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C^*-ALGEBRAS OF UNITARY RANK TWO

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1. INTRODUCTION

The Russo-Dye Theorem from 1966 [22] states that the convex hull of the unitary elements in a unital C^*-algebra A is norm-dense in its unit ball. For each $a \in A_1 = \{a \in A \mid \|a\| \leqslant 1\}$ let $u(a)$, the unitary rank of a, be the least number of unitary elements in a convex combination of unitaries representing a, and let $u(a) = \infty$ if no such representation exists. It is well known that if $\|a\| < 1$, then a is in the convex hull of $U(A)$, the unitaries in A, and so $u(a) < \infty$ (cf. [13], [8], [20] and [4]). This was improved in [16] and [9] to give upper bounds for $u(a)$ depending on $\|a\|$; the closer $\|a\|$ is to 1, the more unitaries are needed in general. In [12] and [23] the unitary rank $u(a)$ is expressed as a function of the distance $\alpha(a)$, from a to $A_{\rm inv}$, the invertible elements in A. More precisely, if $u(a) \leqslant n$ and $n \geqslant 2$, then $\alpha(a) \leqslant 1 - \frac{2}{n}$; and if $\alpha(a) < 1 - \frac{2}{n}$ and $a \in A_1$, then $u(a) \leqslant n$. Hence, if $A_{\rm inv}$ is dense in A — a condition on A which frequently is written $\mathrm{sr}(A) = 1$ where 'sr' is M. Rieffel's stable rank [19] — then $u(a) \leqslant 3$ for all $a \in A_1$. If $A_{\rm inv}$ is not dense in A, then there is $b \in A$ with $\|b\| = \alpha(b) = 1$ and so $u(b) = \infty$, i.e. b is not in the convex hull of $U(A)$ (see [23]). Moreover, $u(tb) = n$ if $1 - \frac{2}{n-1} < t < 1 - \frac{2}{n}$.

Let $u(A)$, the (maximum) unitary rank of A, be $\sup\{u(a) \mid a \in A_1\}$. Then, from the above (see also [23]), $u(A)$ is two or three if $\mathrm{sr}(A) = 1$, and $u(A) = \infty$ is $\mathrm{sr}(A) \neq 1$. As noted in [9], if A is a finite von Neumann algebra, then $u(A) = 2$, and in [18] it is proved that $u(A) = 3$ if A is an infinite dimensional AF-algebra or an irrational rotation algebra (the latter also requires I. Putnam's result that have these stable rank one [17]).

This paper characterizes C^*-algebras of unitary rank two in most cases of interest.

In particular, the following will be established:

THEOREM 1.1.

1. *Every separable unital C^*-algebra of unitary rank two is finite dimensional.*
2. *Every simple, infinite dimensional C^*-algebra of unitary rank two is an AW^*--factor of type II_1.*

2. THE MAIN RESULT

DEFINITION 2.1. Let A be a C^*-algebra.

a) Two elements $x, y \in A$ are orthogonal if $xy = yx = xy^* = x^*y = 0$.

b) A is called σ-finite if any orthogonal family $(x_i)_{i\in I}$ of non-zero elements in A is countable.

DEFINITION 2.2. Following Kaplansky [10], A is said to be an AW^*- algebra if

(α) each maximal abelian subalgebra of A is generated by its projections, and

(β) each orthogonal family of projections in A has at least upper bound.

Moreover, A is called finite if

(γ) $v \in A$ and $v^*v = 1$ implies $vv^* = 1$.

THEOREM 2.3. *Let A be a σ-finite unital C^*-algebra. Then the following three conditions are equivalent.*

(i) *A has unitary rank two, i.e. every $x \in A_1$ can be written as $x = \frac{1}{2}(v_1 + v_2)$ where $v_1, v_2 \in U(A)$.*

(ii) *Every $x \in A$ has a polar decomposition $x = u|x|$ where $u \in U(A)$.*

(iii) *A is a finite AW^*-algebra.*

2.4. We prove here that Theorem 1.1 follows from Theorem 2.3.

1. Assume A is separable and that $(x_i)_{i\in I}$ is an orthogonal family in A with $\|x_i\| = 1$ for all i. Then $\|x_i - x_j\| = 1$ if $i \neq j$, and so I must be countable. It follows from (i) $\Rightarrow$ (iii) in Theorem 2.3 that if also A is of unitary rank two, then A is an AW*- algebra. The conclusion of (1) now follows, because all separable AW*--algebras are finite dimensional.

