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RELATIVE ENTROPY FOR FINITE VON NEUMANN ALGEBRAS

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1. INTRODUCTION

Nakamura and Umegaki [11] extended the notion of the entropy formulated by J. von Neumann [12]. Umegaki [18] introduced the relative entropy as a noncommutative version of the Kullback-Leibler entropy, which is given by the trace of $a \log a - a \log b$, i.e. $s_U(a|b) = \tau(a \log a - a \log b)$ for positive operators a, b affiliated with a semifinite von Neumann algebra.

J. I. Fujii and E. Kamei [4] introduced the relative operator entropy $s(a|b)$ for positive operators a, b as a relative version of the Nakamura-Umegaki operator entropy. In the case where a, b are commutative, this relative operator entropy coincides with the Umegaki relative entropy, but in general they do not coincide. On the other hand Belavkin and Staszewski have defined in [1] a relative entropy s_{BS} in C^*-algebra setting. Hiai and Petz [9] pointed out that $s_{BS}(a, b) = \mathrm{Tr}(s(a, b))$ for density matrices a, b where Tr denotes the usual trace matrices.

In noncomutative probability theory, F. Hiai studied the relative entropy of normal positive functionals in general von Neunann algebras and showed some remarkable results in [8] [9]. Also, the relation between the Umegaki relative entropy s_U and Belavkin and Staszewski relative entropy s_{BS} was investigated.

Now, let M be a finite von Neumann algebra with the faithful normal trace τ, $\tau(1) = 1$, N a von Neumann subalgebra of M and E_N denotes the unique τ--preserving conditional expectation of M onto N. Let X be the set of all families $x_1, \ldots, x_n$ of positive elements in M with $\sum_{i=1}^{n} x_i = 1$ and this be called a partition of the unity in M. Then Connes and Størmer [2] introduced the relative entropy

$H(M|N)$ of M relative to N as follows:

$$H(M|N) = \sup_{(x_i)\in X} \sum_i \tau\eta E_N(x_i) - \tau\eta(x_i),$$

where η is defined by $\eta(t) = -t\log t$ and $\eta(0) = 0$. The entropy $H(M|N)$ was used to study the entropy of Kolmogorov-Sinai type for automorphisms of finite von Neumann algebras. Lately, Pimsner and Popa [13] exactly calculated the entropy $H(M|N)$ of a factor of type II_1 M relative to its subfactor N in terms of Jones' index. The general relation between $H(M|N)$ and Jones' index $[M:N]$ is given by $H(M|N) \leqslant \log[M:N]$, and several characterization for the equality $H(M|N) = \log[M:N]$ were established in [13]. The complete computation of the entropy in the finite dimensional case is also contained in [13].

In the previous paper [16], we introduced an entropy $S(M|N)$ of a finite von Neumann algebra M relative to its subalgebra N as a noncommutative version of the Umegaki relative entropy which is not identical with the Connes Størmer relative entropy $H(M|N)$ [2] and showed a version of the Pimsner-Popa Theorem on the entropy and Jones'index for factors of type II_1. Also we showed that if $M \supset N$ are finite dimensional von Neumann algebras, then

$$(*) \qquad S(M|N) \geqslant H(M|N).$$

In this paper, we shall show that if $M \supset N$ are factors of type II_1, then the relative entropy coincides with Jones' index in the sense that $S(M|N) = \log[M : N]$. In Section 4 we shall exactly estimate the entropy $S(M|N)$ of a finite dimensional von Neumann algebra M relative to subalgebra N in terms of the Pimsner and Popa's generalized index.

2. PROPERTIES OF ENTROPY

Following [3] the relative operator entropy $s(a|b)$ for positive operators a and b is given by

$$s(a|b) = -\lim_{\varepsilon\to 0} a^{1/2}(\log a^{1/2}(b+\varepsilon)^{-1}a^{1/2})a^{1/2}.$$

if the strong limit exists.

The relative operator entropy for noninvertible positive operators does not always exist, but if there exists $\lambda \geqslant 0$ such that $b \geqslant \lambda a$, then $s(a|b)$ exists and

$$(\log\lambda)a \leqslant s(a|b) \leqslant -a\log a + (\log\|b\|)a.$$

First we summarize the basic properties concerning the relative operator entropy and use them frequently. If $s(a|b)$ exists, then

(2–1) monotonity: $b \leqslant c$ imply $s(a|b) \leqslant s(a|c)$.

(2–2) transformer inequality: $x^*s(a|b)x \leqslant s(x^*ax|x^*bx)$ for all x.

(2–3) subadditive: $s(a|c) + s(b|d) \leqslant s(a+b|c+d)$.

(2–4) upper semicontinuity: $b_n \downarrow b$ implies $s(a|b_n) \downarrow s(a|b)$.

where a, b, c and d are positive operators. Note that in (2–2) $x^*s(a|b)x = s(x^*ax|x^*bx)$ for invertible operators x. For some general results on the relative operator entropy, see [3], [4].

