J. OPERATOR THEORY
30(1993), 201–215

CP-DUALITY FOR C^*- AND W^*-ALGEBRAS

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INTRODUCTION

In this paper, we initiate a theory of operator convexity for completely positive maps, and discuss the duality for C^*- and W^*-algebras in the context of this new convexity theory.

The convexity arguments in the theory of operator algebras provide a significant perspective for the subject and clarify the order or Jordan structure of the algebra (e.g., [1], [4]). In this scheme, the self-adjoint part A_{sa} of a unital C^*-algebra A is abstracted as an order unit space and is identified with the $A(K)$-space, as is well known as Kadison's function representation theorem (originally due to [9]), where $A(K)$ denotes the set of all real-valued w^*-continuous affine functions on the state space $K = S(A)$. However, the essential stream of this approach is in the realm of real functional analysis, and the attempt to find the extended expression for the whole C^*-algebra A would reveal the limitation of this formalism, i.e., the complex-valued affine functions on the state space can no more preserve the C^*-product and C^*-norm. Therefore, the refined algebraic arguments which depend essentially on the C^*-product would have eluded from this scheme.

For algebraic duality, we can go back to the duality theorems by M. Takesaki [14] and K. Bichteler [5], where they proved that a C^*-algebra A is $*$-isomorphic to the set of all weakly continuous $B(H)$-valued functions which preserve direct sum and unitary equivalence on the space $\mathrm{Rep}(A : H)$ of all representations of A on a Hilbert space H, where the dimension of H is large enough so that every cyclic representation of A is realized on H. (We call this Takesaki's duality theorem; see Section 2 in details). From these observations, one may naturally be led to wish for a duality theory which would preserve algebraic structure as in Takesaki's duality theorem, provide the geometric perspective as in the convexity scheme, and interpolate these two duality theorems. The intention of this paper, therefore, is to propose and develop the basic theory of a non-commutative $B(H)$-valued functional analysis for operator algebras, which includes and interpolates both aspects of convexity theory and algebraic theory, by letting the $B(H)$-valued completely bounded maps of the algebra play the role of the dual space, instead of the usual complex valued bounded linear functionals.

To realize this idea, for a C^*-algebra A and a Hilbert space H which was posited in Takesaki's duality theorem, we take as our dual object the *CP-state space* $Q_H(A)$ of A, which is defined to be the unit ball of the cone $CP(A, B(H))$ of all completely positive maps from A to $B(H)$. Note that $Q_H(A)$ includes both the quasi-state space $Q(A)$ of A in the scalar convexity theory and the representation space $\mathrm{Rep}(A : H)$ in Takesaki's duality theorem. We then propose the following convexity in $Q_H(A)$: ψ is said to be a *CP-convex combination* of $(\psi_\alpha) \subset Q_H(A)$, if

$$\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \quad \text{with} \quad S_\alpha \in B(H) \quad \text{such that} \quad \sum_\alpha S_\alpha^* S_\alpha \leqslant I_H.$$

The convergence of the above sum is secured from the condition $\sum_\alpha S_\alpha^* S_\alpha \leqslant I_H$ (Proposition 1.2), and this convexity reduces to the scalar convexity when it is restricted to the quasi-state space $Q(A)$; it also describes the direct sum and unitary equivalence on $\mathrm{Rep}(A : H)$ which are essential operations in Takesaki's duality. (We note that the idea of CP-convexity was also motivated from the theory of "operation" in the C^*-algebraic formulation of quantum physics. cf. Remark to Proposition 1.4.)

After discussing some basic properties of CP-convexity in Section 1, we will show in Theorem 2.2 that the original algebra A is *-isomorphic to the set of all weakly continuous $B(H)$-valued "CP-affine" functions on $Q_H(A)$, which generalizes Kadison's function representation theorem with recovering the full C^*-structure. This CP-duality theorem is a natural extension of Takesaki's duality theorem, and can also be derived directly from it. Moreover, it will be shown in Theorem 3.2 that the CP-convexity in CP-state space characterizes the C^*-structure of the algebra, i.e., the CP-state spaces $Q_H(A)$ and $Q_H(B)$ of C^*-algebras A and B are "CP-affine" BW-homeomorphic if and only if A and B are *-isomorphic, which should be compared to Kadison's result [10] that $Q(A)$ and $Q(B)$ are affine w^*-homeomorphic if and only if A and B are Jordan isomorphic.

The notion of CP-convexity was exploited further in [7] to discuss various applications in operator algebras, such as CP-facial structure of CP-state space and duality, CP-duality for JC- and JW-algebras, CP-measure and integration, CP-decomposition and CP-Choquet theorem, CP-orientability, CP-geometric realization of Tomita-Takesaki theory, and Stone-Weierstrass theorem for separable C^*-algebras (which will be submitted for publication elsewhere). The theory of CP-convexity is also expected to find useful applications in mathematical physics. A preliminary report on main results of this paper has already appeared in [8].

1. CP-CONVEXITY

We refer for the definition and basic properties of completely positive maps to the references [3], [12], [13] and [15]. Recall in particular that, as first proven by Stinespring [13], every $\psi \in CP(A, B(H))$ can be represented as

$$\psi(a) = V^*\pi(a)V \quad \text{for all} \quad a \in A,$$

where π is a representation of A on a Hilbert space K and $V \in B(H, K)$ is a bounded linear operator from H to K such that $\|\psi\| = \|V\|^2$. We assume, without loss of generality, the minimal condition $K = [\pi(A)VH]$ (where [] represents the closed linear span) under which the Stinespring representation is unique up to unitary intertwining operators. A bounded net $(\psi_\alpha) \subset CP(A, B(H))$ is defined to converge to $\psi \in CP(A, B(H))$ in *BW-topology* [resp. *BS-topology*] if $\psi_\alpha(a)$ converges weakly [resp. strongly] to $\psi(a)$ in $B(H)$ for all $a \in A$. The BW-topology has been commonly used in literatures, and it is shown in [3] that every bounded (norm closed) ball in $CP(A, B(H))$ is BW-compact.

