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SPECTRAL INVARIANCE AND TAMENESS OF PSEUDO-DIFFERENTIAL OPERATORS ON WEIGHTED SOBOLEV SPACES

JOSEFINA ALVAREZ and JORGE HOUNIE

0. INTRODUCTION

A smooth real function $\gamma \geqslant 1$ on $\mathbf{R}^n$ is called a *weight* if all its derivatives of positive order are bounded. Given any real number t we may consider the density $\mathrm{d}\mu = \gamma^t \mathrm{d}x$ where $\mathrm{d}x$ denotes the Lebesgue measure. The Sobolev spaces $H^{s;t}_{p;\gamma}$ based on this density are called *weighted Sobolev spaces,* where s measures regularity and p indicates the exponent of integrability (see Definition 1.5 for details). They behave quite well under the action of pseudo-differential operators and constitute a suitable framework for the study of linear and nonlinear partial differential equations (we consider pseudo-differential operators in the class $\mathcal{L}^m_{\rho,\delta}$, (cf. [9], [11], [1]), $m \in \mathbf{R}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $\delta \leqslant \rho$). In this article we study systematically two properties for pseudo-differential operators acting on weighted Sobolev spaces:

i) spectral invariance and

ii) tameness of the basic operations needed in order to apply the Nash-Moser implicit function theorem.

Concerning spectral invariance, Schrohe proved in [12] that the $H^{s;t}_{p;\gamma}$-spectrum of an operator in $\mathcal{L}^0_{1,\delta}$ is independent of the choice of $1 < p < \infty$, $s \in \mathbf{R}$, $t \in \mathbf{R}$, and the weight γ. He also showed that the spectrum of an operator in $\mathcal{L}^0_{\rho,\delta}$ does not change with s, t, γ if $\rho < 1$ and $p = 2$. Here we study the dependence on p of the spectrum when $\rho < 1$ and show that there is a well determined interval $I = [p_0, p_0']$ around $p = 2$ such that the spectrum remains invariant for $p \in I$ but changes, in general, for p outside I.

Our study of tameness was motivated by the work of Goodman and Yang [8] on local solvability of general nonlinear operators of real principal type where pseudo-differential and Fourier integral operators with classical symbols are used as right inverses for the linearization of nonlinear differential operators and tame estimates are proved in the context of standard L^2-based Sobolev spaces. Here we show how to extend these properties to pseudo-differential operators in Hörmander's class acting on scales of weighted Sobolev spaces. As an application, we prove local solvability for the semilinear equation with complex coefficients in $\mathbf{R}^2$

$$P(x,t,D_x,D_t)u + F(x,t,u,\ldots,D^{\alpha}_{x,t}u) = f(x,t), \quad |\alpha| \leqslant m-1,$$

where P is a homogeneous linear differential operator of order $m \geqslant 1$ with smooth complex coefficients, F is a complex-valued function, holomorphic in $u,\ldots,D^{\alpha}u$, for $|\alpha| \leqslant m-1$, and smooth in (x,t) and $f \in C^{\infty}_c(\mathbf{R}^2)$. We assume that $P(x,t,D_x,D_t)$ satisfies Trèves'positivity condition $(\mathcal{R})$ [15] which is necessary (but not sufficient) for the hypoellipticity of P. Local solvability for this equation was proved by Dehman [4] under the stronger assumption that P is subelliptic. In our proof, the construction of right inverses for the linearized operator involve pseudo-differential operators in the class $\mathcal{L}^0_{1,1/2}$.

Unless otherwise specified, the functions we will consider are defined on $\mathbf{R}^n$ with complex values. They will be called *smooth* to mean that they are of class C^{∞}. The Bessel potential of order α is denoted by J^{α} and L^p_s indicates the usual Sobolev space of order s and exponent p (L^p_0 will be identified the Lebesgue space L^p, $1 < p < \infty$). As usual, $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'$ its dual, the space of tempered distributions. Given two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of linear, continuous operators from X to Y. When $X = Y$ we write $\mathcal{L}(X)$ rather than $\mathcal{L}(X,X)$. The identity operator will be denoted by I. Given an exponent $1 < p < \infty$, p' denotes the conjugate exponent, $1/p + 1/p' = 1$. The paper is organized as follows:

Section 1. Preliminary results
Section 2. Spectral invariance
Section 3. Holomorphic functional calculus
Section 4. Tame scales of Banach spaces
Section 5. Tame estimates
Section 6. A class of solvable semilinear equations
Section A. A tame right inverse for L

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1. PRELIMINARY RESULTS

We will consider pseudo-differential operators in the class $\mathcal{L}^m_{\rho,\delta}$, (cf. [9], [11], [1]), $m \in \mathbf{R}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$. These are operators L of the form

$$Lf(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi, \quad f \in S, \tag{1.1}$$

where n is the dimension of the euclidean space. The function $p(x,\xi)$, uniquely determined by L and called the symbol of L, is assumed to belong to the class $S^m_{\rho,\delta}$. This means that it is a smooth function satisfying the estimates

$$|D^\alpha_x D^\beta_\xi p(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}. \tag{1.2}$$

THEOREM 1.1. *Let $L \in \mathcal{L}^m_{\rho,\delta}$, $m \in \mathbf{R}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$. Then $L \in \mathcal{L}(L^p)$, provided that, $m \leqslant m_p = -n(|1/p - 1/2|(1-\rho)+\lambda)$, $\lambda = \max(0, (\delta-\rho)/2)$, $1 < p < \infty$.*

We omit the proof of this theorem (cf. [6] when $\delta < \rho$, [1] when $\delta \geqslant \rho$).

COROLLARY 1.2. *Let $L \in \mathcal{L}^m_{\rho,\delta}$, $m \in \mathbf{R}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$. Then*

a) *$L \in \mathcal{L}(L^p_s, L^p_r)$, provided that $m \leqslant s - r + m_p$, m_p as above, $1 < p < \infty$.*

b) *$L \in \mathcal{L}(L^q_s, L^q_r)$, provided that $m \leqslant s - r + m_p$, $1 < p \leqslant 2$, $p \leqslant q \leqslant p'$.*

Proof. a) Let us write $L = J^r(J^{-r}LJ^s)J^{-s}$. The calculus of pseudo-differential operators ([9], [11]) shows that the term between parentheses is in $\mathcal{L}^{m+r-s}_{\rho,\delta}$. Since $m + r - s \leqslant m_p$, this term is bounded in L^p by Theorem 1.1. It only remains to observe that J^t is an isomorphism between L^p and L^p_t.

b) The condition $p \leqslant q \leqslant p'$ implies that $m_p \leqslant m_q$ and then $m \leqslant s - r + m_p \leqslant s - r + m_q$. Thus, $L \in \mathcal{L}(L^q_s, L^q_r)$. ■

DEFINITION 1.3. (cf. [12]) A weight is a smooth function γ satisfying the following conditions

a) $\gamma(x) \geqslant 1$, $x \in \mathbf{R}^n$,

b) $D^\alpha\gamma(x) = O(1)$, $\alpha \neq 0$.

We will also denote by γ the operator of multiplication by γ which is clearly continuous in $\mathcal{S}$. Given linear continuous operators A and $B \in \mathcal{L}(\mathcal{S})$, their commutator is denoted by $[A,B] = AB - BA$.

LEMMA 1.4. (cf. [12]) *Given $L \in \mathcal{L}^m_{\rho,\delta}$, $m \in \mathbf{R}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, the commutator $[\gamma, L]$ belongs to $\mathcal{L}^{m-\rho}_{\rho,\delta}$.*

Proof. Given $f \in S$ we have

$$[\gamma, L]f(x) = (2\pi)^{-n} \iint \exp[i(x-y)\cdot\xi]p(x,\xi)(\gamma(x)-\gamma(y))f(y)dyd\xi,$$

in the sense of oscillatory integrals. Now,

$$\gamma(x)-\gamma(y) = \int_0^1 (x-y)\cdot\nabla\gamma(y+s(x-y))ds,$$

so using this expression in the above integral and integrating by parts yields

$$[\gamma,\ L]f(x) = (2\pi)^{-n} \iint \exp[i(x-y)\cdot\xi]\nabla_\xi p(x,\xi)\cdot \int_0^1 \nabla\gamma(y+s(x-y))ds\, f(y)\, dy\, d\xi.$$

The amplitude in the last integral belongs to $S^{m-p}_{\rho,\delta}$ so it defines an operator in $\mathcal{L}^{m-p}_{\rho,\delta}$. This proves the lemma.

It is clear that if we consider higher order commutators of L with γ, the commutator of order j belong to $\mathcal{L}^{m-j\rho}_{\rho,\delta}$. ■

DEFINITION 1.5. (cf [12]) Given $s,t \in \mathbf{R}$ and a weight γ, we define

$$H^{s,t}_{p,\gamma} = \{\gamma^{-t}f,\ f \in L^p_s\}.$$

It becomes a Banach space with the norm

$$\|g\|_{H^{s,t}_{p,\gamma}} = \|\gamma^t g\|_{L^p_s}.$$

We will often omit the dependence on γ writing just $H^{s,t}_p$. It is clear that if $\gamma \equiv 1$ or $t = 0$ we obtain the usual Sobolev spaces $H^s_p = L^s_p$. The following result extends Theorem 1.7 in [12].

THEOREM 1.6. *Let $L \in \mathcal{L}^m_{\rho,\delta}$, $m \in \mathbf{R}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $\delta \leqslant \rho$. Then $L \in \mathcal{L}(H^{s,t}_{q,\gamma}, H^{r,t}_{q,\gamma})$ provided that $1 < p \leqslant 2$, $m \leqslant s - r + m_p$, $m_p = -n(1-\rho)(1/p - -1/2)$, $p \leqslant q \leqslant p'$, $t \in \mathbf{R}$.*

Proof. Since γ^t is an isometry from $H^{s,t}_{p,\gamma}$ onto L^p_s we need only show that $\gamma^t L\gamma^{-t}$ satisfies the hypothesis of Corollary 1.2. Following [12], assume first that t is an integer. Let us use induction on $k = |t|$ to prove that

$$\gamma^t L \gamma^{-t} \in \mathcal{L}^m_{\rho,\delta}$$

with m, ρ, δ satisfying the hypotheses of Corollary 1.2. If $k = 0$ this is true from the hypothesis on L. Assume it has been proved for $|t| = k$ and let $|t| = k+1$. By the inductive assumptions $L' = \gamma^k L \gamma^{-k}$ and $L'' = \gamma^{-k} L \gamma^k$ belong to $\mathcal{L}^m_{\rho,\delta}$. If $t = k+1$ we write $A = \gamma^t L \gamma^{-t} = L' + [\gamma, L']\gamma^{-1}$ and if $t = -k-1$ we write $A = L'' + \gamma^{-1}[L'', \gamma]$. Now, using Lemma 1.4, $[\gamma,\ L']$ and $[\gamma,\ L'']$ are in $\mathcal{L}^{m-p}_{\rho,\delta}$ and it is also plain that $\gamma^{-1} \in \mathcal{L}^0_{1,0}$. Thus, $A \in \mathcal{L}^m_{\rho,\delta}$. Finally, if t is real, let $k = [|t|] + 1$ be the least integer $\geqslant |t|$. Then, $\gamma^t = \gamma_1^{\pm k}$ with $\gamma_1 = \gamma^{|t|/k}$. Since $0 < |t|/k \leqslant 1$, γ_1 is itself a weight and we may reason as before. The proof also shows that the symbol of $\gamma^t L \gamma^{-t}$ is expressible in terms of the derivatives up to order $[|t|] + 1$ of the symbol of L. ∎

COROLLARY 1.7. *The norms* $\|J^s \gamma^t g\|_{L^p}$ *and* $\|\gamma^t J^s g\|_{L^p}$ *are equivalent on* $H^{s,t}_p$, $1 < p < \infty$.

