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ON INVARIANT SUBSPACES IN THE BITORUS

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1. INTRODUCTION

The classical Theorem of Beurling ([2]) says that every invariant subspace of the Hardy space H^2 on the unit disc is of the form fH^2 with f an inner function. A characterization of all the invariant subspaces of Lebesgue space L^2 on the one dimensional torus $\mathbf{T}$ was obtained by Helson in [8]. Namely, he proved that the doubly invariant subspaces are of the form $\chi_\tau L^2(\mathbf{T})$, where χ_τ is the characteristic function of a Borel subset τ in $\mathbf{T}$, while the simply invariant subspaces are fH^2, with f a unimodular function in $L^2(\mathbf{T})$.

As it is well known, the problem of invariant subspaces of L^2 on the n-dimensional torus or even in the corresponding H^2 is much more complicated. The invariant subspaces in H^2 are no longer of the form fH^2, with f an inner function. Moreover Rudin gave in [14] an example of an invariant subspace in H^2 in two variables which can't be of the form $f_1H^2 + f_2H^2 + \cdots + f_nH^2$, with f_j inner functions. Considerable progress in the study of invariant subspaces in this setting was made in the last few years through the papers [1], [10] and [15]. More recently in [7], [13] and [12] invariant subspaces on which the coordinate multiplication operators are double commuting, have been studied. Here are used techniques of operator theory. Especially [7] and [12] are based on the Wold decomposition of a pair of doubly commuting isometries on a complex Hilbert space, which was obtained in [17] and [11].

In this paper we attempt to describe the structure of invariant subspaces of L^2 in two variables, by using the Wold type decomposition from [5] for a pair of (not necessarily doubly) commuting isometries. In such a framework four types of invariant subspaces appear. The first one is the well known doubly invariant type, whereas the other three we propose are special types of simply invariant subspaces. For the second and the third types we have complete characterizations. The last one seems to have a more complicated structure, which can be elucidated in some particular cases only. The invariant subspaces in H^2 are separately discussed in connection with the so-called Szegö property. Let us finally point out that because of the general decomposition of [6] our results remain true also in several variables.

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2. NOTATION AND PRELIMINARIES

Let $\mathbf{T}^2 := \mathbf{T} \times \mathbf{T}$ be the two dimensional torus (the bitorus) and $L^2(\mathbf{T}^2)$ the usual Lebesgue space with respect to the Haar measure m_2 on $\mathbf{T}^2$. We denote by Z_1, Z_2 the multiplication operators with the coordinate functions on $L^2(\mathbf{T}^2)$, i.e. for $j = 1,2$

$$(Z_j f)(t) = t_j f(t) \quad (f \in L^2(\mathbf{T}^2),\ t = (t_1, t_2) \in \mathbf{T}^2).$$

A closed subspace $\mathcal{M}$ of $L^2(\mathbf{T}^2)$ is *invariant* (respectively $*$-*invariant*), if $Z_j\mathcal{M} \subset \subset \mathcal{M}$ (respectively $Z_j^*\mathcal{M} \subset \mathcal{M}$) for $j = 1,2$. An invariant subspace which is also $*$-invariant is called *doubly invariant*. We shall call a subspace *simply invariant* if it is invariant and not $*$-invariant. It is well known that the doubly invariant subspaces of $L^2(\mathbf{T}^2)$ are of the form $\chi_\sigma L^2(\mathbf{T}^2)$, where $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$—the Borel sets in $\mathbf{T}^2$ (Theorem 7.14 from [18]) or Lemma 3 from [7]). But the simply invariant subspaces of $L^2(\mathbf{T}^2)$ are not fully known, their structure being more complicated. Among them the most important one is the subspace $H^2(\mathbf{T}^2)$ consisting of all $f \in L^2(\mathbf{T}^2)$ for which the Fourier coefficients satisfy $\hat{f}(k,l) = 0$ for $k < 0$ or $l < 0$. Another invariant subspace which together with $H^2(\mathbf{T}^2)$ will play an important role in our approach, is $M^2(\mathbf{T}^2)$, the space of all $f \in L^2(\mathbf{T}^2)$ so that $\hat{f}(k,l) = 0$ for $k, l \leqslant 0$. We also observe that the restrictions of operators Z_j on $H^2(\mathbf{T}^2)$ are doubly commuting isometries, whereas the restrictions of these operators to $M^2(\mathbf{T}^2)$ are not doubly commuting.

Let $\mathcal{M}$ be an invariant subspace and $\mathfrak{M}$ the minimal doubly invariant subspace in $L^2(\mathbf{T}^2)$ containing $\mathcal{M}$. Then $M_j := Z_j|\mathcal{M}$ are commuting isometries on $\mathcal{M}$ and $[\mathfrak{M}, (Z_1|\mathfrak{M}, Z_2|\mathfrak{M})]$ is the minimal unitary extension of $[\mathcal{M}, (M_1, M_2)]$ (for details see [19], [18]).

By the doubly invariance of $\mathfrak{M}$, there is an m_2-essentially uniquely determined $E \in \mathrm{Bor}(\mathbf{T}^2)$ so that $\mathfrak{M} = \chi_E L^2(\mathbf{T}^2)$. We also note that the subspace $\widetilde{\mathcal{M}} := \mathfrak{M} \ominus \mathcal{M}$ is $*$-invariant and $\widetilde{M}_j := Z_j^*|\widetilde{\mathcal{M}},\ j = 1,2$ are commuting isometries on $\widetilde{\mathcal{M}}$. One observes that all functions from $\mathcal{M}$ (as from $\widetilde{\mathcal{M}}$) vanish on each $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ with $m_2(\sigma) > 0$ and $m_2(E \cap \sigma) = 0$. We shall say that a Borel set $E \subset \mathbf{T}^2$ is *the support* of a given set $\mathcal{M} \subset L^2(\mathbf{T}^2)$, if each function from $\mathcal{M}$ vanishes on all Borel sets σ with $m_2(\sigma) > 0$, $m_2(E \cap \sigma) = 0$ and E does not contain any $\tau \in \mathrm{Bor}(\mathbf{T}^2)$ with $m_2(\tau) > 0$ on which all members of $\mathcal{M}$ vanish. It is easy to see that the above conditions determine the support of $\mathcal{M}$ (which we shall denote by $s(\mathcal{M})$) in an m_2-essentially uniquely way. Moreover, it can be easily proved that the support of $\mathcal{M}$ is just the Borel set E for which $\mathfrak{M} = \chi_E L^2(\mathbf{T}^2)$, where $\mathfrak{M} := \bigvee_{(k,l)\in\mathbf{Z}^2} Z_1^k Z_2^l \mathcal{M}$. Let us remark, as W. Hackenbroch pointed out (private communication), that the characteristic function $\chi_{s(\mathcal{M})}$ is the support in the sense of Sakai ([16] Definition 1.21.14 p.54) of the W^*-representation

$$L^\infty(\mathbf{T}^2) \ni h \mapsto M_h \in B(\mathfrak{M}), \quad \text{where } M_h g := hg, \ g \in \mathfrak{M};$$

of the W^*-algebra $L^\infty(\mathbf{T}^2)$ on the Hilbert space $\mathfrak{M}$, which is naturaly associated to $\mathcal{M}$.