Now we introduce our new concept of convexity in the cone $CP(A, B(H))$ as follows.

DEFINITION 1.1 Let $(\psi_\alpha)_{\alpha\in\Lambda}$ be a bounded family in $CP(A, B(H))$. We define a *CP-convex combination* of $(\psi_\alpha)_{\alpha\in\Lambda}$ by

$$\sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha \quad \text{with} \quad S_\alpha \in B(H) \quad \text{such that} \quad \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leqslant I_H,$$

where the sum converges in the BS-topology (cf. Proposition 1.2). A subset $K \subset \subset CP(A, B(H))$ is defined to be *CP-convex* if it is closed under the operation of CP-convex combination. For a bounded subset B in $CP(A, B(H))$, the *CP-convex hull* of B, denoted by *CP-conv* B, is defined to be the set of all CP-convex combinations of bounded families in B, which is the smallest CP-convex set including B.

In the above definition, the convergence of the CP-convex combination automatically follows from the condition $\sum_\alpha S_\alpha^* S_\alpha \leqslant I_H$. For the completness of the definition, we shall prove this fact.

PROPOSITION 1.2. *For any bounded family $(\psi_\alpha)_{\alpha\in\Lambda}$ in $CP(A, B(H))$ and for any family $(S_\alpha)_{\alpha\in\Lambda}$ in $B(H)$ such that $\sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leqslant I_H$, the CP-convex combination*

$\sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha$ *converges in the BS-topology to an element* $\psi \in CP(A, B(H))$ *with* $\|\psi\| \leqslant \sup_{\alpha\in\Lambda} \|\psi_\alpha\|$ $(:= r)$.

Proof: Let $a \in A^+ := \{a \in A; a \geqslant 0\}$ and let $J \subset \Lambda$ be any finite subset. Then,

$$\psi_J(a) := \sum_{\alpha\in J} S_\alpha^*\psi_\alpha(a)S_\alpha \leqslant \sum_{\alpha\in J} S_\alpha^*(\|\psi_\alpha\|\,\|a\|I_H)S_\alpha \leqslant r\|a\|\sum_{\alpha\in J} S_\alpha^* S_\alpha \leqslant r\|a\|I_H.$$

Thus, $(\psi_J(a))_{J\subset\Lambda}$ (J: finite subsets of Λ) is a bounded increasing net in $B(H)$; hence it converges strongly to an element in $B(H)$. Since every element $a \in A$ is decomposed as a linear combination of positive elements, i.e.,

$$a = a_1 - a_2 + i(a_3 - a_4) \quad \text{with} \quad a_i \in A^+ (1 \leqslant i \leqslant 4),$$

we have

$$\|\psi_J(a)\| \leqslant r\sum_{i=1}^{4}\|a_i\| \quad \text{for all} \quad a \in A,$$

so $(\psi_J)_{J\subset\Lambda}$ is a bounded Cauchy net in the BW-topology. Since a bounded ball of $CP(A, B(H))$ is BW-compact, the net $(\psi_J)_{J\subset\Lambda}$ converges to an element $\psi \in$ $\in CP(A, B(H))$ in the BW-topology, i.e., $\psi = \lim_J \psi_J = \sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha$. It is straightforward to see that the net (ψ_J) then converges to ψ in the BS-topology.

It is left to show $\|\psi\| \leqslant r$. Note that

$$\|\psi\|^2 = \sup_{\|a\|\leqslant 1} \|\psi(a)\|^2 = \sup_{\|a\|\leqslant 1} \|\psi(a)^*\psi(a)\| \leqslant \|\psi\| \sup_{\|a\|\leqslant 1} \|\psi(a^*a)\|,$$

where we used the inequality $\psi(a)^*\psi(a) \leqslant \|\psi\|\psi(a^*a)$ for $a \in A$ (e.g., [15; Chapter IV, Corollary 3.8]). From the first part of the proof, we have

$$\psi(a) = \lim_J \psi_J(a) \leqslant r\|a\|I_H \quad \text{for} \quad a \in A^+.$$

Hence, $\sup_{\|a\|\leqslant 1} \|\psi(a^*a)\| \leqslant \sup_{\|a\|\leqslant 1} r\|a^*a\| \leqslant r$, from which $\|\psi\| \leqslant r$ follows. ■

In what follows, throughout this paper, we are mainly concerned with the unit ball of the cone $CP(A, B(H))$ for a large Hilbert space H; note that this is *CP-convex* by Proposition 1.2. We shall use notations Rep(A) [resp. Rep$_c(A)$, Irr(A)] for the set of all [resp. cyclic, irreducible] representations of A. To specify the Hilbert space H on which the representations are confined, we will write Rep$(A : H)$ etc. The notation H_π for $\pi \in$ Rep(A) is reserved to denote the essential subspace of π. (Hence, Rep$(A : H) = \{\pi \in \text{Rep}(A); H_\pi \subset H\}$.) The notations for W^*-algebras are defined similarly, e.g., Rep$_c(M : H)_n$ denotes the set of all normal cyclic representations of M on H, where the suffix n represents the normal part.

