\operatorname{Lat}_W \mathcal{S}^*$. We include it here for convenience.

PROPOSITION 4. *Let $\mathcal{X}$ be a Banach space with dual space $\mathcal{X}^*$. Let $\mathcal{S}$ be a subset of $\mathcal{L}(\mathcal{X})$.*

(a) *If $\mathcal{M} \in \operatorname{Lat}\mathcal{S}$, then $\mathcal{M}^\perp \in \operatorname{Lat}_W \mathcal{S}^*$.*

(b) *If $\mathcal{F} \in \operatorname{Lat}_W \mathcal{S}^*$, then ${}^\perp\mathcal{F} \in \operatorname{Lat}\mathcal{S}$.*

(c) *The map $\eta : \operatorname{Lat}_W \mathcal{S}^* \to \operatorname{Lat}\mathcal{S}$ defined by $\eta(\mathcal{F}) = {}^\perp\mathcal{F}$, $\mathcal{F} \in \operatorname{Lat}_W \mathcal{S}^*$, is a lattice anti-isomorphism with $\eta^{-1}(\mathcal{M}) = \mathcal{M}^\perp$, $\mathcal{M} \in \operatorname{Lat}\mathcal{S}$.*

Proof. First we prove (a). For all $x \in \mathcal{M}$, $f \in \mathcal{M}^\perp$, and $S \in \mathcal{S}$, we have $\langle S^* f, x\rangle = \langle f, Sx\rangle = 0$, because $Sx \in \mathcal{M}$. So $\mathcal{M}^\perp$ is invariant under S^*. A similar computation establishes (b). We now prove (c). It is well known that η interchanges spans and intersections. For $\mathcal{F} \in \operatorname{Lat}_W \mathcal{S}^*$, $({}^\perp\mathcal{F})^\perp = \mathcal{F}$. For $\mathcal{M} \in \operatorname{Lat}\mathcal{S}$, ${}^\perp(\mathcal{M}^\perp) = \mathcal{M}$. Thus the maps η and η^{-1} are indeed inverses. ■

The following application of Proposition 4 yields a description of the invariant subspaces of $\mathcal{A}_R$ in terms of the weak* closed left ideals of $\mathcal{A}$.

THEOREM 5. *Let $\mathcal{A}$ be a dual algebra. Let Λ be the lattice of weak* closed left ideals of $\mathcal{A}$*

(a) *If $\mathcal{I} \in \Lambda$, then ${}^\perp\mathcal{I} \in \operatorname{Lat}\mathcal{A}_R$.*

(b) *If $\mathcal{M} \in \operatorname{Lat}\mathcal{A}_R$, then $\mathcal{M}^\perp \in \Lambda$.*

(c) *The map $\eta : \Lambda \to \operatorname{Lat}\mathcal{A}_R$ defined by $\eta(\mathcal{I}) = {}^\perp\mathcal{I}$, $\mathcal{I} \in \Lambda$, is a lattice anti-isomorphism with $\eta^{-1}(\mathcal{M}) = \mathcal{M}^\perp$, $\mathcal{M} \in \operatorname{Lat}\mathcal{A}_R$.*

Proof. Fix $S \in \mathcal{A}$. For all $[K] \in Q_{\mathcal{A}}$ and $A \in \mathcal{A}$, we have $\langle A, R_S([K])\rangle = \langle SA, [K]\rangle$. Thus $(R_S)^*(A) = SA$ for all $A \in \mathcal{A}$. It follows that $\operatorname{Lat}_W (\mathcal{A}_R)^* = \Lambda$. The result now follows immediately from Proposition 4. ■

Of course, we could also show that the lattice of weak* closed right ideals of $\mathcal{A}$ is anti-isomorphic to $\operatorname{Lat}\mathcal{A}_L$ via the map η. We will now obtain more information about the structure of the weak* closed left ideals of $\mathcal{A}$. For $\mathcal{E} \subset Q_{\mathcal{A}}$, we define the *left annihilator ideal* of $\mathcal{E}$ by $\operatorname{Lan}\mathcal{E} = \{A \in \mathcal{A} : L_A([K]) = 0 \text{ for all } [K] \in \mathcal{E}\}$. It is easy to verify that $\operatorname{Lan}\mathcal{E}$ is a weak* closed left ideal of $\mathcal{A}$. The next result describes the left annihilator ideal of a single element. If $\mathcal{S}$ is a subset of $\mathcal{L}(\mathcal{X})$ and $x \in \mathcal{X}$, then $[\mathcal{S}x]$ denotes the closed linear span of the set $\{Sx : S \in \mathcal{S}\}$.

