J. OPERATOR THEORY
30(1993), 217–226

THE PUNCTURED NEIGHBOURHOOD THEOREM FOR INCOMPLETE SPACES

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The "punctured neighbourhood theorem" of Fredholm theory says that if $T \in BL(X, X)$ is bounded linear operator on a Banach space X and if $\lambda \in \mathbb{C}$ is both an accumulation point and a boundary point of the spectrum of T then ([7] Theorem IV.5.31) $T - \lambda I$ is not semi-Fredholm. In this note we partially extend this result, and also Banach's "closed range" theorem, to incomplete normed spaces.

Recall [4] that if X and Y are normed spaces and if $k > 0$ and if $\|x\| \leqslant k\|Tx\|$ for each $x \in X$ then we call $T \in BL(X, X)$ *bounded below*, if $y \in \{Tx : \|x\| \leqslant k\|y\|\}$ for each $y \in Y$ then we call T *open* and if $y \in \mathrm{cl}\{Tx : \|x\| \leqslant k\|y\|\}$ *almost open*. For example the *kernel*

$$\ker(T) : T^{-1}(0) \to X \tag{0.1}$$

is always bounded below, while the *cokernel*

$$\mathrm{cok}(T) : Y \to Y/\mathrm{cl}\, T(X) \tag{0.2}$$

is always open. The operator $T \in BL(X, Y)$ will be called *relatively open*, respectively *relatively almost open*, if its *truncation*

$$T^{\vee} : X \to T(X) \tag{0.3}$$

is open (respectively, almost open). Thus bounded below is just relatively open one-one, open is the same as relatively open onto, and almost open is relatively almost open dense. In the *canonical factorization* ([5] Theorem 2.3.3)

$$T = \ker(\mathrm{cok}(T)) \circ \mathrm{core}(T) \circ \mathrm{cok}(\ker(T)) \tag{0.4}$$

the uniquely determined middle term

$$\text{core}\,(T) : X/T^{-1}(0) \to \text{cl}\,T(X) \tag{0.5}$$

is always one-one and dense; when it happens to be invertible the operator T is called *proper* ([5] Definition 3.2.7), or *strict* [2]. Evidently

$$\text{proper} \implies \text{relatively open} \implies \text{relatively almost open}; \tag{0.6}$$

conversely T is proper if and only if it is relatively open with closed range $T(X) = = \text{cl}\,T(X)$; when X and Y are both complete then the open mapping theorem shows that an operator with closed range must be relatively open and therefore also proper. Indeed the easy half of the open mapping theorem shows ([5] Theorems 4.4.3, 4.4.4) that if $T \in BL(X,Y)$ then

$$X \text{ complete, } T \text{ relatively almost open} \implies T \text{ proper.} \tag{0.7}$$

We can also see that T is relatively open if and only if core(T) is bounded below, and relatively almost open iff core(T) is almost open.

Banach's "closed range theorem" ([3] Corollary IV.1.8.; [1]) says that, if the spaces X and Y are both complete, then $T \in BL(X,Y)$ has closed range if and only if the same is true of the *dual operator*

$$T^\dagger \in BL(Y^\dagger, X^\dagger) : g \mapsto gT.$$

Towards an extension of Banach's theorem to incomplete spaces, recall ([5] Theorem 5.5.2) the duality

$$T \text{ dense} \iff T^\dagger \text{ one-one} \tag{0.8}$$

and

$$T \text{ almost open} \iff T^\dagger \text{ bounded below :} \tag{0.9}$$

in each case forward implication is elementary, while the reverse uses the Hahn-Banach theorem, in (0.9) in its strong "separation" form. We need another auxiliary operator, dual to the "truncation": write

$$T^\wedge : X/T^{-1}(0) \to Y \tag{0.10}$$

for the "one-one part" of T. Our observation is very elementary; here and elsewhere we write "$\cong$" to indicate isometric isomorphism through the medium of a specific map, obvious from the context:

THEOREM 1. *If $T \in BL(X,Y)$ is arbitrary then*

$$\ker(T^\dagger) = \text{cok}(T)^\dagger \tag{1.1}$$

and

$$(T^\vee)^\dagger \cong (T^\dagger)^\wedge. \tag{1.2}$$

Proof. Equality (1.1) is the easy part of the proof of (0.8); for the isomorphism (1.2) use the Hahn-Banach theorem to identify the dual of the subspace $T(X) \subseteq Y$ with the quotient of $Y^\dagger$ by the annihilator of $T(X)$. ■

The first part of our next result was noticed by Lee ([8] Theorem 1):

