

NEST ALGEBRAS IN c_1

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1. INTRODUCTION

In the present paper we study some basic questions of the Banach space structure of the nest algebras. In particular, we study whether any two nest algebras in c_1 are isomorphic to each other.

The answer to this question is known for the other Schatten p -classes, c_p , and for $B(H)$: all the nest algebras in c_p , $1 < p < \infty$ are isomorphic to c_p . This is an easy consequence of the results of Macaev [17] and Gohberg and Krein [13] that say that the nest algebras in c_p , $1 < p < \infty$ are complemented in c_p . Likewise, all the nest algebras in $B(H)$ are completely isomorphic to each other (see [5]).

The structure of the nest algebras in c_1 is richer. We will show, for instance, that if the complete nest is uncountable, then the nest algebra in c_1 is isomorphic to the continuous nest; and for the countable case there is a natural collection of spaces, indexed by the countable ordinal numbers, that resembles the classification of spaces of continuous functions on countable metric spaces given by Bessaga and Pełczyński [6]; although we can only prove that three of them are not isomorphic to each other.

The Banach space invariant we use is primary, (a Banach space X is primary if whenever $X \approx Y \oplus Z$ then either $X \approx Y$ or $X \approx Z$). As side results we prove that the nest algebras in the trace class have bases and that the trace class, the space of compact operators and the nest algebras in $B(H)$ are primary. J. Arazy, [2], [3], gave an earlier proof (1980-1) of the primaryity of c_1 . Our proof is shorter and extends to $K(H)$; the technique was motivated by a paper of Blower [7].

Section 2 has the preliminaries; we fix the notation and quote the necessary results from operator and Banach space theory needed later. In Section 3 we prove that c_1 is primary. In Section 4 we apply the technique developed in Section 3 to study the nest

algebras in c_1 . In Section 5 we prove that the nest algebras in $B(H)$ are primary. We conclude with applications and open questions in Section 6; in particular, we prove that the nest algebras in the trace class have bases.

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2. PRELIMINARIES

For this paper H denotes a separable Hilbert space and $B(H)$ the set of all linear, bounded operators on H . A complete nest, \mathcal{N} , is a totally ordered family of closed subspaces that contains 0 , H , and is closed under intersections and closed unions. The nest algebra induced by \mathcal{N} is the set of all $T \in B(H)$ that leave invariant the elements of \mathcal{N} , i.e.,

$$\text{Alg } \mathcal{N} = \{T \in B(H) : TN \subseteq N \text{ for every } N \in \mathcal{N}\}.$$

The following examples have motivated a big part of the theory.

EXAMPLE 1. In ℓ_2 let $N_k = \overline{\text{span}}\{e_i\}_1^k$ and $k = 1, 2, \dots, \infty$. Then $\mathcal{N} = \{0\} \cup \{N_k\}_1^\infty$ is a nest and it is easy to see that $\text{Alg } \mathcal{N}$ is the set of upper triangular operators.

EXAMPLE 2. In $L_2(0, 1)$ let $N_t = \{f \in L_2 : \text{supp } f \subset [0, t]\}$ for $0 \leq t \leq 1$.

EXAMPLE 3. In $L_2(0, 1) \oplus L_2(0, 1)$ let $M_t = \{f \in L_2 \oplus L_2 : \text{supp } f \subset [0, t] \oplus [0, t]\}$ for $0 \leq t \leq 1$.

The nest algebras were introduced by Kadison and Singer'60 [14] and Ringrose'65 [22]. A central problem was that of classification. Two nest \mathcal{N} and \mathcal{M} are similar (unitarily equivalent) if there exists $T \in B(H)$ invertible (unitary) such that $T\mathcal{N} = \mathcal{M}$; equivalently, $T\text{Alg } \mathcal{N}T^{-1} = \text{Alg } \mathcal{M}$.

It is clear, (by a simple cardinality argument), that the nests of Examples 1 and 2 are not similar; however, it was open for a long time whether the nests of Examples 2 and 3 were. This was answered by Larson'85 [16], who not only proved they are similar but also that any two continuous nests (i.e., those whose index is connected for the order topology) are similar.

Examples 2 and 3 are particular cases of the following natural family of nest algebras: Let $L_2(\mu) = L_2([0, 1], \mu)$ where μ is a positive Borel measure on $[0, 1]$. For $0 \leq t \leq 1$, let $M_t = \{f \in L_2(\mu) : \text{supp } f \subseteq [0, t]\}$, and $M_t^- = \{f \in L_2(\mu) : \text{supp } f \subseteq [0, t]^c\}$. Then $\mathcal{M} = \{M_t, M_t^-\}_{0 \leq t \leq 1}$ is a complete nest called the standard nest.

J. Erdos [11] proved that if \mathcal{N} is a nest in a separable Hilbert space, then there exists a sequence $\mu_1 \gg \mu_2 \gg \dots$ of regular Borel measures on $[0, 1]$ such that $\text{Alg } \mathcal{N}$ is unitarily equivalent to the standard nest on $L_2(\mu_1) \oplus L_2(\mu_2) \oplus \dots$, i.e., for $0 \leq t \leq 1$ and $f \in L_2(\mu_1) \oplus L_2(\mu_2) \oplus \dots$, we have that $f \in M_t$ if and only if $\text{supp } f \subset [0, t] \oplus [0, t] \oplus \dots$, and $f \in M_t^-$ if and only if $\text{supp } f \subset [0, t) \oplus [0, t) \oplus \dots$.

For $1 \leq p < \infty$ let c_p , the Schatten p -class, be the set of all $T \in B(H)$ for which $\|T\|_p^p = \text{tr}(T^*T)^{p/2}$ is finite. Given \mathcal{N} , a nest in H , define $(\text{Alg } \mathcal{N})^p = \text{Alg } \mathcal{N} \cap c_p$ to be the corresponding nest algebra in c_p . Macaev [17] and Gohberg and Krein [13] proved that an infinite nest algebra in c_p is complemented in c_p if and only if $1 < p < \infty$. Since this behavior is identical to that of the Hardy spaces H^p in L_p , the nest algebras in c_p are sometimes called the non-commutative H^p -spaces. However, there are many more analogies than that (see for example [12], [21], [4]).

The similarity theorem also extends to the nest algebras in c_p . If \mathcal{N} and \mathcal{M} are continuous nests then we can find $T \in B(H)$ invertible such that $T \text{Alg } \mathcal{N} T^{-1} = \text{Alg } \mathcal{M}$. Which implies, of course, that $\text{Alg } \mathcal{N} \approx \text{Alg } \mathcal{M}$. But since $T, T^{-1} \in B(H)$ we have that $T(\text{Alg } \mathcal{N})^p T^{-1} = (\text{Alg } \mathcal{M})^p$; hence, $(\text{Alg } \mathcal{N})^p \approx (\text{Alg } \mathcal{M})^p$. In particular, up to similarity, there is only one continuous nest in c_1 which we denote by $\mathbf{T}^1(\mathbf{R})$.

Example 1 has been studied from the Banach space point of view where it is denoted by \mathbf{T} (triangular) and $\mathbf{T}^p = \mathbf{T} \cap c_p$. It is a particular case of [13] that \mathbf{T}^p is complemented in c_p if and only if $1 < p < \infty$. This fact was stressed by Arazy [3] who proved that \mathbf{T}^1 is not isomorphic to a complemented subspace of c_1 , we will use this fact in the proof of Proposition 11. Another important fact, proved by Kwapien and Pelczyński [15], says that c_1 does not embed into \mathbf{T}^1 .

We will use repeatedly Pelczyński's decomposition method [20]. The form we use asserts that if X embeds complementably into Y , Y embeds complementably into X and $X \approx \left(\sum \oplus X\right)_p$ for some $1 \leq p \leq \infty$ then $X \approx Y$.

A Banach space X is primary if whenever $X \approx Y \oplus Z$ then either $X \approx Y$ or $X \approx Z$. It is an immediate consequence of Pelczyński's decomposition method that if $X \approx \left(\sum \oplus X\right)_p$ for $1 \leq p \leq \infty$ then X is primary if for any $T : X \rightarrow X$ bounded and linear, the identity on X factors through T or through $I - T$. (I factors through T if we can find $A, B : X \rightarrow X$ bounded and linear for which $I = ATB$).

Finally, the necessary combinatorial results used in Section 5 can be found in [8].

3. c_1 IS PRIMARY

In this section we give a new proof of the primarity of c_1 . The technique of the proof will be used in the next section to distinguish different isomorphic types of nest

algebras in c_1 .

