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ON THE SPATIAL MATRICIAL SPECTRA OF OPERATORS

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INTRODUCTION

Contemporary spectral theory for (bounded linear) operators on Hilbert spaces accomodates a variety of settings where extended notions of spectrum play a key role, be it in the introduction of joint spectra for a several variable theory, or in the notion of the spectra of an algebra. In single operator theory, there are occasions when it is natural to enlarge the notion of spectrum so as to admit matrix or operator values as spectral values. For example, a matrix-valued reducing spectrum is the natural spectral object in the study of n-normal and essentialy n-normal operators [9], [12], [14], and an operator-valued reducing spectrum occurs natturally in the classification of operators up to approximate unitary equivalence [5], [16]. The formal study of non-reducing matricial spectra seems, at this time, to be confined to portions of [1] and [6]; in particular, the conditions defining the full (non-reducing) matricial spectrum have yet to be adequately discussed, though properties of this spectrum have already been partly explored. Therefore, it is the purpose of the present paper to study, somewhat systematically, the general ideas leading to a matrix-valued spectrum for operators on Hilbert spaces. Much of the groundwork for the study of matricial spectra has, it should be noted, already been done before now, most notably in the works of Pearcy and Salinas [11], [12], Bunce and Salinas [1], and Hadwin [5],[6].

The emphasis here will be on a matricial spectrum that arises from spatial rather than algebraic considerations, and on how properties of an operator show up in this spectrum. As is explained by Davis in [3], it is resonable to expect that matrix-valued spectra better reflect the characteristics of an operator than the (numerical) spectrum does — knowing the action of an operator on one of its invariant subspaces is necessarily more informative than knowing only individual eigenvalues, for example.

One program, therefore, is to pass from approximate eigenvalues to finite-dimensional approximate restrictions and obtain matrix-valued analogues of the approximate point spectrum and the defect spectrum; but it is less than clear that the end result yields a suitable notion of spectrum. Another program, stemming from a suggestion credited to Arveson and carried out in part by Bunce and Salinas, considers spectral elements to be approximate compressions to rationally semi-invariant subspaces of finite dimension. However, in the literature there is little indication given to why such an object ought to be considered as the 'correct' definition of a matricial spectrum. One objective, therefore, of this paper is to make sense of this program. Specifically, it will be shown that Arveson's definition of the spatial matricial spectrum can be arrived at in a manner whereby the spectrum is made large enough to possess a hierarchical property, namely that $k \times k$ spectral elements of an $n \times n$ spectral element of T be $k \times k$ spectral elements of T as well whenever $1 \leqslant k \leqslant n$, but is not made so large as to not coincide with the usual numerical spectrum when $n = 1$.

The paper is organised as follows. In Section 1 the matrix-valued spectra that function as analogues of the reducing spectrum and left spectrum are reviewed, along with their spatial components. In Section 2 and Section 3 the spatial matricial spectrum is defined and its fundamental properties are studied. The two main theorems establish the spectral hierarchy of the spatial matricial spectrum and its upper- semi-continuity property. The final section, Section 4, consists of computations of these spectra for certain classes of operators.

PRELIMINARIES AND NOTATION

Throughout, H will denote a complex separable Hilbert space with inner product (x, y), and $B(H)$ will denote the algebra of (bounded linear) operators on H. The spectrum of an element T of a unital Banach algebra A is denoted by $\sigma(T)$, and $C^*(T)$ denotes the unital C^*-algebra generated by T; the C^*-algebra of complex $n \times n$ matrices is denoted by M_n.

We will make use of two notions of equivalence in $B(H)$. Two operators $S, T \in$ $\in B(H)$ are unitarily equivalent, denoted by $S \sim T$, if $S = U^*TU$ for some unitary operator U, and are approximately (unitarily) equivalent, denoted by $S \sim_a T$, if $S = \lim_n U_n^* T U_n$ (norm limit) for some sequence of unitary operators U_n. All of the spectra studied in this paper are invariant under approximate unitary eqivalence.

Elements of the set $\mathbb{N}$ of natural numbers are denoted by i, j, k, m, n, whereas μ, ν will denote cardinal numbers between 1 and countable ∞ inclusive. The Hilbert space $\mathbb{C}^\nu$ is taken to be the finite-dimensional space $\mathbb{C}^n$ if $\nu = n$ and to be $l_2(\mathbb{N})$ if $\nu = \infty$.

If L is a self-adjoint linear manifold in a C^*-algebra A, and if $\varphi : L \to A_0$ is a continuous linear mapping into a C^*-algebra A_0, then let φ_n denote the map $L \otimes M_n \to\to A_0 \otimes M_n$ given by $\varphi_n = \varphi \otimes \mathrm{id}_n$, where id_n is the identity map on M_n. The map φ is called *completely positive* if φ_n is a positivemap for all $n \in \mathbf{N}$. We will have occasion to refer to the following important and well known theorem (see [10]).

EXTENSION AND DILATION THEOREM. *If A is a unital C^*-algebra, if $L \subset A$ is a self-adjoint linear manifold with $1 \in L$, and if $\varphi : L \to B(H)$ is a unital completely positive map, then*

(1) *(Arveson's Extension Theorem) φ has a completely positive extension to A, and*

(2) *(Stinespring's Dilation Theorem) φ dilates to a $*$-homomorphism; that is, there is a Hilbert space H_π, a unital $*$-homomorphism $\pi : A \to B(H_\pi)$, and an isometry $V : H \to H_\pi$ such that $\varphi(a) = V^*\pi(a)V$ for every $a \in A$.*

It is well known that for each completely positive map φ there is a minimal dilation which is unique up to unitary equivalence; we will refer to this dilation as "the Stinespring decomposition" of φ.

For each $T \in A$, denote by $\mathrm{CP}^n(T)$ the set of all unital completely positive maps $C^*(T) \to M_n$. This set is endowed with the BW-topology by identifying $\mathrm{CP}^n(T)$ with a certain weak*-compact subset of the unit ball of the dual of the Banach space $C^*(T) \otimes M_n$ (see [10; Chapter 6] for explicit details); hence, $\mathrm{CP}^n(T)$ is BW-compact. Indeed, the separability of $C^*(T) \otimes M_n$ implies that $\mathrm{CP}^n(T)$ is metrizable in the BW-topology, and so $\mathrm{CP}^n(T)$ is actually sequentially BW-compact. As for convergence in $\mathrm{CP}^n(T)$, we note that the range space of every map in $\mathrm{CP}^n(T)$ is finite-dimensional, and so, by [10;6.3], a sequence $\{\varphi_j\} \subset \mathrm{CP}^n(T)$ converges to $\varphi \in \mathrm{CP}^n(T)$ if and only if for every $A \in C^*(T)$, $\{\varphi_j(A)\}$ converges (in norm) to $\varphi(A)$.