2. Assume now that A is simple and of unitary rank two. Then, as mentioned in the introduction, $\mathrm{sr}(A) = 1$, which again implies that A is stably finite (see [19]). Hence A admits a (faithful) Cuntz dimension function D (see [3]). Let $(x_i)_{i\in I}$ be a family of non-zero orthogonal elements in A. Then $\sum_{i\in I} D(x_i) \leqslant 1$, and $D(x_i) > 0$ for all $i \in I$. Hence I is countable, and A is σ-finite. Again, (i) $\Rightarrow$ (iii) in Theorem 2.3 implies that A is a finite AW*- algebra, which must be a type II_1-factor because A is

simple and infinite dimensional.

COROLLARY 2.5. *Let A be a unital σ-finite C^*-algebra and let $n \in \mathbb{N}$. Then A and $M_n(A)$ have the same unitary rank.*

Proof: Combining Theorem 2.3 with results from [23] mentioned in the introduction, $u(A) = \infty$ if $\mathrm{sr}(A) \neq 1$, $u(A) = 3$ if $\mathrm{sr}(A) = 1$ and A is not a (finite) AW*-algebra, and $u(A) = 2$ if A is a finite AW*-algebra. The latter properties are known to be stable. ∎

2.6. The proof of Theorem 2.3 involves the following fourth property:

(ii)′ Every $x \in A$ is of the form $x = va$ with $v \in U(A)$ and $a = a^* \in A$.

It will be proved that (i), (ii), (ii)′ and (iii) are equivalent for σ-finite C^*-algebras. The critical part lies in proving (ii)′ ⇒ (ii), and this is done in Section 5. That (i) implies (ii)′ is proved in [15]. For completeness the brief proof is included in Section 3 together with the remaining implications of Theorem 2.3.

The assumption that A is σ-finite is used in the proofs of (ii)′ ⇒ (ii) and of (ii) ⇒ ⇒ (iii). For the latter implication, σ-finiteness is necessary as illustrated in Example 3.5. The σ-finiteness is crucial in the present proof of (ii)′ ⇒ (ii). It is not clear to the authors if the implication remains valid without this assumption.

3. PROOF OF THEOREM 2.2, PART I

The implications (ii) ⇒ (iii) ⇒ (ii) ⇒ (i) ⇒ (ii)′ of Theorem 2.3 (cf. 2.6) are proved. It should be stressed that none of these implications are new, and we have included this section only as a service to the reader.

3.1. (ii) ⇒ (iii). This implication is almost contained in Proposition 2.3 of D. Handelman's paper [7], where it is proved that A is an AW^*-algebra if A is σ-finite and is $\aleph_0$-injective (a polar decomposition property). The $\aleph_0$-injectivity can, without changing the proof in [7], be replaced with the following (SAW^*-algebra [14]) condition: For every pair or orthogonal elements x and y in A there is an element p in A such that $xp = x$ and $yp = 0$.

Assume that A satisfies (ii), and let x and y be orthogonal elements in A (which without loss of generality can be assumed to be positive). Then $x - y$ has a polar decomposition $x - y = u(x+y) = (x+y)u^*$, and so $(x+y)p = x$ when $p = \frac{1}{2}(u^* + 1)$.

Hence, by [7], (ii) and σ-finiteness implies that A is an AW^*- algebra. Also, A is finite because if $v^*v = 1$, then $|v| = 1$ and so v is unitary by (ii).

3.2. (iii) $\Rightarrow$ (ii). Assume A is a finite AW^*- algebra and let $x \in A$. From [1, Proposition 21.1], x has a polar decomposition $x = v|x|$, where $v \in A$ is a partial isometry. Because A is finite, v extends to a unitary u in A ([1], Proposition 17.4). that is $uv^*v = v$, and it follows that $x = v|x|$.