THEOREM 2.*If $T \in BL(X,Y)$ there is implication*

$$T \text{ relatively almost open} \iff T^\dagger \text{ relatively open.} \tag{2.1}$$

If T is relatively open then

$$T^{-1}(0)^\circ \subseteq T^\dagger(Y^\dagger) \tag{2.2}$$

and

$$\text{cok}(T^\dagger) \cong \ker(T)^\dagger,$$

and hence

$$\text{core}(T^\dagger) \cong \text{core}(T)^\dagger.$$

Proof. For (2.1) apply (0.9) to the truncation $T^\vee$ together with (1.2). Towards (2.2) suppose $f \in X^\dagger$ is in the annihilator of $T^{-1}(0)$, and define $g_0 : T(X) \to \mathbb{C}$ by setting

$$g_0(Tx) = f(x) \text{ for each } x \in X. \tag{2.5}$$

The status of f ensures that g_0 is well-defined, necessarily linear, and bounded provided T is relatively open: if $x' \in X$ satisfies

$$Tx' = Tx \text{ and } \|x'\| \leqslant k\|Tx\|, \tag{2.6}$$

so that also $f(x') = f(x)$, then

$$|g_0(Tx)| = |f(x')| \leqslant k\|f\|\,\|Tx\| \text{ for each } x \in X.$$

By the Hahn-Banach theorem there is $g \in Y^\dagger$ for which

$$\|g\| = \|g_0\| \leqslant k\|f\| \text{ and } gT = g_0T = f.$$

This proves (2.2), which then gives (2.3); finally (2.4) follows from the uniqueness of the core. ∎

Alternatively, (2.2) can be obtained by applying the dual of (0.9) ([5] Theorem 5.5.3) to the truncation $T^\vee$. Since dual spaces are always complete, Theorem 2 gives the extension of Banach's theorem to incomplete spaces:

$$\text{(2.7)} \qquad T \text{ relatively almost open } \Longleftrightarrow T^\dagger \text{ has closed range.}$$

For an example in which $T \in BL(X,Y)$ is relatively almost open but not relatively open take Y to be complete and $f : Y \to \mathbf{C}$ a discontinuous linear functional, and then take $T = I : X \to Y$ where X is obtained by renorming Y with the aid of f:

$$\text{(2.8)} \qquad \|y\|_X = \|y\|_Y + |f(y)| \text{ for each } y \in Y.$$

It is clear that T is one-one and onto, and therefore by the hard part of the open mapping theorem ([5] Theorem 4.6.2) almost open, but not open and not bounded below; the functional f is in $X^\dagger$ but not $Y^\dagger$. This is an easier realisation of the situation of Theorem 4.7.4 of [5] than (5.5.6.2) of [5]. Relatively almost open is not sufficient for (2.2): in (2.8) the functional f lies in $T^{-1}(0)^\circ$ but not in $T^\dagger(Y^\dagger)$. When T is relatively open but possibly not proper then the canonical factorization of $T^\dagger$ is the dual of the canonical factorization of T; this was incorrectly claimed in Theorem 5.5.5 of [5] for arbitrary T. Without completeness this is of course not enough, as incorrectly claimed in [5], to ensure implication $T^\dagger$ proper $\Longrightarrow T$ proper.

Relative openness, and hence also relative almost openness, can be tested with the (*reduced*) *minimum modulus*

$$\text{(2.9)} \qquad \gamma(T) = \inf\{\|Tx\| : \operatorname{dist}(x, T^{-1}(0)) \geqslant 1\} \text{ if } 0 \neq T \in BL(X,Y);$$

if $T = 0$ we may take $\gamma(T) = \infty$. Evidently

$$\text{(2.10)} \qquad T \text{ relatively open } \Longleftrightarrow \gamma(T) > 0;$$

also ([3] Theorem IV.1.8)

$$\text{(2.11)} \qquad \gamma(T) > 0 \Longrightarrow \gamma(T^\dagger) = \gamma(T).$$

Of course it is possible ([3] Example II.1.10) to have $\gamma(T) = 0 < \gamma(T^\dagger)$; this happens for example in the situation of (2.8). We recall ([4]; [5] Theorems 3.3.3, 3.4.3) that, between incomplete spaces, the bounded below and the almost open operators form open sets:

THEOREM 3. *If S and T are in $BL(X,Y)$ there is implication*

$$(3.1)\qquad T \text{ bounded below}, \|S\| < \gamma(T) \Longrightarrow T - S \text{ bounded below}$$

and

$$(3.2)\qquad T \text{ almost open}, \|S\| < \gamma(T^\dagger) \Longrightarrow T - S \text{ almost open.}$$