1. LEFT AND REDUCING MATRICIAL SPECTRA: A REVIEW

The left spectrum $\sigma_l(T)$ of an element T in a unital C^*-algebra A is the set of all $\lambda \in \mathbf{C}$ such that $T - \lambda 1$ has no left inverse in A. For operators $T \in B(H)$ the left spectrum has a spatial description: $\lambda \in \sigma_l(T)$ if $\inf \|Tx - \lambda x\| = 0$, where the infimum is over all unit vectors $x \in H$. This defines, of course, the approximate point spectrum of T and the definition extends easily to one with matrix values, as will soon be discussed. As a purely algebraic phenomenom, the notion of left invertibility is not immediately suited to carry over with matrix values, for there is no direct way of defining the corresponding one-sided resolvents. Hence, we are forced to view the failure to be left invertible in a different and, perhaps, indirect context. A very nice way of thinking of the left spectrum, which makes no reference whatsoever to left invertibility, was suggested by Bunce and Salinas and is given in the theorem below. By a state, we mean a positive unital linear functional.

THEOREM 1.1 ([1]). *The left spectrum of $T \in A$ is the set of all complex numbers $\varphi(T)$ obtained from all states φ on the C^*-algebra $C^*(T)$ that satisfy $\varphi(T^*T) = = \varphi(T^*)\varphi(T)$.*

Proof: We may assume, by using a faithful unital $*$-representation of $C^*(T)$ as an algebra of operators, that T acts on a Hilbert space H. If $\lambda \in \sigma_l(T)$, then there exist unit vectors $x_k \in H$ such that $\lim_k \|Tx_k - \lambda x_k\| = 0$. Let φ_k be the state on $C^*(T)$ defined by $\varphi_k(A) = (Ax_k, x_k)$ for $A \in C^*(T)$; by the weak*-compactness of the state space, there is a subsequence of states φ_β weak*-convergent to a state φ. Thus, $\varphi(T^*T) = \lim_\beta \varphi_\beta(T^*T) = \lim_\beta \|Tx_\beta\|^2 = |\lambda|^2 = \varphi(T^*)\varphi(T)$.

Conversely, if φ is a state on $C^*(T)$, then there is a unital $*$-representation π of $C^*(T)$ on a Hilbert space H_π and a unit vector $x \in H_\pi$ such that for every $A \in C^*(T)$, $\varphi(A) = (\pi(A)x, x)$. If in addition, $\lambda = \varphi(T)$ and $\varphi(T^*T) = \varphi(T^*)\varphi(T)$, then

$$\|\pi(T)x\|^2 = \varphi(T^*T) = \varphi(T^*)\varphi(T) = |\pi(T)x, x)|^2 \leqslant \|\pi(T)x\|^2,$$

and so $\pi(T)x = \lambda x$ by the Cauchy-Schwarz inequality. Hence, $\lambda \in \sigma_l(T)$. ∎

In moving to matrix values, the role of states in the above theorem must be taken by some suitable class of unital positive linear maps $C^*(T) \to M_n$. The proof suggests that the class $\mathrm{CP}^n(T)$ of unital completely positive maps $C^*(T) \to M_n$ provides an appropiate analogue: the weak*-compactness of the state space is generalized by the BW-compactness of $\mathrm{CP}^n(T)$, and the representation of states as 1-dimensional compressions of $*$-homomorphims is generalized by the Stinespring representation of elements of $\mathrm{CP}^n(T)$ as n-dimensional compressions of $*$-homomorphisms.

DEFINITION. Suppose that $T \in B(H)$, that φ is a unital completely positive map $C^*(T) \to M_n$, and that $\Lambda = \varphi(T)$.

(1) $\Lambda \in \Pi^n(T)$, the $n \times n$ left matricial spectrum of T, if $\varphi(T^*T) = \varphi(T^*)\varphi(T)$. If, in adition, $\varphi(K) = 0$ whenever K is a compact operator in $C^*(T)$, then $\Lambda \in \Pi^n_e(T)$, the left essential matricial spectrum of T. Λ is in the spatial component $\Pi^n_s(T)$ of $\Pi^n(T)$ if $\|TV_k \to V_k\Lambda\| \to 0$ for some sequence of isometries $V_k : \mathbf{C}^n \to H$. See [1].

(2) $\Lambda \in R^n(T)$, the $n \times n$ reducing matricial spectrum of T, if φ is a unital $*$-homomorphism. If, in addition, φ annihilates the compact operators in $C^*(T)$, then $\Lambda \in R^n_e(T)$, the reducing essential matricial spectrum. Λ is in the spatial component $R^n_s(T)$ of $R^n(T)$ if $\|TV_k - V_k\Lambda\| + \|T^*V_k - V_k\Lambda^*\| \to 0$ for some sequence of isometries $V_k : \mathbf{C}^n \to H$. See [11].

The theory of the reducing matricial spectrum was put forth and developed in [11] and [12] by Pearcy and Salinas, and they found that the $n \times n$ reducing essential matricial spectra classify n-normal operators up to unitary equivalence modulo the ideal of compact operators. A spatially defined operator-valued version of the reducing spectrum for Hilbert space operators was introduced by Hadwin; he proves, using Voiculescu's theorem [16], that a reducing spectrum based on operator rather than matrix values contains sufficiently many elements to determine every operator up to approximate unitary equivalence [5;3.6].

The left matricial spectrum is studied extensively in [1]. Elements of $\Pi^n(T)$ are unitarily equivalent to restrictions of representations of T to n-dimensional invariant subspaces: that is, if $\varphi : C^*(T) \to M_n$ is a unital completely positive map such that $\varphi(T^*T) = \varphi(T^*)\varphi(T)$, and if $\varphi = V^*\pi V$ is the Stinespring decomposition of φ, then $V(\mathbb{C}^n)$ is an n-dimensional invariant subspace for $\pi(T)$. The pertinent calculation, due to Choi, is

$$0 = \varphi(T^*T) - \varphi(T^*)\varphi(T) = (VV^*\pi(T)V - \pi(T)V)^*(VV^*\pi(T)V - \pi(T)V),$$

and so $\pi(T)V = VV^*\pi(T)V$. If, in addition, φ is a $*$-homomorphism, then $V(\mathbb{C}^n)$ reduces $\pi(T)$.