Let $(e_i)_{i=1}^\infty$ be an orthonormal basis for H , and let $e_{ij} = e_j \otimes e_i$ be the rank-1 operator sending z to $(z, e_j)e_i$. Let σ and ψ be infinite subsets of \mathbb{N} and define $J_{\sigma,\psi} : c_1 \rightarrow c_1$ and $K_{\sigma,\psi} : c_1 \rightarrow c_1$ by

$$J_{\sigma,\psi}(e_{ij}) = e_{\sigma(i)\psi(j)}, \text{ and}$$

$$K_{\sigma,\psi}(e_{ij}) = \begin{cases} e_{ki}, & \text{if } \sigma(k) = i \text{ and } \psi(l) = j, \\ 0, & \text{otherwise.} \end{cases}$$

These maps were used in [15], however, our notation and motivation comes from a paper of Blower [7] where he proved a finite dimensional analogue of Theorem 1 below.

We will use σ and ψ as subsets or as functions $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ according to our needs: $\sigma(i)$ denotes the i th smallest element of σ . It is easy to see that $J_{\sigma,\psi}$ is an isometric embedding and $K_{\sigma,\psi}J_{\sigma,\psi} = I$, I is the identity of c_1 . Moreover, if σ_i and ψ_i , $i = 1, 2$ are infinite subsets of \mathbb{N} then $J_{\sigma_1,\psi_1}J_{\sigma_2,\psi_2} = J_{\sigma_1\sigma_2,\psi_1\psi_2}$, where $\sigma_1\sigma_2(j) = \sigma_1(\sigma_2(j))$. Similarly, $K_{\sigma_1,\psi_1}K_{\sigma_2,\psi_2} = K_{\sigma_1\sigma_2,\psi_1\psi_2}$.

For this section, Φ , with or without subscripts, denotes a bounded linear operator.

THEOREM 1. *For every $\varepsilon > 0$ and $\Phi : c_1 \rightarrow c_1$, we can find $\sigma, \psi \subset \mathbb{N}$, and $\lambda \in \mathbb{C}$ such that $\|K_{\sigma,\psi}\Phi J_{\sigma,\psi} - \lambda I\| < \varepsilon$. Thus, one of $K_{\sigma,\psi}\Phi J_{\sigma,\psi}$ and $I - K_{\sigma,\psi}\Phi J_{\sigma,\psi}$ is invertible.*

We prove Theorem 1 in 5 steps. Each one of them will be a factorization of the form $K_{\sigma,\psi}\Phi J_{\sigma,\psi}$; yielding new maps Φ_i with nicer properties. The sets σ and ψ are constructed inductively.

REMARK. It might be instructive to consider the following example “far” from a multiplier. Let $\Phi : c_1 \rightarrow c_1$ be the transpose operator; i.e., $\Phi e_{ij} = e_{ji}$; σ the set of even integers and ψ the set of odd integers. Then $K_{\sigma,\psi}\Phi J_{\sigma,\psi} = 0$.

We will use several times the following elementary lemma.

LEMMA 2. *Let N be either finite or infinite, E_n an n -dimensional space, $\varepsilon > 0$ and $T : \ell_2^N \rightarrow E_n$ a bounded, linear map. Then, $\text{card}\{i \leq N : \|Te_i\| > \varepsilon\} \leq n^3\|T\|^2/\varepsilon^2$.*

Proof. Let $\{\tilde{e}_i\}_{i \leq n}$ be an Auberbach basis for E_n ; i.e.,

$$\max_{i \leq n} |a_i| \leq \left\| \sum_{i \leq n} a_i \tilde{e}_i \right\| \leq \sum_{i \leq n} |a_i|.$$

For every $i \leq n$ let $A_i = \{j \leq N : |\tilde{e}_i^*(Te_j)| > \varepsilon/n\}$, where $\{\tilde{e}_i^*\}_{i \leq n}$ is the dual basis in E_n^* . Choose $|\varepsilon_j| = 1$ appropriately so that $\left\| \sum_{j \in A_i} \varepsilon_j e_j \right\| = \sqrt{\text{card } A_i}$ and

$\left\| T \left(\sum_{j \in A_i} \varepsilon_j e_j \right) \right\| \geq \varepsilon \text{card } A_i/n$. Then it is clear that $\text{card}(A_i) \leq n^2 \|T\|^2 / \varepsilon_2$. Hence,

if $j \notin \bigcup_{i \leq n} A_i$ we have that $\|Te_j\| < \varepsilon$ and $\text{card} \left(\bigcup_{i \leq n} A_i \right) \leq n^3 \|T\|^2 / \varepsilon^2$. ■

REMARK. We will not need the estimate of Lemma 2. It suffices to know that $\text{card}\{i \leq N : \|Te_i\| > \varepsilon\}$ is small compared to N and independent of N . Moreover, we will also apply Lemma 2 for $T : X \rightarrow E_n$ when X is just isomorphic, not necessarily isometric, to a Hilbert space. The estimate changes but depends on the Banach-Mazur distance of X to the respective Hilbert space, and not on the dimension of X .

Proof of Theorem 1. We introduce some notation now. For every $n \in \mathbb{N}$ let $F_n = \text{span}\{e_{ij} : \max\{i, j\} = n\}$ and $H_n = \overline{\text{span}}\{e_{ij} : \min\{i, j\} = n\}$. Notice that both $\{F_n\}$ and $\{H_n\}$ form a Schauder decomposition for c_1 . The first one has very nice properties (see [15] and [1]), and $H_n \approx \ell_2$.

We also use $M_n = \text{span}\{e_{ij} : \max\{i, j\} \leq n\}$ with P_n the natural projection onto it, and $E_n = \overline{\text{span}}\{e_{ij} : \min\{i, j\} \leq n\}$, with Q_n the natural projection onto it. E_n is still isomorphic to a Hilbert space (but with an isomorphism constant depending on n).

Step 1. For every $\varepsilon > 0$, there exist $\sigma_1 \subset \mathbb{N}$ and Φ_1 such that:

$$\|\Phi_1 - K_{\sigma_1, \sigma_1} \Phi J_{\sigma_1, \sigma_1}\| < \varepsilon, \text{ and } \Phi_1 M_n \subset M_n, \Phi_1 E_n \subset E_n$$

for every $n \in \mathbb{N}$.

The proof of Step 1 is easy. We present in full detail the construction for M_n and indicate how to do it for E_n .

The key ideas are that if $K \subset c_1$ is compact, then there is some m such that K is essentially inside M_m ; and if $E \approx \ell_2$, then there is some m such that E is essentially inside E_m . (See [2], Proposition 2.2).

Let $\sigma_1(1) = 1$ and assume that we have chosen $\sigma_1(1), \dots, \sigma_1(n)$. Since $\Phi \text{Ball}(M_{\sigma_1(n)})$ is compact we can find $m > \sigma_1(n)$ such that $\sup_{x \in \text{Ball}(M_{\sigma_1(n)})} \|P_m \Phi x - \Phi x\| < \varepsilon_{n+1}$, where $\varepsilon_{n+1} > 0$ is chosen small enough. Then set $\sigma_1(n+1) = m+1$. Proceeding this way we construct σ_1 .

Let $x \in F_n$. Then $J_{\sigma_1, \sigma_1} x \in F_{\sigma_1(n)}$; and hence, $\|\Phi J_{\sigma_1, \sigma_1} x - P_m \Phi J_{\sigma_1, \sigma_1} x\| \leq \varepsilon_n \|x\|$, where $m = \sigma_1(n+1) - 1$. It is easy to check that $K_{\sigma_1, \sigma_1} P_m = P_{\sigma_1(n)} K_{\sigma_1, \sigma_1}$. Therefore,

$$\|K_{\sigma_1, \sigma_1} \Phi J_{\sigma_1, \sigma_1} x - P_{\sigma_1(n)} K_{\sigma_1, \sigma_1} \Phi J_{\sigma_1, \sigma_1} x\| < \varepsilon_n \|x\|.$$

Define Φ' by $\Phi'x = P_{\sigma_1(n)}K_{\sigma_1,\sigma_1}\Phi J_{\sigma_1,\sigma_1}x$ for $x \in F_n$. Then, if $\sum^n \varepsilon_n < \varepsilon$ is small enough, Φ' is well defined and satisfied $\|K_{\sigma_1,\sigma_1}\Phi J_{\sigma_1,\sigma_1} - \Phi'\| < \varepsilon$ and $\Phi'M_n \subset M_n$ for every $n \in \mathbb{N}$.

Repeat the process for Φ' with respect to the E_n 's, doing the perturbation argument along the H_n 's and finish.

Step 2. For every $\varepsilon > 0$ there exist $\sigma_2, \psi_2 \subset \mathbb{N}$ and $\Phi_2 \in B(c_1)$ such that $\|\Phi_2 - K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2}\| < \varepsilon$, and $\Phi_2 F_n \subset F_n$, $\Phi_2 E_n \subset E_n$ for every $n \in \mathbb{N}$.

We construct σ_2, ψ_2 satisfying

$$(1) \quad \sigma_2(1) \leq \psi_2(1) < \sigma_2(2) \leq \psi_2(2) < \sigma_2(3) \leq \psi_2(3) \dots,$$

to guarantee that M_n and E_n are invariant for $K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2}$.