There is also implication

$$(3.3)\qquad \|S\| < \gamma(T) \Longrightarrow \dim(T-S)^{-1}(0) \leqslant \dim T^{-1}(0)$$

and

$$(3.4)\qquad \|S\| < \gamma(T^\dagger) \Longrightarrow \dim Y/\mathrm{cl}\,(T-S)(X) \leqslant \dim Y/\dim T(X).$$

Proof. For (3.1) follow the proof of Theorem 3.3.3 of [5], and for (3.2) apply (3.1) to the duals of S and T. (3.3) is quoted by Goldberg ([3] Theorem V.1.2) for unbounded operators, and also by Lindenstrauss/Tzafriri ([10] Proposition 2.c.9); then (3.4) is (3.3) applied to $S^\dagger$ and $T^\dagger$. ∎

Both Goldberg [3] and Lindenstrauss/Tzafriri [10] rely on Borsuk's antipodal lemma, and the statement in Lindenstrauss/Tzafriri is restricted to complete spaces, although not the proof. The argument in Harte ([5], 6.10, 2.9) avoids Borsuk's lemma, but misses the numerical precision above. Of course (3.3) and (3.4) are vacuously true if the operator T fails to be relatively open, or relatively almost open.

We recall ([5] Definition 6.10.1) that the operator $T \in BL(X,Y)$ is called *almost upper semi-Fredholm* if it is relatively open with finite dimensional null space, and *almost lower semi-Fredholm* if it is relatively almost open with the closure of its range of finite co-dimension. If T is either almost upper or almost lower semi-Fredholm we shall call it *almost semi-Fredholm*, and *almost Fredholm* if it is both; if it is almost Fredholm and proper we shall call it *Fredholm*. The *index* of an almost (semi-)Fredholm operator is given by

$$(3.5)\qquad \mathrm{index}\,(T) = \dim T^{-1}(0) - \dim Y/\mathrm{cl}\,T(X).$$

These concepts are also ([5] (6.12.1.19)) dual to one another:

$$(3.6)\qquad T \text{ almost lower semi-Fredholm} \Longleftrightarrow T^\dagger \text{ almost upper semi-Fredholm.}$$

For incomplete spaces (contradicting [5] Theorem 6.10.2) the almost upper semi-Fredholm operators need not form an open set. For example if $X = Y = c_{00} \subseteq \ell_\infty$ is the space "terminating" sequences (sup norm) and $T = V : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$ the backward shift then $T - \lambda I$ is almost right invertible ([5] Definition 3.7.1.) whenever $|\lambda|| < 1$, but if $0 < |\lambda| < 1$ then $T - \lambda I$ is one-to-one but not bounded below: the index of $T - zI$ is not a continuous function at $z = 0$.

When $T \in BL(Y, X)$ for which

$$(3.7.) \qquad T = TT'T.$$

In an algebraic sense the Fredholm operators form an open set on which the index is continuous: if $T = TT'T$ is Fredholm and if $K \in BL(X, Y)$ satisfies

$$(3.8) \qquad I + T'K \text{ invertible in } BL(X, Y)$$

then ([5] Theorem 6.4.5, 6.5.5)

$$(3.9) \qquad T + K \text{ is Fredholm with index}(T + K) = \text{index}(T);$$

we have as well $\dim(T+K)^{-1}(0) \leqslant \dim T^{-1}(0)$ and hence also $\dim Y/\text{cl}(T+K)(X) \leqslant \dim Y/\text{cl}T(X)$.

The restriction of an almost upper semi-Fredholm operator to a closed subspace is again almost upper semi-Fredholm:

THEOREM 4. *If $T \in BL(X, Y)$ is almost upper semi-Fredhom and $M \subseteq X$ a linear subspace then*

$$(4.1) \qquad T_M \in BL(M, Y) \textit{ is relatively almost open}$$

and there is implication

$$(4.2) \qquad M \textit{ closed} \Longrightarrow T_M \textit{ relatively open.}$$

Proof. Beginning with the closed case (4.2), suppose W is a (closed) complement for $M \cap T^{-1}(0)$ in M and claim

$$T \text{ relatively open} \Longrightarrow T_W \text{ bounded below:}$$

for if T_W is not bounded below there in (x_n) in W for which $\|x_n\| = 1$ with $\|Tx_n\| \to 0$, giving if also T is relatively open $\text{dist}(x_n, T^{-1}(0)) \to 0$. If $(e_j)_{j \in J}$ is a basis for the finite dimensional space $T^{-1}(0)$ there must be scalars (α_{jn}) for which

$$\left\| x_n - \sum_{j \in J} \alpha_{jn} e_j \right\| \to 0 \quad \left\| \sum_{j \in J} \alpha_{jn} e_j \right\| \to 1.$$