It should be mentioned that, unlike the case $n = 1$, not all of the elements of $\Pi^n(T)$ need lie in the spatial component $\Pi^n_s(T)$; some special cases are studied in [1]. It is important to note, however, that the left essential matricial spectrum and the reducing essential matricial spectrum *do* lie in the respective spatial components of $\Pi^n(T)$ and $R^n(T)$ and that the isometries that define the elements of these essential matricial spectra can be chosen so that they converge weakly to zero [1], [11].

For most of the remainder of the paper, we will concern ourselves with the spatial components of the above sets.

Plainly, the set $\Pi^n_s(T)$ funcions as a matrix-valued analogue of the approximate point spectrum of the operator T. Observe that if $V : \mathbb{C}^n \to H$ is an isometry and $\Lambda \in M_n$ is such that $\|TV - V\Lambda\| < \varepsilon$, then the n-dimensional subspace $V(\mathbb{C}^n)$ is approximately invariant under T in the following sense: if $P \in B(H)$ is the projection of H onto $V(\mathbb{C}^n)$ (i.e., $P = VV^*$), then

$$\|(1-P)TP\| = \|TVV^* - VV^*TVV^*\| \leqslant \|TV - VV^*TV\|\,\|V^*\| =$$

$$= \|TV - VV^*(TV - V\Lambda + V\Lambda)\| \leqslant \|TV - VV^*V\Lambda\| + \|VV^*\|\,\|TV - V\Lambda\| < 2\varepsilon.$$

The "action" of T on the range of V is approximately the action of Λ on $\mathbf{C}^n$. Similarly, if $\Lambda \in M_n$ and if $V : \mathbf{C}^n \to H$ such that $\|TV - V\Lambda\| < \varepsilon$ and $\|T^*V - V\Lambda^*\| < \varepsilon$, then $V(\mathbf{C}^n)$ approximately reduces T : if $P = VV^*$, then P projects H onto $V(\mathbf{C}^n)$, and, by a computation similar to the one above, $\|TP - PT\| < 2\varepsilon$, and the action of T and T^* on the range of V is approximately that of Λ and Λ^* on $\mathbf{C}^n$.

The terminology employed in [6] calls elements of $\Pi_s^n(T)$ approximate restrictions and elements of $R_s^n(T)$ approximate summands of T. Bona fide restrictions and summands are found in the sets $\Pi_0^n(T)$ and $R_0^n(T)$, which are the natural analogues of the point spectrum and the set of reducing eigenvalues.

Finally, all of the spectra defined up to this point enjoy a hierarchical property. For the spatially defined spectra, this is to say that $R_s^k(\Lambda) \subset R_s^k(T)$ and $\Pi_s^k(\Omega) \subset \Pi_s^k(T)$ for all $\Lambda \in R_s^n(T)$ and all $\Omega \in \Pi_s^n(T)$, and for all $1 \leqslant k \leqslant n$.

We conclude this section with an elementary theorem which illustrates that the sets $\Pi_s^n(\cdot)$ do indeed behave very much like the approximate point spectrum.

THEOREM 1.2. *The spatial component of the left matricial spectrum is compact and nonempty. In fact when H has infinite dimension, $\Lambda \otimes 1_k \in \Pi_s^{kn}(T)$ for every $\Lambda \in \Pi_e^n(T)$ and for every $k \in \mathbf{N}$. In addition, if T is compact, then invertible elements of $\Pi_s^n(T)$ are elements of $\Pi_0^n(T)$.*

Proof: Since $\Pi_s^n(T)$ is contained in the ball centred at $0 \in M_n$ of radius $\|T\|$, to show the compactness of $\Pi_s^n(T)$ it is enough to show that the complement of $\Pi_s^n(T)$ is open. This is achieved by following the scalar version verbatim: if $\Lambda \notin \Pi_s^n(T)$, then there is a $\delta > 0$ such that $\|TV - V\Lambda\| \geqslant \delta$ for every isometry $V : \mathbf{C}^n \to H$, and so if $\varepsilon = \frac{1}{2}\delta$, then the ε-ball centred at Λ does not intersect $\Pi_s^n(T)$; thus, the complement of $\Pi_s^n(T)$ is open.

Suppose that T is compact and that $\Lambda \in \Pi_s^n(T)$ is invertible. There exist isometries $V_j : \mathbf{C}^n \to H$ such that $0 = \lim_j \|TV_j - V_j\Lambda\|$. For the basis vector e_1 of $\mathbf{C}^n$, the sequence $\{TV_je_1\}_j$ has a convergent subsequence $\{TV_{\beta,1}e_1\}$ converging to a vector $\omega_1 \in H$ (because T is compact). The equation $0 = \lim_{\beta,1} \|TV_{\beta,1} - V_{\beta,1}\Lambda\|$ implies that $V_{\beta,1}\Lambda e_1 \to \omega_1$ as $\beta, 1 \to \infty$. Similarly, for e_2, the sequence $\{TV_{\beta,1}e_2\}_{\beta,1}$ has a convergent subsequence $\{TV_{\beta,2}e_2\}$ converging to a vector ω_2, and likewise, $V_{\beta,2}\Lambda e_2 \to \omega_2$ as $\beta, 2 \to \infty$. By a continuation of this process until the n-th basis vector e_n of $\mathbf{C}^n$ is reached, we conclude that there is a sequence $\{V_\beta\}$ of isometric maps of $\mathbf{C}^n$ into H such that

$$\lim_\beta V_\beta \Lambda e_k = \omega_k = \lim_\beta TV_\beta e_k \quad \text{for all} \quad k = 1, 2, \ldots, n,$$

and such that $0 = \lim_\beta \|TV_\beta - V_\beta\Lambda\|$. Because Λ is invertible, $\{\Lambda e_k\}_{k=1}^n$ is a linear basis for $\mathbb{C}^n$, and so a map $V : C^n \to H$ can be defined on every $\xi = \sum_k \xi_k \Lambda e_k \in \mathbb{C}^n$ by

$$V\xi = \lim_\beta V_\beta \xi = \sum_{k=1}^n \xi_k \omega_k;$$

the operator V is an isometry, since $V_\beta \to V$ strongly. Moreover, for every $\xi \in \mathbb{C}^n$,

$$||TV\xi - V\Lambda\xi|| = \lim_\beta ||TV_\beta\xi - V_\beta\Lambda\xi|| \leqslant$$

$$\leqslant \lim_\beta(||T|| \, ||V\xi - V_\beta\xi|| + ||TV_\beta - V_\beta\Lambda|| \, ||\xi|| + ||V_\beta(\Lambda\xi) - V(\Lambda\xi)||) = 0.$$

Thus, $TV = V\Lambda$.