Let $\sigma_2(1) = \psi_2(1) = 1$ and assume that we have chosen $\{\sigma_2(1), \dots, \sigma_2(n)\}$ and $\{\psi_2(1), \dots, \psi_2(n)\}$ satisfying $\sigma_2(1) \leq \psi_2(1) < \dots < \sigma_2(n) \leq \psi_2(n)$.

Let $N > \psi_2(n)$ be a "large" number and consider

$$A = \{j > N : \text{for } 1 \leq i \leq N, \|P_{\psi_2(n)}\Phi_1 e_{i,j}\| < \varepsilon_{n+1}\}.$$

By Lemma 2, A^c is finite. Choose $\psi_2(n+1) = \min A$, and since N is large enough, find $i_0, \psi_2(n) < i_0 \leq N$ satisfying: for all $1 \leq j \leq n$, $\|P_{\psi_2(n)}\Phi_1 e_{i_0,\psi_2(j)}\| \leq \varepsilon_{n+1}$. Then set $\sigma_2(n+1) = i_0$. Proceeding this way we construct σ_2, ψ_2 .

Summarizing we have: If $e_{ij} \in F_n$, then $\|P_{\psi_2(n-1)}\Phi_1 J_{\sigma_2,\psi_2} e_{ij}\| < \varepsilon_n$. It is easy to see that (1) implies $K_{\sigma_2,\psi_2} P_{\psi_2(n-1)} = P_{n-1} K_{\sigma_2,\psi_2}$; hence, $\|P_{n-1} K_{\sigma_2,\psi_2} \Phi_1 J_{\sigma_2,\psi_2} e_{ij}\| < \varepsilon_n$.

Define $\Phi_2 e_{ij} = (P_n - P_{n-1})K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2} e_{ij}$, where $e_{ij} \in F_n$ (i.e., the projection onto F_n . Recall that $K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2} M_n \subset M_n$). Therefore,

$$\|(K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2} - \Phi_2)e_{ij}\| < \varepsilon_n,$$

Since the $(2n - 1)$ -dimensional space F_n has a 1-basis consisting of e_{ij} 's we conclude that if $x \in F_n$, then $\|(K_{\sigma_2,\psi_2}\Phi_1 J_{\sigma_2,\psi_2} - \Phi_2)x\| < (2n - 1)\varepsilon_n\|x\|$.

If we choose $\sum_n (2n - 1)\varepsilon_n < \varepsilon$ small enough we finish.

Step 3. For every $\varepsilon > 0$ there exist $\sigma_3, \psi_3 \subset \mathbb{N}$, and $\Phi_3 \in B(c_1)$ such that $\|\Phi_3 - K_{\sigma_3,\psi_3}\Phi_2 J_{\sigma_3,\psi_3}\| < \varepsilon$, and $\Phi_3 H_n \subset H_n$ and $\Phi_3 F_n \subset F_n$ for every $n \in \mathbb{N}$.

We will chose σ_3, ψ_3 as in (1) to guarantee that F_n is invariant for Φ_3 . Let $\sigma_3(1) = \psi_3(1) = 1$ and $A(1) = B(1) = \mathbb{N}$. Assume that we have chosen $\sigma_3(1), \dots, \sigma_3(n)$; $\psi_3(1), \dots, \psi_3(n)$ and $A(n), B(n)$, infinite subsets of \mathbb{N} , satisfying: $\sigma_3(1) \leq \psi_3(1) <$

$\dots < \sigma_3(n) \leq \psi_3(n)$. (We will choose $\sigma_3(n+1)$ from $A(n)$ and $\psi_3(n+1)$ from $B(n)$).

Let $N > \psi_3(n)$ be a "large" number. For every $j \in B(n)$, $j > N$ find $i(j)$, $\psi_3(n) < i(j) < N$ such that $\|Q_{\sigma_3(n)} \Phi_2 e_{i(j)j}\| < \varepsilon_{n+1}$, (Apply Lemma 2 to $Q_{\psi_3(n)} \Phi_2 : F_j \rightarrow Q_{\psi_3(n)} F_j$, notice that $\dim(Q_{\psi_3(n)} F_j) = 2\psi_3(n)$). Let $B(n+1) \subset B(n)$ be an infinite subset of those j 's with common $i(j) = i_0$ and set $\sigma_3(n+1) = i_0$. Exchanging the roles of $A(n), B(n)$ with $B(n), A(n+1)$ we find $A(n+1) \subset A(n)$, and $\psi_3(n+1)$ such that $\|Q_{\psi_3(n)} \Phi_2 e_{i\psi_3(n+1)}\| < \varepsilon_{n+1}$ for every $i \in A(n+1)$. Proceeding this way we construct σ_3, ψ_3 .

Summarizing we have: If $e_{ij} \in H_n$, then $\|Q_{\psi_3(n-1)} \Phi_2 J_{\sigma_3, \psi_3} e_{ij}\| < \varepsilon_n$. It is easy to check that $K_{\sigma_3, \psi_3} P_{\psi_3(n-1)} = P_{n-1} K_{\sigma_3, \psi_3}$; hence, $\|Q_{n-1} K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3} e_{ij}\| < \varepsilon_n$.

Define $\Phi_3 e_{ij} = (Q_n - Q_{n-1}) K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3} e_{ij}$, where $e_{ij} \in H_n$ (i.e., the projection onto H_n . Recall that $K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3} E_n \subset E_n$). Therefore, $\|(K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3} - \Phi_3) e_{ij}\| < \varepsilon_n$. Since E_n is K_n -isomorphic to ℓ_2 , and $Q_n K_{\sigma_3, \psi_3} \Phi_3 J_{\sigma_3, \psi_3} : H_{n+1} \rightarrow E_n$ is "diagonal" with respect to the decompositions: $(H_n \cap F_j)_j$ for H_n , and $(E_n \cap F_j)_j$ for E_n ; we see that if $x \in H_n$, then $\|(K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3} - \Phi_3)x\| < K_n \varepsilon_n \|x\|$.

Since the H_n 's form a Schauder decomposition, it is enough to choose $\sum_n K_n \varepsilon_n < \varepsilon$ small enough to finish.

Step 4. Find σ_4, ψ_4 such that $\Phi_4 = K_{\sigma_4, \psi_4} \Phi_3 J_{\sigma_4, \psi_4}$ satisfies $\Phi_4 e_{ij} = \lambda_{ij} e_{ij}$ for some $\lambda_{ij} \in \mathbb{C}$.

Just take $\sigma_4 = \{1, 3, 5, \dots\}$ and $\psi_4 = \{2, 4, 6, \dots\}$. To see that it suffices, notice that if $i < n$ then $H_i \cap F_n = [e_{in}, e_{ni}]$; hence, $\Phi_3 e_{in} = c_1 e_{in} + c_2 e_{ni}$ for some constants c_1, c_2 .

Step 5. For every $\varepsilon > 0$, there exist $\sigma_5, \psi_5 \subset \mathbb{N}$; $\Phi_5 \in B(c_1)$; and $\lambda \in \mathbb{C}$ such that $\|\Phi_5 - K_{\sigma_5, \psi_5} \Phi_4 J_{\sigma_5, \psi_5}\| < \varepsilon$, and $\Phi_5 e_{ij} = \lambda e_{ij}$ for every $i, j \in \mathbb{N}$.

Look at the upper part of $\{\lambda_{ij}\}_{i < j}$. By a standard diagonal argument we find a subsequence $\sigma_5(1) < \psi_5(1) < \sigma_5(2) < \psi_5(2) < \sigma_5(3) < \psi_5(3) < \dots$ such that for some $\lambda_i, \lambda \in \mathbb{C}$, $|\lambda_{\sigma_5(i), \psi_5(j)} - \lambda_i| < \varepsilon/2^{i+j}$ and $|\lambda_i - \lambda| < \varepsilon/2^i$. Which roughly speaking says that the upper triangular part of $\tilde{\Phi}_5$ is essentially λ . We order them as in (1) to preserve the upper triangular structure of $\{\lambda_{ij}\}$. Do the same for the lower part, and assume that it is "essentially" $\mu \in \mathbb{C}$.

Define $\tilde{\Phi}_5 = K_{\sigma_5, \psi_5} \Phi_4 J_{\sigma_5, \psi_5}$. Hence, $\tilde{\Phi}_5$ has essentially upper triangular part λ and lower triangular part μ . Since $\tilde{\Phi}_5$ is bounded they must agree. Otherwise, $(\tilde{\Phi}_5 - \mu I)/(\lambda - \mu)$ would be like the upper triangular projection; and the latter one is known to be unbounded (see [13]).

Let $\Phi_5 = \lambda I$, and notice that if $e_{ij} \in H_n$, then $\|\tilde{\Phi}_5 e_{ij} - \Phi_5 e_{ij}\| < \varepsilon \left(\frac{1}{2^{i+j}} + \frac{1}{2^i} \right)$.