It is clear that, when $\mathcal{M}$ is an invariant subspace and $E = s(\mathcal{M})$ then $\chi_E L^2(\mathbf{T}^2)$ is just the space of minimal unitary extension of $[\mathcal{M}, (M_1, M_2)]$. Moreover $s(\widetilde{\mathcal{M}}) \subseteq s(\mathcal{M})$, the equality being true iff $\mathcal{M}$ does not contain any non-trivial doubly invariant subspace. Now we associate to the invariant subspace $\mathcal{M}$ with $s(\mathcal{M}) = E$ the subspace

$$\mathcal{N} := \{f \in L^2(\mathbf{T}^2) : \chi_E f \in \mathcal{M}\}.$$

It is easy to observe that $\mathcal{N}$ is also (closed) invariant, $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{M} = \chi_E \mathcal{N}$. Evidently $s(\mathcal{N}) = \mathbf{T}^2$. So it can be said that by this procedure $\mathcal{M}$ is "enlarged" to a subspace $\mathcal{N}$, which is supported on the whole $\mathbf{T}^2$. We shall say that such an $\mathcal{N}$ is *maximally supported.*

Now if $\mathcal{M}$ is again a given invariant subspace, by applying to the pair (M_1, M_2) the Wold decomposition ([5], Theorem 5) we obtain the uniquely determined orthogonal decomposition

$$\mathcal{M} = \mathcal{M}_u \oplus \mathcal{M}_t \oplus \mathcal{M}_m \oplus \mathcal{M}_e, \tag{1}$$

where the subspaces $\mathcal{M}_\alpha$; $\alpha = u, t, m, e$ reduce M_1 and M_2, $M_j (j = 1,2)$ are both unitary on $\mathcal{M}_u$; (M_1, M_2) is a shift pair on $\mathcal{M}_t$, is a modified shift pair on $\mathcal{M}_m$ and it is ultraevanescent on $\mathcal{M}_e$. The subspaces $\mathcal{M}_t$ and $\mathcal{M}_m$ have the following structure

$$\mathcal{M}_t = \bigoplus_{k,l\geqslant 0} M_1^k M_2^l \mathcal{R} \quad \text{and} \quad \mathcal{M}_m = \bigoplus_{\substack{k>0 \\ \text{or} \\ l>0}} Z_1^k Z_2^l \mathcal{L}, \tag{2}$$

where $\mathcal{R} := \mathcal{M} \ominus P_{\mathcal{M}}\left(\bigvee_{\substack{k>0 \\ \text{or} \\ l>0}} Z_1^k Z_2^l \mathcal{M}\right)$ and $\mathcal{L} := \widetilde{\mathcal{M}} \ominus P_{\widetilde{\mathcal{M}}}\left(\bigvee_{\substack{k<0 \\ \text{or} \\ l<0}} Z_1^k Z_2^l \widetilde{\mathcal{M}}\right)$ are the *right defect* respectively the *left defect* spaces for (M_1, M_2). Evidently $\mathcal{M} = \mathcal{M}_u$ iff $\mathcal{M}$ is *doubly invariant*. We shall say that the invariant subspace $\mathcal{M}$ is of H^2-*-type*, of M^2*-type*, or of *ultraevanescent type*, if $\mathcal{M} = \mathcal{M}_t$, $\mathcal{M} = \mathcal{M}_m$ or $\mathcal{M} = \mathcal{M}_e$ respectively. So we have a classification of invariant subspaces in $L^2(\mathbf{T}^2)$ with respect to the Wold-type decomposition (1).

It is obvious that $H^2(\mathbf{T}^2)$ is an invariant subspace of H^2-type, $M^2(\mathbf{T}^2)$ is of M^2-type and, as it was implicitely observed in [5] Example 2, the space $H_1 = H_0^2(\mathbf{T}^2)$ consisting of all members of $H^2(\mathbf{T}^2)$ whose mean value vanishes, is an ultraevanescest type invariant subspace.

The first main theorem is a Helson type result. Its statement is the following

THEOREM 1. *Every invariant subspace $\mathcal{M}$ of $L^2(\mathbf{T}^2)$ has one of the following forms*

$$\mathcal{M} = fH^2(\mathbf{T}^2) \text{ or,} \tag{3}$$

$$\mathcal{M} = gM^2(\mathbf{T}^2) \text{ or,} \tag{4}$$

$$\mathcal{M} = \chi_\sigma L^2(\mathbf{T}^2) \oplus \chi_\omega \mathcal{N}, \tag{5}$$

where $\sigma, \omega \in \mathrm{Bor}(\mathbf{T}^2)$, $\sigma \cap \omega = \emptyset$, $f \in \mathcal{M}$ and $g \in \widetilde{\mathcal{M}}$ are unimodular functions (i.e. $|f| = |g| = 1$ m_2-a.e.), and $\mathcal{N}$ is a maximally supported invariant ultraevanescent type subspace of $L^2(\mathbf{T}^2)$.

The second main theorem is a Beurling type result.

THEOREM 2. *Each invariant subspace $\mathcal{M}$ of $H^2(\mathbf{T}^2)$ is either of H^2-type and in this case $\mathcal{M} = fH^2(\mathbf{T}^2)$ with f an inner function, or is of ultraevanescent type (maximally) supported on $\mathbf{T}^2$ and then M_1, M_2 are non-doubly commuting shifts on $\mathcal{M}$.*

The proofs will be given in Section 5.