Let M be a finite von Neumann algebra with the faithful normal trace τ, $\tau(1) = 1$, N a von Neumann subalgebra of M and E_N denotes the unique τ-preserving conditional expectation of M onto N. Let X be the set of all families $x_1, \dots, x_n$ of positive elements in M with $\sum_{i=1}^{n} x_i = 1$ and this be called a partition of the unity in M. Then the relative entropy $S(M|N)$ is defined by

$$S(M|N) = \sup_{(x_i)\in X} \sum_i -\tau(s(x_i|E_N(x_i))).$$

Also, Pimsner and Popa introduced the generalized index $\lambda(M, N)$ as a more analytical characterization of Jones'index $[M : N]$ as follows:

$$\lambda(M,N) = \max\{\lambda \geqslant 0 : E_N(x) \geqslant \lambda x \text{ for all } x \in M_+\}.$$

In [16], we showed the relation between the relative entropy $S(M|N)$ and the generalized index $\lambda(M, N)$, namely $S \leqslant -\log \lambda$.

We collect the basic properties of the generalized index.

PROPOSITION 1. *If $M \supset N \supset L$ are finite von Neumann algebras, then*

(1) $\lambda(M, M) = 1$

(2) $0 \leqslant \lambda(M, N) \leqslant 1$

(3) $\lambda(M, L) \geqslant \lambda(M, N)\lambda(N, L)$

(4) $\lambda(M, L) = 1$ *implies* $M = N$

(5) $\lambda(M_1 \otimes M_2, N_1 \otimes N_2) \geqslant \lambda(M_1, N_1)\lambda(M_2, N_2)$

(6) $\lambda(M_1 \oplus M_2, N_1 \oplus N_2) = \min\{\lambda(M_1, N_1),\ \lambda(M_2, N_2)\}$

Proof. (1), (2), (5) and (6) are clear by the definition. In (3), let $E_L : M \to L$, $E_N : M \to N$ and $E_L^N : N \to L$ be the expectation. Then we get

$$E_L(x) = E_L^N E_N(x) \geqslant \lambda(N, L)E_N(x) \geqslant \lambda(N, L)\lambda(M, N)x.$$

Therefore we get (3).

If $\lambda(M,N)=1$, then $E_N(p)\geqslant p$ for all projections $p\in M$. Since $E_N(1-p)\geqslant 1-p$, we have $p\geqslant E_N(p)$, so that $E_N(p)=p$. Hence $M=N$. ■

We present general properties of entropy $S(M|N)$.

Proposition 2. *Let $M=\sum_k^{\oplus} M_k$ and $N=\sum_k^{\oplus} N_k$ for countable families $\{M_k\}$ and $\{N_k\}$ of finite von Neumann algebras with $M_k\supseteq N_k$. Then*

$$S(M|N)=\sum_k S(M_k|N_k).$$

Proof. By the definition of the relative operator entropy, we have $s(a|b)=\sum_k^{\oplus} s(a_k|b_k)$ with $a=\sum_k^{\oplus} a_k$, $b=\sum_k^{\oplus} b_k$. For $x_i=\sum_k^{\oplus} x_{ik}$ with $x_{ik}\in M_k$, $\sum_i x_i=1$, we have

$$\sum_i -\tau_M(s(x_i|E_N(x_i)))=\sum_i\sum_k -\tau_{M_k}(s(x_{ik}|E_N(x_{ik})))\leqslant \sum_k S(M_k|N_k).$$

Hence $S(M|N)\leqslant \sum_k S(M_k|N_k)$. The reverse inequality is similarly shown. ■

We examine how the entropy behaves under tensor products.

Lemma 3.

$$s(a\otimes c|b\otimes d)=s(a|b)\otimes c+a\otimes s(c|d)$$

Proof. By the upper continuity of the relative operator entropy, we can assume that the operators a,b,c and d are invertible. Since

$$\log(a\otimes b)=\log(a\otimes 1)(1\otimes b)=\log(a\otimes 1)+\log(1\otimes b)=(\log a)\otimes 1+1\otimes(\log b),$$

we have

$$\begin{aligned}
&s(a\otimes c|b\otimes d)=\\
&=(a^{1/2}\otimes c^{1/2})(\log(a^{1/2}\otimes c^{1/2})(b^{-1}\otimes d^{-1})(a^{1/2}\otimes c^{1/2}))(a^{1/2}\otimes c^{1/2})=\\
&=(a^{1/2}\otimes c^{1/2})(\log a^{1/2}b^{-1}a^{1/2}\otimes c^{1/2}d^{-1}c^{1/2})(a^{1/2}\otimes c^{1/2})=\\
&=(a^{1/2}\otimes c^{1/2})(\log a^{1/2}b^{-1}a^{1/2}\otimes 1+1\otimes\log c^{1/2}d^{-1}c^{1/2})(a^{1/2}\otimes c^{1/2})=\\
&=s(a|b)\otimes c+a\otimes s(c|d).
\end{aligned}$$