Moreover, it is clear that if $x \in H_n$, then $\|\Phi_5 x_{ij} - \tilde{\Phi}_5 x\| < \frac{\varepsilon}{2^{n-1}} \|x\|$. Since the H_n 's form a Schauder decomposition, we finish. ■

COROLLARY 3. (J. Arazy) c_1 is primary.

Proof. It follows from Theorem 1 that I , the identity on c_1 , factors through Φ or through $I - \Phi$. This implies that if $c_1 \approx X \oplus Y$ then c_1 embeds complementably into X or Y . Since $c_1 \approx \left(\sum \oplus c_1\right)_1$, the Pełczyński decomposition method gives the result. ■

Notice that $J_{\sigma,\psi}$ and $K_{\sigma,\psi}$ can be defined in $K(H)$, the space of compact operators in the Hilbert space H . Moreover, it is easy to see that $J_{\sigma,\psi}^* = K_{\sigma,\psi}$, and $K_{\sigma,\psi}^* = J_{\sigma,\psi}$. Hence, one gets the equivalence of Theorem 1 and,

COROLLARY 4. $K(H)$ is primary.

REMARKS. (1) The proof of Theorem 1 works in more general situations. For instance, if $\Phi : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ and we make sure that all of the $\sigma_i, \psi_i, i = 1, \dots, 5$ respect triangularity, (i.e., they satisfy (1)), then the same result holds, giving another proof of the fact that \mathbb{T}^1 is primary. It also works for $\mathbb{T}^p, 1 < p < 2$ giving the same conclusion (both results are proved by J. Arazy [2]). And for \mathbb{T}_E if E is a 1-symmetric sequence space of type $p, p < 2$.

(2) Some steps of the proof can be adapted to more general subspaces $\mathcal{S} \subset c_1$ provided we can find enough $\sigma, \psi \subset \mathbb{N}$ satisfying $J_{\sigma,\psi} \mathcal{S} \subset \mathcal{S}$ and $K_{\sigma,\psi} \mathcal{S} \subset \mathcal{S}$. This fact will be essential in the next section.

4. NEST ALGEBRAS IN c_1

In this section we study the isomorphism types of the nest algebras in c_1 . Notice that (\mathcal{N}, \leq) is a compact space with the order topology.

The results we obtain are:

THEOREM 5. *If \mathcal{N} is an uncountable nest then $(\text{Alg } \mathcal{N})^1$ is isomorphic to $T^1(\mathbb{R})$, the continuous nest in c_1 .*

For \mathcal{N} countable we have the following natural class. Let α be a countable ordinal number, index the canonical basis of ℓ_2 by $\{e_\beta : \beta \leq \alpha\}$ and let $\mathbb{T}^1(\alpha)$ be the nest algebra in c_1 associated to the nest of subspaces $\{N_\beta : \beta \leq \alpha\}$ where $N_\beta = [e_\gamma : \gamma \leq \beta]$.

THEOREM 6. *No two of the following nest algebras are isomorphic to each other: $\mathbb{T}^1(\omega), \mathbb{T}^1(2\omega)$, and $\mathbb{T}^1(\omega^2)$.*

If $\alpha \geq \omega^2$ then the intervals of isomorphism of the $\mathbb{T}^1(\alpha)$'s are at most like those for the $C(\alpha)$'s.

PROPOSITION 7. *If $\omega^2 \leq \alpha \leq \beta < \alpha^\omega$ then $\mathbb{T}^1(\alpha) \approx \mathbb{T}^1(\beta)$.*

The proof of Theorem 5 will consist of two parts, (Lemma 8 and 9). The first one shows that $\mathbb{T}^1(\mathbb{R})$ embeds complementably into $(\text{Alg } \mathcal{N})^1$, and the second shows that $(\text{Alg } \mathcal{N})^1$ embeds complementably into $\mathbb{T}^1(\mathbb{R})$. Since $\mathbb{T}^1(\mathbb{R}) \approx \left(\sum \oplus \mathbb{T}^1(\mathbb{R})\right)_1$, the proof follows from Pelczyński's decomposition method [20].

LEMMA 8. *If \mathcal{N} is uncountable then $\mathbb{T}^1(\mathbb{R})$ embeds complementably into $(\text{Alg } \mathcal{N})^1$.*

Proof. It is a consequence of the Similarity Theory [10] that \mathcal{N} is similar to a nest $\tilde{\mathcal{N}}$ with a continuous part. By [11], this one comes from a continuous measure μ supported on $[0,1]$. Let P be the orthogonal projection onto $L_2([0,1], \mu)$; then $\Phi(T) = PTP$ sends $\text{Alg } \mathcal{N}$ onto a continuous nest. Moreover, Φ is a projection and also sends $(\text{Alg } \mathcal{N})^1$ to a continuous nest in c_1 . ■

LEMMA 9. *$(\text{Alg } \mathcal{N})^1$ embeds complementably into $\mathbb{T}^1(\mathbb{R})$.*

Proof. The proof uses Erdos representation Theorem [11]. For clarity we will prove it for multiplicity free nests but the proof extends easily to the general case.

Assume that \mathcal{N} is the standard nest on $L_2([0,1], \mu)$ where $\mu = \mu_c + \mu_d$ and μ_c is continuous and μ_d is discrete with atoms at $\{d_n\}_n \subset [0,1]$.

We "split" every atom d_n into d_n^- and d_n^+ and insert a copy of $[0,1]$ in between, 0 corresponding to d_n^- and 1 to d_n^+ . More formally, if $I_n = [0,1] \times \{n\}$ for every n then

$$\Omega = \left([0,1] \setminus \bigcup_n \{d_n\} \right) \bigcup_n I_n$$

with the natural order. It is easy to see that Ω is then a compact connected space. Define a measure ν on Ω by $\nu = \mu_c$ on $[0,1] \setminus \bigcup_n \{d_n\}$ and Lebesgue on $\bigcup_n I_n$. Then ν is continuous and we define the standard continuous nest on $L_2(\Omega, \nu)$, which we denote $\text{Alg } \tilde{\mathcal{N}}$.

To take care of the atoms consider $x_n = \sqrt{2}\chi_{[1/2,1]}$ and $y_n = \sqrt{2}\chi_{[0,1/2]}$ supported on I_n . Then for every n , $x_n \otimes y_n \in \text{Alg } \tilde{\mathcal{N}}$. ($x \otimes y$ denotes the rank-1 map that sends $h \rightarrow (h, x), y$).

Let P be the orthogonal projection on $L_2(\Omega, \nu)$ onto $L_2([0,1] \setminus \bigcup_n \{d_n\}, \mu_c)$; P_x the orthogonal projection onto $[x_n]$, and P_y onto $[y_n]$. It is clear that they are orthogonal from each other.

Define Φ on $B(L_2(\Omega, \nu))$ by $\Phi(T) = (P + P_x)T(P + P_y)$. Notice that Φ is a projection and its range is isomorphic to $\text{Alg } \mathcal{N}$. They have the same continuous part:

PTP , the same atomic part P_xTP_y , and they interact in the same way. Moreover, Φ is also defined on $c_1(L_2(\Omega, \nu))$, giving the result.

If \mathcal{N} is not multiplicity free, then represent it as in [11], make the “enlargement” on every interval and proceed as before. ■

The proof of Theorem 6 is more involved. We start with a concrete representation of $\mathbb{T}^1(\omega)$, (which will be denoted from now on by \mathbb{T}^1), $\mathbb{T}^1(2\omega)$ and $\mathbb{T}^1(\omega^2)$. Notice that the last one is isomorphic to $\mathbb{T}^1 \otimes c_1 = \overline{\text{span}}\{e_{ij} \otimes e_{kl} : i \leq j; k, l = 1, 2, \dots\}$.

$$\mathbb{T}^1(\omega) = \begin{pmatrix} * & * & \cdots \\ & * & \cdots \\ & & \ddots \end{pmatrix}, \quad \mathbb{T}^1(2\omega) = \begin{pmatrix} * & * & \cdots & * & * & \cdots \\ & * & \cdots & * & * & \cdots \\ & & \ddots & \vdots & \vdots & \ddots \\ & & & * & * & \cdots \\ & & & & * & \cdots \\ & & & & & \ddots \end{pmatrix} = \begin{pmatrix} \mathbb{T}^1 & c_1 \\ & \mathbb{T}^1 \end{pmatrix}, \quad \text{and}$$

$$\mathbb{T}^1(\omega^2) = \begin{pmatrix} \mathbb{T}^1 & c_1 & c_1 & \cdots \\ & \mathbb{T}^1 & c_1 & \cdots \\ & & \mathbb{T}^1 & \cdots \\ & & & \ddots \end{pmatrix} \approx \begin{pmatrix} c_1 & c_1 & \cdots \\ & c_1 & \cdots \\ & & \ddots \end{pmatrix} = \mathbb{T}^1 \otimes c_1.$$

It is clear that \mathbb{T}^1 embeds complementably into $\mathbb{T}^1(2\omega)$ and this one embeds complementably into $\mathbb{T}^1(\omega^2)$; however, the reverse complemented embeddings do not hold (see Lemmas 9 and 10 below). The key point is the decomposition

$$\mathbb{T}^1(2\omega) \approx \mathbb{T}^1 \oplus c_1.$$

LEMMA 10. $\mathbb{T}^1(2\omega)$ does not embed into \mathbb{T}^1 .