3. INVARIANT SUBSPACES OF H^2- AND M^2-TYPES

Two invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of $H^2(\mathbf{T}^2)$ will be called *unitarily equivalent*, if there exists a unitary operator U from $\mathcal{M}$ to $\mathcal{N}$ so that $UM_j = N_jU$, where $N_j := Z_j|\mathcal{N}$; $j = 1, 2$. It results from [5] (Theorem 10) that the unitary equivalence preserves the decomposition (1) of an invariant subspace. Consequently the type of an invariant subspace (doubly invariant, H^2-type, M^2-type, ultraevanescent) is an "invariant" with respect to the unitary equivalence.

For the given invariant subspace $\mathcal{M}$ we use one of the following notation above mentioned: $M_j = Z_j|\mathcal{M}$; $\mathfrak{M}$ the minimal unitary extension space, $\widetilde{\mathcal{M}} = \mathfrak{M} \ominus \mathcal{M}$ and $\mathcal{R}, \mathcal{L}$ the corresponding (wandering) defect spaces.

The H^2-type invariant subspaces in $L^2(\mathbf{T}^2)$ will be characterized in the following theorem. For completness we include in the statement some known equivalent conditions from [7] (or [12]).

THEOREM 3. *Let $\mathcal{M}$ be a non-trivial invariant subspace of $L^2(\mathbf{T}^2)$. The following are equivalent:*

(i) $\mathcal{M} = fH^2(\mathbf{T}^2)$, *with* $f \in \mathcal{M}$ *and* $|f| = 1$ m_2-a.e. *on* $\mathbf{T}^2$;

(ii) $\mathcal{M}$ *is unitarily equivalent with* $H^2(\mathbf{T}^2)$;

(iii) $\mathcal{M}$ *is of* H^2-*type;*

(iv) *the shift part of* $\mathcal{M}$ *is non-trivial;*

(v) *the right defect space for* (M_1, M_2) *is one dimensional*

(vi) $M_2(\mathcal{M} \ominus M_1\mathcal{M}) \subsetneqq \mathcal{M} \ominus M_1\mathcal{M}$

Under these conditions the space of minimal unitary extension of (M_1, M_2) *is the whole* $L^2(\mathbf{T}^2)$.

Proof. If $\mathcal{M} = fH^2(\mathbf{T}^2)$ with f unimodular, then the multiplication with f is a unitary equivalence between $\mathcal{M}$ and $H^2(\mathbf{T}^2)$. Thus (i) $\Rightarrow$ (ii). Now if U is a unitary equivalence between $H^2(\mathbf{T}^2)$ and $\mathcal{M}$, then

$$\mathcal{M} = U\left(\bigoplus_{k,l\geqslant 0} Z_1^k Z_2^l \mathbf{C}\right) = \bigoplus_{k,l\geqslant 0} M_1^k M_2^l (U\mathbf{C}),$$

therefore $\mathcal{M}$ is of H^2-type. Thus (ii) $\Rightarrow$ (iii), while (iii) $\Rightarrow$ (iv) is trivial. Suppose now that (iv) holds, i.e. $\mathcal{M}_t \neq \{0\}$ in the decomposition (1) of $\mathcal{M}$. It is not difficult to check (see also Theorem 1 from [17]) that the restrictions of M_j to $\mathcal{M}_t$ are doubly commuting shifts. Consequently by Corollary 4 from [7] there is a unimodular function $f \in \mathcal{M}$ so that

$$\mathcal{M}_t = fH^2(\mathbf{T}^2). \tag{6}$$

It now follows clearly that $\mathcal{R} = f\mathbf{C}$ and therefore $\dim \mathcal{R} = 1$. This proves (iv) $\Rightarrow$ (v). We assume now that $\mathcal{R}$ is one dimensional. Then by choosing $f \in \mathbf{R}$ of norm one, from the orthogonality between $Z_1^k Z_2^l \mathcal{R}$ and $\mathcal{R}$ for $(k,l) \neq (0,0)$, it follows that f is a unimodular function on $\mathbf{T}^2$. Thus we obtain (6) and since $\mathcal{R}$ is the left defect space for $[\mathcal{M},(M_1,M_2)]$ (see also the relation (6) from [3]) we deduce that

$$(\widetilde{\mathcal{M}})_m := \bigoplus_{\substack{k<0 \\ \text{or} \\ l<0}} Z_1^k Z_2^l (f\mathbf{C}) = f\overline{M^2(\mathbf{T}^2)}.$$

Thus the space of minimal unitary extension $\mathfrak{M}$ contains the subspace $\mathcal{M}_t \oplus (\widetilde{\mathcal{M}})_m = = fL^2(\mathbf{T}^2) = L^2(\mathbf{T}^2)$, whence $\mathfrak{M} = L^2(\mathbf{T}^2)$ and consequently $\mathcal{M} = \mathcal{M}_t = = fH^2(\mathbf{T}^2)$. Thus (v) $\Rightarrow$ (i) is proven. The equivalence (i) $\Leftrightarrow$ (vi) is contained in the Corollaries 4 and 7 from [7] and in Theorem 5 from [12]. The proof is finished. ∎

Similar characterizations hold for the M^2-type invariant subspaces.

THEOREM 4. *Let $\mathcal{M}$ be a non-trivial invariant subspace of $L^2(\mathbf{T}^2)$. The following are equivalent*

(i) $\mathcal{M} = gM^2(\mathbf{T}^2)$, *with* $g \in \widetilde{\mathcal{M}}$ *and* $|g| = 1$ m_2-a.e. *on* $\mathbf{T}^2$;
(ii) $\mathcal{M}$ *is unitarily equivalent with* $M^2(\mathbf{T}^2)$;
(iii) $\mathcal{M}$ *is of* M^2-*type;*
(iv) *the modified shift part of* $\mathcal{M}$ *is non-trivial;*
(v) *the left defect space for* (M_1, M_2) *is one dimensional;*
(vi) $\mathcal{M} \ominus M_1\mathcal{M} \subsetneq M_2(\mathcal{M} \ominus M_1\mathcal{M})$.

Under these conditions the space of minimal unitary extension of (M_1, M_2) *is the whole* $L^2(\mathbf{T}^2)$.