Now assume T is arbitrary. If $\Lambda \in \Pi_e^n(T)$, then, by [1;4.8], there exist isometries $V_j : \mathbb{C}^n \to H$ with mutually orthogonal ranges such that $\lim_j ||TV_j - V_j\Lambda|| = 0$. Fix $k \in \mathbb{N}$ and, for each $m \in \mathbb{N}$, let $W_m : \mathbb{C}^{kn} \to H$ be the isometry $V_m \oplus \ldots \oplus V_{k+m}$; then for every $\xi = \xi_1 \oplus \ldots \otimes \xi_k \in \mathbb{C}^n \oplus \mathbb{C}^k$,

$$\lim_{m\to\infty} ||TW_m\xi - W_m(\Lambda \otimes 1_k)\xi|| \leqslant \lim_{m\to\infty} \sum_{i=1}^n ||\xi_i|| \, ||(TV_{m+i} - V_{m+i}\Lambda|| \leqslant$$

$$\leqslant \lim_{m\to\infty} \left(\sum_{i=1}^n ||\xi_i||^2\right)^{\frac{1}{2}} \left(\sum_{i=m+1}^{m+n} ||TV_i - V_i\Lambda||^2\right)^{\frac{1}{2}} = 0.$$

Since the domain of the strongly convergent sequence of operators $TW_m - W_m(\Lambda \otimes 1_k)$ (with strong limit 0) is finite-dimensional, the sequence in fact converges uniformly to zero, and so $\Lambda \otimes 1_k \in \Pi_s^{kn}(T)$. The nonemptiness of each $\Pi_s^n(T)$ follows, therefore, from the fact that the left essential spectrum of an operator is nonempty. ■

2. ON THE DEFINITION AND PROPERTIES OF $\sigma_s^\nu(\cdot)$

Let us now extend $\sigma(\cdot)$ itself to a matrix-valued spectrum. The definition of the matricial spectrum arrived at here is precisely the one put forth in [1], where Bunce and Salinas attribute the definition to Arveson. There seems to be no good explanation in the literature for why the proposed definition is a suitable one and so it is hoped that this section fills this gap. In what follows, the definition of the matricial analogue of $\sigma(\cdot)$ is reached by approaching the definition with the intent of establishing a spectral hierarchy.

Since $\sigma(T) = \Pi(T) \cup \Pi(T^*)^*$, a natural suggestion is that the $n \times n$ spatial matricial spectrum be defined by $\Pi_s^n(T) \cup \Pi_s^n(T^*)^*$. In doing so, we find that the definition reproduces T's spectrum when $n = 1$, and for every $\Lambda \in \Pi_s^n(T) \cup \Pi_s^n(T^*)^*$

there is a spectral inclusion $\sigma(\Lambda) \subset \sigma(T)$. Unfortunately, there is no higher spectral inclusion: there are $T \in B(H)$ and $\Lambda \in \Pi_s^k(T) \cup \Pi_s^k(T^*)^*$ such that

$$\Pi_s^k(\Lambda) \cup \Pi_s^k(\Lambda^*)^* \not\subset \Pi_s^k(T) \cup \Pi_s^k(T^*)^*$$

for some $1 < k < n$.

EXAMPLE (CH. DAVIS). *With the operators*

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \Omega = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we have that $\Lambda \in \Pi_s^3(T), \Omega \in \Pi_s^2(\Lambda^*)^*$, *but* $\Omega \notin \Pi_s^2(T) \cup \Pi_s^2(T^*)^*$.

Proof: The assertions $\Lambda \in \Pi_s^3(T)$ and $\Omega \in \Pi_s^2(\Lambda^*)^*$ are obvious. To prove the final assertion, we shall show that every nilpotent in $\Pi_s^2(T) \cup \Pi_s^2(T^*)^*$ has norm $\frac{2}{\sqrt{3}}$ (the norm of Ω is 1).

Suppose that $x, y \in \mathbb{C}^4$ are orthonormal with $x \in \ker T$ and such that x and y span a 2-dimensional T-invariant subspace L on which T acts as

$$\begin{pmatrix} 0 & (Ty, x) \\ 0 & 0 \end{pmatrix}.$$

Since $\ker T$ is 1-dimemsional, the eigenvector $x \in L$ corresponding to the eigenvalue 0 must be a multiple of $\sqrt{\frac{1}{2}}(e_1 - e_2)$. Since y is orthogonal to x, and because $T^2 y = 0$ (since $y \in L$), a short computation shows that y must be a multiple of $\sqrt{\frac{1}{6}}(e_1 + e_2) - \sqrt{\frac{4}{6}} e_3$. Thus, the norm of each nilpotent in $\Pi_s^2(T)$ is $|(Ty, x)| = \frac{2}{\sqrt{3}} \neq 1$.

If $\Gamma \in \Pi_s^2(T^*)^*$ is nilpotent, then $\|\Gamma\| = \frac{2}{\sqrt{3}}$ as well, since Γ^* is a nilpotent of $\Pi_s^2(T)$. Hence, Ω cannot be an element of $\Pi_s^2(T) \cup \Pi_s^2(T^*)^*$. ∎

In the example above, for Ω to be a 2×2 spectral element of T it is necesary to define $\sigma_s^n(\cdot)$ so that it is large enough to include compressions of T to n-dimensional semi-invariant subspaces. Recall that a subspace L of H is semi-invariant for $T \in B(H)$ if H admits a decomposition $H = M \oplus L \oplus N$, where the subspaces M and $M \oplus L$ are T-invariant. This concept was introduced by Sarason in [15], where he shows that semi-invariant subspaces are naturally associated with power dilations; his fundamental lemma is stated below.

SARASON'S LEMMA. *A subspace L is semi-invariant for $T \in B(H)$ if and only if $PT^m_{|L} = (PT_{|L})^m$ for every $m \in \mathbf{N}$, where P is the projection of H onto L.*

In this description, T has an operator matrix with respect to $H = M \oplus L \oplus N$ of the form

$$\begin{pmatrix} * & * & * \\ 0 & PT_{|L} & * \\ 0 & 0 & * \end{pmatrix},$$

if and only if L is semi-invariant for T.