DEFINITION 1.3. A CP-map $\psi \in CP(A, B(H))$ is called a *CP-state* if it is a contraction, i.e., $\|\psi\| \leqslant 1$. We denote by $Q_H(A)$ the set of all *CP-states* from A to $B(H)$, i.e.,

$$Q_H(A) = \{\psi \in CP(A, B(H)); \|\psi\| \leqslant 1\},$$

and call $Q_H(A)$ the *CP-state* space of A for H. We define the *cyclic dimension* $\alpha_c(A)$ by

$$\alpha_c(A) := \sup\{\dim H_\pi; \pi \in \mathrm{Rep}_c(A)\}.$$

Similarly, for a W^*-algebra M, we define the *normal CP-state space* of M for H by

$$Q_H(M)_n = \{\psi \in CP(M, B(H))_n; \|\psi\| \leqslant 1\},$$

and the *normal cyclic dimension* $\alpha_c(M)_n$ by

$$\alpha_c(M)_n := \sup\{\dim H_\pi; \pi \in \mathrm{Rep}_c(M)_n\}.$$

Note that $Q_H(A)$ [resp. $Q_H(M)_n$] generalizes the quasi-state space $Q(A)$ [resp. the normal quasi-state space $Q(M)_n$] of the scalar theory (where $H = \mathbb{C}$).

PROPOSITION 1.4.A. *Let A be a C^*-algebra and H be a Hilbert space with $\dim H \geqslant \alpha_c(A)$. Then,*

$$Q_H(A) = CP\text{-}conv\,\mathrm{Rep}_c(A : H).$$

Proof: Let $\psi \in Q_H(A)$, and let $\psi = V^*\pi V$ be the Stinespring representation of ψ where $\pi \in \mathrm{Rep}(A)$ and $V \in B(H, H_\pi)$. Since π is non-degenerate from the minimal condition, π can be decomposed into a direct sum of cyclic representations $(\pi_\alpha)_{\alpha\in\Lambda}$, i.e., $\pi = \oplus_{\alpha\in\Lambda}\pi_\alpha$([6; 2.2.7]). Let us denote by p_α the projection of H_π onto the representation space H_{π_α} of π_α. Then,

$$\psi = V^*\pi V = V^*(\underset{\alpha\in\Lambda}{\oplus} p_\alpha)^*(\underset{\alpha\in\Lambda}{\oplus} \pi_\alpha)(\underset{\alpha\in\Lambda}{\oplus} p_\alpha)V = \sum_{\alpha\in\Lambda} V^*p_\alpha^*\pi_\alpha p_\alpha V.$$

From the assumption $\dim H \geqslant \alpha_c(A)$, there exists a partial isometry $W_\alpha : H \longrightarrow H_{\pi_\alpha} \subset \subset H_\pi$ from H onto H_{π_α} for each $\alpha \in \Lambda$. In this case $W_\alpha W_\alpha^* = p_\alpha$, so we have

$$\psi = \sum_{\alpha\in\Lambda} V^*(W_\alpha W_\alpha^*)^*\pi_\alpha(W_\alpha W_\alpha^*)V = \sum_{\alpha\in\Lambda}(W_\alpha^*V)^*(W_\alpha^*\pi_\alpha W_\alpha)(W_\alpha^*V).$$

Note here that $W_\alpha^*\pi_\alpha W_\alpha \in \mathrm{Rep}_c(A : H)$, and that $W_\alpha^*V \in B(H)$ satisfies

$$\sum_{\alpha\in\Lambda}(W_\alpha^*V)^*(W_\alpha^*V) = \sum_{\alpha\in\Lambda} V^*(W_\alpha W_\alpha^*)V = \sum_{\alpha\in\Lambda} V^*p_\alpha V =$$

$$= V^*(\underset{\alpha\in\Lambda}{\oplus} p_\alpha)V = V^*V \leqslant \|V\|^2 I_H \leqslant I_H.$$

Thus we proved that ψ is a CP-convex combination of cyclic representations of A on H, i.e., $Q_H(A) \subset CP\text{-}conv\,\mathrm{Rep}_c(A : H)$. Since the inverse inclusion is obvious, the proposition is proved. ■

By similar arguments, we can show

PROPOSITION 1.4.B. *Let M be a W^*-algebra, and H be a Hilbert space with* $\dim H \geqslant \alpha_c(M)_n$. *Then,*

$$Q_H(M)_n = CP\text{-}conv\,\mathrm{Rep}_c(M : H)_n.$$

REMARK. In the above proposition, we can replace $\mathrm{Rep}_c(A : H)$ [resp. $\mathrm{Rep}_c(M : H)_n$] by $\mathrm{Irr}(A : H)$ [resp. $\mathrm{Irr}(M : H)_n$] if and only if A [resp. M] is a scattered C^*-algebra [resp. atomic W^*-algebra] (cf. [7]). K. Kraus [11] considered the particular case of $M = B(H)$ and obtained this pure decomposition, where a CP-state acquires the physical meaning of "operation" which describes the change of observables caused by an interaction of a physical system with exterior. In the theory of operation, the coefficient "$V_\alpha^*(\cdot)V_\alpha$" is called "effect", which represents the weight of the pure operation, and in this sense CP-convexity can be considered as a "quantization" of scalar convexity. The idea of this generalization of probability measure was developed into the theory of CP-measure and integration and generalization of Choquet's theorem in [7].