■

PROPOSITION 4. *Let $M \supseteq N$ and $P \supseteq Q$ be two pairs of finite von Neumann algebras. Then*

$$S(M \otimes P|N \otimes Q) \geqslant S(M|N) + S(P|Q).$$

Proof. Let $a_i \in M_+$ with $\sum_i a_i = 1$ and $b_j \in P_+$ with $\sum_j b_j = 1$. Then $\sum_{ij} a_i \otimes b_j = 1 \otimes 1$ and we have

$$S(M \otimes P|N \otimes Q) \geqslant -\sum_{ij} \tau_{M\otimes P}(s(a_i \otimes b_j | E_N(a_i) \otimes E_Q(b_j))) =$$
$$= -\sum_{ij} \tau_M(s(a_i|E_N(a_i))\tau_P(b_j) - \sum_{ij} \tau_M(a_i)\tau_P(s(b_j|E_Q(b_j))) =$$
$$= -\sum_i \tau_M(s(a_i|E_N(a_i)) - \sum_j \tau_P(s(b_j|E_Q(b_j))).$$

Hence $S(M \otimes P|N \otimes Q) \leqslant S(M|N) + S(P|Q)$. ∎

REMARK. The above equality does not hold in the finite dimensional case. For instance, let $M_n = M_n(\mathbb{C})$ be the $n \times n$ matrix algebra with the normalized trace τ_n. As for the conditional expectation $\tau_m \otimes \mathrm{id}_{M_n} : M_m \otimes M_n \to \mathbb{C} \otimes M_n$ with respect to $\tau_m \otimes \tau_n$, we have by [15 Corollary 10]

$$S(M_m \otimes M_n|\mathbb{C} \otimes M_n) = \begin{cases} \log m^2, & m \leqslant n, \\ \log mn, & m > n, \end{cases}$$

which is not equal to $S(M_m|\mathbb{C}) = \log m$ whenever $m, n > 1$.

LEMMA 5. *Let $B \subset M$ be arbitrary finite von Neumann algebras and $\{f_n\}_{n\geqslant 1}$ a sequence of projections in $B' \cap M$ with $\|f_n - 1\|_2 \to 0$. Let $M_{f_n} = M_n, B_{f_n} = B_n$ and $\lambda(M_n, B_n) \neq 0$ for all n. Then*

$$S(M|B) \geqslant \limsup S(M_n|B_n).$$

Proof. Let $\lambda = \limsup_n S(M_n|B_n)$ and $\varepsilon > 0$. Suppose that $\lambda < \infty$. There exists a sufficently many n such that

$$-\sum_i \tau_{f_n}(s(x_i|E_{B_n}(x_i))) \geqslant \lambda - \varepsilon,$$

where x_i is a partition of $f_n, x_i = f_n y_i f_n$ for some $y_i \in M$.

Since $E_{B_n}(x_i) = \frac{1}{\tau(f_n)} f_n E_B(y_i) f_n$ and the relative operator entropy satisfies the transformer inequality [3], we have

$$-\sum_i \tau_{f_n}(s(x_i|E_{B_n}(x_i))) = -\sum_i \frac{1}{\tau(f_n)}\tau\left(s\left(f_n y_i f_n \Big| \frac{1}{\tau(f_n)} f_n E_B(y_i) f_n\right)\right) \leqslant$$

$$\leqslant -\sum_i \frac{1}{\tau(f_n)}\tau\left(f_n s\left(y_i \Big| \frac{1}{\tau(f_n)}E_B(y_i)\right)\right) \to -\sum_i \tau(s(y_i|E_B(y_i))).$$

Hence $\lambda - \varepsilon \leqslant S(M|B)$ and as $\varepsilon > 0$ is arbitrary we get

$$S(M|B) \geqslant \limsup_n S(M_{f_n}|B_{f_n}).$$

Also, if $\lambda = \infty$, then for any $\varepsilon > 0$ there exists a sufficently many n such that

$$-\sum_i \tau_{f_n}(s(x_i|E_{B_n}(x_i))) \geqslant \varepsilon.$$

By above discussion we get the same consequence. ■

The next lemma is similar and has the same proof as Lemma 4.2 in [1]. We reproduce the proof though, for the reader's convenience.