Proof. If $\mathcal{M} = gM^2(\mathbf{T}^2)$ with g a unimodular function, then the multiplication with g is a unitary equivalence between $\mathcal{M}$ and $M^2(\mathbf{T}^2)$. Thus (i) $\Rightarrow$ (ii). Now if U is a unitary equivalence between $M^2(\mathbf{T}^2)$ and $\mathcal{M}$, then

$$\mathcal{M} = U\left(\bigoplus_{\substack{k>0 \\ \text{or} \\ l>0}} Z_1^k Z_2^l \mathbf{C}\right) = \bigoplus_{\substack{k>0 \\ \text{or} \\ l>0}} Z_1^k Z_2^l (U\mathbf{C}),$$

i.e. $\mathcal{M}$ is of M^2-type and (ii) $\Rightarrow$ (iii) is proven. (iii) $\Rightarrow$ (iv) being trivial let suppose that (iv) holds. This means $\mathcal{M}_m \neq \{0\}$ and consequently the left defect space $\mathcal{L}$ of the pair (M_1, M_2) is non-null. By Lemma 4 from [3] we have that $(\widetilde{\mathcal{M}})_t := \bigoplus_{k,l\geqslant 0} Z_1^k Z_2^l \mathcal{L}$ is a reducing subspace for $\widetilde{M}_1, \widetilde{M}_2$. We check that $\dim \mathcal{L} = 1$. For this aim let $h_1, h_2 \in \mathcal{L}$. From the wandering property of $\mathcal{L}$ it results that for all $(k,l) \in \mathbf{Z}^2 \setminus \{(0,0)\}$

$$\int h_1 \bar{h}_2 t_1^k t_2^l \mathrm{d}m_2 = (Z_1^k Z_2^l h_1, h_2) = 0.$$

Therefore the Fourier coefficients (excepting the (0,0) one) of the function $h_1\bar{h}_2$ vanish. It results that $h_1\bar{h}_2$ is m_2-a.e. constant. This constant is zero whenever h_1 and h_2 are orthogonal. Then it follows $0 = |h_1 h_2|^2 = |h_1|^2|h_2|^2$ and, since $h_1\bar{h}_1$ and $h_2\bar{h}_2$ are also constant, we finally have that either h_1 or h_2 is m_2-a.e. null. This means that $\dim \mathcal{L} = 1$, i.e. (v) holds. Assume now (v). Then $\mathcal{L} = g\mathbf{C}$ with a unimodular function g from $\widetilde{\mathcal{M}}$ and the subspaces $\mathcal{M}_m$, $(\widetilde{\mathcal{M}})_t$ become:

$$\mathcal{M}_m = gM^2(\mathbf{T}^2), \quad (\widetilde{\mathcal{M}})_t = g\overline{H^2(\mathbf{T}^2)},$$

which leads to $\mathcal{M}_m \oplus (\widetilde{\mathcal{M}})_t = L^2(\mathbf{T}^2)$. But $\mathcal{M}_m \oplus (\widetilde{\mathcal{M}})_t \subset \mathcal{M} \oplus \widetilde{\mathcal{M}} = L^2(\mathbf{T}^2)$, which obviously implies $\widetilde{\mathcal{M}} = (\widetilde{\mathcal{M}})_t$, and consequently $\widetilde{M}_1, \widetilde{M}_2$ are doubly commuting shifts. By denoting $\mathcal{L}_j := \widetilde{\mathcal{M}} \ominus \widetilde{M}_j\widetilde{\mathcal{M}}$ $(j = 1,2)$, from [17] or [5] we have $\widetilde{M}_2\mathcal{L}_1 \subset \mathcal{L}_1$. If $\widetilde{M}_2\mathcal{L}_1 = \mathcal{L}_1$ then

$$\mathcal{L}_2 = \widetilde{\mathcal{M}} \ominus \widetilde{M}_2\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}} \ominus \widetilde{M}_2\mathcal{L}_1 = \widetilde{\mathcal{M}} \ominus \mathcal{L}_1$$

from which $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthogonal. On the other hand by comparing Slocinski's Wold type decomposition with ours (Theorem 8 from [5]) we have $\widetilde{\mathcal{M}} = \bigoplus_{k,l\geqslant 0} Z_1^k Z_2^l \cdot$ $\cdot(\mathcal{L}_1 \cap \mathcal{L}_2)$, which contradicts the orthogonality of $\mathcal{L}_1$ and $\mathcal{L}_2$. Thus $\widetilde{M}_2\mathcal{L}_1 \subsetneqq \mathcal{L}_1$, i.e. (vi) holds. If (vi) holds then $\widetilde{M}_2$ is not unitary on $\mathcal{L}_1$, therefore $\widetilde{M}_2$ has a non-trivial shift part in $\mathcal{L}_1$. At the same time it is easy to see that the defect space of $\widetilde{M}_2|\mathcal{L}_1$ is $\mathcal{L}_1 \cap \mathcal{L}_2$, and consequently it is non-trivial. But again by Theorem 8 from [5] $\mathcal{L}_1 \cap \mathcal{L}_2$ is the right defect space of the pair $(\widetilde{M}_1, \widetilde{M}_2)$ and then it is obvious just $\mathcal{L}$. This implies that $\mathcal{M}_m$ is non-trivial, which by the above proof of the implication (v) $\Rightarrow$ (vi) leads to $\mathcal{M} = gM^2(\mathbf{T}^2)$, with g a unimodular function in $\widetilde{\mathcal{M}}$. Thus (i) holds. We also note that (as in the preceding theorem) the minimal unitary extension space for $\mathcal{M}$ is the whole $L^2(\mathbf{T}^2)$. The proof is finished. ∎

Now by connecting the Wold decomposition of an invariant subspace $\mathcal{M}$ with those of its "dual" $\widetilde{\mathcal{M}}$ we obtain

COROLLARY 1. *An invariant subspace $\mathcal{M}$ of $L^2(\mathbf{T}^2)$ is of H^2-type (respectively of M^2-type) iff the $*$-invariant subspace $\widetilde{\mathcal{M}}$ is of M^2-type (respectively of H^2-type).*

COROLLARY 2. *An invariant subspace $\mathcal{M}$ of $L^2(\mathbf{T}^2)$ is neither of H^2-type nor of M^2-type iff the $*$-invariant subspace $\widetilde{\mathcal{M}}$ is neither of H^2-type nor of M^2-type.*

The following proposition characterizes the invariant subspaces $\mathcal{M}$ satisfying the condition of Corollary 2 and for which M_1 and M_2 doubly commute. For completness we also include here some known characterizations from [7] and [12].