To conclude this section, we shall briefly note the situation of the quasi-state space $Q(A)$ being embedded in the CP-state space $Q_H(A)$. Let us denote by $Q_{H,1}(A)$ the set of all one-dimensional CP-states, i.e.,

$$Q_{H,1}(A) := \{\psi = V^*\pi V \in Q_H(A); \dim VH = 1\}.$$

If $\psi = V^*\pi V \in Q_{H,1}(A)$, then, since $\dim VH = 1$, V must have the form

$$V = \xi \otimes \overline{h} = (\cdot, h)\xi \quad \text{for} \quad \xi \in H_\pi \quad \text{and} \quad \overline{h} \in \overline{H},$$

where $\overline{H}$ denotes the complex conjugate of H. Then, we can easily check that

$$\psi(a) = (\xi \otimes \overline{h})^*\pi(a)(\xi \otimes \overline{h}) = (\pi(a)\xi, \xi)(h \otimes \overline{h}) = \omega(a)P_h \quad \text{for all} \quad a \in A,$$

where $\omega := (\pi(\cdot)\xi, \xi) \in Q(A)$ and $P_h := h \otimes \overline{h}$ is the projection of H onto $[h]$. Conversely, each $\omega \in Q(A)$ defines a one dimensional CP-state by the above equality. Hence, we have shown that there exists a one to one correspondence

$$\omega \in Q(A) \quad \longleftrightarrow \quad \psi_\omega = \omega(\cdot)P \in Q_{H,1}(A) \quad (\mathrm{mod}\, P)$$

where $(\mathrm{mod}\, P)$ represents the unitary equivalence class of the one dimensional *$CP-states$*. This also implies that $Q(A) = Q_C(A)$. The similar argument for the normal quasi-states of a W^*-algebra is straighforward.

2. CP-DUALITY FOR C^*- AND W^*-ALGEBRAS

We shall establish a duality for C^*- and W^*-algebras in the context of CP-convexity, where we take as our dual object the CP-state space $Q_H(A)$ [resp. the normal CP-state space $Q_H(M)_n$] of a C^*-algebra A [resp. a W^*-algebra M]. We first generalize the notion of affine functions in the classical scalar convexity theory for our CP-convexity context.

DEFINITION 2.1. Let A be a C^*-algebra. A function $\gamma : Q_H(A) \to B(H)$ is defined to be *CP-affine*, if

$$\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \text{ with } \psi_\alpha \in Q_H(A) \text{ and } S_\alpha \in B(H) \text{ such that } \sum_\alpha S_\alpha^* S_\alpha \leqslant I_H$$

implies that

$$\gamma(\psi) = \sum_\alpha S_\alpha^* \gamma(\psi_\alpha) S_\alpha.$$

γ is defined to be *bounded* if $\|\gamma\| = \sup\{\|\gamma(\psi)\|; \psi \in Q_H(A)\} < \infty$. We denote by $AC(Q_H(A), B(H))$, or precisely by $AC_w(Q_H(A), B(H))$ or $AC_s(Q_H(A), B(H))$, the set of all BW-w or BS-s continuous CP-affine functions from $Q_H(A)$ to $B(H)$. For a W^*-algebra M, we denote by $AB(Q_H(M)_n, B(H))$ the set of all bounded CP-affine functions from $Q_H(M)_n$ to $B(H)$.

Obviously $AC(Q_H(A), B(H))$ is a linear space with pointwise addition and scalar multiplication. Assuming now that H is a Hilbert space with $\dim H \geqslant \alpha_c(A)$, we shall try to define a product in $AC(Q_H(A), B(H))$ as follows: for $\gamma_1, \gamma_2 \in AC(Q_H(A), B(H))$ and $\psi = V^*\pi V \in Q_H(A)$ with CP-decomposition

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \quad \text{where} \quad \pi_\alpha \in \mathrm{Rep}_c(A : H) \quad \text{with} \quad \pi \simeq \bigoplus_\alpha \pi_\alpha$$

and $V_\alpha \in B(H)$ such that $\sum_\alpha V_\alpha^* V_\alpha \leqslant I_H$ (cf. Proposition 1.4.A), we define the product of γ_1 and γ_2 by

$$(\gamma_1 \cdot \gamma_2)(\psi) := \sum_\alpha V_\alpha^* \gamma_1(\pi_\alpha) \cdot \gamma_2(\pi_\alpha) V_\alpha.$$

We can define a $*$-operation by $\gamma^*(\psi) := \gamma(\psi)^*$.

It will be shown in the subsequent theorems that the product defined above is well-defined, with which $AC(Q_H(A), B(H))$ [resp. $AB(Q_H(M)_n, B(H))$] has the natural structure as a C^*-algebra [resp. W^*-algebra]. The following duality theorems generalize Kadison's function representation theorem [9] and have the advantage that they recover the full C^*-structure of the original algebra, i.e., C^*-product and C^*-norm; this structure would have escaped our grasp had we considered only the set of affine functions on the quasi-state space as is customarily done in the scalar theory.

THEOREM 2.2.A. *Let A be a C^*-algebra and H be a Hilbert space with* $\dim H \geqslant \geqslant \alpha_c(A)$. *Then, $AC(Q_H(A), B(H))$ is a C^*-algebra with the operations defined above, and we have*

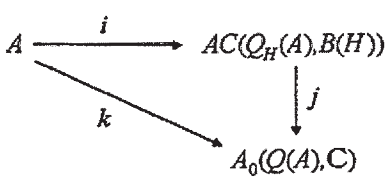
$$A \cong AC(Q_H(A), B(H)) \quad (^*\text{-isomorphism}).$$

THEOREM 2.2.B. *Let M be a W^*-algebra and H be a Hilbert space with* $\dim H \geqslant \geqslant \alpha_c(M)_n$. *Then, $AB(Q_H(M)_n, B(H))$ is a W^*-algebra with the operations defined above, and we have*

$$M \cong AB(Q_H(M)_n, B(H)) \quad (^*\text{-isomorphism}).$$

Proof: The proofs of the above theorems proceed in parallel, and Theorem 2.2.B is obtained in the same manner using the normal part of the CP-state space, so that it will suffice to prove Theorem 2.2.A. We also note that the proofs of the isomorphisms for $AC_w(Q_H(A), B(H))$ and $AC_s(Q_H(A), B(H))$ will be discussed uniformly.