Therefore, if for $T \in B(H)$ a matrix-valued spectrum is defined to be the set of all $\Lambda \in M_n$ such that there exist isometries $V_j : \mathbf{C}^n \to H$ satisfying for every $m \in \mathbf{N}$ the equation $0 = \lim_j \|V_j^* T^m V_j - \Lambda^m\|$, then elements such as Ω in the above example will appear as matricial spectral elements. However, a definition such as this does not always reproduce the (ordinary) spectrum of an operator. For example, for the bilateral shift B on $l_2(\mathbf{Z})$, the point 0 would appear in a spectrum defined in this way since every standard orthonormal basis vector for $l_2(\mathbf{Z})$ spans a 1-dimensional semi-invariant subspace for B on which the compression of B to it is 0. This problem is easily accounted for and remedied as follows. Note that this definition of a matrix-valued spectrum involves only powers of the operator T; that is, this spectrum depends only upon the unital commutative Banach algebra generated by T. But the spectrum of an element T in a unital (Banach) subalgebra $A \subset B(H)$ is assured to agree with the spectrum of T as an element of $B(H)$ only if A is a full algebra (i.e., A is full in the sense that if an element of $X \in A$ is invertible in $B(H)$, then $X^{-1} \in A$). Therefore, in order to have agreement with $\sigma(T)$ in the case $n = 1$, the definition of $\sigma_s^n(T)$ should involve the least full unital commutative Banach algebra generated by T: namely, the closure of the algebra of rational functions in T. For a compact set $X \subset \mathbf{C}$, let $\mathrm{Rat}(X)$ denote the algebra of complex rational functions with poles off X. By the rational functional calculus, $r(T)$ is a well-defined bounded operator on H for every $r \in \mathrm{Rat}\sigma(T)$, provided that $\sigma(T) \subset X$.

Let $\mathcal{R}(T)$ and $\mathcal{R}_e(T)$ denote, respectively, the algebras of operators of the form $r(T)$ and $r_e(T)$, where $r \in \mathrm{Rat}(\sigma(T))$ and $r_e \in \mathrm{Rat}(\sigma_e(T))$.

At this time we consider as well the spectrum that arises by allowing operators on $l_2(\mathbf{N})$ to be spectral values.

DEFINITION. Suppose that $T \in B(H)$ and $1 \leqslant \nu \leqslant \infty$.

(1) $\Lambda \in \sigma_s^\nu(T)$ if there exist isometries $V_j : \mathbf{C}^\nu \to H$ satisfying $\lim_j \|V_j^* r(T) V_j - r(\Lambda)\| = 0$ for every $r \in \mathrm{Rat}\sigma(T)$.

(2) $\Lambda \in \sigma^\nu(T)$ if $\Lambda = \varphi(T)$ for some unital completely positive map $\varphi : C^*(T) \to B(\mathbf{C}^\nu)$ that is multiplicative on the algebra $\mathcal{R}(T)$.

(3) $\Lambda \in \sigma_e^\nu(T)$ if $\Lambda = \varphi(T)$ for some unital completely positive map $\varphi : C^*(T) \to$ $\to B(\mathbb{C}^\nu)$ that is multiplicative on the algebra $\mathcal{R}_e(T)$ and satisfies $\varphi(K) = 0$ whenever K is a compact operator in $C^*(T)$.

REMARK. The definition of $\sigma_e^\nu(T)$ given above differs from one put forth by Bunce and Salinas in that they require φ to be multiplicative only on the (smaller) algebra $\mathcal{R}(T)$, which leads to a larger essential matricial spectrum. Indeed, the referee has pointed out that the 1×1 essential matricial spectrum of Bunce and Salinas produces the closed unit disc in the case of the unilateral shift operator, when the usual essential spectrum produces only the unit circle. With the definition given in (3) above, $\sigma_e^1(T)$ and $\sigma_e(T)$ do coincide for all operators T.

REMARK. Operator-valued versions of $R_s^n(\cdot)$ and $\Pi_s^n(\cdot)$ can be (and have been) defined by enlarging the domain of the isometries to include $\mathbb{C}^\nu$ for every $1 \leqslant \nu \leqslant \infty$.

THEOREM 2.1. *Suppose that $T \in B(H)$, $n \in \mathbb{N}$, and $1 \leqslant \nu \leqslant \infty$.*
(1) *$\sigma_s^n(T)$ is compact, non-empty, and contains $\Pi_s^n(T) \cup \Pi_s^n(T^*)^*$.*
(2) *$\sigma_s^1(T) = \sigma(T)$ and $\sigma_e^1(T) = \sigma_e(T)$.*
(3) *(Spectral hierarchy) $\sigma_s^\mu(\Lambda) \subset \sigma_s^\mu(\Lambda)$ for all $1 \leqslant \mu \leqslant \nu$ and all $\Lambda \in \sigma_s^\nu(T)$.*
(4) *(Invariance) $\sigma_s^\nu(S) = \sigma_s^\nu(T)$ whenever S is approximately equivalent to T.*
(5) *$\sigma_e^\nu(T) \subset \sigma_s^\nu(T) \subset \sigma^\nu(T)$.*

Proof: (1) If $\Lambda \in \Pi_s^n(T)$, then there is a sequence of isometries $V_j : \mathbb{C}^n \to H$ with $0 = \lim_j \|TV_j - V_j\Lambda\|$. The maps $\varphi_j(\cdot) = V_j^*(\cdot)V_j$ on $C^*(T)$ are unital completely positive, and so by the sequential BW-compactness of $\mathrm{CP}^n(T)$, $\{\varphi_j\}$ has a BW-convergent subsequence $\{\varphi_\beta\}$ converging to $\varphi \in \mathrm{CP}^n(T)$. Let $\varphi = V^*\pi V$ be the Stinespring decomposition of φ; then $\Lambda = \varphi(T)$ and $V\Lambda = \pi(T)V$ (because Λ is a spectral element). Now suppose that $r \in \mathrm{Rat}\sigma(T)$. Because $r(\Lambda)$ is completely determined by the values of $r, r', \ldots, r^{n-1}$ on the finite set $\sigma(\Lambda)$, any polynomial p which satisfies $p^{(k)}(\zeta) = r^{(k)}(\zeta)$ for all $1 \leqslant k \leqslant n-1$ and for all $\zeta \in \sigma(\Lambda)$ (such as the Lagrange- -Sylvester interpolating polynomial) necessarily satisfies $p(\Lambda) = r(\Lambda)$. Thus,

$$\begin{aligned} r(\Lambda) = p(\Lambda) &= \varphi(p(T)) \\ &= V^* r(\pi(T))V \qquad \text{(because } V(\mathbb{C}^n) \text{ has finite dimension)} \\ &= \varphi(r(T)) = \lim_{\beta\to\infty} V_\beta^* r(T) V_\beta \,, \end{aligned}$$

and so, $\Lambda \in \sigma_s^n(T)$. Similarly, $\Pi_s^n(T^*)^* \subset \sigma_s^n(T)$. By Theorem 1.2, this shows that $\sigma_s^n(T)$ is non-empty.