Proof. If $\mathbb{T}^1(2\omega)$ embedded into \mathbb{T}^1 then we would have that c_1 embeds into \mathbb{T}^1 . But this is impossible as we stated in the preliminaries (see [15]). ■

The next proposition says that $\mathbb{T}^1(\omega^2)$ does not embed complementably into $\mathbb{T}^1(2\omega)$. Nevertheless, since $c_1 \subset \mathbb{T}^1(2\omega)$, it does embed.

PROPOSITION 11. $\mathbb{T}^1(\omega^2)$ does not embed complementably into $\mathbb{T}^1(2\omega)$.

The idea of the proof is that we cannot take away one c_1 from $\mathbb{T}^1 \otimes c_1$ in such a way that what we have left is just one \mathbb{T}^1 .

To formalize this we first prove that we can replace any bounded linear operator Φ on $\mathbb{T}^1 \otimes c_1$ by a multiplier; then we will show that for this simple type of operator it is not possible to have $\Phi(\mathbb{T}^1 \otimes c_1) \approx c_1$ and $(I - \Phi)(\mathbb{T}^1 \otimes c_1) \approx \mathbb{T}^1$.

A multiplier on $\mathbb{T}^1 \otimes c_1$ is a bounded linear operator, Φ , that satisfies: for every $i \leq j$,

$$\Phi e_{ij} \otimes e_{kl} = \lambda_{ijkl} e_{ij} \otimes e_{kl},$$

for some $\lambda_{ijkl} \in \mathbb{C}$.

To make the replacement we reduce the problem from $c_1 \otimes c_1$ to c_1 and then adapt the steps of the proof of Theorem 1.

Let $\varphi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a one-to-one and onto map, and define $S : c_1 \otimes c_1 \rightarrow c_1$ by $S(e_{ij} \otimes e_{kl}) = e_{\varphi(i,k)\varphi(j,l)}$. It is easy to see that S is an isometry onto. This allows us to define on $c_1 \otimes c_1$ the equivalent maps for $J_{\sigma,\psi}$ and $K_{\sigma,\psi}$ as follows:

$$\widehat{\sigma}(i, j) = (\varphi^{-1}\sigma\varphi)(i, j) \quad \text{and} \quad \widehat{\psi}(k, l) = (\varphi^{-1}\psi\varphi)(k, l).$$

Then $J_{\widehat{\sigma},\widehat{\psi}} : c_1 \otimes c_1 \rightarrow c_1 \otimes c_1$, and $K_{\widehat{\sigma},\widehat{\psi}} : c_1 \otimes c_1 \rightarrow c_1 \otimes c_1$ are well defined and have the same properties. Moreover, we have

$$S^{-1}J_{\sigma,\psi} = J_{\widehat{\sigma},\widehat{\psi}}S^{-1} \quad \text{and} \quad K_{\sigma,\psi}S = SK_{\widehat{\sigma},\widehat{\psi}}.$$

Let $\mathcal{S} = S(\mathbb{T}^1 \otimes c_1)$. Notice that \mathcal{S} is a $*$ -diagram, i.e., for i, j fixed, either $e_{ij} \in \mathcal{S}$ or for every $A \in \mathcal{S}$, $(Ae_j, e_i) = 0$.

Let π_1 be the projection onto the first coordinate of $\mathbb{N} \times \mathbb{N}$ and define

$$r(i) = \pi_1\varphi^{-1}(i).$$

It is clear that $e_{ij} \in \mathcal{S}$ if and only if $r(i) \leq r(j)$. Moreover, it is very important to notice that for i fixed there are infinitely many j 's satisfying $r(i) = r(j)$.

LEMMA 12. *With the above notation, a necessary and sufficient condition for $J_{\sigma,\psi}\mathcal{S} \subset \mathcal{S}$ and $K_{\sigma,\psi}\mathcal{S} \subset \mathcal{S}$ is that $r(i) \leq r(j)$ if and only if $r(\sigma(i)) \leq r(\psi(j))$. In particular, this is true if $r(i) = r(\sigma(i)) = r(\psi(i))$ for every i .*

The proof of Lemma 12 follows immediately from the definitions.

LEMMA 13. *For every $\varepsilon > 0$ and $\tilde{\Phi} : \mathbb{T}^1 \otimes c_1 \rightarrow \mathbb{T}^1 \otimes c_1$ a bounded operator, there exists σ and ψ , as Lemma 12, and a multiplier Φ_1 on $\mathbb{T}^1 \otimes c_1$ satisfying $\|K_{\widehat{\sigma},\widehat{\psi}}\tilde{\Phi}J_{\widehat{\sigma},\widehat{\psi}} - \Phi_1\| < \varepsilon$.*

Proof of Lemma 13. The proof mimics the one of Theorem 1, and we solve it on \mathcal{S} instead. Let $M_n(\mathcal{S})$, $E_n(\mathcal{S})$, $F_n(\mathcal{S})$ and $H_n(\mathcal{S})$ be as in Theorem 1 with the natural modifications, e.g, $M_n(\mathcal{S}) = \text{span}\{e_{ij} \in \mathcal{S} : \max\{i, j\} \leq n\}$, etc..

Let $\Phi : \mathcal{S} \rightarrow \mathcal{S}$. To simplify notation whenever we say that $\Phi \approx \Psi$ we mean that they are arbitrarily close.

Step 1. Find σ_1 with $r(i) = r(\sigma(i)) = r(\psi(j))$ for every i , such that $\Phi_1 \approx K_{\sigma_1, \sigma_1} \Phi J_{\sigma_1, \sigma_1}$, and it satisfies $\Phi_1(M_n(\mathcal{S})) \subset M_n(\mathcal{S})$ and $\Phi_1(E_n(\mathcal{S})) \subset E_n(\mathcal{S})$.

This is easy. Repeat the proof in Step 1. Theorem 1 but choose $\sigma(i)$ satisfying $r(i) = r(\sigma(i))$. This is always possible because there are infinitely many j 's with $r(j) = r(i)$.

Step 2. Find σ_2, ψ_2 with $r(i) = r(\sigma_2(i)) = r(\psi_2(i))$ for all i , such that $\Phi_2 \approx K_{\sigma_2, \psi_2} \Phi_1 J_{\sigma_2, \psi_2}$ and it satisfies $\Phi_2(F_n(\mathcal{S})) \subset F_n(\mathcal{S})$.

The proof is very similar to the one of Step 2, Theorem 1. We only have to take $\psi_2(n+1) \in A$ with $r(\sigma_2(n+1)) = r(n+1)$ and make sure that $\{i : \psi_2(n) < i \leq N, r(i) = r(n+1)\}$ is "large enough" to extract $\sigma_2(n+1)$ from it.

Step 3. Find σ_3, ψ_3 as in Lemma 12 such that $\Phi_3 \approx K_{\sigma_3, \psi_3} \Phi_2 J_{\sigma_3, \psi_3}$ and satisfies $\Phi_3(H_n(\mathcal{S})) \subset H_n(\mathcal{S})$.

The proof is more delicate now. If we repeat the proof of Step 3, Theorem 1 we may end up with all the elements in $A(n)$ with constant $r(i)$.

We need to find σ_3, ψ_3 as in Lemma 12. This means that if we have chose $\sigma_3(i), \psi_3(i)$ for $i \leq n$, then $\sigma_3(n+1)$ and $\psi_3(n+1)$ have the following constrains: If $k \leq n$ then

$$r(k) \leq r(n+1) \implies r(\sigma_3(k)) \leq r(\psi_3(n+1)), \text{ and}$$

$$r(k) > r(n+1) \implies r(\sigma_3(k)) > r(\psi_3(n+1)),$$

we also have similar conditions for $\sigma_3(n+1)$ in addition to $r(\sigma_3(n+1)) \leq r(\psi_3(n+1))$. However, we will see that there is a lot of room and we will not worry much.

Let $R_k = \{i : r(i) = k\}$. We will choose σ_3 and ψ_3 in such a way that if $r(i) = r(j)$ then $r(\sigma_3(i)) = r(\sigma_3(j))$ and $r(\psi_3(i)) = r(\psi_3(j))$. Therefore, once we choose an element from R_k we must be able to continue selecting elements from the same R_k .