We consider the following diagram:



where $A_0(Q(A), \mathbb{C})$ denotes the set of all w^*-continuous complex valued affine functions on $Q(A)$ vanishing at 0; i assigns the evaluation map at each element of A, i.e.,

$$i(a) : \psi \longmapsto \psi(a) \quad \text{for} \quad a \in A \quad \text{and} \quad \psi \in Q_H(A);$$

j is defined for $\gamma \in AC(Q_H(A), B(H))$ by

$$j(\gamma)(\omega) = (\gamma(\psi_\omega)h, h) \quad \text{for} \quad \omega \in Q_H(A) \quad \text{with} \quad h \in PH, \|h\| = 1,$$

where $\psi_\omega := \omega(\cdot)P$ is the corresponding CP-state of ω and P is a one dimensional projection on H; and k is the Kadison's isomorphism. We note that the above diagram commutes, i.e., $j \circ i = k$. Indeed, let $a \in A$ and $\omega \in Q(A)$, then

$$j(i(a))(\omega) = (i(a)(\psi_\omega)h, h) = (\psi_\omega(a)h, h) = (\omega(a)Ph, h) = \omega(a) := k(a)(\omega).$$

It is immediate to see that i is injective, since, if $i(a) = 0$ on $Q_H(A)$, then $i(a) = 0$ on $Q_{H,1}(A)$, hence on $Q(A)$, which implies $a = 0$. We will show that j is injective, from which, combined with the commutativity of the diagram, we can conclude that i is surjective.

Let $\gamma \in AC(Q_H(A), B(H))$ be arbitrary such that $\gamma \neq 0$. Then, there exist $\varphi \in Q_H(A)$ and $h_0 \in H$ such that $(\gamma(\varphi)h_0, h_0) \neq 0$. Define

$$\omega(\varphi; h_0) := (\varphi(\cdot)h_0, h_0) \in Q(A),$$

and observe that, for $h \in H$ with $\|h\| = 1$,

$$\psi_{\omega(\varphi;h_0)} = \omega(\varphi; h_0)P_h = (\varphi(\cdot)h_0, h_0)P_h = (h_0 \otimes \overline{h})^* \varphi(h_0 \otimes \overline{h}).$$

Then,

$$j(\gamma)(\omega(\varphi; h_0)) = (\gamma(\psi_{\omega(\varphi;h_0)})h, h) = ((h_0 \otimes \overline{h})^* \gamma(\varphi)(h_0 \otimes \overline{h})h, h) =$$

$$= ((\gamma(\varphi)h_0, h_0)P_h h, h) = (\gamma(\varphi)h_0, h_0) \neq 0,$$

i.e., $j(\gamma) \neq 0$. Hence, j is injective, so that i is bijective.

Now let $\gamma_1, \gamma_2 \in AC(Q_H(A), B(H))$, then there exist $a_1, a_2 \in A$ such that $\gamma_1 = i(a_1)$ and $\gamma_2 = i(a_2)$. Assume that $\psi \in Q_H(A)$ has a CP-decomposition

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \text{ where } \pi_\alpha \in \mathrm{Rep}_c(A : H) \text{ and } V_\alpha \in B(H) \text{ with } \sum_\alpha V_\alpha^* V_\alpha \leqslant I_H.$$

Then, by definition,

$$(\gamma_1 \cdot \gamma_2)(\psi) = \sum_\alpha V_\alpha^* \gamma_1(\pi_\alpha)\gamma_2(\pi_\alpha)V_\alpha = \sum_\alpha V_\alpha^* \pi_\alpha(a_1)\pi_\alpha(a_2)V_\alpha =$$

$$= \sum_\alpha V_\alpha^* \pi_\alpha(a_1 a_2)V_\alpha = \psi(a_1 a_2).$$

It follows from this that $\gamma_1 \cdot \gamma_2 = i(a_1 a_2) \in AC(Q_H(A), B(H))$ and that the definition of the product is well defined, i.e., it does not depend on any particular CP-decomposition of ψ. It also implies $i(a_1) \cdot i(a_2) = i(a_1 a_2)$, so that, since $i(a)^* = i(a^*)(a \in A)$ and the linearity is obvious, i is a *-isomorphism as *-algebras.

We shall next show that i is an isometry with the norm of CP-affine functions which was defined in Definition 2.1. In fact, let $\gamma \in AC(Q_H(A), B(H))$ correspond to

$a_\gamma \in A$. Then, it is easily observed that $\|a_\gamma\| = \|\gamma\|_c \leqslant \|\gamma\| \leqslant \|a_\gamma\|$, where $\|\gamma\|_c := \sup\{\|\gamma(\pi)\|; \pi \in \mathrm{Rep}_c(A : H)\}$ and the first equality follows from [6; Theorem 2.7.3]. Hence we have $\|i(a_\gamma)\| = \|\gamma\| = \|a_\gamma\|$, i.e., i is an isometry. This completes the proof. ∎

REMARK 1. We note that Kadison's isomorphism $k : A \cong A_0(Q(A), \mathbb{C})$ for a C^*-algebra A in scalar convexity theory is order and norm preserving for the self-adjoint part A_{sa}, but it is not isometry for the whole C^*-algebra A. (One can easily check counter examples and inequality $\frac{1}{4}\|a\| \leqslant \|k(a)\| \leqslant \|a\|$.) Theorem 2.2. shows that our CP-duality recovers the full C^*-structure, i.e., C^*-product and C^*-norm.