3.3. (ii) $\Rightarrow$ (i). This is an elementary and classical fact: Let $x \in A$ with $\|x\| \leqslant 1$ be given. Write $x = u|x|$ for some unitary $u \in A$. Then

$$x = \frac{1}{2}u\left(|x| + \mathrm{i}(1-|x|^2)^{\frac{1}{2}}\right) + \frac{1}{2}u\left(|x| - \mathrm{i}(1-|x|^2)^{\frac{1}{2}}\right)$$

is the mean of two unitaries.

3.4. (i) $\Rightarrow$ (ii)$'$. It suffices to write all $x \in A$ with $\|x\| < 1$ as $x = ua$ with $u \in U(A)$ and $a = a^* \in A$. By assumption there are unitaries v_1 and v_2 in A such that $x = \frac{1}{2}(v_1 + v_2)$. Set

$$c = \frac{1}{2\mathrm{i}}(v_1 - v_2),$$

and check that $|c|^2 = 1 - |x|^2$ and $|c^*|^2 = 1 - |x^*|^2$. Conclude that c is invertible and $u = c|c|^{-1}$ is unitary. From $c^*x = x^*c$ one obtains $u^*x = x^*u$, and so $a = u^*x$ is self-adjoint. This yields $x = ua$ as required.

Note that in fact (i) and (ii)$'$ are equivalent for all unital C^*-algebras.

EXAMPLE 3.5. The implication (ii) $\Rightarrow$ (iii) without the assumption that A is σ-finite is not true in general as this example shows:

Consider the extension

$$0 \to c_0(\mathbb{N}) \to \ell^\infty(\mathbb{N}) \to \frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})} \to 0.$$

Property (ii) holds for $\ell^\infty(\mathbb{N})$ because $\ell^\infty(\mathbb{N})$ is a finite von Neumann algebra. It follows that (ii) also holds in the quotient $\frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})}$. But $\frac{\ell^\infty(\mathbb{N})}{c_0(\mathbb{N})}$ is not an AW^*-algebra. This follows by the same argument as in the proof that the Calkin Algebra $\frac{B(H)}{K(H)}$ is not an AW^*-algebra given in [11] p.222.

G. Robertson proves in [21] that if A is abelian, then conditions (i) and (ii) are equivalent for A, and they are again equivalent to the spectrum $\hat{A}$ of A being an F-space of dimension at most 1 (by definition, $\hat{A}$ is an F-space if disjoint cozero sets of $\hat{A}$ have disjoint closures).

4. THE RELATIVE POINT SPECTRUM

A key step in proving (ii)′ ⇒ (ii) lies in the observation that if A is σ-finite and $a \in A$ is normal, then there are at most countably many $\lambda \in \mathbb{C}$ such that $ax = \lambda x$ for some non-zero $x \in A$.

DEFINITION 4.1. Let A be a C^*-algebra, and let $a \in A$.

a) For $\lambda \in \mathbb{C}$ set

$$E(a,\lambda) = \{x \in A \,|\, ax = \lambda x\}.$$

b) The set

$$\Lambda(a) = \{\lambda \in \mathbb{C} \,|\, E(a,\lambda) \neq \{0\}\}$$

will be called the point spectrum of a relative to A.

c) Say that a has pure point spectrum relative to A if $x \in A$ and $xy = 0$ for all $y \in \cup_{\lambda \in \mathbb{C}} E(a,\lambda)$ implies $x = 0$.

Note that $E(a,\lambda)$ is a closed right-ideal in A, and that $\Lambda(a)$ is contained in the spectrum of a. In $B(H)$, the algebra of all bounded operators on a Hilbert space H, $\Lambda(a)$ is the (usual) point spectrum of $a \in B(H)$.