LEMMA 6. *Let $N \subset M$ be finite von Neumann algebras with a nonzero generalized index λ. If $q \in M$ is a projection such that $E_{N'\cap M}(q) = cf$ for some scalar c and some projection $f \in N' \cap M$, then*

$$S(M|N) \geqslant -\frac{1}{c}\tau(s(q|E_N(q))).$$

Proof. For any $\varepsilon > 0$ there exist unitary element $v_1, \ldots, v_n$ in N such that $\|y - f\|_2 < \varepsilon\|f\|_2$ with $y = \dfrac{1}{cn}\sum v_i q v_i^*$. Let $\delta > 0$ and denote by p the spectral projection of y corresponding to $[0, 1+\delta]$ in the algebra fMf. Put

$$x_i = \frac{1}{(1+\delta)cn} p \wedge v_i q v_i^*, z_i = v_i q v_i^* - p \wedge v_i q v_i^*.$$

Since $\sum x_i \leqslant f, \tau(z_i) \leqslant (\varepsilon\delta^{-1})^2\tau(f)$, and $S(v_i q v_i^*|E_N(v_i q v_i^*)) = v_i s(q|E_N(q))v_i^*$, we have

$$-\sum \tau(s(x_i|E_N(x_i))) \geqslant$$
$$\geqslant -\sum \frac{1}{(1+\delta)cn}\tau(s(v_i q v_i^*|E_N(v_i q v_i^*))) + \frac{1}{(1+\delta)cn}\sum \tau(s(z_i|E_N(z_i))) \geqslant$$
$$\geqslant -\frac{1}{(1+\delta)c}\tau(s(q|E_N(q))) + \frac{1}{(1+\delta)c}(\log\lambda)(\varepsilon\delta^{-1})^2\tau(f).$$

Letting first $\varepsilon \to 0$ and then $\delta \to 0$, then we have $S(M|N) \geqslant -\dfrac{1}{c}\tau(s(q|E_N(q)))$. ■

PROPOSITION 7. *Let M be a finite factor. If $\{e_k\}$ is a set of projections in M with $\sum e_k = 1$, then*

$$S\left(M\Big|\sum_k^{\oplus} M_{e_k}\right) = \log\lambda\left(M, \sum_k^{\oplus} M_{e_k}\right)^{-1}.$$

Proof. Put $N = \sum_k^{\oplus} M_{e_k}$. Suppose that $\{e_k\}$ is a finite set $\{e_1, \ldots, e_m\}$. Let $0 \neq q_i \leqslant e_i$ be some mutually equivalent projections and choose $\{v_{ij}\}_{1 \leqslant i,j \leqslant m}$ a set of matrix units such that $v_{ii} = q_i$. If $q = \sum_{ij} (\tau(e_i)\tau(e_j))^{1/2} v_{ij}$ then q is a projection of the same trace as q_i.

$$
\begin{aligned}
q(E_N(q)+\varepsilon)^{-1}q &= q\left(\sum_{k=1}^{m} (\tau(e_k)+\varepsilon)^{-1} v_{kk} + \varepsilon^{-1}\left(\sum_{k=1}^{m} v_{kk}\right)^{\perp}\right) q = \\
&= \sum_{it}\sum_{k} (\tau(e_i)\tau(e_t))^{1/2}\tau(e_k)(\tau(e_k)+\varepsilon)^{-1} v_{it} = \\
&= \left(\sum_{k=1}^{m} \tau(e_k)(\tau(e_k)+\varepsilon)^{-1}\right) q.
\end{aligned}
$$

Hence $\lim_{\varepsilon\to 0} q(E_N(q)+\varepsilon)^{-1}q = mq$. Therefore we have

$$
s(q|E_N(q)) = s - \lim_{\varepsilon\to 0} -q(\log q(E_N(q)+\varepsilon)^{-1}q)q = -q(\log mq)q = -q\log m.
$$

Since $E_{N'\cap M}(q) = \sum_k \frac{\tau(e_k q)}{\tau(e_k)} e_k = \sum_k \tau(q_k)e_k = \tau(q)1$, it follows from Lemma 6 that

$$
S(M|N) \geqslant -\frac{1}{\tau(q)}\tau(s(q|E_N(q)) = \frac{1}{\tau(q)}\tau(q)\log m = \log m.
$$

On the other hand, for $x = (e_i x e_j) \in M_+$ and $\xi = (\xi_i)$, we have

$$
((mE_N(x)-x)\xi,\xi) = \sum_{i=1}^{m}((m-1)(e_i x e_i \xi_i, \xi_i) - \sum_{j\neq i}^{m}(e_i x e_j \xi_j, \xi_i)) =
$$

$$
= \sum_{i=1}^{m}((m-1)(x\xi_i,\xi_i) - \sum_{j\neq i}^{m}(x\xi_i,\xi_i)) = \sum_{i<j}^{m} \left\|x^{1/2}(\xi_i - \xi_j)\right\|^2 \geqslant 0.
$$

Hence $E_N(x) \geqslant \frac{1}{m}x$ for all $x \in M_+$. Therefore $\lambda(M,N) \geqslant \frac{1}{m}$. Hence

$$
\log\lambda(M,N)^{-1} \leqslant \log m \leqslant S(M|N) \leqslant \log\lambda(M,N)^{-1},
$$

so that $S(M|N) = \log\lambda(M,N)^{-1}$.