Assume that we have chosen $\sigma_3(i), \psi_3(i)$ for $i \leq n$ and that we have $A(n), B(n)$ subsets of \mathbb{N} satisfying:

$$(2) \quad \begin{aligned} \text{card}(A_n \cap R_{r(\psi_3(i))}) &= \aleph_0 \text{ for } i \leq n, \\ \text{card}\{i : \text{card}(A(n) \cap R_i) = \aleph_0\} &= \aleph_0, \\ \text{card}(B(n) \cap R_{r(\sigma_3(i))}) &= \aleph_0 \text{ for } i \leq n, \\ \text{card}\{i : \text{card}(B(n) \cap R_i) = \aleph_0\} &= \aleph_0. \end{aligned}$$

The first and third conditions are the ones that allow us to choose elements from the previously chosen R_i 's and the others are similar to those of Step 3, Theorem 1.

Suppose we have to choose $\sigma_3(n+1)$ from R_m , here m is one of the elements of the fourth line of (2). Then let $\rho \subset B(n) \cap R_m$ be such that $\text{card}(\rho) = N = N(n)$, where N is a very large number.

We want to take $A(n+1) \subset A(n)$ satisfying (2), but first we find $A_1(n+1) \subset A(n)$ satisfying (2), and $\rho_1 \subset \rho$ very large such that for $i \leq n$

$$(3) \quad k \in \rho_1 \text{ and } j \in A_1(n) \cap R_{r(\psi_3(i))} \implies \|Q_n e_{kj}\| \leq \varepsilon_{n+1}.$$

To check (3) it is enough to do if for only one. Lemma 2 gives that for every $j \in A(n) \cap R_{r(\psi_3(1))}$, $\text{card}\{i \in \rho : \|Q_n e_{kj}\| \geq \varepsilon_{n+1}\}$ is very small. Hence, we have find a large $\rho_1 \subset \rho$ and $F_1 \subset A(n) \cap R_{r(\psi_3(1))}$ infinite such that if $j \in F_1$ and $k \in \rho_1$ then $\|Q_n e_{kj}\| \leq \varepsilon_{n+1}$. Let

$$A_1(n) = [A(n) \setminus (A(n) \cap R_{r(\sigma_3(1))})] \cup F_1.$$

It is clear that this does it. Moreover, it is also clear that we have complete control over finitely many R_k 's. This is the "room" we mentioned before.

So, assume that $A_1(n)$ satisfies (2) and (3). For every $j \in A_1(n)$ there exists $k(j) \in \rho_1$ such that $\|Q_n e_{k(j)j}\| \leq \varepsilon_{n+1}$. Since ρ_1 is finite we choose $\sigma_3(n+1) \in \rho$ such that $A(n+1) = \{j \in A_1(n) : j > \max \rho \text{ and } \|Q_n e_{\sigma_3(n+1)j}\| \leq \varepsilon_{n+1}\}$ satisfies (2). Find $\psi_3(n+1)$ similarly.

Step 4. Find σ_4, ψ_4 as in Lemma 12 such that $\Phi_4 = K_{\sigma_4, \psi_4} \Phi_2 J_{\sigma_4, \psi_4}$ satisfies $\Phi_4 e_{ij} = \lambda_{ij} e_{ij}$.

This is just like Step 4 of Theorem 1.

We finish now the proof of Lemma 13. If $\tilde{\Phi} \in B(\mathbb{T}^1 \otimes c_1)$ then $S\tilde{\Phi}S^{-1} : \mathcal{S} \rightarrow \mathcal{S}$ Combining Steps 1 through 4 we find σ, ψ satisfying the condition of Lemma 12 and $\tilde{\Phi}_1$ a multiplier such that

$$\|K_{\sigma, \psi} S\tilde{\Phi}S^{-1} J_{\sigma, \psi} - \tilde{\Phi}_1\| \leq \varepsilon.$$

Since $K_{\sigma, \psi} S = SK_{\sigma, \psi}$ and $S^{-1} J_{\hat{\sigma}, \hat{\psi}} = J_{\hat{\sigma}, \hat{\psi}} S^{-1}$ we obtain the result. ■

Proof of Proposition 11. Suppose that $\mathbb{T}^1 \otimes c_1 \approx \mathbb{T}^1 \oplus c_1$. Then find $\Phi \in B(\mathbb{T}^1 \otimes c_1)$ such that $\Phi(\mathbb{T}^1 \otimes c_1) \approx c_1$ and $(I - \Phi)(\mathbb{T}^1 \otimes c_1) \approx \mathbb{T}^1$. Therefore,

$I_{\mathbb{T}^1}$ does not factor through Φ , and

I_{c_1} does not factor through $I - \Phi$.

We will show that this leads to a contradiction.

Find, as in Lemma 13, σ_1, ψ_1 and a multiplier Φ_1 such that $\|\Phi_1 - K_{\hat{\sigma}_1, \hat{\psi}_1} \Phi_1 J_{\hat{\sigma}_1, \hat{\psi}_1}\| \leq \varepsilon$ and for $i \leq j$, $\Phi_1 e_{ij} \otimes e_{kl} = \lambda_{ijkl} e_{ij} \otimes e_{kl}$.

By a standard diagonal argument we can assume that

$$\begin{aligned} \lim_{l \rightarrow \infty} \lambda_{ijkl} &= \lambda_{ijk}, \\ \lim_{k \rightarrow \infty} \lambda_{ijkl} &= \bar{\lambda}_{ijl}, \\ \lim_{k \rightarrow \infty} \lambda_{ijk} &= \lambda_{ij}, \\ \lim_{l \rightarrow \infty} \bar{\lambda}_{ijl} &= \bar{\lambda}_{ij}. \end{aligned}$$

Claim 1. $\lambda_{ij} = \bar{\lambda}_{ij}$.

The proof of this is essentially Step 5 of Theorem 1. If for some i, j we had $\lambda_{ij} \neq \bar{\lambda}_{ij}$, then looking at the c_1 at the i, j position we would find a block projection with upper triangular part λ_{ij} and lower $\bar{\lambda}_{ij}$. And this would be unbounded.

We can assume moreover that all of the λ_{ij} essentially agree; i.e., for some $\lambda \in \mathbb{C}$, $|\lambda_{ij} - \lambda| < \varepsilon \left(\frac{1}{2^i} + \frac{1}{2^j} \right)$.

Claim 2. $|1 - \lambda| \leq 2\varepsilon$.

If not, define $J_2 : c_1 \rightarrow \mathbb{T}^1 \otimes c_1$ and $K_2 : \mathbb{T}^1 \otimes c_1 \rightarrow c_1$ by $J_2 e_{ij} = e_{ij} \otimes e_{11}$ and $K_2(e_{ij} \otimes e_{kl}) = e_{ij}$ only if $k = l = 1$. Then $\Phi_2 = K_2 \Phi_1 J_2 : c_1 \rightarrow c_1$; also notice that $I_{c_1} - \Phi_2 = K_2(I - \Phi_1)J_2$.

As in Step 5 of Theorem 1 find J_3, K_3 in c_1 such that $\Phi_3 = K_3 \Phi_2 J_3$ satisfies $\|\Phi_3 - \lambda_{11} I_{c_1}\| < \varepsilon$. Hence, if $K = K_3 K_2 K_{\hat{\sigma}_1, \hat{\psi}_1}$ and $J = J_{\hat{\sigma}_1, \hat{\psi}_1} J_2 J_3$ we have that $\|K \Phi J - \lambda I_{c_1}\| < 2\varepsilon$. Therefore,

$$\|K(I - \Phi)J - (1 - \lambda)I_{c_1}\| < 2\varepsilon.$$

And since $|1 - \lambda| > 2\varepsilon$ we have that $K(I - \Phi)J : c_1 \rightarrow c_1$ is invertible. This implies that I_{c_1} factors through $I - \Phi$, a contradiction.

It remains to prove that both Claims 1 and 2 contradict our assumption.

For this it will be enough to find σ, ψ such that for $i \leq j$, $|\lambda_{ij\sigma(i)\psi(j)}| \geq 1/2$. Because once we have this we define $J : \mathbb{T}^1 \rightarrow \mathbb{T}^1 \otimes c_1$ and $K : \mathbb{T}^1 \otimes c_1 \rightarrow \mathbb{T}^1$ by $J e_{ij} = e_{ij} \otimes e_{\sigma(i)\psi(j)}$ and K such that $KJ = I_{\mathbb{T}^1}$. Then $K \Phi_1 J : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is a multiplier with big elements; hence, an adaptation of the proof of Step 5 of Theorem 1 implies that $I_{\mathbb{T}^1}$ factors through Φ giving a contradiction.

The existence of σ, ψ will follow from the next claim which we prove only in the first row of c_1 's.