Proof. One must show that the operator $L = \gamma^{-t} J^{-s} \gamma^t J^s$ is an isomorphism of $H^{s,t}_p$. Since $J^{-s} \in L^{-s}_{1,0}$, the proof above shows that $\gamma^{-t} J^{-s} \gamma^t \in L^{-s}_{1,0}$. Hence $L \in L^0_{1,0}$ which implies that $L \in \mathcal{L}(H^{s,t}_p)$. In the same way, $L^{-1} \in \mathcal{L}(H^{s,t}_p)$.

When $s = k$ is a nonnegative integer one checks that the norms $\sum_{|\alpha| \leqslant k} \|D^\alpha(\gamma^t g)\|_{L^p}$ and $\sum_{|\alpha| \leqslant k} \|\gamma^t D^\alpha g\|_{L^p}$ also define the topology of $H^{k,t}_p$. ∎

2. SPECTRAL INVARIANCE

Let us first recall the action of two basic commutators (cf. [2]). Given a linear continuous operator $L : \mathcal{S} \to \mathcal{S}'$ we consider

$$P_j L = [D_j, L] = D_j L - L D_j, \tag{2.1}$$

$$Q_j L = [-\mathrm{i}x_j, L] = \mathrm{i}L x_j - \mathrm{i}x_j L. \tag{2.2}$$

If $L \in \mathcal{L}^m_{\rho,\delta}$ and $f \in \mathcal{S}$ then

$$P_j L f(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} \frac{\partial p}{\partial \xi_j}(x,\xi) \hat{f}(\xi) \mathrm{d}\xi$$

$$Q_j L f(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} \frac{\partial p}{\partial x_j}(x,\xi) \hat{f}(\xi) \mathrm{d}\xi$$

If α, β are multi-index it turns out that

$$Q^\alpha P^\beta L = Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_n^{\alpha_n} P_1^{\beta_1} \cdots P_n^{\beta_n} L$$

belongs to $\mathcal{L}^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$. The next theorem of J. Ueberberg [16] generalizes a famous characterization of pseudo-differential operators due to R. Beals [2].

THEOREM 2.1. ([16, p.463]) *Let $A : S \to S'$ be a linear and continuous operator. Suppose that for some $m \in \mathbf{R}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $1 < p < \infty$,*

$$Q^\alpha P^\beta A \in \mathcal{L}(H_p^{s+m-\rho|\alpha|+\delta|\beta|},\ H_p^s)$$

for all $s \in \mathbf{R}$ and all multi-indexes α, β. Then, $A \in \mathcal{L}^m_{\rho,\delta}$.

It is important to point out that, because in Corollary 1.2 the hypotheses on the order of L are sharp, the converse of Theorem 2.1 is not true unless $p = 2$ or $\rho = 1$. The equivalence for $p = 2$ was proved by Beals [2] for a wider class of pseudo-differential operators.

PROPOSITION 2.2. (*Spectral invariance for fixed γ, p.*) *Let $A \in \mathcal{L}^m_{\rho,\delta}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $m \leqslant m_p = -n(1-\rho)|1/p - 1/2|$, $1 < p < \infty$. If $A - \lambda I$ is invertible in $\mathcal{L}(H^{s_0,t_0}_{p,\gamma})$ for some $\lambda \in \mathbf{C}$, s_0, $t_0 \in \mathbf{R}$, γ a weight, then $A - \lambda I$ is invertible in $\mathcal{L}(H^{s,t}_{p,\gamma})$ for every $s, t \in \mathbf{R}$.*

The proof follows from an obvious modification of Theorem 1.8 in [12] using Corollary 1.2 a).

PROPOSITION 2.3. *Let $A \in \mathcal{L}^m_{\rho,\delta}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $m \leqslant m_p = -n(1-\rho)|1/p - 1/2|$, $1 < p < \infty$. If $A - \lambda I$ is invertible in $\mathcal{L}(H^{s_0,t_0}_{p,\gamma})$ for some $\lambda \in \mathbf{C}$, s_0, $t_0 \in \mathbf{R}$, γ a weight, then $(A - \lambda I)^{-1}$ belongs to $\mathcal{L}^0_{\rho,\delta}$.*

Proof. Applying Proposition 2.2 with $t = 0$ we derive that $R = (A - \lambda I)^{-1} \in \mathcal{L}(H^s_p)$ for any $s \in \mathbf{R}$. Using the notation in (2.1) we have

$$P_j R = -RP_j(A - \lambda I)R = -RP_j(A)R$$

with a similar formula for $Q_j R$. Thus, given multi-indexes α, β, $Q^\alpha P^\beta R$ can be expressed as a sum of products having as factors R and commutators with A. Induction on $|\alpha| + |\beta|$ shows that $Q^\alpha P^\beta R$ belongs to $\mathcal{L}(H_p^{s+m-\rho|\alpha|+\delta|\beta|},\ H_p^s)$ which, in Theorem 2.1, implies that $R = (A - \lambda I)^{-1} \in \mathcal{L}^0_{\rho,\delta}$. ∎

REMARK 2.4. *No operator A will satisfy the hypothesis of Proposition 2.3 with $\lambda = 0$ when $\rho < 1$ and $p \neq 2$. Indeed, in this case, $I = AA^{-1}$ would belong to $\mathcal{L}^m_{\rho,\delta}$ with $m < 0$, a contradiction.*

Schrohe [12] proved that the spectrum of A is also independent of γ if $p = 2$ and independent of γ and $1 < p < \infty$ if $\rho = 1$. The next theorem extends these results.

THEOREM 2.5. (*Global spectral invariance*) *Let $A \in \mathcal{L}^m_{\rho,\delta}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $m \leqslant m_{p_0} = -n(1-\rho)|1/p_0 - 1/2|$, $1 < p_0 \leqslant 2$. If $A - \lambda I$ is invertible in $\mathcal{L}(H^{s_0,t_0}_{p_0,\gamma_0})$ for some $\lambda \in \mathbf{C}$, $s_0, t_0 \in \mathbf{R}$, γ_0 a weight, then*

a) $A - \lambda I$ is invertible in $\mathcal{L}(H^{s,t}_{p,\gamma})$ for any $s, t \in \mathbf{R}$, $p_0 \leqslant p \leqslant p_0'$, γ a weight,

b) *if $\rho < 1$ the range of p in* a) *is optimal.*

Proof: If $\rho = 1$ or $p_0 = 2$, a) follows from Proposition 2.3 and Corollary 1.2. Let's assume then that $\rho < 1$ and $1 < p_0 < 2$. We will prove

$$(A - \lambda I)^{-1} \in \mathcal{L}^0_{1,0} + \mathcal{L}^m_{\rho,\delta} \tag{2.3}$$

Since $\mathcal{L}^0_{1,0}$ and $\mathcal{L}^m_{\rho,\delta}$ are both contained in $\mathcal{L}(H^{s,t}_{p,\gamma})$ (2.3) proves a). To show (2.3) observe that, since the symbol $a(x,\xi)$ of A has negative order and by Remark 2.4 $\lambda \neq 0$, there exists $M > 0$ such that

$$|a(x,\xi)| \leqslant |\lambda|/2, \quad x \in \mathbb{R}^n, \ |\xi| \geqslant M.$$

Let $\varphi(\xi)$ be a smooth cut-off function equal to 1 for $|\xi| > 2M$ and vanishing for $|\xi| \leqslant M$. Then,

$$b_\lambda(x,\xi) \stackrel{\text{def}}{=} \frac{\varphi(\xi)}{a(x,\xi) - \lambda} = -\frac{\varphi(\xi)}{\lambda}(1 + \lambda^{-1}a(x,\xi) + c_\lambda(x,\xi))$$

with $c_\lambda(x,\xi) \in S^{2m}_{\rho,\delta}$. In particular, $b_\lambda \in S^0_{1,0} + S^m_{\rho,\delta}$. Furthermore, since $\nabla\varphi$ is compactly supported, $\nabla_\xi b_\lambda \in S^{m-\rho}_{\rho,\delta}$. Let $B_\lambda = b_\lambda(x,D)$ be the operator with symbol b_λ and let $d(x,\xi)$ be the symbol of the composition $B_\lambda(A-\lambda I)$. By the symbolic calculus of pseudo-differential operators

$$d(x,\xi) = b_\lambda(x,\xi)(a(x,\xi) - \lambda) + r(x,\xi) = \varphi(\xi) + r(x,\xi) \tag{2.4}$$

where $r(x,\xi)$ is given by the oscillatory integral

$$r(x,\xi) = \frac{1}{\mathrm{i}(2\pi)^n} \int_0^1 \iint \mathrm{e}^{-\mathrm{i}(x-z)\cdot(\xi-\eta)} \nabla_\xi b_\lambda(x, \xi + s(\xi - \eta)) \cdot \nabla_z a(z,\xi) \mathrm{d}z\mathrm{d}\eta\mathrm{d}s. \tag{2.5}$$

Since $\nabla_\xi b_\lambda \in S^{m-\rho}_{\rho,\delta}$, $\nabla_x a \in S^{m+\delta}_{\rho,\delta}$ and $\rho \geqslant \delta$, it follows that $r \in S^{2m}_{\rho,\delta}$. It is also clear that $\varphi - 1 \in S^{-\infty}$ so (2.4) and (2.5) show that $d(x,\xi) \in 1 + S^{2m}_{\rho,\delta}$ or, equivalently, $B_\lambda(A - \lambda I) \in I + \mathcal{L}^{2m}_{\rho,\delta}$. This gives

$$B_\lambda - (A - \lambda I)^{-1} \in \mathcal{L}^{2m}_{\rho,\delta}(A - \lambda I)^{-1} \subset \mathcal{L}^{2m}_{\rho,\delta}\mathcal{L}^0_{\rho,\delta} \subset \mathcal{L}^{2m}_{\rho,\delta}$$

where we have used Proposition 2.3. Since $B_\lambda \in \mathcal{L}^0_{1,0} + \mathcal{L}^m_{\rho,\delta}$ this proves (2.3).