PROPOSITION 1. *Let $\mathcal{M}$ be an invariant subspace in $L^2(\mathbf{T}^2)$ such that $\mathcal{R}_j :=$ $:= \mathcal{M} \ominus M_j\mathcal{M} \neq \{0\}$ j being either 1 or 2. The following are equivalent:*

(i) *$\mathcal{M}$ is not of H^2-type and M_1, M_2 are doubly commuting;*
(ii) *$\mathcal{M}$ is not of M^2-type and M_1, M_2 are doubly commuting;*
(iii) *$M_k \mathcal{R}_j = \mathcal{R}_j$ $(k \neq j)$;*
(iv) *M_k is a unitary operator $(k \neq j)$;*
(v) *$\widetilde{M}_k$ is a unitary operator $(k \neq j)$;*
(vi) *$\mathcal{M} = \mathcal{M}_{uu} \oplus \mathcal{M}_{su}$ if $j = 1$, and $\mathcal{M} = \mathcal{M}_{uu} \oplus \mathcal{M}_{us}$ if $j = 2$,*

where the summands $\mathcal{M}_{uu}, \mathcal{M}_{su}, \mathcal{M}_{us}$ are those from Slocinski's Wold type decomposition ([17]).

Proof. The equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) are contained in Corollary 7 from [7] and Theorem 5 from [12]. By inspecting the structure of the minimal unitary extension space it is easy to see the equivalence between (iv) and (v). For the rest of the proof we choose $j = 1$ and $k = 2$. If (v) holds, it is clear that $\widetilde{M}_1$ and $\widetilde{M}_2$ are doubly commuting and, since M_1 is non-unitary, $\widetilde{M}_1$ is non-unitary too. Now we apply (iv) $\Rightarrow$ (i) for the isometric pair $(\widetilde{M}_1, \widetilde{M}_2)$ instead of (M_1, M_2). It results that $(\widetilde{M}_1, \widetilde{M}_2)$ is not a shift pair, which is equivalent to the fact that (M_1, M_2) is not of modified shift pair (see [5]). This means that $\mathcal{M}$ is not of M^2-type. Thus we obtain (ii). Supposing (ii) we have that $(\widetilde{M}_1, \widetilde{M}_2)$ is not a shift pair but a doubly commuting one. Using now (i) $\Rightarrow$ (iv) for the pair $(\widetilde{M}_1, \widetilde{M}_2)$ again, we have just (v). The proof is complete. ∎

REMARK 1. As was observed in Theorem 8 of [5] the space $\mathcal{M}_{su} \oplus \mathcal{M}_{us}$ is the ultraevanescent part of $\mathcal{M}$. With its structure given in [7] the description of invariant subspaces of ultraevanescent type in the "doubly commuting" case is complete. The preceding proposition says even more; an ultraevanescent type invariant "doubly commuting" subspace is either of the form $\mathcal{M}_{su}$ or of the form $\mathcal{M}_{us}$.

Let us recall that in the structure of $\mathcal{M}_{su}$ and $\mathcal{M}_{us}$ the essential role is played by the following two standard invariant subspaces

$$\mathcal{H}^1 := \{f \in L^2(\mathbf{T}) : \hat{f}(k,l) = 0,\ k < 0\}$$

$$\mathcal{H}^2 := \{f \in L^2(\mathbf{T}) : \hat{f}(k,l) = 0,\ l < 0\}$$

respectively. It is easy to see that both $\mathcal{H}^1$ and $\mathcal{H}^2$ are ultraevanescent type invariant subspaces.

4. SUBSPACES WITH SZEGÖ PROPERTY

A closed subspace $\mathcal{M}$ of $L^2(\mathbf{T})$ has the *Szegö property* or is a *Szegö-subspace* if the following condition holds

(Sz) *Every function vanishing on a set of m_2-positive measure, vanishes m_2-almost everywhere on $\mathbf{T}^2$.*

Evidently, each Szegö subspace $\mathcal{M}$ has the support $s(\mathcal{M}) = \mathbf{T}^2$ in other words it is maximally supported.

First we prove

THEOREM 5. *For a given non-trivial closed subspace $\mathcal{M}$ of $L^2(\mathbf{T}^2)$ the Szegö--property is equivalent to each of the following two conditions:*

(i) *if $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ and $\chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M} \neq \{0\}$, then $m_2(\sigma) = 1$;*

(ii) *if $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ and $\mathcal{N}$ is some subspace of $L^2(\mathbf{T}^2)$ so that $\mathcal{M} \supset \chi_\sigma\mathcal{N} \neq \{0\}$, then $m_2(\sigma) = 1$.*

Proof. Let $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ so that $\chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M} \neq \{0\}$ and $0 < m_2(\sigma) < 1$. If $h \in \chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M}$, $h \neq 0$, then $h = \chi_\sigma h$. Therefore $h = 0$ on $\sigma' := \mathbf{T}^2 \setminus \sigma$ and $m_2(\sigma') > 0$. This contradicts (Sz). Thus (Sz) $\Rightarrow$ (i). Now it is clear that if $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ and $\mathcal{N}$ is a subspace of $L^2(\mathbf{T}^2)$ so that $\mathcal{M} \supset \chi_\sigma\mathcal{N} \neq \{0\}$, then $\chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M} \neq \{0\}$. So (i) $\Rightarrow$ (ii). Finally, let $h \in \mathcal{M}$, $h \neq 0$ and $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ with $m_2(\sigma) > 0$ such that $h = 0$ on σ. Then by denoting $\sigma_0 := \{t \in \mathbf{T}^2 : h(t) = 0\}$, we have $m_2(\sigma_0) > 0$ and $h = \chi_{\sigma_0'} h$. Thus $\mathcal{M} \supset \chi_{\sigma_0'}(h\mathbf{C}) \neq \{0\}$ and $m_2(\sigma_0') < 1$. Consequently (ii) $\Rightarrow$ (Sz). ∎

From Theorem 5 we deduce the following

COROLLARY 3. *Every non-trivial (closed) subspace of a Szegö subspace is a Szegö subspace too.*

COROLLARY 4. *A Szegö subspace does not contain any non-trivial doubly invariant subspace.*

COROLLARY 5. *The space of the minimal unitary extension of an invariant or $*$-invariant Szegö subspace is the whole $L^2(\mathbf{T}^2)$.*