It follows (cf. [5] Theorem 6.2.2) that there is $i \in J$ for which $\liminf_n |\alpha_{in}| > 0$, and hence, passing to subsequences $(\beta_{jn}) = (\alpha_{j\varphi(n)})$ of (α_{jn}),

$$\left\| e_i - \beta_{in}^{-1} \left(x_n - \sum_{j \neq i} \beta_{jn} e_j \right) \right\| \to 0,$$

giving (cf. [5] Theorem 6.2.1)

$$e_i \in \mathrm{cl} \left(W + \sum_{j \neq i} \mathbb{C} e_j \right) = W + \sum_{j \neq i} \mathbb{C} e_j ,$$

a contradiction, which establishes (4.2).

To derive (4.1) apply (4.2) to the closure of M together with one of the "Riesz lemmas" of Harte ([5] Theorem 1.5.1), which says that $x \in \mathrm{cl}(M)$ can be approximated by (x_n) in M with $\|x_n\| \leqslant \|x\|$. ∎

When X and Y are the same space then we can introduce ([5] Definition 7.8.1) the *hyperrange* of $T \in BL(X, Y) = BL(X, X)$:

$$T^\infty(X) = \bigcap_{n=1}^{\infty} T^n(X).$$

If S comutes with T, so that also $ST^\infty(X) \subseteq T^\infty(X)$, we shall write

$$S \ : T^\infty(X) \to T^\infty(X) \tag{4.3}$$

for the operator introduced by S. If in particular S is invertible and comutes with T then ([5] Theorem 7.8.3)

$$(T - S)^{-1}(0) \subseteq T^\infty(X), \tag{4.4}$$

so that the null space of $T - S$ is the same as the null space of $(T - S)$. We recall also ([5] Theorem 7.8.3) the familiar implication

$$T^{-1}(0) \text{ finite dimensional } \Longrightarrow T \ \text{ onto.} \tag{4.5}$$

Theorem 5. *If $T \in BL(X, X)$ there is implication*

$$T \text{ almost upper semi-Fredholm } \Longrightarrow T \ \text{ almost open.} \tag{5.1}$$

If T is Fredholm then T is open.

Proof. The argument for (4.5) is familar: if $y \in T^\infty(X)$ then there is $(x_n) \in X$ for which $y = T^n x_n$ $(n \in \mathbf{N})$, and by finite dimensionality of $T^{-1}(y)$ there is $N \in \mathbf{N}$ (depending on y) for which

$$x_N \in T^{-1}y \cap T^N X = T^{-1}y \cap T^\infty(X). \tag{5.2}$$

To convert this to (5.1) apply (4.1) with $M = T^\infty(X)$. When also $T(X)$ is closed (in particular, when T is Fredholm) then (4.2) says that T is open ∎

It is sufficient, for (4.5) and hence also for (5.1), that the finite dimensionality of the null space of T be weakened ([9] Lemma 1) to

$$\dim T^{-1}(0) \cap T^n(X) < \infty \text{ for some } n \in \mathbf{N}. \tag{5.2}$$

For a "punctured neighbourhood theorem" in the spirit of Harte ([5] Theorem 7.8.4) we might look for implication

$$\max(\|S\|, \|S'\|) < \gamma(T) \Leftrightarrow \dim(T - S)^{-1}(0) = \dim(T - S')^{-1}(0), \tag{5.4}$$

assuming T is almost upper semi-Fredholm and S and S' are invertible and commute with T, arguing

$$\dim(T - S)^{-1}(0) = \dim(T - S)^{\sim -1}(0) = \operatorname{index}(T - S)^{\sim} = \operatorname{index} T^{\sim}. \tag{5.5}$$

If T is almost upper semi-Fredholm then by (5.1) the operator $T^{\sim}$ is almost open. By (3.2) if $\|S\| < \gamma(T)$ then $(T - S)^{\sim}$ is almost open and in particular dense. This gives the second equality in (5.5); the first equality is (4.4), and the third would be by the continuity of the index. Unfortunately, as we saw before, this fails for the backward shift on c_{00}.