We now prove b). To simplify the notation we write p instead of p_0. We will consider the multiplier of Hardy-Littlewood-Hirschman-Wainger (cf. [17])

$$a(\xi) = \psi(\xi)^{m_p} \exp(\mathrm{i}|\xi|^{1-\rho}) \in S^{m_p}_{\rho,0},$$

where $m_p = -n(1-\rho)(1/p - 1/2)$, $1 < p \leqslant 2$, $0 < \rho < 1$ and ψ^{m_p} is a smooth function vanishing for $|\xi| \leqslant 1$ and equal to $|\xi|^{m_p}$ for $|\xi| \geqslant 2$. Assume first that $p < 2$. Consider some $0 \neq \lambda \in \mathbb{C}$ for which the operator with symbol $a(\xi) - \lambda$ is invertible in $\mathcal{L}(H^{s_0,t_0}_{p,\gamma_0})$ for some choice of the parameters. By the first part of the theorem it will also be invertible in $\mathcal{L}(H^s_p)$ for every $s \in \mathbb{R}$. Hence, $a(\xi) - \lambda$ is bounded away from zero and we have

$$\frac{1}{a(\xi)-\lambda} = -\frac{1}{\lambda}(1 + \lambda^{-1}a(\xi) + c_\lambda(\xi))$$

where $c_\lambda = a^2\lambda^{-2}/(a-\lambda) \in S^{2m_p}_{\rho,0}$. Let us fix $1 < p_1 < p$. According to Corollary 1.2 the operator with symbol c_λ will belong to $\mathcal{L}(H^s_{p_1})$ provided that $2m_p \leqslant m_{p_1}$. This inequality holds for $p_1 = p - \varepsilon$, $0 < \varepsilon \leqslant p(2-p)/(4-p)$, $1 < p < 2$. Since $a(\xi)$ defines an operator that is unbounded in $H^0_{p_1} = L^{p_1}$ for $p_1 < p_2$, we conclude that the operator with symbol equal to $a(\xi) - \lambda$ is unbounded in $H^0_{p_1}$, for values of $p_1 < p$ arbitrarily close to p. Using interpolation we conclude that it is also unbounded for any $1 < p_1 < p$. The same argument applies to $p' < p_1 < \infty$.

Finally, assume that $p = 2$. This implies that $m_p = 0$. Modifying slightly $a(\xi)$ we may assume that it is bounded away from zero (and equal to $\exp(\mathrm{i}|\xi|^{1-\rho})$ for large $|\xi|$). Hence 0 is not in the H^s_2-spectrum of the operator defined by $a(\xi)$ and the inverse has a symbol equal to $\exp(-\mathrm{i}|\xi|^{1-\rho})$ for large $|\xi|$. In particular, it is unbounded in H^0_p for $p \neq 2$. ∎

3. HOLOMORPHIC FUNCTIONAL CALCULUS

The methods of the previous section can be used to precise a holomorphic functional calculus for pseudo-differential operators in appropriate classes. Indeed, let $A \in \mathcal{L}^m_{\rho,\delta}$, $\delta \leqslant \rho$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $m \leqslant m_{p_0} = -n(1-\rho)|1/p_0 - 1/2|$, $1 < p_0 \leqslant 2$. Under these conditions, Theorem 2.5 shows that the spectrum $\sigma(A)$ as an operator in $\mathcal{L}(H^{s;t}_{p;\gamma})$ is independent of s, t, γ, p provided that $p_0 \leqslant p \leqslant p_0'$. The resolvent function $R(z;A) = (zI - A)^{-1}$ defines a holomorphic function on the resolvent set $\rho(A)$ with values in $\mathcal{L}(H^{s;t}_{p;\gamma})$. Let $f(z)$ be a holomorphic function defined in a neighborhood of $\sigma(A)$ and let U be an open subset of the domain of f containing $\sigma(A)$. Assume further that the boundary ∂U of U consists of a finite number of rectifiable Jordan curves, counter-clockwisely oriented. Then, the Dunford integral ([18, p.225])

$$f(A) = \frac{1}{2\pi\mathrm{i}} \int_{\partial U} f(z)R(z;A)\mathrm{d}z \tag{3.1}$$

defines an operator $f(A) \in \mathcal{L}(H^{s;t}_{p;\gamma})$, $p_0 \leqslant p \leqslant p_0'$. By Cauchy's theorem, $f(A)$ is independent of the choice of U. The map $(f, A) \to f(A)$ given by (3.1) enjoys the typical properties of a functional calculus.

LEMMA 3.1. *Under the above hypotheses,* $f(A) \in \mathcal{L}^0_{\rho,\delta}$.

Proof. By Proposition 2.3 $R(z; A)$ is a continuous function on ∂U with values in $\mathcal{L}^0_{\rho,\delta}$ so (3.1) implies the lemma. ■

THEOREM 3.2. *Under the above hypotheses,* $f(A) \in \mathcal{L}^0_{1,0} + \mathcal{L}^m_{\rho,\delta}$.

Proof. If $\rho = 1$ and $m = 0$ we have $\mathcal{L}^0_{1,0} + \mathcal{L}^m_{\rho,\delta} = \mathcal{L}^0_{1,\delta}$ so the result follows from Lemma 3.1 with $\rho = 1$. If $m < 0$ (this is implied by the hypothesis when $\rho < 1$) we know (cf. Remark 2.4) that 0 is in $\sigma(A)$. Thus, we can write $f(z) = f(0) + zg(z)$ obtaining (cf.[18])

$$f(A) = f(0)I + Ag(A).$$

Applying Lemma 3.1 to g, we see that this is a decomposition in $\mathcal{L}^0_{1,0} + \mathcal{L}^m_{\rho,\delta}$. ■

4. TAME SCALES OF BANACH SPACES

A scale of Banach spaces $\mathcal{H} = \{\mathcal{H}^k\}$, $k = 0, 1, 2, \dots$ is a collection of Banach spaces such that $\mathcal{H}^{k+1} \subset \mathcal{H}^k$ and $||h||_k \leqslant ||h||_{k+1}$, $h \in \mathcal{H}^{k+1}$. The intersection $\bigcap_k \mathcal{H}^k$ is denoted $\mathcal{H}^\infty$ and becomes a Frechet space with the projective topology.

DEFINITION 4.1. We say that $\mathcal{H}$ satisfies a convexity condition if for all $j < l \in \mathbb{Z}^+$ there exist $C = C(j, l)$ such that if $\mathbb{Z} \ni k = \alpha j + (1 - \alpha) l$, $0 < \alpha < 1$, then

$$||h||_k \leqslant C ||h||_j^\alpha ||h||_l^{1-\alpha}, \quad h \in \mathcal{H}^l \tag{4.1}$$

REMARKS.

i) *Observe that when* $C = 1$, (4.1) *means that* $j \to \ln ||h||_j$ *is a convex function of* j.

ii) *When proving inequalities* (4.1) *it is enough to check the cases where* j, k, l *are contiguous, i.e, when* $j = k - 1$, $l = k + 1$, $\alpha = 1 - \alpha = 1/2$, *because then the general case follows by iteration.*

DEFINITION 4.2. We say that $\mathcal{H}$ is tame if there exists a 1-parameter family of linear smoothing operators

$$S_\theta : \mathcal{H}^0 \to \mathcal{H}^\infty, \quad \theta \geqslant 1,$$

satisfying

T1) $\|S_\theta u\|_k \leqslant C_k \theta^{k-j} \|u\|_j,\ u \in \mathcal{H}^j,\ j \leqslant k,$

T2) $\|u - S_\theta u\|_j \leqslant C_k \theta^{j-k} \|u\|_k,\ u \in \mathcal{H}^k,\ j \leqslant k,$

T3) $\lim_{\theta \to \infty} \|u - S_\theta u\|_k = 0,\ u \in \mathcal{H}^k.$

The following two propositions will be useful in the sequel. For the proof we refer, for instance, to [8].

PROPOSITION 4.3. *If $\mathcal{H}$ possesses a 1-parameter family of smoothing operators satisfying* T1) *and* T2), *then it verifies a convexity condition.*

PROPOSITION 4.4. *Let $\mathcal{H}, \mathcal{G}$ be scales of Banach spaces satisfying a convexity condition. If $j < k$, $l < m$ are such that $j + m = k + l$ then*

$$\|u\|_k \|v\|_l \leqslant C_m (\|u\|_j \|v\|_m + \|u\|_m \|_j), \quad u \in \mathcal{H}^m,\ u \in \mathcal{G}^m. \tag{4.2}$$

LEMMA 4.5. *Let γ be a weight* (cf. Definition 1.3). *There is $C > 0$ such that*

$$\gamma(x + y) \leqslant (1 + C|y|)\gamma(x), \quad x, y \in \mathbf{R}^n.$$

If $t \in \mathbf{R}$ we have

$$\gamma^t(x) \leqslant (1 + C|y|)^{|t|} \gamma^t(x - y), \quad x, y \in \mathbf{R}^n.$$

Proof. By Taylor's formula and the fact that $\gamma \geqslant 1$ has bounded derivatives

$$\gamma(x + y) \leqslant \gamma(x) + C|y| \leqslant \gamma(x) + C\gamma(x)|y| = \gamma(x)(1 + C|y|).$$

The second estimate follows from the first one in a standard way. ∎

PROPOSITION 4.6. *The scale $\{\mathcal{H}_k = H^{k,t}_{p,\gamma}\}$, $k = 0, 1, \ldots$, is tame if $1 \leqslant p < \infty$ and satisfies conditions* T1) *and* T2) *of* Definition 4.2 *if $p = \infty$. In particular, it verifies a convexity condition.*

Proof. Let $\varphi \geqslant 0 \in \mathcal{S}$ have a compactly supported Fourier transform $\hat{\varphi}(\xi)$ equal to 1 in a neighborhood of the origin. In particular,

$$\int \varphi(x)\mathrm{d}x = 1, \quad \int x^\alpha \varphi(x)\mathrm{d}x = 0,\ \alpha \neq 0.$$