THEOREM 6. *Let $\mathcal{A}$ be a dual algebra. For all $[K] \in Q_{\mathcal{A}}$, $\operatorname{Lan}\{[K]\} = [\mathcal{A}_R[K]]^\perp$.*

Proof. Fix $[K] \in Q_{\mathcal{A}}$. Let $S \in \operatorname{Lan}\{[K]\}$. Then for all $A \in \mathcal{A}$, $\langle S, R_A([K])\rangle = \langle A, L_S([K])\rangle = 0$. Thus $S \in [\mathcal{A}_R[K]]^\perp$. Now suppose that $S \in [\mathcal{A}_R[K]]^\perp$. The same computation then shows that $S \in \operatorname{Lan}\{[K]\}$. ■

Let $\mathcal{S}$ be a subset of $\mathcal{L}(\mathcal{X})$. A vector $x \in \mathcal{X}$ is *separating* for $\mathcal{S}$ if $Sx = 0$ implies $S = 0$ for every $S \in \mathcal{S}$. A vector x is *cyclic* for $\mathcal{S}$ if $[\mathcal{S}x] = \mathcal{X}$. When $\mathcal{S}$ is a von Neumann algebra on a Hilbert space, a vector is separating for $\mathcal{S}$ if and only if it is cyclic for $\mathcal{S}'$. We also obtain this result for $\mathcal{A}_L$. Of course, there is a similar version for $\mathcal{A}_R$.

COROLLARY 6. *Let $\mathcal{A}$ be a dual algebra. The element $[K]$ is separating for $\mathcal{A}_L$ if and only if it is cyclic for $\mathcal{A}_R$.*

The next corollary describes the weak* closed ideals of a dual algebra in terms of the left annihilator ideals of single vectors.

COROLLARY 7. *Let $\mathcal{A}$ be a dual algebra with weak* closed left ideal $\mathcal{I}$. Then $\mathcal{I} = \bigcap_{[K]\in {}^{\perp}\mathcal{I}} \operatorname{Lan}\{[K]\}$.*

Proof. Let $\mathcal{M} = {}^{\perp}\mathcal{I}$. Now $\mathcal{M} \in \operatorname{Lat}\mathcal{A}_R$, so $\mathcal{M} = \bigvee_{[K]\in\mathcal{M}} [\mathcal{A}_R[K]]$. Thus $\mathcal{I} = \mathcal{M}^{\perp} = \bigcap_{[K]\in\mathcal{M}} [\mathcal{A}_R[K]]^{\perp} = \bigcap_{[K]\in {}^{\perp}\mathcal{I}} \operatorname{Lan}\{[K]\}$ by Theorem 6. ■

4. REFLEXIVITY

For $\mathcal{S}$ a subset of $\mathcal{L}(\mathcal{X})$, $\operatorname{Ref}\mathcal{S} = \{T \in \mathcal{L}(\mathcal{X}) : Tx \in [\mathcal{S}x] \text{ for all } x \in \mathcal{X}\}$. It is easy to see that $\operatorname{Ref}\mathcal{S}$ is a weakly closed subspace of $\mathcal{L}(\mathcal{X})$. The set $\mathcal{S}$ is *reflexive* if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$. When $\mathcal{A}$ is a unital subalgebra of $\mathcal{L}(\mathcal{X})$, $\operatorname{Ref}\mathcal{A} = \operatorname{AlgLat}\mathcal{A} = \{T \in \mathcal{L}(\mathcal{X}) : T\mathcal{M} \subset \mathcal{M} \text{ for all } \mathcal{M} \in \operatorname{Lat}\mathcal{A}\}$. We will establish two sufficient conditions for the reflexivity of $\mathcal{A}_R$ and $\mathcal{A}_L$. First we require the following result.

PROPOSITION 8. *The maps $\pi_L\colon(\mathcal{A}, \text{weak}^*) \to (\mathcal{A}_L, \text{weak})$ and $\pi_R\colon(\mathcal{A}, \text{weak}^*) \to (\mathcal{A}_R, \text{weak})$ are homeomorphisms.*

Proof. Suppose $A_i \to A$ weak * in $\mathcal{A}$. Then $\langle BA_i, [K]\rangle \to \langle BA, [K]\rangle$ for every $[K] \in Q_{\mathcal{A}}$ and $B \in \mathcal{A}$. Thus $\langle B, [A_iK]\rangle \to \langle B, [AK]\rangle$, and so π_L is continuous.