Claim 3. Let B be an infinite subset of \mathbb{N} , and for every $k = 1, 2, \dots$ let

$$A_k = \{j \in B : \text{card}\{l : |\lambda_{1jkl}| \geq 1/2\} = \aleph_0\}.$$

Then for some k , $\text{card}A_k = \aleph_0$.

We know that if we look at the $(c_1)_{1j}$ then there are plenty of λ_{1jkl} 's in the "lower-right" corner that satisfy $|\lambda_{1jkl}| \approx 1$. More specifically, given l large enough, there exists k_0 such that for $k \geq k_0$, $\lambda_{1jkl} \approx \lambda_{1j} \approx \lambda$.

If Claim 3 were false, for every k fixed, we would have $|\lambda_{1jkl}| < 1/2$ "eventually". Hence, one can extract arbitrarily large blocks that look essentially like

$$\begin{pmatrix} \lambda & \mu_{12} & \cdots & \mu_{1N} \\ \lambda & \lambda & \cdots & \mu_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & \lambda \end{pmatrix}.$$

Where $|\mu_{ij}| < 1/2$. Then, Ramsey's Theorem, used as in Proposition 17 gives us a large submatrix with upper triangular part μ , $|\mu| < 1/2$ and lower λ with $|\lambda| > 1 - 2\epsilon$. Since N is arbitrarily large and the latter matrices are not uniformly bounded we conclude that Φ is not bounded. A clear contradiction. \blacksquare

We finally start the proof of Proposition 7. For this we need the following lemma.

LEMMA 14. *If $\omega^2 \leq \alpha$ then $T^1(\alpha) \approx T^1(\alpha^2)$.*

Proof of Proposition 7. It is clear that Lemma 14 gives that for every $n \in \mathbb{N}$, $T^1(\alpha) \approx T^1(\alpha^n)$. Hence, if $\alpha \leq \beta < \alpha^\omega$ we can find n such that $\alpha \leq \beta < \alpha^n$. Therefore, we have that $T^1(\alpha)$ embeds complementably into $T^1(\beta)$, and this one into $T^1(\alpha^n)$. Since the latter is isomorphic to $T^1(\alpha)$ and we have that $T^1(\alpha) \approx \left(\sum \oplus T^1(\alpha)\right)_1$, we finish the proof using Pelczyński's decomposition theorem. \blacksquare

Before the proof of Lemma 14 we prove this simpler case.

LEMMA 15. *If $\omega^2 \leq \alpha$ then $T^1(\alpha\omega) \approx T^1(\alpha)$.*

Proof. Notice that $(0, \alpha\omega) = \bigcup_{0 \leq n < \omega} I_n$ where $I_n = (\alpha n, \alpha(n+1)]$. Therefore, $\ell_2(\alpha\omega) = \left(\sum_n \oplus H_n\right)_2$ where $H_n = [e_\eta : \alpha n < \eta \leq \alpha(n+1)]$. Taking the "diagonal" of this decomposition, (which is clearly complemented), we obtain

$$\begin{aligned} T^1(\alpha\omega) &\approx \left(\sum_n \oplus T^1(\alpha)\right)_1 \oplus [T^1 \otimes c_1] \approx \\ &\approx T^1(\alpha) \oplus T^1(\omega^2) \approx T^1(\alpha). \end{aligned}$$

\blacksquare

Proof of Lemma 14. Notice that $(0, \alpha^2) = \bigcup_{0 \leq \xi < \alpha} I_\xi$ where $I_\xi = (\alpha\xi, \alpha(\xi + 1)]$.

Then repeating a similar argument as in Lemma 15 we have that

$$\mathbf{T}^1(\alpha^2) \approx \left(\sum_{0 \leq \xi < \alpha} \mathbf{T}^1(\alpha) \right)_1 \oplus [\mathbf{T}^1 \otimes c_1] \approx \mathbf{T}^1(\alpha) \oplus [\mathbf{T}^1(\alpha) \otimes c_1].$$

And we finish the proof if we prove.

Claim 1. If $\omega^2 \leq \alpha$ then $\mathbf{T}^1(\alpha) \otimes c_1 \approx \mathbf{T}^1(\alpha)$.

Notice first that $(0, \omega\alpha) = \bigcup_{0 \leq \xi < \alpha} I_\xi$ where $I_\xi = (\omega\xi, \omega(\xi + 1)]$. Then we have that

$$\mathbf{T}^1(\omega\alpha) \approx \mathbf{T}^1 \oplus [\mathbf{T}^1(\alpha \otimes c_1)] \approx \mathbf{T}^1(\alpha) \otimes c_1.$$

Hence, it is enough to prove.

Claim 2. If $\omega^1 \leq \alpha$ then $\mathbf{T}^1(\alpha) \approx \mathbf{T}^1(\omega\alpha)$.

For $\omega^2 \leq \alpha < \omega^\omega$ it is enough to take $\alpha = \omega^n$. And for this case Lemma 15 gives the result. For $\alpha = \omega^\omega$ we have that $\omega\omega^\omega = \omega^\omega$. Actually, if $\gamma \geq \omega$ then $\omega\omega^\gamma = \omega^\gamma$. Hence, for $\alpha \geq \omega^\omega$ we have $\omega\alpha < \alpha\omega$. (Just take $\alpha = \omega^\gamma k + \delta$ for some $\delta < \omega^\gamma$ and $\gamma \geq \omega$). This implies that $\mathbf{T}^1(\omega\alpha)$ embeds complementably into $\mathbf{T}^1(\alpha\omega)$, and now Lemma 15 finishes the proof. ■

5. PRIMARITY OF OPERATOR SPACES

In this section we show that if \mathcal{N} is an infinite nest in a separable Hilbert space then $\text{Alg } \mathcal{N}$ and $B(H)/\text{Alg } \mathcal{N}$ are primary. These are non-commutative versions of theorems proved by Bourgain [9] and Müller [18].

The technique we use was developed by Bourgain [9] to prove that H^∞ is primary. It allows to obtain the general theorem from its finite dimensional version.

To prove that $\text{Alg } \mathcal{N}$ is primary we use the decomposition, from [5],

$$\text{Alg } \mathcal{N} \approx \left(\sum \oplus \mathbf{T}_n \right)_\infty.$$

where \mathbf{T}_n is the set of all $n \times n$ upper triangular matrices in M_n . Then we modify slightly the combinatorial argument used by Blower [7] to prove that $B(H)$ is primary. It is the arguments in [7] which motivated our work above.

THEOREM 16. *Alg \mathcal{N} is primary.*

The proof of the theorem will follow from the finite dimensional case as Bourgain [9] indicated. Since the proof is a slight modification of [7] we use the notation employed there.

If $\sigma = \{\sigma(1), \dots, \sigma(n)\}$ and $\psi = \{\psi(1), \dots, \psi(n)\}$ are finite subsets of $\{1, \dots, N\}$, define $J_{\sigma, \psi} : M_n \rightarrow M_n$ and $K_{\sigma, \psi} : M_n \rightarrow M_n$ by

$$J_{\sigma, \psi}(e_{ij}) = e_{\sigma(i), \psi(j)}, \text{ and}$$

$$K_{\sigma, \psi}(e_{ij}) = \begin{cases} e_{kl}, & \text{if } \sigma(k) = i \text{ and } \psi(l) = j, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $J_{\sigma, \psi}$ is an isometric embedding and $K_{\sigma, \psi}J_{\sigma, \psi} = I$, where I is the identity of M_n .

Moreover, if

$$(4) \quad \sigma(1) < \psi(1) < \sigma(2) < \psi(2) < \dots < \sigma(n) < \psi(n),$$

then $K_{\sigma, \psi}(\mathbb{T}_N) \equiv \mathbb{T}_n$.

Then we obtain a proposition similar to the one in [7].

PROPOSITION 17. *Given $n, \varepsilon > 0$ and $K < \infty$ there exists N_0 such that if $N > N_0$ and $T \in B(\mathbb{T}_N, \mathbb{T}_N)$ with $\|T\| \leq K$, then there exist subsets σ and ψ of $\{1, \dots, N\}$ of cardinality n such that $\sigma(1) < \psi(2) < \dots < \sigma(n) < \psi(n)$ and a constant λ such that*

$$\|K_{\sigma, \psi}TJ_{\sigma, \psi} - \lambda I_n\| < \varepsilon.$$

Thus, one of $K_{\sigma, \psi}TJ_{\sigma, \psi}$ and $K_{\sigma, \psi}(I_n - T)J_{\sigma, \psi}$ is invertible.

REMARK. The previous was proved by Blower [7] without the assumption that σ and ψ satisfy (4). In fact, he proved it for $\max \sigma < \min \psi$. However, since we need to preserve the triangular structure we modify the argument from [7] to obtain the desired result.

Sketch of the proof. Just as in [7] find a large $\sigma_1 \subset \{1, \dots, N\}$ and $\lambda \in \mathbb{C}$ such that if $i < j$ are in σ_1 then $|T(e_{ij})_{ij} - \lambda| < \varepsilon n^{-6}/4$.