PROPOSITION 4.2. *Assume A is a σ-finite C^*-algebra and $a \in A$ is normal. Then $\Lambda(A)$ is countable.*

Proof: An easy computation shows that if $a \in A$ is normal and $ax = \lambda x$ for some $x \in A$ and $\lambda \in \mathbb{C}$, then $a^* x = \bar{\lambda} x$. For each $\lambda \in \Lambda(a)$ choose a non-zero $x_\lambda \in E(a,\lambda)$ and set $z_\lambda = x_\lambda x_\lambda^*$. If $\lambda, \mu \in \Lambda(a)$, then

$$\mu z_\lambda z_\mu = z_\lambda a z_\mu = (a^* z_\lambda)^* z_\mu = (\bar{\lambda} z_\lambda)^* z_\mu = \lambda z_\lambda z_\mu.$$

Hence $(z_\lambda)_{\lambda \in \Lambda(a)}$ is an orthogonal family, and therefore $\Lambda(a)$ is countable. ■

LEMMA 4.3. *Let a_+ and a_- be orthogonal positive elements in a C^*-algebra A. Then*

$$\Lambda(a_+ - a_-) \cup \{0\} = \Lambda(a_+) \cup -\Lambda(a_-).$$

Proof: Assume first that $(a_+ - a_-)x = \lambda x$ for some non-zero scalar λ and some non-zero $x \in A$. Then either $a_+ x$ or $a_- x$ is non-zero. Multiplying $(a_+ - a_-)x = \lambda x$ from the left with a_+ and a_- yields

$$a_+(a_+ x) = \lambda a_+ x \quad \text{and} \quad a_-(a_- x) = -\lambda a_- x.$$

Hence either $\lambda \in \Lambda(a_+)$ or $-\lambda \in \Lambda(a_-)$.

Suppose $a_+x = \lambda x$ and $a_-y = \mu y$ for some non-zero scalars λ and μ, and some non-zero $x, y \in A$. Then $a_-x = 0 = a_+y$. So

$$(a_+ - a_-)x = \lambda x \quad \text{and} \quad (a_+ - a_-)y = -\mu y,$$

which implies that λ and $-\mu$ are in $\Lambda(a_+ - a_-)$. ■

5. PROOF OF THEOREM 2.2, PART II

This section contains the proof of (ii)′ ⇒ (ii). The proof uses the concept of relative spectrum discussed above. Throughout this section A will be assumed to be a unital σ-finite C^*-algebra satisfying property (ii)′.

LEMMA 5.1. *Assume that $a = a^* \in A$ has a pure point spectrum relative to A, and that $\Lambda(a) \cap \Lambda(-a) \subseteq \{0\}$. Then $a = u|a|$ for some unitary u in A.*

Proof: Since (ii)′ holds in A, thare is a unitary v in A such that

$$a_+^{\frac{1}{2}} + \mathrm{i}a_-^{\frac{1}{2}} = vb,$$

where $b = b^* \in A$. Note that $b^2 = |a|$, and because vb is normal, v commutes with $b^2 = |a|$ and with $a^2 = |a|^2$.

Let $\lambda \in \Lambda(a)$ and let $x \in E(a, \lambda)$ so that $ax = \lambda x$. Then

$$(a + \lambda 1)(a - \lambda 1)vx = (a^2 - \lambda^2 \cdot 1)vx = v(a^2 - \lambda^2 \cdot 1)x = v(a + \lambda 1)(a - \lambda 1)x = 0.$$

If $\lambda \neq 0$, then $-\lambda \notin \Lambda(a)$, and so $(a - \lambda 1)vx = 0$. This proves

$$avx = \lambda vx = vax,$$

and hence $(av - va)x = 0$ for all $x \in \cup_{\lambda \in \mathbf{C}} E(a, \lambda)$. This entails $av = va$ by the assumption that a has pure point spectrum relative to A.

Since $vb = a_+^{\frac{1}{2}} + \mathrm{i}a_-^{\frac{1}{2}}$ is a function of a, v also commutes with b. Thus

$$a = \left(a_+^{\frac{1}{2}} + \mathrm{i}a_-^{\frac{1}{2}}\right)^2 = vbvb = v^2b^2 = v^2|a|,$$

and we may take $u = v^2$. ■

LEMMA 5.2. *Let $a = a^* \in A$, and let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(t) > 0$ for all $t > 0$, $f(0) = 0$, and $f(t) < 0$ for all $t < 0$. Assume $f(a) = u|f(a)|$ for some unitary u in A. Then $a = u|a|$.*

Proof: Let B_+ and B_- be the norm-closed hereditary subalgebras of A generated by $f(a)_+$, respectively $f(a)_-$. If $f(a) = u|f(a)|$, then $ub = b$ for all $b \in B_+$, and $ub = -b$ for all $b \in B_-$. The assumptions on f imply that $a_+ \in B_+$ and $a_- \in B_-$. Hence $a = u|a|$. ∎