Next suppose that $\{e_k\}$ is a infinite set. Put $f_n = \sum_{i=1}^{n} e_i$. Then $f_n \in N' \cap M$ with $\|f_n - 1\|_2 \to 0$. Since $\lambda(M_n, N_n) \geqslant \frac{1}{n}$, it follows from Lemma 5 that

$$
S(M|N) \geqslant \limsup S(M_n|N_n) =
$$

$$= \limsup \log \lambda(M_n, N_n)^{-1} = \limsup \log n.$$

Hence we have $S(M|N) = \log \lambda(M, N)^{-1} = \infty$. ■

3. COMPUTATION OF $S(M|N)$ FOR FACTORS OF TYPE II_1

In this section we shall obtain a formula for the relative entropy $S(M|N)$ in the case M and N are factors of type II_1.

THEOREM 8. *Let M be a factor of type II_1 and N a subfactor of M. Then*

$$S(M|N) = \log[M : N].$$

Proof. The proof in the following is a slight modification of that of [13, Theorem 4.4]. Suppose that $[M : N] < \infty$, it follows from Jones'result [10: Corollary 2.2.3] that $N' \cap M$ is finite dimensional. Let $\{f_k\}_{0 \leqslant k \leqslant m}$ be the atoms in $N' \cap M$ such that $\sum_{k=0}^{m} f_k = 1$. Let $M_{f_k} = M_k$, $N_{f_k} = N_k$, $\tau_{f_k} = \tau_k$ and E_{N_k} the τ_k-preserving conditional expectation of M_k onto N_k. Fix the projection $f_0 \in \{f_k\}_k$ and denote by $\theta_j : N_0 \to N_j$ the isomorphism making commutative the diagram.

$$\begin{array}{ccc} N & & \\ \downarrow & \searrow & \\ N_0 & \xrightarrow{\theta_j} & N_j \end{array}$$

Note that if $x_j \in N_j$ then $E_N(x_j) = \tau(f_j) \sum_i \theta_i \theta_j^{-1}(x_j)$. Let r_j be mutually orthogonal projection in N_0 with $\sum r_j = 1_{N_0} = f_0$ and $\tau(r_j) = \dfrac{\tau(f_0)[M_j : N_j]}{\tau(f_j)[M : N]}$. Denote by $p_j = \theta_j(r_j) \in N_j$ and $s_j = \sum_i \theta_i(r_j)$. Then $E_N(p_j) = \tau(f_j)s_j$ and $\sum s_j = 1_N$. There exist projections $e_j \in M_j$ such that $E_{N_j}(e_j) = [M_j : N_j]^{-1} 1_N$, and $[e_j, p_j] = 0, \tau_j(e_j p_j) = \tau_j(e_j)\tau_j(p_j)$.

Put $q_j = p_j e_j$ and it follows that $E_{N_j}(q_j) = [M_j : N_j]^{-1} p_j$ and $\tau(q_j) = [M : N]^{-1}$. Then we get that q_j are mutually equivalent projections in M so that there exists a set of matrix units in M, $\{v_{ij}\}_{0 \leqslant i,j \leqslant m}$ having q_j as diagonal $q_j = v_{jj}$. Put $q = \sum_{ij} (\tau(f_i)\tau(f_j))^{1/2} v_{ij}$, then $\tau(q) = [M : N]^{-1}$, $E_N(q) = \sum_k \dfrac{\tau(f_k)^2}{[M_k : N_k]} s_k$ and $E_{N' \cap M}(q) = \tau(q) 1_M$.

Since a partial isometry v_{ij} : ran $q_j \to$ ran q_i satisfies $v_{ij}v_{ij}^* = q_i, v_{ij}^*v_{ij} = q_j, v_{ij}q_j = v_{ij}$ and $q_jv_{ij} = v_{ij}$, we have $v_{ij}s_k = 0$ for $j \neq k$ and $v_{ij}s_j = v_{ij}\sum_l \theta_l(r_j) = v_{ij}\theta_j(r_j) = v_{ij}p_j$. Also since

$$v_{ij} = v_{ik}v_{kj} = v_{ik}q_kv_{kj} = v_{ik}p_ke_kv_{kj} \leqslant v_{ik}p_kv_{kj} \leqslant v_{ij},$$

then we have $v_{ij} = v_{ik}p_kv_{kj}$.

In [10] Jones showed the relation between Jones'index and the local index, namely if $[M : N] < \infty$ and $f_k \in N' \cap M$ are projections with $\sum f_k = 1$ then

$$[M : N] = \sum \frac{[M_k : N_k]}{\tau(f_k)}.$$

Since $E_N(q)^{-1} = \sum_k \frac{[M_k : N_k]}{\tau(f_k)^2}s_k$, then we have the following:

$$qE_N(q)^{-1}q = \sum_{i,j}\sum_k\sum_{st}(\tau(f_i)\tau(f_j))^{1/2}\frac{[M_k : N_k]}{\tau(f_k)^2}(\tau(f_s)\tau(f_t))^{1/2}v_{ij}s_kv_{st} =$$

$$= \sum_{i,t}\sum_k(\tau(f_i)\tau(f_k))^{1/2}\frac{[M_k : N_k]}{\tau(f_k)^2}(\tau(f_k)\tau(f_t))^{1/2}v_{ik}p_kv_{kt} =$$