Now suppose $\pi_L(A_i) \to \pi_L(A)$ weakly in $\mathcal{A}_L$. Then $\langle I, [A_iK]\rangle \to \langle I, [AK]\rangle$ for every $[K] \in Q_{\mathcal{A}}$. Thus $\langle A_i, [K]\rangle \to \langle A, [K]\rangle$, and so π_L is a homeomorphism. The proof for π_R is similar. ■

The next theorem provides a sufficient condition for the reflexivity of $\mathcal{A}_L$ and $\mathcal{A}_R$.

THEOREM 9. *If the algebra generated by the idempotents of $\mathcal{A}$ is weak* dense in $\mathcal{A}$, then $\mathcal{A}_L$ and $\mathcal{A}_R$ are reflexive.*

Proof. Let E be an idempotent in $\mathcal{A}$. Then L_E and L_{I-E} are idempotents in $\mathcal{A}_L$ with $L_E L_{I-E} = L_{I-E} L_E = 0$. It is easy to see that $L_E Q_{\mathcal{A}}$ is an element of Lat $\mathcal{A}_R$. Suppose $G \in \mathrm{AlgLat}\,\mathcal{A}_R$. Then

$$L_E G([K]) = L_E G L_E([K]) + L_E G L_{I-E}([K]) = G L_E([K]) \quad \text{for } [K] \in Q_{\mathcal{A}},$$

because $GL_E([K]) \in L_E Q_{\mathcal{A}}$ and $GL_{I-E}([K]) \in L_{I-E} Q_{\mathcal{A}}$. Thus G commutes with L_E for every idempotent E in $\mathcal{A}$. Now the algebra generated by the set $\{L_E : E \in \in \mathcal{A},\ E^2 = E\}$ is weakly dense in $\mathcal{A}_L$ by Proposition 8. It follows that $G \in (\mathcal{A}_L)' = = \mathcal{A}_R$. The proof for $\mathcal{A}_R$ is similar. ■

An element $[M]$ in $Q_{\mathcal{A}}$ is *multiplicative* if

$$\langle AB, [M]\rangle = \langle A, [M]\rangle\langle B, [M]\rangle \quad \text{for all } A, B \in \mathcal{A}$$

We establish some useful properties of multiplicative elements in the following proposition.

PROPOSITION 10. *Let $[M]$ be a multiplicative element of $Q_{\mathcal{A}}$. Then $L_A([M]) = = R_A([M]) = \langle A, [M]\rangle[M]$ for $A \in \mathcal{A}$. If $[M]$ is nonzero, then $\langle I, [M]\rangle = 1$.*

Proof. Fix $A \in \mathcal{A}$. If $[M]$ is a multiplicative element of $Q_{\mathcal{A}}$, then $\langle B, L_A([M])\rangle = = \langle BA, [M]\rangle = \langle B, [M]\rangle\langle A, [M]\rangle = \langle AB, [M]\rangle = \langle B, R_A([M])\rangle$ for all $B \in \mathcal{A}$. It follows that $L_A([M]) = R_A([M])$. Also, $\langle B, L_A([M])\rangle = \langle B, [M]\rangle\langle A, [M]\rangle = = \langle B, \langle A, [M]\rangle[M]\rangle$ for all $B \in \mathcal{A}$. Thus $L_A([M]) = \langle A, [M]\rangle[M]$. In particular, $[M] = \langle I, [M]\rangle[M]$. So if $[M]$ is nonzero, then $\langle I, [M]\rangle = 1$. ■

Thus the multiplicative elements of $Q_{\mathcal{A}}$ are eigenvectors for the elements of $\mathcal{A}_L$ and $\mathcal{A}_R$. We can now prove a second reflexivity result.

THEOREM 11. *If $Q_{\mathcal{A}}$ is the closed linear span of its multiplicative elements, then $\mathcal{A}$ is abelian, and both $\mathcal{A}_L$ and $\mathcal{A}_R$ are reflexive.*

Proof. First we will show that $\mathcal{A}_L = \mathcal{A}_R$. Let $[M]$ be a multiplicative element of $Q_{\mathcal{A}}$. By Proposition 10, $L_A([M]) = R_A([M])$ for $A \in \mathcal{A}$. Because the multiplicative elements span $Q_{\mathcal{A}}$, we have $L_A = R_A$ for $A \in \mathcal{A}$. Thus $\mathcal{A}_L = \mathcal{A}_R$. By Theorem 3 (e), $\mathcal{A}$ is abelian.