It is important to remember that $(T - S)^{\sim}$ is the restriction of $T - S$ to the hyperrange of T rather than of $T - S$. In the corresponding assertion (7.8.4.4) of Harte [5] the operator $T \in BL(X, X)$ is assumed to be Fredholm, and therefore to have a bounded *generalised inverse* $T' \in BL(X, X)$ for which $T = TT'T$, and the operators S and S' supposed to be such that $I + T'S$ and $I + T'S'$ are invertible: no actual smallness of norm is assumed. Whether or not Theorem 7.8.4 of [5] is true as stated remains an open problem. One hope is to find a generalised inverse T' for T which leaves invariant the hyperrange $T^\infty(X)$:

THEOREM 6. *If $T \in BL(X, X)$ is Fredholm then it has a generalised inverse $T' \in BL(X, X)$ for which*

$$T'T^\infty(X) \subseteq T^\infty(X). \tag{6.1}$$

Proof. By the finite dimensionality of the null space there is $N \in \mathbf{N}$ for which $T^N(X) \cap T^{-1}(0) = T^\infty(X) \cap T^{-1}(0)$ as in (5.2), and also $W_n \subseteq X$ for each $n \in \mathbf{N}$ with

(6.2) $$T^n(X) = W_n \oplus (T^n(X) \cap T^{-1}(0)) \text{ and } W_{n+1} \subseteq W_n.$$

Since T^n is also Fredholm its range, and hence the sum $T^n(X)+T^{-1}(0)$, is finite codimensional and hence complemented: thus there are finite dimensional $Z_n \subseteq X$ for which

(6.3) $$T^{-1}(0) \oplus W_n \oplus Z_n = X.$$

Now choose (cf. [5] Theorem 3.8.2) $T' \in BL(X,X)$ for which

(6.4) $$T = TT'T \text{ and } T'T(X) = W_N + Z_N.$$

With $W = \bigcap_{n=N}^{\infty} W_n$ we claim

(6.5) $$W \subseteq W_N \subseteq T'T(X) \text{ and } W \subseteq T^\infty(X);$$

(6.6) $$T'T^\infty(X) \subseteq T^\infty(X) + T^{-1}(0) \subseteq W + T^{-1}(0);$$

(6.7) $$T'T(X) \cap (W + T^{-1}(0)) \subseteq W.$$

(6.5) is clear; towards (6.6) argue that if $y = T'T^{n+1}x_{n+1}$ for each n then $Ty = T^{n+1}x_{n+1}$ giving $y \in T^nX+T^{-1}(0)$ and hence in the intersection of the $T^nX+T^{-1}(0)$: but since (Theorem 5) T is onto this is $T^\infty(X)+T^{-1}(0)$. For the second version of (6.6) we argue

(6.8) $$T^\infty(X) \subseteq W + T^{-1}(0):$$

if $y \in T^\infty(X)$ and $n \geqslant N$ then there are $w_n \in W_n$ and $z_n \in T^NX \cap T^{-1}(0) = T^\infty(X) \cap T^{-1}(0)$ with $y = w_n + z_n$. By finite dimensionality there is a convergent subsequence $(z'_n) = (z_{\varphi(n)})$ of (z_n), with limit z'_∞ say, so that also $w'_n = w_{\varphi(n)}$ converges to $w'_\infty = y - z'_\infty$, giving

$$y = w'_\infty + z'_\infty \in \bigcap_{n=N}^{\infty} W_{\varphi(n)} + T^\infty(X) \cap T^{-1}(0) \subseteq W + T^\infty(X) \cap T^{-1}(0).$$

For (6.7) we argue that if $y = w + z \in T'T(X)$ with $w \in W$ and $z \in T^{-1}(0)$ then $y - w = z \in (T'TX + W) \cap T^{-1}(0) = \{0\}$, giving $y = w \in W$. From (6.6) and (6.7) it follows

$$T'T^{\infty}(X) \subseteq W \subseteq T^{\infty}(X),$$

giving (6.1). ■

With such a choice of generalized inverse, we have a nearness criterion giving a punctured neighbourhood theorem for Fredholm operators:

THEOREM 7. *Suppose $T \in BL(X,X)$ is Fredholm, with generalized inverse $T' \in BL(X,X)$ for which $T'T^{\infty}(X) \subseteq T^{\infty}(X)$: then if S and S' in $BL(X,X)$ are invertible and commute with T, and if*

(7.1) $$(I + T'S) \text{ and } (I + T'S') \text{ are invertible}$$

then there is equality

(7.2) $$\dim(T - S)^{-1}(0) = \dim(T - S')^{-1}(0)$$

and hence also

(7.3) $$\dim X/(T - S)X = \dim X/(T - S')X.$$

Proof. We claim that equality (5.5) really does hold in the case: the first two equalities follow from (4.4) and (5.1), while the third is (3.9). ■

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Received December 7, 1990; revised November 20, 1991.