As usual, we set $\varphi_\theta(x) = \theta^n \varphi(\theta x)$, $\theta \geqslant 1$, and define

$$S_\theta u = \varphi_\theta * u \tag{4.3.}$$

To prove T2) consider two non-negative integers $j < k$ and $u \in \mathcal{S}$. By Corollary 1.7, an equivalent norm in $\mathcal{H}_j = H^{j,t}_p$ is given by

$$\sum_{|\alpha| \leqslant j} \|\gamma^t D^\alpha u\|_{L^p}.$$

Set $v = D^\alpha u$ for $|\alpha| \leqslant j$ and write, according to (4.3),

$$S_\theta v(x) = \int v(x - y/\theta)\varphi(y)\mathrm{d}y.$$

Expanding v in Taylor series up to order $k - j$ around x we get

$$S_\theta v(x) = \sum_{|\beta|<k-j} \frac{\partial^\beta v}{\partial x^\beta}(x)\frac{(-\theta)^{-|\beta|}}{\beta!}\int y^\beta\varphi(y)\mathrm{d}y +$$

$$+\frac{(-\theta)^{-(k-j)}}{(k-j-1)!}\sum_{|\beta|=k-j}\int_0^1\int (1-s)^{k-j}y^\beta\varphi(y)\frac{\partial^\beta v}{\partial x}(x - sy/\theta)\mathrm{d}y\mathrm{d}s.$$

The first sum reduces to $v(x)$ so, using Lemma 4.5, we estimate $|\gamma^t D^\alpha(S_\theta u(x) - -u(x))|$ by

$$C\theta^{j-k}\sum_{|\beta|=k-j}\int_0^1\int (1-s)^{k-j}|y^\beta\varphi(y)(1 + C|sy/\theta|)^{|t|}\gamma^t(x - sy/\theta)D^\beta v(x - sy/\theta)\mathrm{d}y\mathrm{d}s.$$

Since $(1+C|ys/\theta|)^{|t|} \leqslant (1+C|y|)^{|t|}$ for $0 \leqslant s \leqslant 1$, $\theta \geqslant 1$ we obtain, by a variation of Young's inequality that

$$(4.4),\qquad \|\gamma^t D^\alpha(S_\theta u - u)\|_{L^p} \leqslant C\theta^{j-k}\sum_{|\beta|=k-j}\|\psi_\beta\|_{L^1}\|\gamma^t D^\beta v\|_{L^p} \leqslant C\theta^{j-k}\|u\|_{\mathcal{H}^k},$$

where we have written $\psi_\beta(y) = y^\beta(1 + C|y|^{|t|}\varphi(y)$. Adding estimates (4.4) over all $|\alpha| \leqslant j$ we obtain

$$\|S_\theta u - u\|_{\mathcal{H}^j} \leqslant C\theta^{j-k}\|u\|_{\mathcal{H}^k}$$

as required.

The proof of T1) is similar and simpler: when differentiating $u * \varphi_\theta$ one lets act at most $k - j$ derivatives on φ_θ and at most j derivatives on u. To prove T3) for $p < \infty$ it is enough to check that $S_\theta u \to u$ in $\mathcal{H}^k$ for $u \in \mathcal{S}$ and then use the density of $\mathcal{S}$ in $\mathcal{H}^k$.

The following result shows that the Gagliardo-Nirenberg inequality is valid in the scale $\{H^{k,t}_{p,\gamma}\}$ of weighted Sobolev spaces. ■

PROPOSITION 4.7. *Let γ be a weight, $t \in \mathbb{R}$. If $1 \leqslant q,\ r \leqslant \infty$ are real numbers, $l \leqslant j \leqslant k$ are integers and we write*

$$j = al + (1-a)k,$$

$$\frac{1}{p} = a\frac{1}{q} + (1-a)\frac{1}{r},$$

there exists a positive constant $C = C(q, r, k, t)$ *such that*

$$\|f\|_{H_p^{j,t}} \leqslant C\|f\|^a_{H_q^{l,pt/q}}\|f\|^{(1-a)}_{H_r^{k,pt/r}}, \quad f \in \mathcal{S} \tag{4.4}$$

Proof. The result follows by induction in k once it has been proved for $k = 2$. When $k = 2$ only the case $l = 0,\ j = 1,\ k = 2$ is relevant. Hence, $a = 1/2$ and $1/p = 1/2q + 1/2r$. We must show that

$$\|f\|_{L_1^p} \leqslant C\|f\|^{1/2}_{L^q}\|f\|^{1/2}_{L_2^r}$$

where the Sobolev norms are taken with respect to the measure $d\mu = \gamma^{tp}dx$. Since the argument is essentially one-dimensional we give the proof for $n = 1$ to simplify the notation. If $f \in \mathcal{S}(\mathbf{R})$ and $1 < p < \infty$ we have, in the sense of distributions,

$$\frac{d}{dx}(f|f'|^{p-2}\overline{f'}\gamma^{pt}) = |f'|^p\gamma^{pt} + (p-2)f|f'|^{p-4}\mathrm{Re}\,(f'\overline{f''})\overline{f'}\gamma^{pt} +$$

$$+ f|f'|^{p-2}\overline{f''}\gamma^{pt} + f|f'|^{p-2}\overline{f'}pt\gamma^{pt-1}\gamma'.$$

Integrating this with respect to dx we obtain

$$\int |f'|^p d\mu \leqslant (p-1)\int |f||f'|^{p-2}|f''|d\mu + Cp|t|\int |f||f'|^{p-1}\gamma^{-1}d\mu$$

$$= I_1 + CI_2,$$

with C depending only on γ. Since $1/q + 1/r + (p-2)/p = 1$, an application of Hölder's inequality gives

$$I_1 \leqslant (p-1)\|f\|_{L^q(\mu)}\|f'\|^{p-2}_{L^p(\mu)}\|f''\|_{L^r(\mu)} \leqslant (p-1)\|f\|_{L^q(\mu)}\|f\|^{p-2}_{L_1^p(\mu)}\|f\|_{L_2^r(\mu)}.$$

Estimating the integrand of I_2 by the triple product $|f||f'|^{p-2}|f''|$ and reasoning as before we get

$$I_2 \leqslant p|t|\|f\|_{L^q(\mu)}\|f'\|^{p-2}_{L^p(\mu)}\|f'\|_{L^r(\mu)} \leqslant p|t|\|f\|_{L^q(\mu)}\|f\|^{p-2}_{L_1^p(\mu)}\|f\|_{L_2^r(\mu)}.$$

Adding the estimates obtained for I_1 and I_2 yields

$$\|f'\|^p_{L^p(\mu)} \leqslant (p(C|t|+1)-1)\|f\|_{L^q(\mu)}\|f\|^{p-2}_{L_1^p(\mu)}\|f\|_{L_2^r(\mu)}. \tag{4.5}$$

On the other hand $1/p = 1/2p + 1/2r$ so an application of Hölder inequality gives

$$\|f\|_{L^p(\mu)} \leqslant \|f\|^{1/2}_{L^q(\mu)}\|f\|^{1/2}_{L^r(\mu)} \leqslant \|f\|^{1/2}_{L_q(\mu)}\|f\|^{1/2}_{L_2^r(\mu)}$$

which implies

$$\|f\|^p_{L^p(\mu)} \leqslant \|f\|_{L^q(\mu)} \|f\|^{p-2}_{L^p_1(\mu)} \|f\|_{L^r_2(\mu)}. \tag{4.6}$$

Adding (4.5) and (4.6) gives

$$\|f\|^p_{L^p_1(\mu)} \leqslant p(C|t|+1) \|f\|_{L^q(\mu)} \|f\|^{p-2}_{L^p_1(\mu)} \|f\|_{L^r_2(\mu)}$$

which implies

$$\|f\|_{L^p_1(\mu)} \leqslant (p(C|t|+1))^{1/2} \|f\|^{1/2}_{L^q(\mu)} \|f\|^{1/2}_{L^r_2(\mu)}. \tag{4.7}$$

Estimate (4.7) can be easily extended by density to arbitrary $f \in \mathcal{S}$. A limiting argument shows that (4.7) is also valid for $p = 1$. Finally, assume that $p = \infty$. We have $q = r = \infty$. Starting from the well known inequality $\sup|g'(x)| \leqslant C \sup|g(x)|^{1/2} \sup|g''(x)|^{1/2}$, $g \in \mathcal{S}$, and letting $g = \gamma^t f$ we easily obtain (4.4) for $k = 2$, $j = 1$, $l = 0$. This completes the proof. ∎

We now state explicity a particular case of Proposition 4.7 that we will need later. It is obtained setting $q = \infty$ and $l = 0$ in (4.4). Thus,

$$\|f\|_{H^{j,t}_p} \leqslant C \|f\|^{1-b}_{L^\infty(\mathrm{d}x)} \|f\|^b_{H^{k,pt/r}_r}, \quad b = \frac{j}{k} = \frac{r}{p}, \quad f \in \mathcal{S}. \tag{4.8}$$

We now consider the scale of symbols $\mathcal{S}^k = \mathcal{S}^{m,k}_{\rho,\delta}$ defined by

$$\mathcal{S}^k = \{a(x,\xi) : (1+|\xi|)^{-m-\delta|\alpha|+\rho|\beta|} D^\alpha_x D^\beta_\xi a(x,\xi) \in L^\infty,\ |\alpha|+|\beta| \leqslant k\}. \tag{4.9}$$

endowed with the obvious norm. The norm in $\mathcal{S}^k = \mathcal{S}^{m,k}_{\rho,\delta}$ will be denoted by $|||\cdot|||_{m,k}$ or just by $|||\cdot|||_k$ if there is no possibility of confusion about the order of the symbols. Notice that $\bigcap_k \mathcal{S}^{m,k}_{\rho,\delta} = \mathcal{S}^{m,\infty}_{\rho,\delta}$ is just the usual space of smooth symbols $S^m_{\rho,\delta}$.