$$= \sum_{i,t}\sum_k\frac{[M_k : N_k]}{\tau(f_k)}(\tau(f_i)\tau(f_t))^{1/2}v_{i,t} =$$

$$= [M : N]\sum_{i,t}(\tau(f_i)\tau(f_t))^{1/2}v_{i,t} = [M : N]q.$$

Hence $-\tau(s(q|E_N(q))) = \tau(q\log[M : N]q) = \tau(q)\log[M : N]$. Since $q \in M$ is a projection such that $E_{N'\cap M}(q) = \tau(q)1_M$, it follows from lemma 6 that we have

$$S(M|N) \geqslant -\frac{1}{\tau(q)}\tau(s(q|E_N(q))) = \log[M : N]$$

Therefore we have $S(M|N) = \log[M : N]$. Next we consider the case when $[M : N] = \infty$. If $N' \cap M$ has a completely nonatomic part, then by [16: Theorem 6] we have $S(M|N) = \infty$, so that $S(M|N) = \log[M : N]$.

Next assume that $N' \cap M$ is atomic and $\{f_k\}_k$ are atoms in $N' \cap M$ such that $\sum f_k = 1$.

If $\{f_k\}_k = \{f_0, \ldots, f_m\}$ is a finite set, then there exists some projection $f \in \{f_k\}_k$ such that $[M_f : N_f] = \infty$. . Then for any $\varepsilon > 0$ there exists a projection $e \in M_f$ such that $E_{N_f}(e) \leqslant \varepsilon$. Thus $E_N(e) = E_{N_f}(e) \leqslant \varepsilon$ and $E_{N'\cap M}(e) = \frac{\tau(fef)}{\tau(f)}f = \tau(e)\tau(f)^{-1}f$. Then we have

$$S(M|N) \geqslant -\frac{\tau(f)}{\tau(e)}\tau(e)\log\|E_N(e)\| = -\tau(f)\log\|E_N(e)\| \geqslant \tau(f)\log\varepsilon^{-1} \to \infty.$$

Next let $\{f_k\}_k$ is an infinite set with $[M_k : N_k] < \infty$ for all k. Put $g_n = \sum_{k=0}^{n} f_k$. Since $(N_{g_n})' \cap M_{g_n} = (N' \cap M)_{g_n}$ is finite dimensional by [15: Lemma 2.1] and $[M_{g_n} : N_{g_n}] < \infty$, it follows from the above discussion that $S(M_{g_n}|N_{g_n}) = \log[M_{g_n} : N_{g_n}]$. Since

$$0 = [M : N]^{-1} \geqslant \limsup [M_{g_n} : N_{g_n}]^{-1} \geqslant \liminf [M_{g_n} : N_{g_n}]^{-1} \geqslant 0,$$

we have the follows:

$$\begin{aligned}\log[M : N] \geqslant S(M|N) \geqslant \limsup S(M_{g_n}|N_{g_n}) = \limsup \log[M_{g_n} : N_{g_n}] = \\ = \limsup - \log [M_{g_n} : N_{g_n}]^{-1} = - \liminf \log [M_{g_n} : N_{g_n}]^{-1} = \\ = - \log \liminf [M_{g_n} : N_{g_n}]^{-1} = \infty.\end{aligned}$$

■

By combining Theorem 8 and $H(M|N) \leqslant \log[M : N]$, we shall denote the relation between the Connes-Størmer relative entropy $H(M|N)$ and the relative entropy $S(M|N)$ as follows:

Corollary 9. *If $M \supset N$ are factors of type II_1, then*

$$H(M|N) \leqslant S(M|N).$$

4. COMPUTATION OF S FOR FINITE DIMENSIONAL ALGEBRAS

In this section we shall calculate the relative entropy $S(M|N)$ of a finite dimensional von Neumann algebra M relative to subalgebra N in terms of the Pimsner and Popa's generalized index.

Let M will be a finite dimensional von Neumann algebra with faithful trace τ, $\tau(1) = 1$, and $N \subset M$ a von Neumann subalgebra. Thus $M = \bigoplus_{l \in L} M_l$, $N = \bigoplus_{k \in K} N_k$ where M_l is the algebra of $m_l \times m_l$ matrices, N_k the algebra of $n_k \times n_k$ matrices and the set of indices L and K are finite. We denote by $A = (a_{kl})_{k \in K, l \in L}$ the embedding matrix of N in M and by t_l respectively s_k the traces of the minimal projections in M_l respectively N_k. Thus if $m = (m_l)$, $n = (n_k)$, $t = (t_l)$, $s = (s_k)$ are column vectors, then $At = s$, $A^t n = m$ (A^t is the transpose of A). We denote by e^k and f^l the minimal central projections in N and respectively M (e^k is the support of N_k in N and f^l the support of M_l in M).