THEOREM 6. *The Szegö-property of invariant subspaces is invariant with respect to the unitary equivalences.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two unitarily equivalent invariant subspaces of $L^2(\mathbf{T}^2)$ and assume that $\mathcal{M}$ is a Szegö-subspace. If U is a unitary equivalence between $\mathcal{M}$ and $\mathcal{N}$, then by Corollary 5 and by the lifting commutants theorem (see for example [18] Theorem 10.9 p.260) U can be extended to a unitary operator $\widetilde{U}$ from $L^2(\mathbf{T}^2)$ onto the space $\chi_F L^2(\mathbf{T}^2)$ of the minimal unitary extension corresponding to $\mathcal{N}$, so that $\widetilde{U}Z_j = Z_j\widetilde{U}$, $j = 1, 2$. Since $\widetilde{U}\chi_F L^2(\mathbf{T}^2) = \chi_F L^2(\mathbf{T}^2)$, it follows that $\widetilde{U}$ is unitary on $\chi_F L^2(\mathbf{T}^2)$, which implies $\widetilde{U}(1 - \chi_F)L^2(\mathbf{T}^2) = \widetilde{U}(L^2(\mathbf{T}^2) \ominus \chi_F L^2(\mathbf{T}^2)) = \{0\}$. Hence

$\chi_F = 1$ m_2-a.e. on $\mathbf{T}^2$. Thus $\widetilde{U}$ is a unitary operator on $L^2(\mathbf{T}^2)$ and consequently it is given by the multiplication with a unimodular function, say f. Therefore $\mathcal{N} = f\mathcal{M}$ and so $\mathcal{N}$ is a Szegö subspace too. ■

Now we characterize the reducing subspaces with respect to a given invariant subspace.

PROPOSITION 2. *Let $\mathcal{M}$ be an invariant subspace of $L^2(\mathbf{T}^2)$. Every non-trivial reducing subspace $\mathcal{N}$ for the pair (M_1, M_2) has the form $\mathcal{N} = \chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M}$, where σ is the support of $\mathcal{N}$.*

Proof. Let $\mathfrak{N}$ be the space of the minimal unitary extension for $[\mathcal{N}, (N_1, N_2)]$. Then $\mathfrak{N} = \bigvee_{k,l\in\mathbf{Z}} Z_1^k Z_2^l \mathcal{N} = \bigvee_{\substack{k\leqslant 0\\ \text{or}\\ l\leqslant 0}} Z_1^k Z_2^l \mathcal{N}$ and $\mathfrak{N} = \chi_\sigma L^2(\mathbf{T}^2)$, where $\sigma = s(\mathcal{N})$. Now we go to calculate $P_\mathcal{M}\mathfrak{N}$. To this aim it suffices to calculate $P_\mathcal{M} Z_1^k Z_2^l n$; $n \in \mathcal{N}$ with $l < 0$ (the case $k < 0$ being similarly). Since Z_1 and Z_2 are doubly commuting we have $P_\mathcal{M} Z_1^k Z_2^l n = P_\mathcal{M}(Z_2^*)^{-1} Z_1^k n = P_\mathcal{M}(Z_2^*)^{-1} N_1^k n = (M_2^*)^{-1} M_1^k n$, which by the doubly invariance property of $\mathcal{N}$ lies in $\mathcal{N}$. It follows $P_\mathcal{M}\mathfrak{N} \subset \mathcal{N}$. But, obviously, $\mathcal{N} \subset \mathfrak{N} \cap \mathcal{M} \subset P_\mathcal{M}\mathfrak{N}$, from which the equality $\mathcal{N} = P_\mathcal{M}\mathfrak{N} = \mathfrak{N} \cap \mathcal{M} = \chi_\sigma L^2(\mathbf{T}^2) \cap \mathcal{M}$ holds. ■

Before applying this result to the Szegö invariant subspaces let us also prove

PROPOSITION 3. *If $\mathcal{M}$ is a non-trivial invariant subspace of $L^2(\mathbf{T}^2)$ without proper reducing subspaces, so that $\chi_\sigma\mathcal{M} \subset \mathcal{M}$, for some $\sigma \in$ Bor$(\mathbf{T}^2)$, then either $m_2(\sigma) = 0$ or $m_2(\sigma) = 1$.*

Proof. It is simply first to see that, since $\mathcal{M}$ is closed and $\chi_\sigma\mathcal{M} \subset \mathcal{M}$, $\chi_\sigma\mathcal{M}$ is closed too. Next we show that $\mathcal{N} := \chi_\sigma\mathcal{M}$ is reducing for the pair (M_1, M_2). Obviously it suffices to verify $M_j^*\mathcal{N} \subset \mathcal{N}$ $(j = 1, 2)$. For this aim let $n \in \mathcal{N}$ and decompose $Z_j^* n = n_j' + n_j''$ with $n_j' \in \mathcal{N}$ and $n_j'' \in \chi_\sigma L^2(\mathbf{T}^2) \ominus \mathcal{N}$. Now since $\int m n_j'' \mathrm{d}m_2 = \int (\chi_\sigma m) n_j'' \mathrm{d}m_2 = 0$ for each $m \in \mathcal{M}$, we have $n_j'' \perp \mathcal{M}$. Thus $M_j^* n = P_\mathcal{M} Z_j^* n = P_\mathcal{M}(n_j' + n_j'') = P_\mathcal{M} n_j' = n_j' \in \mathcal{N}$. But then by hypothesis either $\mathcal{N} = \mathcal{M}$ or $\mathcal{N} = \{0\}$ which leads to the conclusion of the proposition. ■

By applying these results to the Szegö invariant subspaces we obtain

COROLLARY 6. *Each invariant Szegö subspace does not contain any non-trivial reducing subspace.*

We examine now the Szegö property for the standard subspaces appearing in the structure of invariant subspaces from Theorems 1,2 and 4 (see Remark 1).

EXAMPLE 1. $H^2(\mathbf{T}^2)$ is a Szegö subspace and consequently $H_0^2(\mathbf{T}^2)$ is a Szegö subspace too.

This was indirectly observed by Helson and Lowdenslager in [9]. We shall verify that the condition (ii) from Theorem 5 is satisfied for $H^2(\mathbf{T}^2)$. For, let $\mathcal{N}$ be a subspace of $L^2(\mathbf{T}^2)$ and $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$ so that $H^2(\mathbf{T}^2) \supset \chi_\sigma\mathcal{N} \neq \{0\}$. Then $h \in \chi_\sigma\mathcal{N}$, $h \neq 0$ implies $\chi_\sigma^n h \in H^2(\mathbf{T}^2)$ for $n \in \mathbf{N}$. By applying Corollary 1 from [10] it results that χ_σ itself is in $H^2(\mathbf{T}^2)$ and thus χ_σ is a constant function. Since $m_2(\sigma) > 0$, we surely have $m_2(\sigma) = 1$.