PROPOSITION 4.7. *The scale (4.9) verifies a convexity condition.*

PROOF. It is enough to show that $|||a|||^{1/2}_0 |||a|||^{1/2}_2$, for this implies the general case. Taking the Taylor expansion of order two in the variable ξ_j and keeping the other variables fixed we get

$$a(x,\eta) = a(x,\xi) + \frac{\partial a}{\partial \xi_j}(x,\xi)(\eta_j - \xi_j) + \frac{1}{2}\frac{\partial^2 a}{\partial \xi_j^2}(x,\xi+\theta(\eta-\xi))(\eta_j-\xi_j)^2$$

where $0 < \theta < 1$. This implies

$$\left|\frac{\partial a}{\partial \xi_j}(x,\xi)\right| \leqslant 2|||a|||_0 |\eta_j - \xi_j|^{-1} + \frac{1}{2}\left|\frac{\partial^2 a}{\partial \xi_j^2}(x,\xi+\theta(\eta-\xi))\right| |\eta_j - \xi_j|. \tag{4.10}$$

Given ξ select η such that

$$\begin{aligned} &\xi_k = \eta_k, \quad k \neq j, \\ (4.11) \qquad &\xi_j = \eta_j + \frac{1}{2}(1+|\xi|)^{\rho}|||a|||_0^{1/2}|||a|||_2^{-1/2}. \end{aligned}$$

Notice that $|\eta - \xi| \leqslant (1+|\xi|)^{\rho}/2 \leqslant (1+|\xi|)/2$ so $(1+|\xi|) \sim 1 + |\xi + \theta(\eta-\xi)|$. Hence, substitution of (4.11) into (4.10) gives

$$(1+|\xi|)^{\rho}\left|\frac{\partial a}{\partial \xi_j}(x,\xi)\right| \leqslant C|||a|||_0^{1/2}|||a|||_2^{1/2}, \quad j = 1,\dots,n.$$

Similary,

$$(1+|\xi|)^{-\delta}\left|\frac{\partial a}{\partial x_j}(x,\xi)\right| \leqslant C|||a|||_0^{1/2}|||a|||_2^{1/2}, \quad j = 1,\dots,n.$$

Since $|a(x,\xi)| \leqslant |||a|||_0^{1/2}|||a|||_2^{1/2}$ trivially, we obtain $|||a|||_1 \leqslant C|||a|||_0^{1/2}|||a|||_2^{1/2}$. ■

5. TAME ESTIMATES

We now consider tame maps.

DEFINITION 5.1. Let $\mathcal{H} = \{\mathcal{H}^k\}$, $\mathcal{F} = \{\mathcal{F}^k\}$ be scales of Banach spaces, $\Omega \subset \mathcal{H}^0$. A (possibly non-linear) map $T : \Omega \to \mathcal{F}^0$ is said to be tame if there exist integers τ, k_0, $\tau \leqslant k_0 \geqslant 0$, a subset U of Ω open in $\mathcal{H}^{k_0}$ and a sequence of positive constants (C_k), such that

$$(5.1) \qquad ||Th||_k \leqslant C_k||h||_{k+\tau}, \quad h \in U \cap \mathcal{H}^{k+\tau},\ k \geqslant k_0 - \tau,$$

(this requires $T(U \cap \mathcal{H}^{k+\tau}) \subseteq \mathcal{F}^k$, $k \geqslant k_0 - \tau$).

Consider a map of two variables defined on scales, i.e., $T : \mathcal{H}^0 \times \mathcal{G}^0 \to \mathcal{F}^0$ where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are scales of Banach spaces. The usual way of proving that T is tame is to obtain estimates of the form

$$(5.2) \qquad ||T(h,g)||_k \leqslant C_k(||h||_\tau||g||_{k+\tau} + ||h||_{k+\tau}||g||_\tau), \quad k \geqslant k_0 - \tau,$$

with τ fixed. Indeed, on the set $U = \{||h||_\tau + ||g||_\tau < R\}$ (5.2) implies (5.1) for pairs (h,g).

For simplicity we shall denote the norm in $H^{k,t}_{p,\gamma}$ by $||\cdot||_{p,k}$ without explicit reference to t and γ.

PROPOSITION 5.2. *The map*

$$\{H^{k,t}_{p,\gamma} \times L^\infty_k\} \ni (u,v) \to uv \in \{H^{k,t}_{p,\gamma}\}$$

is tame. More precisely, there exists a positive constant $C_k = C_k(n,p)$ *such that*

$$\|uv\|_{p,k} \leqslant C_k(\|u\|_{p,k}\|v\|_{L^\infty} + \|u\|_{p,0}\|v\|_{L^\infty_k} \quad u \in H^{k,t}_{p,\gamma},\ v \in L^\infty_k. \tag{5.3}$$

Proof. Indeed, using Leibniz rule and Proposition 4.4

$$\|uv\|_{p,k} \leqslant C_k \sum_{j=0}^{k} \|u\|_{p,k-j}\|v\|_{L^\infty_j} \leqslant C_k(\|u\|_{p,k}\|v\|_{L^\infty} + \|u\|_{p,0}\|v\|_{L^\infty_k}). \quad \blacksquare$$

By Sobolev's imbedding theorem, $\|v\|_{L^\infty} \leqslant C\|v\|_{L^p_\tau}$ if $\tau > n/p$. Furthermore, if $t \geqslant 0$, $\|v\|_{L^p_\tau(\mathrm{d}x)} \leqslant \|v\|_{L^p_\tau(\gamma^{pt}\mathrm{d}x)} \leqslant C\|v\|_{p,\tau}$. This observation leads to

COROLLARY 5.3. Let $t \geqslant 0$. *The map*

$$\{H^{k,t}_{p,\gamma} \times H^{k,t}_{p,\gamma}\} \ni (u,v) \to uv \in \{H^{k,t}_{p,\gamma}\}$$

is tame and there are estimates

$$\|uv\|_{p,k} \leqslant C_k(\|u\|_{p,k+\tau}\|v\|_{p,\tau} + \|u\|_{p,\tau}\|v\|_{p,k+\tau}), \qquad u,v \in H^{k+\tau,t}_{p,\gamma},$$

with $\tau = [n/p] + 1$.

We now study the tameness of the composition $\varphi \circ u$ when $\varphi \in L^\infty_k$ and $u \in H^{k,t}_{p,\gamma}$. If we wish to allow φ to depend on several variables it is convenient to consider vector valued weighted Sobolev spaces which will be denoted $H^{k,t}_{p,\gamma}(\mathbf{R}^n, \mathbf{R}^m)$. Let Ω be an open subset of $\mathbf{R}^m$ containing the origin. We will assume that $\varphi : \Omega \to \mathbf{C}$ has bounded derivatives of order $\leqslant k$, $\varphi(0) = 0$, and $u \in H^{k,t}_{p,\gamma}(\mathbf{R}^n, \mathbf{R}^m)$ verifies $u(\mathbf{R}^n) \subset \Omega$. Under these conditions the map $u \to \varphi \circ u$ takes $H^{k,t}_{p,\gamma}$ into $H^{k,t}_{p,\gamma}$ and the composition is a tame map. In fact we have

PROPOSITION 5.4. *Under the above hypothesis the following estimate holds*

$$\|\varphi \circ u\|_{p,k} \leqslant C_k(1 + \|\nabla u\|_{L^\infty})^{k-1}(\|\varphi\|_{L^\infty_1}\|u\|_{p,k} + \|\varphi\|_{L^\infty_k}\|u\|_{p,1}) \tag{5.4}$$

Proof. Consider a ball B of radius r centered at the origin and contained in Ω. We have

$$\varphi(u) = \sum_{j=1}^{m} u_j \psi_j(u), \quad u \in B,\ \|\psi_j\|_{L^\infty} \leqslant \|\nabla\varphi\|_{L^\infty}.$$

Thus, using a cut-off function supported in B we may write $\varphi = \varphi_1 + \varphi_2$ with $|\varphi_1(u)| \leqslant C\|\nabla\varphi\|_{L^\infty}|u|$, $\|\varphi_2\|_{L^\infty} \leqslant \|\varphi\|_{L^\infty}$ and φ_2 supported in B. If $u \in L^p(\mathbf{R}^n, \mathbf{R}^m)$ the measure of $\{|u| < r\}$ is bounded by $r^{-p}\|u\|^p_{L^p}$. It is now easy to conclude that

$$\|\varphi \circ u\|_{L^p} \leqslant C\|\varphi\|_{L^\infty_1}\|u\|_{L^p}. \tag{5.5}$$

Let α be a multi-index of length $k > 0$. Then $D_x^\alpha(\varphi \circ u) = \sum_{j=1}^{k} F_j$ where F_j is a sum of terms of the form

$$\sum_{\substack{|\beta_1|+\cdots+|\beta_j|=k \\ |\gamma|=j,\ |\beta_l|>0}} C_{\beta,\gamma,j}(D^\gamma\varphi) \circ u D^{\beta_1}u_{i_1}\cdots D^{\beta_j}u_{i_j},$$

where $i_l \in \{1, \ldots, m\}$. Applying Hölder inequality to each term of the sum we get

$$\|D_x^\alpha(\varphi \circ u)\|_{L^p(\gamma^{tp}dx)} \leqslant C\sum_{j,\delta}\|\varphi\|_{L_j^\infty(dx)}\|D^{\delta_1}\nabla u\|_{L^{q_1}(\gamma^{tp}dx)}\cdots\|D^{\delta_j}\nabla u\|_{L^{q_j}(\gamma^{tp}dx)},$$

where $|\delta_l| = |\beta_l| - 1$ and $q_l = p(k-j)/|\delta_l|$. The inequality holds because $1/q_1 + \cdots + +1/q_j = 1/p$. Now, with the notation of weighted Sobolev spaces and taking advantage of (4.8) we have the estimate

$$\|D^{\delta_l}\nabla u\|_{L^{q_l}(\gamma^{tp}dx)} \leqslant \|\nabla u\|_{H_{q_l}^{|\delta_l|,tp/q_l}} \leqslant C\|\nabla u\|_{L^\infty(dx)}^{1-b_l}\|\nabla u\|_{H_p^{k-j,t}}^{b_l},$$

where $b_l = |\delta_l|/(k-j) = p/q_l$. Notice that the sum $\sum_{l=1}^{j}|\delta_l| = \sum_{l=1}^{j}(|\beta_l| - 1) = k - j$ so it turns out that $b_1 + \cdots + b_j = 1$. This implies

$$\begin{aligned}(5.6)\qquad \|D^\alpha(\varphi \circ u)\|_{L^p(\gamma^{tp}dx)} &\leqslant C\sum_{j=1}^{|\alpha|}\|\varphi\|_{L_j^\infty}\|\nabla u\|^{j-1}\|\nabla u\|_{H_p^{k-j,t}} \leqslant \\ &\leqslant C(1 + \|\nabla u\|_{L^\infty})^{|\alpha|-1}(\|\varphi\|_{L_1^\infty}\|u\|_{p,k} + \|\varphi\|_{L_k^\infty}\|u\|_{p,1}),\end{aligned}$$

where we have used convexity estimates (4.2) to obtain the second inequality. Adding inequalities (5.6) for $0 < |\alpha| \leqslant k$ and using (5.5) we get (5.4). ∎