Now we show that $\mathcal{A}_L$ is reflexive. If $Q_{\mathcal{A}}$ contains no nonzero multiplicative elements, then $Q_{\mathcal{A}} = (0)$ and the result is trivial. So we may suppose there are nonzero multiplicative elements in $Q_{\mathcal{A}}$. Let $G \in \mathrm{AlgLat}\,\mathcal{A}_L$. Then $\langle I, G([K])\rangle$ defines a continuous linear functional on $Q_{\mathcal{A}}$. So there is a $T \in \mathcal{A}$ such that $\langle T, [K]\rangle = = \langle I, G([K])\rangle$ for all $[K] \in Q_{\mathcal{A}}$. If $[M]$ is a nonzero multiplicative element of $Q_{\mathcal{A}}$, then $G([M]) = \lambda[M]$ for some $\lambda \in \mathbb{C}$ by Proposition 10. Thus $\langle A, G([M])\rangle = \lambda\langle A, [M]\rangle =$

$= \lambda\langle A,[M]\rangle\langle I,[M]\rangle = \langle A,[M]\rangle\langle I,\lambda[M]\rangle = \langle A,[M]\rangle\langle I,G([M])\rangle = \langle A,[M]\rangle\langle T,[M]\rangle =$ $= \langle AT,[M]\rangle = \langle A, L_T([M])\rangle$ for $A \in \mathcal{A}$. Consequently, $G([M]) = L_T([M])$ for every multiplicative element $[M]$. Because these elements span $Q_{\mathcal{A}}$, $G = L_T$. ■

We recall from [4] that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{X})$ is *weakly elementary* if for every weakly continuous linear functional φ on $\mathcal{A}$ there exist vectors $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $\varphi(A) = \langle f, Ax\rangle$ for all $A \in \mathcal{A}$.

THEOREM 12. *The algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ are weakly elementary.*

Proof. Let φ be a weakly continuous linear functional on $\mathcal{A}_L$. Then there exist vectors $[K_1],\dots,[K_n]$ in $Q_{\mathcal{A}}$ and operators $A_1,\dots,A_n$ in $\mathcal{A}$ such that $\varphi(G) = \sum\langle A_i, G([K_i])\rangle$ for $A \in \mathcal{A}$. Thus

$$\varphi(L_A) = \sum\langle A_i,[AK_i]\rangle = \sum\langle I,[AK_iA_i]\rangle = \left\langle I, L_A\left(\sum[K_iA_i]\right)\right\rangle.$$

The proof for $\mathcal{A}_R$ is similar. ■

COROLLARY 13. *If $\mathcal{A}_L$ (respectively, $\mathcal{A}_R$) is reflexive, then every weakly closed subspace of $\mathcal{A}_L$ (respectively, $\mathcal{A}_R$) is reflexive.*

Proof. Let $\mathcal{S}$ be a weakly closed subspace of $\mathcal{A}_L$. Because $\mathcal{A}_L$ is weakly elementary, $\operatorname{Ref}\mathcal{S} \cap \mathcal{A}_L = \mathcal{S}$ by [4, Theorem 4]. Clearly $\operatorname{Ref}\mathcal{S} \subset \operatorname{Ref}\mathcal{A}_L$, and $\operatorname{Ref}\mathcal{A}_L = \mathcal{A}_L$, so $\mathcal{S} = \operatorname{Ref}\mathcal{S}\cap\mathcal{A}_L = \operatorname{Ref}\mathcal{S}$. ■

5. APPLICATIONS

We will now apply the previous results to the action of a von Neumann algebra on its predual. In particular, we can characterize the invariant subspaces of this action. The reader should consult [2] for the theory of von Neumann algebras.

THEOREM 14. *Let $\mathcal{A}$ be a von Neumann algebra. Then $\operatorname{Lat}\mathcal{A}_R = \{L_EQ_{\mathcal{A}} : E \in \mathcal{A}$ and $E = E^* = E^2\}$, and every weakly closed subspace of $\mathcal{A}_R$ is reflexive.*

Proof. Let E be a projection in $\mathcal{A}$. Then L_E is an idempotent in $\mathcal{L}(Q_{\mathcal{A}})$, so $L_EQ_{\mathcal{A}}$ is a closed subspace. Clearly $L_EQ_{\mathcal{A}}$ is invariant for $\mathcal{A}_R$. We now show every invariant subspace is of this form. Let $\mathcal{M} \in \operatorname{Lat}\mathcal{A}_R$. By Theorem 5, $\mathcal{M} = {}^\perp\mathcal{I}$ for some weak* closed left ideal $\mathcal{I}$. By [2, Part I, Chapter 3, Corollary 3] there is a projection E in $\mathcal{A}$ such that $\mathcal{I} = \{AE : A \in \mathcal{A}\}$. Thus

$$\mathcal{M} = {}^\perp\mathcal{I} = \{[K] : \langle AE,[K]\rangle = 0,\ A\in\mathcal{A}\} = \{[K] : \langle A,[EK]\rangle = 0,\ A \in \mathcal{A}\} =$$

$$= \{[K] : L_E([K]) = 0\} = \{[K] : L_{I-E}([K]) = [K]\} = L_{I-E}Q_{\mathcal{A}}.$$