EXAMPLE 2. $\mathcal{H}^1$ and $\mathcal{H}^2$ do not have the Szegö property.

A little modification of the example given by Helson and Lowdenslager in [9] p. 176 will be sufficient to our aim. For, let α be an open arc in the one dimensional torus $\mathbf{T}$ and $g \in L^2(\mathbf{T})$ vanishing on α. Then the series

$$\sum_{k=1}^{\infty}\sum_{l\in\mathbf{Z}}(1/k)\hat{g}(l)t_1^k t_2^l$$

converges in $L^2(\mathbf{T}^2)$, and thus it defines a function f in $L^2(\mathbf{T}^2)$ having the norm $\left(\sum_{k=1}^{\infty}1/k^2\right)^{1/2}\|g\|_{L^2(\mathbf{T})}$. It is clear that $\hat{f}(k,l) = (1/k)\hat{g}(l)$ for $k > 1$, and $\hat{f}(k,l) = 0$ if $k \leqslant 0$. This means that $f \in Z_1\mathcal{H}^1$. On the other hand f vanishes on $\alpha \times \mathbf{T}$, and $m_2(\alpha \times \mathbf{T}) = m(\alpha) > 0$. Thus $\mathcal{H}^1$ does not have the Szegö property. Similarly H^2 is not a Szegö subspace.

EXAMPLE 3. $M^2(\mathbf{T}^2)$ does not have the Szegö property.

It suffices to observe that $Z_1\mathcal{H}^1$ or/and $Z_2\mathcal{H}^2$ are contained in $M^2(\mathbf{T}^2)$.

Since all H^2-type invariant subspaces and all M^2-type invariant subspaces are unitarily equivalent with $H^2(\mathbf{T}^2)$ and $M^2(\mathbf{T}^2)$ respectively, we have the following

COROLLARY 7. *Each H^2-type invariant subspace is a Szegö subspace, while all the M^2-type invariant subspaces do not have the Szegö property.*

COROLLARY 8. *An invariant Szegö subspace can be either of H^2-type or of ultraevanescent (maximally supported) type.*

By the above Examples 1 and 2 it also follows

COROLLARY 9. *There are ultraevanescent type invariant Szegö subspaces and there also are ultraevanescent type subspaces without the Szegö property.*

5. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Let us consider the decomposition (1) of $\mathcal{M}$. Since $\mathcal{M}_u$ reduces Z_1 and Z_2, we have $\mathcal{M}_u = \chi_\sigma L^2(\mathbf{T}^2)$ with $\sigma \in \mathrm{Bor}(\mathbf{T}^2)$, $\sigma \subset E$, where E is the support of $\mathcal{M}$. If $\mathcal{M}_t \neq \{0\}$ or $\mathcal{M}_m \neq \{0\}$ then by Theorem 2 (respectively by Theorem 3) $\mathcal{M} = fH^2(\mathbf{T}^2)$, $\mathcal{M} = gM^2(\mathbf{T}^2)$ with f, g unimodular functions from $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ respectively. So we have for $\mathcal{M}$ the form (3) respectively (4). Assume now that $\mathcal{M}_t = \mathcal{M}_m = \{0\}$. Then $\mathcal{M}$ becomes $\chi_\sigma L^2(\mathbf{T}^2) \oplus \mathcal{M}_e$. Suppose $\mathcal{M}_e \neq \{0\}$. It is clear that the support of $\mathcal{M}_e$, say ω, satisfies $\omega = E \setminus \sigma$ and $\mathcal{M}_e$ can be written as $\mathcal{M}_e = \chi_\omega \mathcal{N}$, where

$$\mathcal{N} := \{h \in L^2(\mathbf{T}^2) : \chi_\omega h \in \mathcal{M}_e\},$$

is a maximally supported invariant subspace. We now apply to $\mathcal{N}$ the decomposition (1). Since $\chi_\omega \mathcal{N}_u$ is doubly invariant and contained in $\mathcal{M}_e$, we have $\chi_\omega \mathcal{N}_u = \{0\}$. This means $\chi_\omega \mathcal{N} = \chi_\omega(\mathcal{N} \ominus \mathcal{N}_u)$. We may therefore suppose $\mathcal{N}_u = \{0\}$. If $\mathcal{N}_t \neq \{0\}$, then by Theorem 2 we have $\mathcal{N} = \mathcal{N}_t = fH^2(\mathbf{T}^2)$, with f an unimodular function in $\mathcal{N}$. But $\mathcal{N} \supset \mathcal{M}_e$, i.e. $fH^2(\mathbf{T}^2) \supset \chi_\omega f H^2(\mathbf{T}^2)$ which contradicts the Szegö property of $H^2(\mathbf{T}^2)$. Hence $\mathcal{N}_t = \{0\}$. If $\mathcal{N}_m \neq \{0\}$, then by Theorem 3 we have $\mathcal{N} = \mathcal{N}_m = gM^2(\mathbf{T}^2)$ with g unimodular in $\widetilde{\mathcal{N}}$. But since the left defect space is one dimensional we may apply Proposition 4 from [4]. Thus $gM^2(\mathbf{T}^2)$ has not proper reducing subspaces and by using Proposition 3 we deduce that $\mathcal{M}_e = \chi_\omega g M^2(\mathbf{T}^2)$ is either $\{0\}$ or $gM^2(\mathbf{T}^2)$, which contradicts the fact that $\mathcal{M}_e \neq \{0\}$ and the ultraevanescence property of $\mathcal{M}_e$, respectively. We then have $\mathcal{N}_m = \{0\}$. It remains that $\mathcal{N}$ coincides with $\mathcal{N}_e$ and is maximally supported, therefore we have the form (5). The proof is finished. ■

Proof of Theorem 2. If $\mathcal{M}$ is an invariant subspace (necessarily Szegö) of $H^2(\mathbf{T}^2)$, then by Corollary 8 and Theorem 3 and 4, $\mathcal{M}$ has one of the following forms: $\mathcal{M} = fH^2(\mathbf{T}^2)$, $\mathcal{M} = \mathcal{M}_e$ and is maximally supported with f unimodular functions. In the first case it clearly results that f is an inner function.