We now consider coordinate changes. Let $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism with inverse Ψ and assume that $\det \Psi' > c > 0$. If Φ has bounded derivatives of any order the same happens to Ψ. Set $\gamma_1 = \gamma \circ \Psi$. Then γ_1 is a weight if γ is a weight. Furthermore, the change of variables $f \to f \circ \Phi$ takes $H_{p,\gamma_1}^{k,t}$ into $H_{p,\gamma}^{k,t}$ and the composition is tame. We have

PROPOSITION 5.5. *If Φ and f are as above there are estimates*

$$(5.7)\quad \|f \circ \Phi\|_{H_{p,\gamma}^{k,t}} \leqslant C_k(1 + \|\nabla\Phi\|_{L^\infty})^{k-1}(\|f\|_{H_{p,\gamma_1}^{0,t}}\|\nabla\Phi\|_{L_k^\infty} + \|f\|_{H_{p,\gamma_1}^{k,t}}\|\nabla\Phi\|_{L^\infty})$$

Proof. If $|\alpha| = k > 0$ we know that $D^\alpha(f \circ \Phi)$ is a sum of terms of the form

$$\sum_{\substack{|\beta_1|+\cdots+|\beta_j|=k \\ |\gamma|=j,\ |\beta_l|>0}} C_{\beta,\gamma,j}(D^\gamma f) \circ \Phi D^{\beta_1}\Phi_{i_1}\cdots D^{\beta_j}\Phi_{i_j},$$

where $\Phi_1, \ldots, \Phi_n$ are the components of Φ and $1 \leqslant j \leqslant k$. Hence,

$$|D^\alpha(f \circ \Phi)(x)| \leqslant C \sum_{j,\gamma,\beta} \|\nabla\Phi\|_{L^\infty_{|\beta_1|-1}} \cdots \|\nabla\Phi\|_{L^\infty_{|\beta_j|-1}} |(D^\gamma f) \circ \Phi(x)|.$$

Set $\delta_l = |\beta_l| - 1$ so $\delta_1 + \cdots + \delta_j = k - j$. Using convexity estimates

$$\|\nabla\Phi\|_{L^\infty_{\delta_l}} \leqslant C \|\nabla\Phi\|_{L^\infty}^{1-\delta_l/(k-j)} \|\nabla\Phi\|_{L^\infty_{k-j}}^{\delta_l/(k-j)}$$

we get

$$\begin{aligned} |D^\alpha(f \circ \Phi)(x)| &\leqslant C \sum_{j=1}^{k} \sum_{|\gamma|=j} \|\nabla\Phi\|_{L^\infty}^{j-1} \|\nabla\Phi\|_{L^\infty_{k-j}} |(D^\gamma f) \circ \Phi(x)| \leqslant \\ &\leqslant C(1 + \|\nabla\Phi\|_{L^\infty})^{k-1} \sum_{j=1}^{k} \sum_{|\gamma|=j} \|\nabla\Phi\|_{L^\infty_{k-j}} |(D^\gamma f) \circ \Phi(x)|. \end{aligned} \tag{5.8}$$

Set $g = f \circ \Phi$. Then

$$\|g\|^p_{H^{k,t}_{p,\gamma}} \leqslant C \sum_{|\alpha| \leqslant k} \int |D^\alpha g(y)|^p \gamma^{tp}(y) \mathrm{d}y.$$

Estimating the integrand with (5.8) and performing the change of variables $y = \Phi(x)$ we obtain right away

$$\|g\|^p_{H^{k,t}_{p,\gamma}} \leqslant C(1 + \|\nabla\Phi\|_{L^\infty})^{p(k-1)} \sum_{j=0}^{k} \|f\|^p_{H^{j,t}_{p,\gamma_1}} \|\nabla\Phi\|^p_{L^\infty_{k-j}}$$

which after the usual convexity estimates yields (5.7). ■

Definition 5.6. A weight γ is called stable if for any diffeomorphism Φ of $\mathbb{R}^n$ with inverse Ψ, satisfying the hypotheses of Proposition 5.5, the ratios $\gamma \circ \Psi/\gamma$ and $\gamma \circ \Phi/\gamma$ remain bounded. In particular, the weights γ and $\gamma_1 = \gamma \circ \Psi$ define the same weighted Sobolev spaces which become invariant under composition with Φ.

Example: The weight $\gamma(\xi) = (1 + |\xi|^2)^{1/2}$ defined on R^n is stable.

Let's now return to the scale of symbols $\mathcal{S}^k = \mathcal{S}^{m,k}_{\rho,\delta}$ introduced at the end of the last section. If $a \in \mathcal{S}^{m,k}_{\rho,\delta}$, $b \in \mathcal{S}^{m',k}_{\rho,\delta}$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $\delta \leqslant \rho$, we have the bilinear map $(a, b) \to a \circ b$, where $a \circ b$ is the symbol of the composition $a(x, D) \circ b(x, D)$ of the pseudo-differential operators with symbols a and b. The composite symbol is thus given by the integral

$$a \circ b(x, \xi) = \frac{1}{(2\pi)^n} \iint \mathrm{e}^{-\mathrm{i}(x-z)\cdot(\xi-\eta)} a(x, \eta) b(z, \xi) \mathrm{d}z \mathrm{d}\eta, \tag{5.9}$$

which is absolutely convergent if a and b are, for instance, compactly supported, and can be given an oscillatory meaning in the general case. Furthermore, there exists $\tau = \tau(\delta, n)$ and a positive constant C such that

$$(5.10) \qquad (1+|\xi|)^{-m-m'}|(a \circ b)(x,\xi)| \leqslant C|||a|||_{m,\tau}|||b|||_{m',\tau}.$$

Using the "Leibniz rule"

$$D_x(a \circ b) = (D_x a) \circ b + a \circ D_x b,$$

$$D_\xi(a \circ b) = (D_\xi a) \circ b + a \circ D_\xi b,$$

we obtain by induction from (5.10) and the convexity properties of Proposition 4.7

PROPOSITION 5.7. *Assume that* $0 < \rho \leqslant 1,\ 0 \leqslant \delta < 1,\ \delta \leqslant \rho$. *The bilinear map*

$$\{S^{m,k}_{\rho,\delta} \times S^{m',k}_{\rho,\delta}\} \ni (a,b) \to a \circ b \in \{S^{m+m',k}_{\rho,\delta}\}$$

is tame and there are estimates

$$(5.11) \qquad |||a \circ b|||_{m+m',k} \leqslant C_k(|||a|||_{m,k+\tau}|||b|||_{m',\tau} + |||a|||_{m,\tau}|||b|||_{m',k+\tau},$$

for $a \in S^{m,k+\tau}_{\rho,\delta}$ *and* $b \in S^{m',k+\tau}_{\rho,\delta}$.

We now consider the bilinear map $(a,f) \to a(x,D)f$ where $a(x,D)$ is the pseudo-differential operator with symbol $a(x,\xi) \in S^{m,k}_{\rho,\delta}$ and f is a function in a weighted Sobolev space. Let $1 < p < \infty$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $\delta \leqslant \rho$, $m \in \mathbf{R}$, and set $m_p = -n(1-\rho)|1/p - 1/2|$. Then, by Theorem 1.6, if $a \in S^m_{\rho,\delta}$, $a(x,D)$ maps continuously $\mathcal{H}^k = H^{k,t}_{p,\gamma}$ into $\mathcal{H}^{k-m+m_p}$. Furthermore, tracking the steps of the proof one determines $\tau = \tau(n,m,\delta,p,\gamma,t)$ and positive C (also depending on these parameters) such that

$$(5.12) \qquad \|a(x,D)f\|_0 \leqslant C|||a|||_{m,\tau}\|f\|_{m-m_p},$$

where $\|\ \|_s$ denotes the norm in $\mathcal{H}^s$. Differentiating under the integral sign and using the Leibniz rule yields

$$(5.13) \qquad D^\gamma(a(x,D)f) = \sum_{\alpha+\beta=\gamma} c_{\alpha\beta}a_\alpha(x,D)D^\beta f$$

where $a_\alpha(x,D)$ is the pseudo-differential operator with symbol $D_x a(x,\xi) \in S^{m+\delta|\alpha|}_{\rho,\delta}$. Applying (5.12) to each term of (5.13) and using the convexity properties of Proposition 4.7 we get

Proposition 5.8. *Let $1 < p < \infty$, $0 < \rho \leqslant 1$, $0 \leqslant \delta < 1$, $\delta \leqslant \rho$, $m, t \in \mathbf{R}$, and set $m_p = -n(1-\rho)|1/p - 1/2|$, γ a weight. The bilinear map*

$$\{S^{m,k}_{\rho,\delta} \times H^{k,t}_{p,\gamma}\} \ni (a, f) \to a(x, D)f \in \{H^{k,t}_{p,\gamma}\}$$

is tame and there are estimates

$$\|a(x,D)f\|_k \leqslant C_k(|||a|||_{m,k+r}\|f\|_{m-m_p} + |||a|||_{m,r}\|f\|_{m-m_p+k}). \tag{5.14}$$

6. A CLASS OF SOLVABLE SEMILINEAR EQUATIONS

Consider the semilinear equation in a neighborhood Ω of the origin in $\mathbf{R}^2$

$$P(x, t, D_x, D_t)u + F(x, t, u, \dots, D^\alpha_{x,t}u) = f(x, t), \quad |\alpha| \leqslant m - 1, \tag{6.1}$$

where P is a homogeneous linear differential operator of order $m \geqslant 1$ with smooth complex coefficients, F is a complex-valued function, holomorphic in $u, \dots, D^\alpha u$ for $|\alpha| \leqslant m-1$ and smooth in (x, t), and $f \in C^\infty_c(\Omega)$. We assume that

$$P(x, t, D_x, D_t) = D^m_t + a_{m-1}(x,t)D^{m-1}_t D_x + \cdots + a_0(x,t)D^m_x$$

with principal symbol

$$p(x, t, \xi, \tau) = \tau^m + a_{m-1}(x,t)\tau^{m-1}\xi + \ldots + a_0(x,t)\xi^m$$

satisfies Trève's condition $(\mathcal{R})$ (cf. [15]), namely:

For every $(x_0, t_0) \in \Omega$, $(\xi_0, \tau_0) \in \mathbf{R}^2 \setminus \{0\}$ and every complex number z such that

$$p(x_0, t_0, \xi_o, \tau_0) = 0, \quad \nabla_{\xi_0,\tau_0} \operatorname{Re} zp(x_0, t_0, \xi_0, \tau_0) \neq 0,$$

the function $\operatorname{Im} zp$ *does not change sign in a neighborhood of $(x_0, t_0, \xi_0, \tau_0)$ in the set*

$$\Sigma_z = \{(x, t, \xi, \tau) : \operatorname{Re} zp(x_0, t_0, \xi_0, \tau_0) = 0\}.$$

In [4] Dehman proved that equation (6.1) is locally solvable in C^∞ assuming that P is subelliptic, which requires that the restriction of $\operatorname{Im} zp$ to the null bicharacteristics of $\nabla_{\xi_0,\tau_0}\operatorname{Re} zp$ only possesses zeros of *finite* even order. It is easy to see that this condition implies $(\mathcal{R})$. On the other hand, if $P = D_t + ib(x,t)D_x$ with $b \geqslant 0$ vanishing of infinite order at the origin it will satisfy $(\mathcal{R})$ but will not be subelliptic. Here we prove, THEOREM 6.1. *If $P(x,t,D_x,D_t)$ verifies condition $(\mathcal{R})$ the semilinear equation (6.1) is locally solvable in C^∞ at the origin.*

As remarked in [4] it is enough to prove the theorem for $m = 1$. We may also assume that $F(x,t,u)$ vanishes identically outside a compact subset of Ω. The method of proof is in application of the Nach-Moser implicit function theorem ([8], [13], [14]) to the map $u \to \Phi(u) = Pu + F(u)$ acting on a suitable scale with loss of one derivative. One possible option is the scale of Sobolev spaces $L^2_k(\Omega)$. One takes $k \geqslant 2$ so that all functions are bounded. The map Φ is tame and twice Fréchet differentiable in $B_2 \cap L^2_k$, where B_2 is a small ball of F^2_2 which insures that the composition $F(u,x,t)$ can be defined. The hypothesis in the Nash-Moser theorem that requires more care is the existence of a tame right inverse for the linearization of Φ. Taking $\overline{\Omega} = [-T,T] \times$ $\times [-T,T]$ and using an extension operator from $[-T,T]$ to $\mathbf{R}$ we can inject the scale $L^2_k(\Omega)$, into the scale $L^2_k(\Omega_T)$, with $\Omega_T = \mathbf{R} \times [-T,T]$. The latter can be imbedded into the scale $\mathcal{F}^k$ given by (A.8) (see the appendix) with, say, $p = 2$ and $\gamma \equiv 1$. In this way we may take advantage of the results of the appendix.

The linearization $\Phi'(u)v$ of Φ is given by

$$\Phi'(u)v = Pv + F_u(x,t,u)v.$$

After a suitable local change of coordinates and division by a nonvanishing factor we may assume without loss of generality that

$$P(x,t,D_x,D_t) = L = \frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x},$$

with $b(x,t)$ real valued. Notice that condition $(\mathcal{R})$ implies that b does not change sign in Ω, say $b \geqslant 0$. Modifying b outside a neighborhood of the origin we may assume that it is compactly supported (in particular, it is defined throughout Ω_T and it is bounded with bounded derivatives). Let Q be the tame right inverse of L described in Theorem A.1 and set

$$\Psi(u)f = \exp[-Q(F_u)]\; Q(f \exp[Q(F_u)]).$$

Of course, $\Psi(u)f$ is linear in f and it is readily verified that

$$\Psi'(u)\Psi(u)f = f.$$

Because of the way the scale $\{\mathcal{F}^k\}$ is built out of the scale $\{H^{k,t}_{p,\gamma}\}$ the tameness properties for the product and composition valid for the latter (guaranteed by Propositions 5.2, 5.3 and 5.4) carry over to the scale $\{\mathcal{F}^k\}$. Then, the same estimates (A.12) for Q imply tame estimates

$$(6.2) \qquad \|\Psi(u)f\|_{\mathcal{F}^k} \leqslant C_k(\|u\|_{\mathcal{F}^k}\|f\|_{\mathcal{F}^2} + \|u\|_{\mathcal{F}^2}\|f\|_{\mathcal{F}^k}), \quad u \in \mathcal{F}^k,\ k = 2,3,\ldots$$

By the Nash-Moser theorem there is an integer k_0 and a positive ϵ such that the equation

$$Lu + F(x,t,u) = f, \quad f \in \mathcal{F}^k, \tag{6.3}$$

can be solved in $\mathcal{F}^{k-k_0}$ provided $k \geqslant k_0$ and there exist $u_0 \in \mathcal{F}^\infty$ such that $\|Lu_0 + +F(x,t,u_0) - f\|_{\mathcal{F}^{k_0}} < \epsilon$. Furthermore, the solution is in $\mathcal{F}^\infty$ if $f \in \mathcal{F}^\infty$. Thus, to finish the proof it is enough to construct an approximate solution u_0 when the right hand side of (6.3) is compactly supported. This is done in a standard way by the power series method. Set $U(x,t) = \sum_{j=1}^{\infty} u_j(x)t^j$ and formally determine the smooth functions $u_j(x)$ by plugging U into equation (6.3). Each function $u_j(x)$ is compactly supported in $\mathbb{R}$ because $U(x,0) \equiv 0$ and, for large x, (6.3) reduces to $U_t = 0$. Choose a function $u_0(x,t) \in C_c^\infty(\mathbb{R} \times [-T,T])$ whose Taylor series at $(x,0)$ is given by the (formal) series $\sum_{j=1}^{\infty} u_j(x)t^j$ (use Borel's lemma). Then all derivatives of $Lu_0 + F(x,t,u_0) - f(x,t)$ up to order k_0 are uniformly small for small t. Modifying f outside a neighborhood of $\{t = 0\}$ we may achieve the same for all t. Hence, we can make $\|Lu_0 + F(x,t,u_0) - f\|_{\mathcal{F}^{k_0}}$ as small as we wish for the modified f and therefore solve the equation (6.3) in $\mathcal{F}^\infty$. This also solves the original equation in a neighborhood of the origin and proves the theorem.

A. A TAME RIGHT INVERSE FOR L

Consider the first-order linear differential operator in two variables

$$L = \frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}, \quad x \in \mathbb{R},\ |t| < T. \tag{A.1}$$

We write $\Omega_T = \mathbb{R} \times [-T,T]$ and assume that

i) *$b(x,t)$ is real and nonnegative,*

ii) *all derivatives of b are bounded, i.e., belong to $L^\infty(\Omega_T)$.*

The size of T will be decreased a number of times. We also write

$$B(x,t,t') = \int_{t'}^{t} b(x,s)ds.$$

The next lemma describes a function which is central to the construction of a parametrix for L. This parametrix (with minor modifications) was used to study the global hypoellipticity of L in [10]. The proof is a routine modification of the results in [10] and will be left to the reader. The only novelty here is that x is allowed to vary unboundedly but the estimates remain uniform because of the hypotheses on b.

LEMMA A.1. *Let L be as above. There exists a function $\varphi(x,t,t')$ in $\Omega_T\times[-T,T]$, such that $x-\varphi(x,t,t')$ is bounded with bounded derivatives and such that*

$$|D_x^\alpha D_t^\beta D_{t'}^\gamma(L\varphi(x,t,t'))| \leqslant C(N,\alpha,\beta,\gamma)|B(x,t,t')|^N, \quad N=0,1,\dots \tag{A.2}$$

and

$$\varphi(x,t',t')=x, \quad (x,t')\in\Omega_T. \tag{A.3}$$

Furthermore, if T is decreased conveniently we also obtain that

$$\begin{aligned}\frac{1}{2}B(x,t,t') \leqslant \operatorname{Im}\varphi(x,t,t') \leqslant \frac{3}{2}B(x,t,t'), \quad t\geqslant t',\\ \frac{3}{2}B(x,t,t') \leqslant \operatorname{Im}\varphi(x,t,t') \leqslant \frac{1}{2}B(x,t,t'), \quad t\leqslant t',\end{aligned} \tag{A.4}$$

and

$$|\operatorname{Re}\varphi_x(x,t,t')-1| < 1/2. \tag{A.5}$$

Now we consider a function $0\leqslant\eta^+(\xi)\leqslant 1\in C^\infty(\mathbf{R})$ such that $\eta^+(\xi)=0$ if $\xi\leqslant -1$ and $\eta^+(\xi)=1$ if $\xi\geqslant 1$ and set $\eta^-=1-\eta^+$.

LEMMA A.2.

i) For $-T\leqslant t'\leqslant t\leqslant T$ the function

$$a^+(x,\xi,t,t')=\eta^+(\xi)\exp(-\operatorname{Im}\varphi(x,t,t')\xi)$$

is a symbol of class $S^0_{1,1/2}(\mathbf{R})$ as a function of (x,ξ) depending continuosly on the parameters t,t'. More generally,

$$D_t^j D_{t'}^k a^+(x,\xi,t,t')\in S^{j+k}_{1,1/2}(\mathbf{R}) \quad j,k=0,1,\dots$$

uniformly and continuously on t,t' for $t'\leqslant t$.

ii) Similarly,

$$a^-(x,\xi,t,t')=\eta^-(\xi)\exp(-\operatorname{Im}\varphi(x,t,t')\xi)$$

satisfies, as a function of (x,ξ),

$$D_t^j D_{t'}^k a^-(x,\xi,t,t')\in S^{j+k}_{1,1/2}(\mathbf{R}) \quad j,k=0,1,\dots$$

uniformly and continuously on $t\leqslant t'$.