We now prove that $\sigma_s^n(T)$ is compact. Consider the BW-closure $\mathcal{V}$ of the set of spatial completely positive maps $\varphi_V \in \mathrm{CP}^n(T)$. (Here, V is an isometry $V : \mathbb{C}^n \to H$

and $\varphi_V(A) = V^*AV$ for every $A \in C^*(T)$.) Now intersect $\mathcal{V}$ with the (BW-closed) set of those $\varphi \in \mathrm{CP}^n(T)$ which are multiplicative on the algebra generated by all rational functions in T; the resulting set $\mathcal{R}_T$ is BW-compact and every $\psi \in \mathcal{R}_T$ induces an element $\psi(T) \in \sigma_s^n(T)$. Conversely, if $\Lambda \in \sigma_s^n(T)$, then there is a sequence of spatial completely positive maps $\varphi_k(\cdot) = V_k^*(\cdot)V_k$ such that $\lim_k \varphi_k(r(T)) = r(\Lambda)$ for every $r \in \mathrm{Rat}\sigma(T)$. Because $\mathcal{R}_T$ is sequentially compact in the BW-topology, this sequence of spatial maps has a BW-convergent subsequence converging to a map $\psi \in \mathcal{R}_T$; hence, $\psi(r(T)) = r(\Lambda)$ for every $r \in \mathrm{Rat}\,\sigma(T)$. This shows that $\sigma_s^n(T)$ is the image of the compact space $\mathcal{R}_T$ through the continuous map $\psi \mapsto \psi(T)$, and therefore, $\sigma_s^n(T)$ is compact.

(2) By (1), $\sigma(T) = \Pi(T) \cup \Pi(T^*)^* \subset \sigma_s^1(T) \subset \sigma(T)$. By definition, $\sigma_e^\nu(T)$ is the spectrum $\sigma^\nu(\tilde{T})$ of the image $\tilde{T}$ of T in the Calkin algebra. Hence, $\sigma_e^1(T) = \sigma^1(\tilde{T}) = \sigma_e(T)$.

(3) Suppose that $\Lambda \in \sigma_s^\nu(T), \mu \leqslant \nu$, and $\Omega \in \sigma_s^\mu(\Lambda)$. Thus, there are sequences of isometries $\{V_j\}, \{W_j\}$ such that $\|V_j^* r(T)V_j - r(\Lambda)\| \to 0$ for all $r \in \mathrm{Rat}\sigma(T)$ and $\|W_j^* f(\Lambda)W_j - f(\Omega)\| \to 0$ for all $f \in \mathrm{Rat}\sigma(\Lambda)$. Because V_jW_j is isometric for all j and because $\mathrm{Rat}\sigma(T) \subset \mathrm{Rat}\sigma(\Lambda) \subset \mathrm{Rat}\sigma(\Omega)$, it follows from

$$\|W_j^*V_j^*r(T)V_jW_j - r(\Omega)\| \leqslant \|W_j^*(V_j^*r(T)V_j - r(\Lambda))W_j\| + \|W_j^*r(\Lambda)W_j - r(\Omega)\|$$

that $\|W_j^*V_j^*r(T)V_jW_j - r(\Omega)\| \to 0$ for every $r \in \mathrm{Rat}\sigma(T)$, whence $\Omega \in \sigma_s^\mu(T)$.

(4) We are to prove that $S \sim_a T$ implies $\sigma_s^\nu(S) = \sigma_s^\nu(T)$. It is straightforward that $\Pi_s^\nu(S) = \Pi_s^\nu(T)$ and $\Pi_s^\nu(S^*)^* = \Pi_s^\nu(T^*)^*$ for every ν, from which one conclusion is that $\mathrm{Rat}\sigma(S) = \mathrm{Rat}\sigma(T)$. In addition, by approximate equivalence once again, there is a sequence of unitaries U_j such that $\|U_j^*r(T)U_j - r(S)\| \to 0$ for every $r \in \mathrm{Rat}(X)$, where $X = \sigma(T) = \sigma(S)$. If $\Lambda \in \sigma_s^\nu(T)$ with $\|V_j^*r(T)V_j - r(\Lambda)\| \to 0$ for all $r \in \mathrm{Rat}(X)$ and for some fixed sequence of isometries V_j, then

$$\|V_j^*U_j^*r(S)U_jV_j - r(\Lambda)\| \leqslant \|U_j^*r(S)U_j - r(T)\| + \|V_j^*r(T)V_j - r(\Lambda)\|$$

for all j and every $r \in \mathrm{Rat}(X)$ implies that $\Lambda \in \sigma_s^\nu(S)$. The proof of the reverse inclusion is analogous.

(5) The non-trivial inclusion is $\sigma_e^\nu(T) \subset \sigma_s^\nu(T)$. This is proved in [1;5.3] when $\nu = n \in \mathbb{N}$ and in [6;2.11(b)] when $\nu = \infty$. ■

Part (1) of the above Theorem seems to have been anticipated by Hadwin in [6]; in fact, he shows there that the assertion is false if n is replaced by $\nu = \infty$. Also in [6], one find descriptions of the reducing and the left matrix- and operator-valued spatial spectra of T in terms of certain operators approximately equivalent to T. Items (3) and (4) of the following theorem complete the picture; the proof is based on the same methods used by Hadwin in [6].