Because the algebra generated by the projections in $\mathcal{A}$ is weak* dense in $\mathcal{A}$, $\mathcal{A}$ and all its weakly closed subspaces are reflexive by Theorem 9 and Corollary 13. ■

We now apply the previous results to the algebra H^∞. The reader should consult [3] for the theory of H^∞. It is known that H^∞ is a dual algebra on H^2 with predual isometrically isomorphic to L^1/H_0^1 (cf. [1, Chapter IV]). Se we can use the previous results to characterize the invariant subspaces of H^∞ acting on L^1/H_0^1. Let $\mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ and let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. For $h \in H^\infty$, let $\hat{h}$ denote the analytic extension of h to $\mathbf{D}$. For $f \in L^1$, $[f]$ will denote the equivalence class of f in L^1/H_0^1, and $[\overline{\phi} H_0^1] = \{[f] \in L^1/H_0^1 : f = \overline{\phi} g \text{ with } g \in H_0^1\}$.

THEOREM 15. *Let* $\mathcal{A} = H^\infty$. *Then*

(a) $\mathcal{A}_L = \mathcal{A}_R$.

(b) $\operatorname{Lat} \mathcal{A}_R = \{[\overline{\phi} H_0^1] : \phi \text{ is inner}\}$.

(c) *Every weakly closed subspace of* $\mathcal{A}_R$ *is reflexive.*

Proof. Because H^∞ is abelian, part (a) follows from Theorem 3. Let $\mathcal{M} \in \operatorname{Lat} H^\infty$. Then $\mathcal{M} = {}^\perp\mathcal{I}$ for some weak* closed ideal $\mathcal{I}$ in H^∞. Now $\mathcal{I} = \phi H^\infty$ for some inner function ϕ by [3, Chapter II, Theorem 7.5]. Thus

$$\begin{aligned}\mathcal{M} = {}^\perp(\phi H^\infty) &= \{[f] \in L^1/H_0^1 : \langle \phi h, [f]\rangle = 0 \text{ for all } h \in H^\infty\} = \\ &= \{[f] \in L^1/H_0^1 : \langle h, [\phi f]\rangle = 0 \text{ for all } h \in H^\infty\} = \\ &= \{[f] \in L^1/H_0^1 : [\phi f] = 0\} = \{[f] \in L^1/H_0^1 : \phi f \in H_0^1\} = \\ &= \{[f] \in L^1/H_0^1 : f = \overline{\phi} g \text{ with } g \in H_0^1\} = [\overline{\phi} H_0^1]\end{aligned}$$

It is easy to see that every set of this form is in $\operatorname{Lat} \mathcal{A}_R$. This establishes part (b).

Finally, we show part (c). For each $\lambda \in \mathbf{D}$, let

$$P_\lambda(z) = (1 - |\lambda|^2)|1 - \overline{\lambda} z|^{-2}, \quad z \in \mathbf{T}.$$

Then $\langle h, [P_\lambda]\rangle = \hat{h}(\lambda)$, $h \in H^\infty$. Thus $[P_\lambda]$ is a multiplicative element for each $\lambda \in \mathbf{D}$. If $0 = \langle h, [P_\lambda]\rangle = \hat{h}(\lambda)$ for each $\lambda \in \mathbf{D}$, then $h = 0$. Thus the multiplicative elements span L^1/H_0^1, and so H^∞ is reflexive by Theorem 11. Statement (c) now follows from Corollary 13. ■

We conclude with a question and some remarks. Does every dual algebra have a non-trivial weak* closed left ideal? In view of Theorem 5, this is equivalent to solving the invariant subspace problem for $\mathcal{A}_R$. Thus a negative solution would provide a new transitive algebra on a Banach space. A singly generated dual algebra with only trivial weak* closed ideals would provide a new example of a transitive operator on a Banach space. An affirmative answer to the question would lead to invariant subspaces for certain dual algebras via [4, Proposition 5].

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Received July 17, 1991.