Proof. It is enough to prove i), for the proof of ii) is analogous. Clearly, (A.4) shows that $\operatorname{Im}\varphi(x,t,t') \geqslant 0$ for $t' \leqslant t$ and the reverse inequality holds for $t \leqslant t'$. Thus, $|a^+(x,\xi,t,t')| \leqslant C$ because φ is bounded. Consider the function $\sqrt{\operatorname{Im}\varphi(x,t,t')}$ for $t' \leqslant t$. By a result of Glaeser ([7], [5]) it is continuously differentiable and any first order derivative is uniformly bounded (recall that all derivatives of φ are bounded). This gives the estimate

$$|D_x \operatorname{Im}\varphi(x,t,t')| \leqslant C|\sqrt{\operatorname{Im}\varphi(x,t,t')}|$$

which implies for $t' \leqslant t$

$$|D_x a^+(x,\xi,t,t') \leqslant C(1+|\xi|)^{1/2}$$

using the trivial estimate $\sqrt{s}e^{-s} \leqslant C,\ s \geqslant 0$. Similarly, the estimate $se^{-s} \leqslant C,\ s \geqslant 0$, yields

$$|D_\xi a^+(x,\xi,t,t')| \leqslant C(1+|\xi|)^{-1}$$

and by induction one gets $|D_x^j D_\xi^k a^+| \leqslant C_{jk}(1+|\xi|)^{j/2-k}$ for $j,k = 0,1,\dots$ and $t' \leqslant t$. The estimates for the derivatives of a^+ with respect to t and t' can be treated in the same way. Now set for $f \in C_c^\infty(\Omega_T)$

$$K^+ f(x,t) = \frac{1}{2\pi}\int_{-T}^{t}\int_{-\infty}^{\infty} e^{i\varphi(x,t,t')\xi}\eta^+(\xi)\widehat{f}(\xi,t')d\xi dt',$$

$$K^- f(x,t) = \frac{1}{2\pi}\int_{T}^{t}\int_{-\infty}^{\infty} e^{i\varphi(x,t,t')\xi}\eta^-(\xi)\widehat{f}(\xi,t')d\xi dt',$$

where $\widehat{f}(\xi,t')$ indicates the partial Fourier transform of the function $f(x,t')$ with respect to the first variable. If we write

$$R^+ f(x,t) = \frac{1}{2\pi}\int_{-T}^{t}\int_{-\infty}^{\infty} e^{i\varphi(x,t,t')\xi} i\xi L\varphi(x,t,t')\eta^+(\xi)\widehat{f}(\xi,t')d\xi dt',$$

$$R^- f(x,t) = \frac{1}{2\pi}\int_{T}^{t}\int_{-\infty}^{\infty} e^{i\varphi(x,t,t')\xi} i\xi L\varphi(x,t,t')\eta^-(\xi)\widehat{f}(\xi,t')d\xi dt',$$

it follows from direct computation that

$$\begin{aligned} LK^+ f &= \eta^+(D)f + R^+ f, \\ LK^- f &= \eta^-(D)f + R^- f. \end{aligned} \tag{A.6}$$

Observe that, in view of (A.2), we have for any $N = 0, 1, \ldots$ and $t' \leqslant t$

$$|\exp(-\operatorname{Im}\varphi(x,t,t')\xi)L\varphi(x,t,t')\eta^+(\xi)| \leqslant$$

$$\leqslant C_N \exp(-B(x,t,t')\xi/2)B(x,t,t')^N\eta^+(\xi) \leqslant C_N(1+|\xi|)^{-N},$$

and similar estimates hold for $D_x^j D_t^k D_{t'}^l \exp(-\operatorname{Im}\varphi\xi)L\varphi(x,t,t')\eta^+(\xi)$. Hence, we may regard R^+ (resp.R^-) as a smooth function of t and t' with values in the space of regularizing operators $L^{-\infty}(\mathbf{R})$ that map $\mathcal{S}'(\mathbf{R})$ into $\mathcal{S}(\mathbf{R})$. Writing $K = K^+ + K^-$ and $R = R^+ + R^-$ we obtain

$$LKf = f + Rf$$

since by construction $\eta^+(D) + \eta^-(D) = I$.

For fixed t and t' the function $x \to \operatorname{Re}\varphi(x,t,t')$ is a diffeomorphism of $\mathbf{R}$ with bounded derivatives by Lemma A.1. Let's denote $\psi(x,t,t')$ its inverse. It follows that the composite

$$\tilde{a}^+(x,\xi,t,t') = a^+(\psi(x,t,t'),\xi,t,t')$$

is also a symbol in $S^0_{1,1/2}$ with the same properties as a^+. If $A^+_{t,t'}$ denotes the pseudodifferential operator with symbol $\tilde{a}^+(x,\xi,t,t')$ depending on t, t' as parameters, then

$$K^+f(x,t) = \int_{-T}^{t} (A^+_{t,t'}f) \circ \operatorname{Re}\varphi dt', \tag{A.7}$$

with an analogous formula for K^-. By Theorem 1.6 $A^+_{t,t'}$ (resp. $A^-_{t,t'}$) maps the weighted Sobolev space $H^{s,t}_{p,\gamma}$ of Definition 1.5 into itself, provided that $1 < p < \infty$. On the other hand, composition with $\operatorname{Re}\varphi$ takes $\mathcal{H}^s = H^{s,t}_{p,\gamma}$ into $\mathcal{H}^s$ if γ is stable, which we shall assume from now on (cf. Definition 5.6 and Proposition 5.5). Now it follows from (A.7) that if $f \in C^0([-T,T];\ H^{s,t}_{p,\gamma})$ so does K^+f (resp. K^-f). More generally, consider the tame scale

$$\mathcal{F}^k = \{f(x,t) \in C^0([-T,T]; H^{k,t}_{p,\gamma}) : D_t^j f \in H^{k-j,t}_{p,\gamma}, \quad 0 \leqslant j \leqslant k\} \tag{A.8}$$

for a certain choice of $1 < p < \infty$, $t \in \mathbf{R}$ and γ a stable weight, endowed with the obvious norm. One checks that the operators K^+, K^- and K map each $\mathcal{F}^k$ into itself for $k = 0, 1, \ldots$, and the operators R^+, R^- and R map $\mathcal{F}^k$ into $\mathcal{F}^\infty = \bigcap_k \mathcal{F}^k$. Furthermore, we have tame estimates

$$\|Kf\|_{\mathcal{F}^k} \leqslant C_k\|f\|_{\mathcal{F}^k} \quad f \in \mathcal{F}^k,\ k = 0, 1, \ldots \tag{A.9}$$

$$\|Rf\|_{\mathcal{F}^j} \leqslant TC_{kj}\|f\|_{\mathcal{F}^k} \quad f \in \mathcal{F}^k,\ k, j = 0, 1, \ldots \tag{A.10}$$

Let's explain the presence of the factor T in (A.10). Writing

$$R^{\pm}f(x,t) = \int_{\mp T}^{t} (B^{\pm}_{t,t'}f) \circ \operatorname{Re}\varphi dt'$$

it follows from (A.2) and the definition of $R^{\pm}$ that the symbol of $D_t^j B^{\pm}_{t,t'}$ vanishes identically for $t = t'$ and all j. Thus,

$$D_t^j R^{\pm} f(x,t) = \int_{\mp T}^{t} D_t^j[(B^{\pm}_{t,t'}f) \circ \operatorname{Re}\varphi] dt'$$

and the weighted Sobolev norm of the left hand side can be estimated by the lenght of the interval, which does not exceed $2T$, times the supremum in t' of the norm of the integrand.

Let's now fix a positive integer k. In virtue of (A.10), we may choose T so that the operator norm of R in $\mathcal{F}^k$ is $< 1/2$. In particular, we may invert $I + R$ in $\mathcal{F}^k$ with norm < 2. In this case, the operator norm of $(I+R)^{-1}$ in $\mathcal{F}^k$ has a bound independent of k but T may shrink when $k \to \infty$. To prove that tame estimates for R carry over to tame estimates for $S = (I+R)^{-1}$ one follows the usual inductive procedure ([4], [8]). Suppose, for instance, that we start at $k = 0$ and the operator norms of R and S in $\mathcal{F}^0$ are respectively $< 1/2$ and < 2. If $v = Su$ we have $v = u - Rv$ and $||v||_{\mathcal{F}^0} \leqslant 2||u||_{\mathcal{F}^0}$. Now,

$$||v||_{\mathcal{F}^1} = \sup_t [||v(\cdot,t)||_{\mathcal{H}^0} + ||v_x(\cdot,t)||_{\mathcal{H}^0} + ||v_t(\cdot,t)||_{\mathcal{H}^0}],$$

using the norm $||v||_{\mathcal{H}^1} = ||v||_{\mathcal{H}^0} + ||D_x v||_{\mathcal{H}^0}$ in $\mathcal{H}^1$. To estimate

$$||v||_{\mathcal{H}^1} \leqslant ||u||_{\mathcal{H}^1} + ||Rv||_{\mathcal{H}^1}$$

write

$$||D_x(Rv)||_{\mathcal{H}^0} \leqslant ||RD_x v||_{\mathcal{H}^0} + ||R_x v||_{\mathcal{H}^0} \leqslant ||RD_x v||_{\mathcal{H}^0} + C||v||_{\mathcal{H}^0},$$

where we have used that R_x is an operator with the same continuity properties as R. Then,

$$\sup_t ||v(\cdot,t)||_{\mathcal{H}^1} \leqslant C||u||_{\mathcal{F}^1} + ||R(D_x v)||_{\mathcal{F}^0} \leqslant C||u||_{\mathcal{F}^1} + (1/2)\sup_t ||D_x v||_{\mathcal{H}^0}.$$

Similarly,

$$\sup_t ||v(\cdot,t)||_{\mathcal{H}^0} \leqslant C||u||_{\mathcal{F}^1} + (1/2)\sup_t ||D_t v||_{\mathcal{H}^0}.$$

Adding these inequalities we obtain $||v||_{\mathcal{F}^1} \leqslant C||u||_{\mathcal{F}^1} + (1/2)||v||_{\mathcal{F}^1}$ which implies $||Su||_{\mathcal{F}^1} \leqslant 2C||u||_{\mathcal{F}^1}$. Keeping up this procedure we get

(A.11) $$||Su||_{\mathcal{F}_k} \leqslant C_k||u||_{\mathcal{F}^k}, \quad k = 0, 1, \ldots$$

Notice that the choice of T was done once and for all at the first step.

We may now define $Q = KS = K(I+R)^{-1}$ which is a right inverse for L and, being the composition of the tame operators K and S, will satisfy the tame estimates

$$\text{(A.12)} \qquad \|Qu\|_{\mathcal{F}_k} \leqslant C_k \|u\|_{\mathcal{F}^k}, \quad u \in \mathcal{F}^k,\ k = 0, 1, \ldots$$

We have proved

THEOREM A.1. *Let L be the operator* (A.1) *satisfying the conditions* i) *and* ii) *and let $1 < p < \infty$, $t \in \mathbf{R}$, γ a stable weight. For T small enough, there exists an operator Q continuous on each space $\mathcal{F}^k$ given by* (A.8) *and such that*

$$LQf = f, \quad f \in \mathcal{F}^k,\ k = 0, 1, \ldots$$

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JOSEFINA ALVAREZ
Department of Mathematical Sciences
New Mexico State University,
Las Cruces, New Mexico 88003-0001,
U.S.A.

JORGE HOUNIE
Departamento de Matemática,
UFPE,
50.739 Recife, Pernambuco,
Brasil.

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