REMARK 2. The CP-duality theorems (Theorem 2.2.A and B) can be considered as natural extensions of Takesaki's and Bichteler's duality thorems for C^*-algebras and their enveloping W^*-algebras ([5], [14]), i.e.,

$$A \cong A_c^F(\mathrm{Rep}(A : H), B(H)) \quad (^*\text{-isomorphism})$$

$$A^{**} \cong A^F(\mathrm{Rep}(A : H), B(H)) \quad (^*\text{-isomorphism})$$

where $A^F(\mathrm{Rep}(A : H), B(H))$ denotes the set of all functions $\gamma : \mathrm{Rep}(A : H) \to$ $\to B(H)$ which satisfy the following conditions

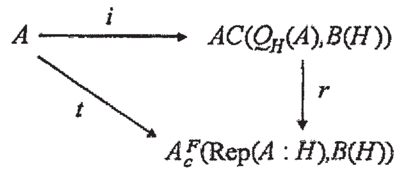
(i) $\gamma(u^*\pi u) = u^*\gamma(\pi)u$ for all partial isometry u on H such that $uu^* \geqslant p_\pi$,

(ii) $\gamma(\pi_1 \oplus \pi_2) = \gamma(\pi_1) \oplus \gamma(\pi_2)$ if $\pi_1 \oplus \pi_2 \in \mathrm{Rep}(A : H)$,

(iii) $\sup\{\|\gamma(\pi)\|; \pi \in \mathrm{Rep}(A : H)\} < \infty$,

and $A_c^F(\mathrm{Rep}(A : H), B(H))$ denotes the set of all weakly (c=w) or strongly (c=s) continuous elements, where note that the BW- and BS-topologies coincide on $\mathrm{Rep}(A : H)$ (cf. [5], [14]). The condition (i) includes, as particular cases, $(\mathrm{i})_1$ $\gamma(\pi) = p_\pi\gamma(\pi)p_\pi$ and $(\mathrm{i})_2$ $\gamma(u^*\pi u) = u^*\gamma(\pi)u$ for all unitary u on H, which were the original definitions by M. Takesaki in [14] for the duality of separable C^*-algebras. (i) was introduced by K. Bichteler in [5] for the non-separable generalization.

We note that Theorem 2.2.A can be proved directly from the above Takesaki's duality theorem for C^*-algebras by considering the following diagram



where t denotes the Takesaki's duality, and r is the restriction map. Indeed, it is enough to check that r is injective as in the proof of Theorem 2.2, which follows immediately. Our method of CP-convexity, however, simplifies the arguments and allows us to state the duality for general W^*-algebras. The CP-duality, which interpolates Kadison's duality and Takesaki's duality (cf. the preceding diagrams), subsumes both the convexity theoretical aspect and the algebraic aspect of C^*-algebras.

3. CP-AFFINE ISOMORPHISMS BETWEEN CP-STATE SPACES

We next discuss the dual maps between CP-state spaces induced by *-isomorphisms. We begin with the following definition.

DEFINITION 3.1 Let A and B be C^*-algebras and let H be a Hilbert space. A map $\Phi : Q_H(A) \to Q_H(B)$ is defined to be CP-affine, if

$$\psi = \sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \text{ with } \psi_\alpha \in Q_H(A) \text{ and } S_\alpha \in B(H) \text{ such that } \sum_\alpha S_\alpha^* S_\alpha \leqslant I_H,$$

then we have

$$\Phi(\psi) = \sum_\alpha S_\alpha^* \Phi(\psi_\alpha) S_\alpha.$$

For the simplicity of our arguments, we shall assume that every CP-affine map preserves minimal condition without loss of generality, i.e., if ψ is expresed in minimal, then so is $\Phi(\psi)$.

In scalar convexity theory, every affine w^*-homeomorphism between quasi-state spaces induces a Jordan isomorphism of the algebras [10]. In our setting of CP-convexity, we can show the following improvement.

THEOREM 3.2.A. *Let A and B be C^*-algebras, and H be a Hilbert space with* $\dim H \geqslant \sup\{\alpha_c(A), \alpha_c(B)\}$. *Then, $Q_H(A)$ and $Q_H(B)$ are CP-affine BW- (or BS-) homeomorphic if and only if A and B are *-isomorphic.*

THEOREM 3.2.B. *Let M and N be W^*-algebras, and H be a Hilbert space with* $\dim H \geqslant \sup\{\alpha_c(M)_n, \alpha_c(N)_n\}$. *Then, $Q_H(M)_n$ and $Q_H(N)_n$ are CP-affine isomorphic if and only if M and N are *-isomorphic.*

Schematically, Kadison's theorem and the above theorems establish the following correspondences:

Jordan structure of the algebra $\longleftrightarrow$ scalar convexity in the state space

C^*-structure of the algebra $\longleftrightarrow$ CP-convexity in the CP-state space.

In order to prove the above theorems, we need the following propositions.

PROPOSITION 3.3.A. *Let A and B be C^*-algebras, and H be a Hilbert space with* $\dim H \geqslant \sup\{\alpha_c(A), \alpha_c(B)\}$. *Then every CP-affine isomorphism* $\Theta : Q_H(A) \to$

$Q_H(B)$ maps Rep$(A : H)$ *[resp.* Rep$_c(A : H)$, Irr$(A : H)$*] onto* Rep$(B : H)$ *[resp.* Rep$_c(B : H)$, Irr$(B : H)$*] bijectively.*

PROPOSITION 3.3.B. *Let M and N be W^*-algebras, and H be a Hilbert space with* dim$H \geqslant \sup\{\alpha_c(M)_n, \alpha_c(N)_n\}$. *Then every CP-affine isomorphism* $\Theta : Q_H(M)_n \to Q_H(N)_n$ *maps* Rep$(M : H)_n$ *[resp.* Rep$_c(M : H)_n$, Irr$(M : H)_n$*] onto* Rep$(N : H)_n$ *[resp.* Rep$_c(N : H)_n$, Irr$(N : H)_n$*] bijectively.*

Proof. We prove Part A, and Part B will be proven similarly.