LEMMA 5.3. *Assume that $a = a^* \in A$ has pure point spectrum relative to A. Then $a = u|a|$ for some unitary u in A.*

Proof: By Proposition 4.2 the sets $\Lambda(a_+)$ and $\Lambda(a_-)$ are countable. Hence

$$\Lambda(a_+) \cap t\Lambda(a_-) \subseteq \{0\}$$

for some $t \in (0,1)$. For this t, use Lemma 4.3 to see that

$$\Lambda(a_+ - ta_-) \cap -\Lambda(a_+ - ta_-) \subseteq \{0\}.$$

Lemma 5.1 now produces a unitary u in A such that $a_+ - ta_- = u|a_+ - ta_-|$, and by Lemma 5.2, $a = u|a|$. ∎

5.4. CANTOR SETS AND CANTOR FUNCTIONS. Let $S_0 \subseteq \left[0, \frac{1}{3}\right]$ be the (non--standard) Cantor set

$$S_0 = \left\{ \sum_{j=1}^{\infty} b_j 4^{-j} \mid b_j = 0 \text{ or } b_j = 1 \right\},$$

and recall that S_0 is compact with no interior points. Let $f_0 : \left[0, \frac{1}{3}\right] \to [0,1]$ be the corresponding Cantor function which is the unique increasing continuous extension of the function that on S_0 is

$$f_0\left(\sum_{j=1}^{\infty} b_j 4^{-j}\right) = \sum_{j=1}^{\infty} b_j 2^{-j}, \quad b_j \in \{0,1\}.$$

Note that $S_0 - S_0 \subseteq \left[-\frac{1}{3}, \frac{1}{3}\right]$ is homeomorphic to $\{0,1,2\}^{\mathbf{N}}$ and therefore also compact without interior points (cf. [6], proof of Lemma 2.2). Let $S = S_0 + \mathbf{Z}$. Then S and $S - S$ are closed and have no interior points. Extend f_0 to a continuous increasing function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(t) = n, \quad t \in [n + \frac{1}{3}, n+1], \quad n \in \mathbf{Z},$$

$$f(t+n) = f_0(t) + n, \quad t \in [0,1], \quad n \in \mathbf{Z},$$

and note that $f(n) = n$ for all $n \in \mathbf{Z}$. Moreover, f is constant on each connected component of the complement of S.

LEMMA 5.5. *With the notation of 5.4, there is an uncountable set $\Gamma \subseteq \mathbf{R}$ such that for all distinct γ_1 and γ_2 in Γ,*

$$(S+\gamma_1)\cap(S+\gamma_2)=\emptyset.$$

Proof: Let $\Gamma \subseteq \mathbf{R}$ be maximal with respect to

$$(\Gamma-\Gamma)\cap(S-S)=\{0\}.$$

Then, clearly, $(S+\gamma_1)\cap(S+\gamma_2)=\emptyset$ for distinct $\gamma_1,\gamma_2 \in \Gamma$. Suppose Γ is countable. Then by Baire's theorem

$$\cup_{\gamma\in\Gamma}(S-S)+\gamma \neq \mathbf{R}.$$

Choose $\gamma_0 \notin \cup_{\gamma\in\Gamma}(S-S)+\gamma$, and set $\Gamma_1 = \Gamma\cup\{\gamma_0\}$. Note that $\gamma_0 \notin \Gamma$, because $0 \in S-S$. By the choice of γ_0,

$$(\gamma_0-\Gamma)\cap(S-S)=\emptyset \quad \text{and} \quad (\Gamma-\gamma_0)\cap(S-S)=\emptyset.$$

Hence $(\Gamma_1-\Gamma_1)\cap(S-S)=\{0\}$ which contradicts the maximality of Γ. ∎

5.6. With the notation of 5.4, set

$$F_0(t)=\exp(f(\log(t))), \quad t\in\mathbf{R}^+,$$

and extend F_0 to an increasing continuous function $F:\mathbf{R}\to\mathbf{R}$ by setting $F(0)=0$ and $F(-t)=F_0(t)$ for $t\in\mathbf{R}^+$. Notice that $F(t)>0$ for all $t>0$, and $F(t)<0$ for all $t<0$. Put

$$L=\exp(S)\cup\{0\}\cup-\exp(S).$$

Then, by construction, L is closed, and F is constant on each connected component of the complement of L. Set $T=\exp(\Gamma)\subseteq\mathbf{R}^+$.