The goal now is to find a large $\tilde{\sigma} \subset \sigma_1$ such that if $i < j$ and $k < l$ are in $\tilde{\sigma}$ and satisfy $(i, j) \neq (k, l)$ then $|(Te_{ij})_{kl}| < \delta$, where $\delta = \varepsilon n^{-6}/4$. Then $\sigma = \{\tilde{\sigma}(2i - 1)\}_{i=1}^n$ and $\psi = \{\tilde{\sigma}(2i)\}_{i=1}^n$ will do it.

To find $\tilde{\sigma}$ we find first, as in [7], a large $\sigma_2 \subset \sigma_1$ such that if $i < k < j < l$ are in σ_2 then $|(Te_{ij})_{kl}| < \delta$. Then find a large $\sigma_3 \subset \sigma_2$ such that if $k < i < j < l$ are in σ_3 then $|(Te_{ij})_{kl}| < \delta$. After this find a large $\sigma_4 \subset \sigma_3$ such that if $i = k < j < l$ are in σ_4 one has $|(Te_{ij})_{kl}| < \delta$. Proceeding in this way we finish. ■

LEMMA 18. *Given $n \in \mathbb{N}$, $\varepsilon > 0$, there exists an $N'(n, \varepsilon)$ such that if $N > N'(n, \varepsilon)$ and E is an n -dimensional subspace of \mathbb{T}_N then there exists a subspace F of \mathbb{T}_N and a block projection q , satisfying (2), from \mathbb{T}_N onto F such that $\|qx\| \leq \varepsilon\|x\|$ for $x \in E$.*

Proof. It is enough to show that if $x \in \mathbf{T}_n$, $\|x\| = 1$ then we can find q , a large block projection that respects triangularity, such that $\|q(x)\| \leq \varepsilon$. Then take an ε -net of the unit sphere of E , $\{x_i\}_{i=1}^M$. Find q_1 a large block projection satisfying (4) such that $\|q_1(x_1)\| < \varepsilon$; after this find q_2 a block projection contained in the range of q_1 such that $\|q_2(q_1x_2)\| < \varepsilon$. Proceeding in this way we get $q = q_M \dots q_2q_1$; and q does it.

To check the previous remark let $x \in \mathbf{T}_N$, $\|x\| = 1$, $\delta > 0$ (to be fixed latter), and let

$$\{i, j\} \text{ is bad if } i < j \text{ and } |x_{ij}| \geq \delta.$$

Ramsey's Theorem gives us a large monochromatic subset ρ . If ρ were *bad* we would set $i = \min \rho$ and then

$$\|x(e_i)\|^2 \geq \sum_{j \in \rho \setminus \{i\}} |x_{ji}|^2 \geq (|\rho| - 1)\delta^2.$$

Since, $\|x(e_i)\| \geq 1$, the right choice of δ would give us a contradiction, hence, ρ is *good*. ■

We conclude with some comments on the proof of the following proposition,

PROPOSITION 18. $B(H)/\text{Alg } \mathcal{N}$ is primary.

The proof is similar to the one for $\text{Alg } \mathcal{N}$. We have an isomorphic representation for $B(H)/\text{Alg } \mathcal{N}$ similar to the one for $\text{Alg } \mathcal{N}$; i.e.,

$$B(H)/\text{Alg } \mathcal{N} \approx \left(\sum_{n=1}^{\infty} \oplus M_n/\mathbf{T}_n \right)_{\infty},$$

where \mathbf{T}_n is the algebra of all upper triangular $n \times n$ matrices. This follows from the proof of the main result in [5]. The reason for this is that the maps $\varphi_n : \text{Alg } \mathcal{N} \rightarrow \mathcal{A}_n$ and $\psi_n : \mathcal{A}_n \rightarrow \text{Alg } \mathcal{N}$ from [19] are restrictions from $\tilde{\varphi} : B(H) \rightarrow \mathcal{B}_n$ and $\tilde{\psi}_n : \mathcal{B}_n \rightarrow B(H)$, where \mathcal{B}_n is the enveloping algebra of \mathcal{A}_n . Therefore, this induces maps $\varphi'_n : B(H)/\text{Alg } \mathcal{N} \rightarrow \mathcal{B}_n/\mathcal{A}_n$ and $\psi'_n : \mathcal{B}_n/\mathcal{A}_n \rightarrow B(H)/\text{Alg } \mathcal{N}$ with the right properties.

Once we have the decomposition, the combinatorial argument is essentially the same.

6. APPLICATIONS AND OPEN QUESTIONS

We conclude this paper with some applications and open questions. The first one is that the nest algebras in the trace class have bases.

It is well known that c_1 has a basis. We take the elements along the “shell” decomposition $\{F_n\}_n$ (See the proof of Theorem 1 for the notation) of c_1 ; i.e.

$$e_{11}, e_{12}, e_{22}, e_{21}, e_{13}, e_{23}, e_{33}, e_{32}, e_{31}, \dots \text{ etc.}$$

Therefore, if $\mathcal{S} \subset c_1$ is a $*$ -diagram, (i.e., for i, j fixed either $e_{ij} \in \mathcal{S}$ or for every $A \subset \mathcal{S}$ we have $(Ae_j, e_i) = 0$), it has a basis. This is so because we are just taking a subsequence of the basis.

This is the principle we use to prove:

THEOREM 19. *The nest algebras in c_1 have bases.*

Proof. We divide the proof in two cases: If the nest is uncountable we use Theorem 7 and the following representation. Index the basis of ℓ_2 by the rational numbers in $[0,1]$, $\{e_r\}_r$. Then for $0 \leq t \leq 1$, let $N_t = [e_r : r \leq t]$ and $N_t^- = [e_r : r < t]$. It is clear that $\mathcal{N} = \{N_t, N_t^-\}_{0 \leq t \leq 1}$ is an uncountable nest algebra. Therefore, if $\varphi : \mathbb{N} \rightarrow \mathbb{Q} \cap [0, 1]$ is a one-to-one and onto map and $U : \ell_2 \rightarrow \ell_2(\mathbb{Q} \cap [0, 1])$ is defined by $Ue_i = e_{\varphi(i)}$, we have that $U^{-1}(\text{Alg } \mathcal{N})^1 U$ is a $*$ -diagram in c_1 . Therefore, it has a basis.

If \mathcal{N} is countable we use the representation theorem from [11]. For simplicity we do it only for the multiplicity free case, although the proof for the general case is basically the same.

Assume then that our nest is the standard nest on $L_2([0, 1], \mu)$ for some measure μ . Since \mathcal{N} is countable we have that μ is totoally atomic. Therefore, if we do a similar construction as above, we see that it is unitarily equivalent to a $*$ -diagram and therefore it has a basis. ■

REMARK. Most of the results of this paper work in the space of compact operators with some minor notational changes. In particular, Theorem 5, part of Theorem 6, Proposition 7 and Theorem 19 all hold in $K(H)$.

We conclude this three questions.

QUESTION 1. If \mathcal{N} is a countable nest, does $(\text{Alg } \mathcal{N})^1$ correspond to some $T^1(\alpha)$?

This question is motivated by the analogy between the classification of the space of continuous functions on compact metric spaces and the nest algebras in c_1 . Recall that (\mathcal{N}, \leq) is a compact space that can be taken inside $[0,1]$. If \mathcal{N} is uncountable, then $C(\mathcal{N}) \approx C([0, 1])$ and $(\text{Alg } \mathcal{N})^1 \approx T^1(\mathbb{R})$. If \mathcal{N} is countable, then $C(\mathcal{N}) \approx C(\omega^\alpha)$ where α is the smallest ordinal number for which $\mathcal{N}^{(\alpha)}$, (the α th derived set of \mathcal{N}), is finite. We cannot reproduce the previous result exactly, because for the nest algebra

case it matters if the limit points are one-sided or two-sided. For example, if we take $A_1 = \{1/2 - 1/n\}_n$ and $A_2 = \{1/2 \pm 1/n\}_n$ then $C(A_1) \approx C(A_2)$ but $\mathbf{T}^1(A_1) \approx \mathbf{T}^1(\omega)$ and $\mathbf{T}^1(A_2) \approx \mathbf{T}^1(2\omega)$. Nevertheless, they correspond to some $\mathbf{T}^1(\alpha)$. The problem seems to be at the limit points, i.e., if $\mathcal{N}(\alpha)$ is finite and α is a limit point, say ω .

QUESTION 2. Are there uncountable many non-isomorphic $\mathbf{T}^1(\alpha)$'s?

In particular we are asking if $\mathbf{T}^1(\alpha) \approx \mathbf{T}^1(\alpha^\omega)$. A first step to question 2 is

QUESTION 3. is $\mathbf{T}^1(\omega^2) \approx \mathbf{T}^1(\omega^\omega)$?

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