From now on the inclusion $N \subset M$ will be described in the following way: For each $k \in K$, $l \in L$ we fix a finite set $A_{k,l}$ of cardinal a_{kl} and identify $[1, m_l]$ with $\bigcup_k (a_{kl} \times [1, n_k]$, where the intervals are integer valued and the $A_{k,l}$ are supposed to be disjoint. According to the above decomposition we shall fix a system of martix units for M denoted by $(f^l_{(a,i)(b,j)})$, $l \in L$, $a \in A_{k_1,l}$, $b \in A_{k_2,l}$, $1 \leqslant i \leqslant n_{k_1}$, $1 \leqslant j \leqslant n_{k_2}$, and a system of matrix units (e^k_{ij}), $k \in K$, $1 \leqslant i, j \leqslant n_k$ for N, and express the inclusion $N \subset M$ by the formula

$$e^k_{ij} = \sum_{l \in L} \sum_{a \in A_{k,l}} f^l_{(a,i)(b,j)}.$$

In terms of these matrix units the minimal projections in N and M respectively, i.e. e^k respectively f^l, have the form:

$$e^k = \sum_{i=1}^{n_k} e^k_{ii},$$

$$f^l = \sum_{k \in K} \sum_{i=1}^{n_k} \sum_{a \in A_{k,l}} f^l_{(a,i)(b,j)}.$$

Note also that the conditional expectation E_N acts as follows:

$$E_N(f^l_{(a,i)(a,,j)}) = \frac{t_l}{s_k} e^k_{ij},$$

k being the index such that $a \in A_{k,l}$

$$E_N(f^l_{(a,i)(b,j)}) = 0 \ \text{ if } \ a \neq b.$$

We shall denote by $f^l_a, a \in A_{kl}$, the minimal projections in $N' \cap M$ defined by:

$$f^l_a = \sum_{i=1}^{n_k} f^l_{(a,i)(a,i)} \ \text{ for } \ a \in A_{kl}.$$

Pimsner and Popa defined in [13], [14] the local generalized index $\lambda_l(M, N)$ for $l \in L$ such that

$$\lambda_l(M, N) = \max\{\lambda \geqslant 0 : E_N(x) \geqslant \lambda x, x \in M f^l_{+,1}\}.$$

They showed that

$$\lambda_l(M, N) = \left(\sum_l \frac{b_{kl} s_k}{t_l} \right)^{-1}, \quad b_{kl} = \min\{a_{kl}, n_k\}$$

and

$$\lambda(M,N)^{-1} = \max_l \lambda_l(M,N)^{-1}.$$

Now we shall calculate the relative entropy $S(M|N)$ for finite dimensional von Neumann algebras $M \supset N$ in terms of the generalized index $\lambda_l(M,N)$.

THEOREM 10.

$$S(M|N) = \sum_l m_l t_l \log \lambda_l(M,N)^{-1}.$$

To prove Theorem, we need some preliminaries.

PROPOSITION 11.

$$S(M|N) \leqslant \sum_l m_l t_l \log \lambda_l(M,N)^{-1}.$$

Proof. It is sufficient to consider partitions of the unity in M consisting of positive multiples of miminal projections in some M_l. Let $\{x_{il}\}_{i\in I, l\in L}$ be such a partion and write $x_{il} = c_{il}p_{il}$ with $c_{il} \in \mathbb{R}_+$ and p_{il} a minimal projection in M_l.

Using $\sum_i x_{il}{}^{*} = f^l$ and $\tau(p_{il}) = t_l$, one gets $\sum_i c_{il}t_l = m_l t_l$. Since $E_N(p_{il}) \geqslant$ $\geqslant \lambda_l(M,N)p_{il}$ for each $l \in L$, it follows from the property of relative operator entropy that

$$(\log \lambda_l(M,N))p_{il} \leqslant s(p_{il}|E_N(p_{il})).$$

Therefore we have the follows:

$$\sum_{i,l} -\tau(s(x_{il}|E_N(x_{il}))) = \sum_{i,l} -c_{il}\tau(s(p_{il}|E_N(p_{il}))) \leqslant$$
$$\leqslant \sum_{i,l} -c_{il}\tau(p_{il}) \log \lambda_l(M,N) = \sum_{i,l} c_{il}t_l \log \lambda_l(M,N)^{-1} =$$
$$= \sum_l m_l t_l \log \lambda_l(M,N)^{-1}.$$

■

To prove Theorem is devoted to the proof of the opposite inequality, by exhibiting a partition of the unity in M with entropy equal to the right hand side in Theorem.

Now, we shall compute the relative operator entropy for the projection with the following properties:

For each l, let p be a minimal projection in M_l. For each k, if $e^k f^l \neq 0$, then there exist nonnegative nunbers $\alpha_k \in R_+$ such that $pe^k f^k p = \alpha_k p$. Put $q_k = \dfrac{1}{\alpha_k} e^k f^l p e^k f^l$. By easy calculation and $\tau(q_k) = \tau(p)$, it follows that q_k is a minimal projection smaller than $e^k f^l$.