LEMMA 5.7. *If $t_1,t_2\in T$ are distinct, then*

$$t_1L\cap t_2L=\{0\}.$$

Proof: Set $\gamma_j=\log(t_j)$ so γ_1 and γ_2 are distinct elements of Γ. The intersection of $t_1L\cap t_2L$ with $\mathbf{R}^+$, respectively $\mathbf{R}^-$, are

$$\exp((S+\gamma_1)\cap(S+\gamma_2)) \quad \text{and} \quad -\exp((S+\gamma_1)\cap(S+\gamma_2)).$$

By Lemma 5.5 these two sets are empty. ■

5.8. The next lemma only assumes the σ-finiteness of A.

LEMMA. *Let a be a self-adjoint element in A. Then there is an $s > 0$ such that $F(sa)$ has pure point spectrum relative to A.*

Proof: We may assume $A \subseteq B(H)$ for some Hilbert space H. Set

$$p_t = \chi_{tL}(a) \quad \text{for} \quad t \in T, \quad \text{and} \quad q = \chi_{\{0\}}(a),$$

p_t and q are projections on H. From Lemma 5.7, $p_{t_1}p_{t_2} = q$ for all pairs of distinct $t_1, t_2 \in T$. Set also

$$I_t = \{x \in A \mid xp_t = x\}, \quad t \in T.$$

We claim that $I_t \subseteq E(a,0)^*$ for at least one $t \in T$. Suppose otherwise, and choose $x_t \in I_t \backslash E(a,0)^*$ for each $t \in T$. Then $b_t = ax_t^*x_ta$ is non-zero, $p_tb_t = b_tp_t = b_t$ and $qb_t = b_tq = 0$. Hence $(b_t)_{t\in T}$ is an orthogonal family in A, in contradiction with T being uncountable and A being σ-finite.

Now, choose $t \in T$ such that $I_t \subseteq E(a,0)^*$, and set $s = t^{-1}$. Suppose $x \in A$ is such that

$$xy = 0 \quad \text{for all} \quad y \in \cup_{\lambda \in \mathbf{R}} E(F(sa), \lambda).$$

Let U be a connected component of the open set L^c, and let g be a continuous function supported on U. Then $Fg = \lambda g$ where λ is the constant value F attains on U, and so $g(sa) \in E(F(sa), \lambda)$. Because each $g \in C_c(L^c)$ is a finite sum of functions supported on connected components of L^c, we have

$$xg(sa) = 0 \quad \text{for all} \quad g \in C_c(L^c).$$

Because $1 - p_t = \chi_{L^c}(sa)$, this implies $x(1-p_t) = 0$, and so

$$x^* \in E(a,0) = E(F(sa),0).$$

Thus $xx^* = 0$, so $x = 0$, and $F(sa)$ has pure point spectrum. ■

5.9. Proof of (ii)$'$ $\Rightarrow$ (ii): It suffices to show that each self-adjoint $a \in A$ is of the form $a = u|a|$ for some unitary u in A. By Lemma 5.8 there is $s \in \mathbf{R}^+$ such that $F(sa)$ has pure point spectrum relative to A. From Lemma 5.3 there is a unitary u in A such that $F(sa) = u|F(sa)|$, and by Lemma 5.2, $a = u|a|$ as wanted.

5.10. It should in conclusion be noted that if $x = ub$ in some C^*–algebra A, where x is normal, u is unitary and b is self-adjoint, then it does not follow that u and b must commute. As a counterexample take $A = C([0,1], M_2)$,

$$x(t) = \begin{pmatrix} t & 0 \\ 0 & it \end{pmatrix}, \quad u(t) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \quad \text{and} \quad b(t) = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}.$$

Moreover, in the C^*-algebra generated by x, u and b there is no unitary v such that $x = v|x|$.

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