We first show that if $\pi \in$ Rep$_c(A : H)$, then $\Theta(\pi) \in$ Rep$_c(B : H)$ with $H_{\Theta(\pi)} = = H_\pi$. We note that it suffices to prove this for some $\tilde{\pi} \in$ Rep$_c(A : H)$ which is unitarily equivalent to π; in fact, if $\tilde{\pi} = u^*\pi u$ where u is a partial isometry from $H_{\tilde{\pi}}$ onto H_π, and suppose that $\Theta(\tilde{\pi}) \in$ Rep$_c(B : H)$ with $H_{\Theta(\tilde{\pi})} = H_{\tilde{\pi}}$, then we have $\Theta(\pi) = \Theta(u\tilde{\pi}u^*) = u\Theta(\tilde{\pi})u^* \in$ Rep$_c(B : H)$ and $H_{\Theta(\pi)} = H_\pi$. Hence, if H is infinite dimensional, then by considering some unitary equivalent representation of π, we can assume dim$H \ominus H_\pi =$ dim$H$ without loss of generality.

Let $\xi \in H$ be a cyclic vector for π, and let $\Theta(\pi) = V^*\rho V \in Q_H(B)$ where $\rho \in$ Rep(B) and $V \in B(H, H_\rho)$. Then,

$$\Theta(P_\xi \pi P_\xi) = P_\xi \Theta(\pi) P_\xi = P_\xi V^* \rho V P_\xi,$$

where P_ξ is the projection of H onto $[\xi]$. By our assumption on minimal condition, we have $H_\rho = [\rho(B)VP_\xi H] = [\rho(B)V\xi]$, which shows that ρ is a cyclic representation with the cyclic vector $V\xi$. Let $V = v|V|$ be the polar decomposition of V.Then, since $\rho \in$ Rep$_c(B)$ and dim$H \geqslant \alpha_c(B)$, or dim$H \ominus H_\pi =$ dim$H \geqslant \alpha_c(B)$ if H is infinite dimensional, there exists a partial isometry $\tilde{v} : H \to H_\rho$ from H onto H_ρ which extends v (i.e., $\tilde{v}|_{|V|H} = v$), where we can assume that dim$H \ominus \tilde{v}^* H_\rho =$ dim$H$ if H is infinite dimensional. Then,

$$\Theta(\pi) = V^*\rho V = |V|v^*\rho v|V| = |V|\tilde{v}^*\rho\tilde{v}|V| = |V|\tilde{\rho}|V|,$$

where $\tilde{\rho} := \tilde{v}^*\rho\tilde{v} \in$ Rep$_c(B : H)$. Hence, we can assume without loss of generality that

$$\Theta(\pi) = V\rho V \quad \text{where} \quad \rho \in \mathrm{Rep}_c(B : H) \quad \text{and} \quad V \in B(H)^+,\ \|V\| \leqslant 1,$$

where we can assume that dim$H \ominus H_\rho =$ dim$H$ if H is infinite dimensional.

Now note that $\pi = p_\pi \pi p_\pi$ and that Θ is a CP-affine map, then

$$V\rho V = \Theta(\pi) = p_\pi \Theta(\pi) p_\pi = p_\pi V \rho V p_\pi.$$

Using an approximate unit (e_λ) of A, we have

$$V^2 = \sup_\lambda V\rho(e_\lambda)V = \sup_\lambda \Theta(\pi)(e_\lambda) = \sup_\lambda p_\pi V\rho(e_\lambda)Vp_\pi = p_\pi V^2 p_\pi \leqslant p_\pi,$$

so that $V \leqslant p_\pi$. We note that, since $\rho \in \mathrm{Rep}_c(B : H), \Theta^{-1}$ is CP-affine, and $\dim H \geqslant \alpha_c(A)$, or $\dim H \ominus H_\rho = \dim H \geqslant \alpha_c(A)$ if H is infinite dimensional, we can conclude $\Theta^{-1}(\rho) = W\theta W \in Q_H(A)$ where $\tau \in \mathrm{Rep}_c(A : H)$ and $W \in B(H)^+$ with $\|W\| \leqslant 1$ by the proceding arguments. Then,

$$\pi = \Theta^{-1}(\Theta(\pi)) = \Theta^{-1}(V\rho V) = V\Theta^{-1}(\rho)V = VW\tau WV,$$

so that

$$p_\pi = \sup_\lambda \pi(e_\lambda) = \sup_\lambda VW\tau(e_\lambda)WV = VW^2V \leqslant V^2 \leqslant p_\pi,$$

which implies $V = p_\pi$. Hence, we have $\Theta(\pi) = p_\pi\rho p_\pi$.

Similarly, we have $W = p_\rho$ and $\Theta^{-1}(\rho) = W\tau W = p_\rho\tau p_\rho$, so that

$$\pi = VW\tau WV = p_\pi p_\rho \tau p_\rho p_\pi = p_\pi \tau p_\pi.$$

By the minimal condition and $H_\pi \subset H_\rho \subset H_\tau$, we conclude that $\tau = \pi$ and $H_\rho = H_\pi$. Hence, $\Theta(\pi) = p_\pi\rho p_\pi = \rho \in \mathrm{Rep}_c(B : H)$ and $H_{\Theta(\pi)} = H_\rho = H_\pi$, which proves our assertion.