THEOREM 2.2. *Suppose that $T \in B(H)$ and that $1 \leqslant \nu \leqslant \infty$.*

(1) *$\Lambda \in R_s^\nu(T)$ if and only if there is an operator S approximately equivalent to T and an isometry $V : \mathbb{C}^\nu \to H$ such that $SV = V\Lambda$ and $S^*V = V\Lambda^*$.*

(2) *$\Lambda \in \Pi_s^\nu(T)$ if and only if there is an operator S approximately equivalent to T and an isometry $V : \mathbb{C}^\nu \to H$ such that $SV = V\Lambda$.*

(3) *$\Lambda \in \sigma_s^\nu(T)$ if and only if there is an operator S approximately equivalent to T and an isometry $V : \mathbb{C}^\nu \to H$ such that $r(\Lambda) = V^*r(S)V$ for every $r \in \mathrm{Rat}\,\sigma(T)$; that is, if and only if Λ has a rational dilation to an operator approximately equivalent to T.*

(4) *If $\Lambda \in \sigma_e^\nu(T)$, then there exists an operator A on a separable space L and an isometry $V : \mathbb{C}^\nu \to L$ such that $T \sim_a T \oplus A$ and $r(\Lambda) = V^*r(A)V$ for every $r \in Rat\sigma_e(T)$.*

Proof: Items (1) and (2) are in Hadwin's papers [6]. For (3) suppose that $\Lambda \in \sigma_s^\nu(T)$ and that V_j are isometries yielding $\|V_j^* r(T)V_j - r(\Lambda)\| \to 0$ for all $r \in \mathrm{Rat}\sigma(T)$. Let $\varphi(\cdot) = V_j^*(\cdot)V_j$ for each j. The space $\mathrm{CP}(C^*(T), \mathbb{C}^\nu; 1)$ of unital completely positive maps $C^*(T) \to \mathcal{B}(\mathbb{C}^\nu)$ is BW-compact and, therefore, $\{\varphi_j\}$ has a convergent subnet $\{\varphi_\beta\}$ converging to some $\varphi \in \mathrm{CP}(C^*(T), \mathbb{C}^\nu; 1)$. This is to say that φ is an approximate compression, in the language of [6], of the identity representation *id* of $C^*(T)$. Hence, by [6;2.4], there exists a representation ρ of $C^*(T)$ approximately eqivalent to *id* and such that φ is a compresion of ρ; that is, $\varphi = W^*\rho W$ for some isometry W. If we let $S = \rho(T)$, then $S \sim_a id(T) = T$ and for every $r \in \mathrm{Rat}\sigma(T)$, $\varphi(r(T)) = W^*\rho(r(T))W = W^*r(S)W$. But because $\varphi_\beta(r(T)) \to \varphi(r(T))$ weakly and $\varphi_\beta(r(T)) \to r(\Lambda)$ in norm, we have that $\varphi(r(T)) = r(\Lambda)$; that is, $W^*r(S)W = r(\Lambda)$ as desired. The converse follows from the invariance of the spectra under approximate equivalence.

For (4), suppose that $\Lambda \in \sigma_e^\nu(T)$ arises from a completely positive map φ with canonical decomposition $\varphi = V^*\pi V$ and satisfying the condition that it map $\mathcal{R}_e(T)$ multiplicatively and that it annihilates the compact operators of $C^*(T)$. By [6;2.3], π annihilates the compacts in $C^*(T)$ and so from Voiculescu's theorem it follows that $T \sim_a T \oplus \pi(T)$. Let $A = \pi(T)$ and $L = H_\pi$ and the result follows immediately. ∎

The next result shows that the spectra are uneffected by the passage from the algebra of rational functions in T to the larger algebra of holomorphic functions in T. However, the passage from the algebra in rational functions in T to the C^*-algebra generated by T can affect the matricial spectrum, even though the numerical spectrum is not altered. Example: if N is normal with spectrum the closed unit disc, then the set of matrices $\varphi(N)$ that arise from those $\varphi \in \mathrm{CP}^n(N)$ that are homomorphisms on $\mathcal{R}(N)$ is precisely the closed ball of M_n [6;p.227], whereas the matrices $\rho(N)$ that arises from those $\rho \in \mathrm{CP}^n(N)$ that are homomorphisms on $C^*(N)$, a C^*-algebra containing $\mathcal{R}(N)$, are just the normal contractions in M_n.

If $G \subset \mathbb{C}$ is an open set, then $H(G)$ is to denote the algebra of holomorphic functions on G. Via the holomorphic functional calculus, we consider the commutative algebra $\mathcal{A}_G(T) = \{f(T) : f \in H(G)\}$ whenever $\sigma(T) \subset G$. Of course, $\mathcal{A}_G(T)$ contains the algebra $\mathcal{R}(T)$ of rational functions in T.

THEOREM 2.3. *Suppose that $T \in B(H)$, $G \subset \mathbb{C}$ is an open set containing $\sigma(T)$, and $1 \leqslant \nu \leqslant \infty$.*

(1) *$\sigma^\nu(T)$ is the set of all $\varphi(T)$ that arise from those unital completely positive maps $\varphi : C^*(T) \to B(\mathbb{C}^\nu)$ that are homomorphisms when restricted to $\mathcal{A}_G(T)$.*

(2) *$\sigma_s^\nu(T)$ consists of all operators $\Lambda \in B(\mathbb{C}^\nu)$ for which there exist isometries $V_k : \mathbb{C}^\nu \to H$ satisfying for every $f \in H(G)$ the equation $0 = \lim_k \|V_k^* f(T) V_k - f(\Lambda)\|$.*

Proof: Only the non-trivial inclusions will be proven.

(1) Suppose that $\varphi(T) \in \sigma^\nu(T)$; here, $\varphi|_{\mathcal{R}(T)}$ is a homomorphism. Assume that $f \in H(G)$, and let $\Gamma \subset G$ be closed simple piecewise differential contour in G containing $\sigma(T)$ in its interior. By the Cauchy formula,

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\zeta)(\zeta 1 - T)^{-1} \mathrm{d}\zeta.$$

For each $\zeta \in \Gamma$, the rational function $r_\zeta(z) = (\zeta - z)^{-1}$ is an element of $\mathrm{Rat}\sigma(T)$; hence, $\varphi(r(T)) = r_\zeta(\varphi(T))$ for every $\zeta \in \Gamma$ and thus

$$\varphi(f(T)) = \varphi\left(\frac{1}{2\pi i}\int_\Gamma f(\zeta) r_\zeta(T)\mathrm{d}\zeta\right) = \frac{1}{2\pi i}\int_\Gamma f(\zeta)\varphi(r_\zeta(T))\mathrm{d}\zeta =$$

$$= \frac{1}{2\pi i}\int_\Gamma f(\zeta) r_\zeta(\varphi(T))\mathrm{d}\zeta = \frac{1}{2\pi i}\int_\Gamma f(\zeta)(\zeta 1 - \varphi(T))^{-1}\mathrm{d}\zeta = f(\varphi(T)).$$

Therefore, by the holomorphic functional calculus, the restriction of φ to $\mathcal{A}_G(T)$ is a homomorphism.