Also, if $q_k f_a^l q_k \neq 0$, then there exist nonnegative numbers $u_{ka} \in \mathbb{R}_+$ such that $q_k f_a^k q_k = u_{ka} q_k$.

Suppose that the value u_{ka} does not depend on $a \in A_{kl}$, that is, $u_k = u_{ka}$ for $a \in A_{kl}$. Then it follows that there exist minimal projections r_a^k in Ne^k such that $f_a^l q_k f_a^l = u_k r_a^k f_a^l$. Moreover suppose that r_a^k is an orthogonal projection for $a \in A_{kl}$.

LEMMA 12. *Let the situation be as above. Then*

$$s(p|E_N(p)) = -\left(\log \sum_k \frac{s_k}{u_k t_l}\right) p.$$

Proof. It follows that

$$E_N(q_k) = \sum_{a \in A_{kl}} E_N(f_a^l q_k f_a^l) = \sum_{a \in A_{kl}} u_k r_a^k E_N(f_a^l) = \sum_{a \in A_{kl}} \frac{u_k t_l}{s_k} r_a^k.$$

Since p is a minimal projection in M_l, it follows that $q_k e^k f^l \xi = e^k f^l \xi$ and $r_a^l f_a^l e^k f^l \xi =$ $= f_a^l e^k f^l \xi$ for $k \in K$, $a \in A_{kl}$. Since r_a^k is orthogonal for $a \in A_{kl}$, $\sum_a r_a^k$ is projection. Let $\left(\sum_a r_a^k\right)^\perp$ be an orthogonal complement of $\sum_a r_a^k$. For $k \in K$, we have

$$p\left(\sum_a r_a^k\right)p = p\left(\sum_a r_a^k\right)e^k f^k p = \sum_{a'} p\left(\sum_a r_a^k\right) f_{a'}^l e^k f^l p =$$

$$= p\left(\sum_a r_a^k f_a^l\right) e^k f^l p = p\left(\sum_a f_a^l\right) e^k f^l p = p e^k f^l p = \alpha_k p.$$

Also, $p\left(\sum_a r_a^k\right)^\perp p = 0$. Hence we have

$$p(E_N(p)+\varepsilon)^{-1}p = p\left(\sum_k E_N(e^k f^l p e^k f^l) + \varepsilon\right)^{-1} p =$$

$$= p\left(\sum_k \alpha_k E_N(q_k) + \varepsilon\right)^{-1} p = p\left(\sum_k \sum_a \alpha_k \frac{u_k t_l}{s_k} r_a^k + \varepsilon\right)^{-1} p =$$

$$= p\left(\sum_k \left(\frac{\alpha_k u_k t_l}{s_k} + \varepsilon\right)^{-1} \sum_a r_a^k + \varepsilon^{-1}\left(\sum_a r_a^k\right)^\perp\right) p =$$

$$= \sum_k \left(\frac{\alpha_k u_k t_l}{s_k} + \varepsilon\right)^{-1} \alpha_k p \to \left(\sum_k \frac{s_k}{u_k t_l}\right) p$$

as $\varepsilon \to 0$. Therefore we have

$$s(p|E_N(p)) = \lim_{\varepsilon\to 0} -p(\log p(E_N(p)+\varepsilon)^{-1}p)p =$$

$$= -p\left(\log\left(\sum_k \frac{s_k}{u_k t_l}\right)p\right)p = -\left(\log\sum_k \frac{s_k}{u_k t_l}\right)p. \qquad \blacksquare$$

Proof of Theorem 10. In [13, Theorem 6.2], Pimsner and Popa showed that there exists a partition of the unity in M, $\{x_{il}\}_{i\in I,\, l\in L}$ with the following properties:

(1) $x_{il} = c_{il}p_{il}$, $c_{il} \in \mathbf{R}_+$, p_{il} is a minimal projection in M_l,

(2) $e^k x_{il} e^k = c_{il}(n_k a_{kl}/m_l)q_{ilk}$ with q_{ilk} a minimal projection in M_l.

(3) $q_{ilk} f_a^1 q_{ilk} = \frac{1}{b_{kl}} q_{ilk}$ where $b_{kl} = \min\{a_{kl}, n_k\}$.

(4) $\sum_i c_{il}t_l = m_l t_l$ and $\tau(q_{ilk}) = t_l$.

Then $\{p_{il}\}_{i\in I, l\in L}$ satisfies the condition in Lemma 12. Therefore by Lemma 12 we have

$$\sum_{i,l} -\tau(s(x_{il}|E_N(x_{il}))) = \sum_{i,l} -c_{il}\tau(s(p_{il}|E_N(p_{il}))) =$$

$$= \sum_{i,l} -c_{il}\tau(p_{il})\left(\log\sum_k \frac{b_{kl}s_k}{t_l}\right) = \sum_{i,l} c_{il}t_l \log\lambda_l(M,N)^{-1} =$$

$$= \sum_l m_l t_l \log\lambda_l(M,N)^{-1}.$$

$\blacksquare$

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