We next assume that $\pi \in \mathrm{Rep}(A : H)$ be arbitrary. We can then decompose π into a direct sum of cyclic representations $\pi = \bigoplus_\alpha \pi_\alpha = \sum_\alpha p_\alpha \pi_\alpha p_\alpha$ where p_α is the projection of H onto H_{π_α}, which is a CP-convex combination with $\sum_\alpha p_\alpha^2 = p_\pi \leqslant I_H$, so we can deduce

$$\Theta(\pi) = \Theta(\sum_\alpha p_\alpha \pi_\alpha p_\alpha) = \sum_\alpha p_\alpha \Theta(\pi_\alpha) p_\alpha = \bigoplus_\alpha \Theta(\pi_\alpha) \in \mathrm{Rep}(B : H),$$

since $\Theta(\pi_\alpha) \in \mathrm{Rep}_c(B : H)$ with $H_{\Theta(\pi_\alpha)} = H_{\pi_\alpha}$.

It is straightforward to see that Θ maps $\mathrm{Irr}(A : H)$ into $\mathrm{Irr}(B : H)$ since Θ preserves direct sum of representations as seen above.

Since Θ is an isomorphism, we can conclude that Θ maps $\mathrm{Rep}(A : H)$ [resp. $\mathrm{Rep}_c(A : H), \mathrm{Irr}(A : H)$] onto $\mathrm{Rep}(B : H)$ [resp. $\mathrm{Rep}_c(B : H), \mathrm{Irr}(B : H)$] bijectively. ∎

Proof of Theorem 3.2. We first prove Part A. It is straightforward to see that, if A and B are *-isomorphic, then $Q_H(A)$ and $Q_H(B)$ are CP-affine isomorphic. In fact, let $\theta : A \to B$ be a *-isomorphism. We define $\theta^\natural : Q_H(B) \to Q_H(A)$ by $\theta^\natural(\psi) = \psi \circ \theta$

for $\psi \in Q_H(B)$. Then, $\theta^\natural$ is a CP-affine BW- (or BS-) homeomorphism from $Q_H(B)$ to $Q_H(A)$.

Conversely, let $\Theta : Q_H(A) \to Q_H(B)$ be a CP-affine BW- (or BS-) homeomorphism. By Theorem 2.2.A, we can identify A with $AC(Q_H(A), B(H))$ and B with $AC(Q_H(B), B(H))$. We define $\Theta^\natural : AC(Q_H(B), B(H)) \to AC(Q_H(A), B(H))$ by $\Theta^\natural(\gamma) = \gamma \circ \Theta$ for $\gamma \in AC(Q_H(B), B(H))$. We have to prove that $\Theta^\natural$ is a *-isomorphism.

We only need to check that $\Theta^\natural$ preserves the C^*-product. Indeed let $\psi \in Q_H(A)$ and assume

$$\psi = \sum_\alpha V_\alpha^* \pi_\alpha V_\alpha \text{ with } \pi_\alpha \in \mathrm{Rep}_c(A : H) \text{ and } V_\alpha \in B(H) \text{ such that } \sum_\alpha V_\alpha^* V_\alpha \leqslant I_H.$$

Then, for $\gamma_1, \gamma_2 \in AC(Q_H(B), B(H))$, we obtain upon using the fact that $\Theta(\pi_\alpha) \in$ $\in \mathrm{Rep}_c(B : H)$ by Proposition 3.3.A,

$$\begin{aligned} \Theta^\natural(\gamma_1 \cdot \gamma_2)(\psi) &= (\gamma_1 \cdot \gamma_2)(\Theta(\psi)) = (\gamma_1 \cdot \gamma_2)(\sum_\alpha V_\alpha^* \Theta(\pi_\alpha) V_\alpha) = \\ = \sum_\alpha V_\alpha^* \gamma_1(\Theta(\pi_\alpha)) \cdot \gamma_2(\Theta(\pi_\alpha)) V_\alpha &= \sum_\alpha V_\alpha^* \Theta^\natural(\gamma_1)(\pi_\alpha) \cdot \Theta^\natural(\gamma_2)(\pi_\alpha) V_\alpha = \\ = (\Theta^\natural(\gamma_1) \cdot \Theta^\natural(\gamma_2))(\sum_\alpha V_\alpha^* \pi_\alpha V_\alpha) &= (\Theta^\natural(\gamma_1) \cdot \Theta^\natural(\gamma_2))(\psi). \end{aligned}$$

This completes the proof of Part A.

Part B is proved by the similar arguments using Proposition 3.3.B. ∎

REMARK. The point of Theorem 3.2, which distinguishes itself from the result in the scalar theory, lies on the fact that every CP-affine isomorphism between CP-state spaces is *orientation preserving* for the state spaces in the sense of Alfsen-Shultz theory (cf. [2]). This mechanism was studied in [7] with respect to the orientability condition in the context of CP-convexity.

Note added in proofs. Since this paper was submitted for publication, there have been some progress on the theory and applications of *CP-convexity* based on the results of this paper, which are partly included in the articles [16–19] added in the end of References, and some others, including results in [7], will soon follow. It seems that these results would be enough to support the usefulness of the theory of *CP-convexity*, and also to expect further useful applications in non-commutative analysis and mathematical physics in the future.

Acknowledgement. The content of this paper is a part of my Ph.D. dissertation under the supervision of Professor Gérard G. Emch, to whom I wish to express my sincere gratitude for his valuable advice and encouragement. I am also thankful to Professor Edward G. Effros, Professor Frederic W. Shultz, Professor Stephen J. Summers and the referee for their valuable comments to preliminary versions of this work which improved the manuscript.

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Received October 23, 1990; revised August 6, 1992.