Finally, if $\mathcal{M}$ is of ultraevanescent type let $\mathcal{M} = \mathcal{M}_u^j \oplus \mathcal{M}_s^j$ be the Wold decomposition of $\mathcal{M}$ with respect to the individual isometries M_j ($j = 1, 2$). If $\mathcal{M}_u^j$ were non-trivial for $j = 1$ or 2, then the operator Z_j would be unitary on $\mathcal{M}_u^j$. By choosing $h \in \mathcal{M}_u^j$, $h \neq 0$, we would obtain $\bar{t}_1^k h \in H^2(\mathbf{T}^2)$ for all $k \in \mathbf{Z}_+$, which by Corollary 1 from [10] leads to $\bar{t}_1 \in H^2(\mathbf{T}^2)$, a contradiction. Thus $\mathcal{M}_u^j = \{0\}$ for $j = 1, 2$, or equivalently, $\mathcal{M} = \mathcal{M}_s^1 = \mathcal{M}_s^2$, which proves that M_1 and M_2 are (non doubly) commuting shifts on $\mathcal{M}$. ■

6. CONCLUDING REMARKS

1. Let $\mathcal{M}$ be an invariant subspace in $L^2(\mathbb{T}^2)$ and $E = s(\mathcal{M})$. As was seen in Section 2

$$\mathcal{N} := \{f \in L^2(\mathbb{T}^2),\ \chi_E f \in \mathcal{M}\}$$

is an invariant subspace in $L^2(\mathbb{T}^2)$ such that $\mathcal{M} \subset \mathcal{N}$ and $\mathcal{M} = \chi_E \mathcal{N}$. If $m_2(E) < 1$ and $\mathcal{M}_u = \{0\}$ then $\mathcal{N}$ is of ultraevanescent type only. In other words an invariant subspace without doubly invariant subspaces supported on a set of measure strictly less than one, is of ultraevanescent type and cannot be represented with the aid of any H^2- or M^2-type subspaces. For example if $0 < m_2(\omega) < 1$, then (cl meaning the closure in $L^2(\mathbb{T}^2)$) cl$[\chi_\omega H^2(\mathbb{T}^2)]$, cl$[\chi_\omega M^2(\mathbb{T}^2)]$ are such subspaces. They are ultraevanescent and cl$[\chi_\omega H^2(\mathbb{T}^2)] = \chi_\omega \mathcal{N}_1$, cl$[\chi_\omega M^2(\mathbb{T}^2)] = \chi_\omega \mathcal{N}_2$ with $\mathcal{N}_1$ and $\mathcal{N}_2$ maximally supported ultraevanescent invariant subspaces. This is why *it would be interesting to characterize the maximally supported ultraevanescent invariant subspaces.*

2. Because of the Szegö property all the ultraevanescent invariant subspaces of $H^2(\mathbb{T}^2)$ are maximally supported. By the same reason $H^2(\mathbb{T}^2)$ does not contain any unitarily equivalent subspace to $\mathcal{H}^1$ or to $\mathcal{H}^2$ (they do not have the Szegö property).

3. The Theorem of Beurling in the one dimensional case ([2]) leads to the famous factorization theorem of functions in $H^2(\mathbb{T})$ into an inner function and an outer function. In the two dimensional case our Beurling type Theorem 2 does not imply such a factorization. However, if we adopt the definition of outer functions in the sense of Helson i.e. $h \in H^2(\mathbb{T}^2)$ is an *H-outer function* if

$$S(h) := \bigvee_{k,l \geqslant 0} Z_1^k Z_2^l h = H^2(\mathbb{T}^2),$$

then the following holds: "*a function $g \in H^2(\mathbb{T}^2)$ admits a Beurling factorization (i.e. $g = fh$ with h an H-outer function and f an inner function) iff $S(g)$ is an invariant subspace of H^2-type*".

4. Since $M^2(\mathbb{T}^2)$ plays an important role in our approach it is meaningfull to consider invariant subspaces in $M^2(\mathbb{T}^2)$. Unlike the space $H^2(\mathbb{T}^2)$, which does not have any invariant subspace of M^2-type, the space $M^2(\mathbb{T}^2)$ contains invariant subspaces of H^2-type (for example $Z_1 H^2(\mathbb{T}^2)$).

It also contains ultraevanescent "doubly commuting" type invariant subspaces ($Z_1\mathcal{H}^1$ and $Z_2\mathcal{H}^2$ are contained in $M^2(\mathbb{T}^2)$). It is not difficult to see that a M^2-type invariant subspace $gM^2(\mathbb{T}^2)$ is contained in $M^2(\mathbb{T}^2)$ iff g is an inner function. An interesting question would be *the characterization of those unimodular functions f for which $fH^2(\mathbf{T}^2) \subset M^2(\mathbf{T}^2)$*. By the orthogonality between $M^2(\mathbf{T}^2)$ and $\overline{H^2(\mathbf{T}^2)}$ we can prove that $M^2(\mathbf{T}^2)$ does not contain ultraevanescent subspaces of the form $\mathrm{cl}[\chi_\sigma g M^2(\mathbf{T}^2)]$ or of the form $\mathrm{cl}[\chi_\sigma H^2(\mathbf{T}^2)]$ with $m_2(\sigma) < 1$. It would be interesting *to characterize the ultraevanescent invariant subspaces of* $M^2(\mathbf{T}^2)$. In order to obtain a Beurling type factorization for functions from $M^2(\mathbf{T}^2)$ it *would be desirable to know if there are functions $h \in M^2(\mathbf{T}^2)$ so that $S(h) = M^2(\mathbf{T}^2)$*.

5. Let us suppose that $\mathcal{M}$ is invariant of ultraevanescent type. Then $\widetilde{\mathcal{M}}$ is of ultraevanescent type too. Moreover from Proposition 1 we have that (M_1, M_2) and $(\widetilde{M}_1, \widetilde{M}_2)$ are simultaneously doubly commuting. This means that we cannot obtain new "known" ultraevanescent type subspaces from the corresponding $*$-invariant dual on which the restrictions of Z_1^*, Z_2^* would be doubly commuting.

6. It is possible to make analogous investigations for invariant subspaces in $L^2(\mu)$, with μ a positive measure on the bitorus $\mathbf{T}^2$. The Lebesgue decomposition of μ with respect to m_2 will play an important role in such a framework (see also [4] Theorem 6). In a forthcoming paper we shall study such subspaces in connection with the two-time parameter stationary processes.

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