(2) For $\Lambda \in \sigma_s^\nu(T)$, suppose that the V_k are isometries satisying, for every $r \in \mathrm{Rat}\sigma(T)$, $\lim_k \|v_k^* r(T) V_k - r(\Lambda)\| = 0$. For each $k \in \mathbb{N}$, define $\omega_k : \Gamma \to [0,\infty)$ by $\omega_k(\zeta) = \|V_k^* r_\zeta(T) V_k - r_\zeta(\Lambda)\|$; by hypothesis, $\lim_k \omega_k(\zeta) = 0$ for every $\zeta \in \Gamma$. Moreover, these continuous functions are uniformly bounded: if $M_T = \max\{\|r_\zeta(T)\| : \zeta \in \Gamma\}$ and $M_\Lambda = \max\{\|r_\zeta(\Lambda)\| : \zeta \in \Gamma\}$, then $0 \leqslant \omega_k(\zeta) \leqslant (M_T + M_\Lambda)$ for every $\zeta \in \Gamma$. Therefore, by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_\Gamma \omega_k(\zeta)|\mathrm{d}\zeta| = 0.$$

Let $f \in H(G)$ be the function described in (i); if C denotes the maximum of all $|f(\zeta)|$ as ζ varies through Γ, then

$$\lim_{k\to\infty} \|V_k^* f(T)V_k - f(\Lambda)\| \leqslant \frac{1}{2\pi} \lim_{k\to\infty} \int_\Gamma |f(\zeta)| \, \|V_k^* r_\zeta(T)V_k - r_\zeta(\Lambda)\| \, |d\zeta| \leqslant$$

$$\leqslant \frac{C}{2\pi} \lim_{k\to\infty} \int_\Gamma |\omega_k(\zeta)| \, |d\zeta| = 0.$$

Since our choice of $f \in H(G)$ was arbitrary, the assertion is now proved. ∎

3. THEOREMS ON ORTHOGONAL ADDITION AND UPPER-SEMICONTINUITY

This section is devoted to the proofs of two theorems. The first has to do with the addition of spatial spectral elements using direct sums; the second concerns the upper-semicontinuity of the spatial matricial spectrum. Some care must be taken in these theorems to ensure that the elements eventually constructed are spatial, not just spectral.

We begin with the addition theorem. Bunce and Salinas note in [1;p.773] that a general addition theorem for elements of the spatial matricial spectrum cannot hold because the orthogonal direct sum of two semi-invariant subspaces need not once again be semi-invariant. Nevertheless, essential elements can be added along with a single spatial element.

THEOREM 3.1. *Suppose that $T \in B(H)$, $\Omega_0 \in \Pi_s^{\nu_0}(T)$, $\Lambda_0 \in \sigma_s^{\nu_0}(T)$, and that $\nu_0 + \nu_1 + \cdots + \nu_k = \nu$ (cardinal arithmetic).*

(1) If $\Omega_i \in \Pi_e^{\nu_i}(T)$ for $1 \leqslant i \leqslant k$, then $\Omega_0 \oplus \Omega_1 \oplus \cdots \oplus \Omega_k \in \Pi_s^{\nu}(T)$.

(2) If $\Lambda_i \in \sigma_e^{\nu_i}(T)$ for $1 \leqslant i \leqslant k$, then $\Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_k \in \sigma_s^{\nu}(T)$.

Proof: Only (2) will be proved; the proof of (1) is similar and easier. By hypothesis, for $1 \leqslant i \leqslant k$ there exist unital completely positive maps $\varphi_i : C^*(T) \to \to B(\mathbb{C}^{\nu_i})$ such that the restriction of each φ_i to the algebra $\mathcal{R}_e(T)$ is a homomorphism, $\Lambda_i = \varphi_i(T)$, and $\varphi_i(K) = 0$ whenever $K \in C^*(T)$ is compact. Let $\mu = \nu_1 + \cdots + \nu_k$. The map $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_k$ of $C^*(T) \to B(\mathbb{C}^\mu)$ has the necessary properties to place $\varphi(T) = \Lambda_1 \oplus \cdots \oplus \Lambda_k$ in $\sigma_e^\mu(T)$. Therefore, to complete the proof of the theorem we will show that if $\Lambda \in \sigma_s^{\nu_0}(T)$ and $\Omega \in \sigma_e^\mu(T)$, then $\Lambda \oplus \Omega \in \sigma_s^\nu(T)$, where $\nu = \nu_0 + \mu$.

By replacing T with an appropriate operator $S \sim_a T$, we may assume that there is a single isometry $V : \mathbb{C}^{\nu_0} \to H$ satisfying $V^* r(T) V = r(\Lambda)$ for every $r \in \operatorname{Rat}\sigma(T)$ (Theorem 2.2). Because $\Omega \in \sigma_e^\mu(T)$, there is, by Theorem 2.2, an operator A on a Hilbert space L and an isometry $W : \mathbb{C}^\mu \to L$ satisfying $r(\Omega) = W^* r(A) W$ for every

$r \in \mathrm{Rat}\sigma_e(T)$ and $T \sim_a T \oplus A$. Using the isometry $V \oplus W$, it is readily seen that $\Lambda \oplus \Omega \in \sigma_s^\nu(T \oplus A) = \sigma_s^\nu(T)$. ∎

The second theorem to be proved in this section is stated below.

THEOREM 3.2. *Suppose that $T \in B(H)$. To each open subset $\mathcal{U} \subset M_n$ containing $\sigma_s^n(T)$ there corresponds an $\varepsilon > 0$ such that $\sigma_s^n(S) \subset \mathcal{U}$ whenever $\|S - T\| < \varepsilon$. That is, the set-valued function $\sigma_s^n(\cdot)$ is upper-semicontinuous.*

The theorem is proved by considering first its sequential statement: if $T_j \to T$, if $\Lambda_j \to \Lambda$, and if $\Lambda_j \in \sigma_s^n(T_j)$ for each j, then $\Lambda \in \sigma_s^n(T)$. There is no problem in showing that $\Lambda \in \sigma^n(T)$ (see Lemma 3.5); again, the main difficulty lies in showing that Λ is obtained spatially. The proof of Theorem 3.2 follows the line of reasoning suggested by work of Bunce and Salinas in Section 4 of [1].

LEMMA 3.3 ([4;3.5.6]). *Suppose that $\varepsilon > 0$ and that $\xi_1, \ldots, \xi_n \in H$. There exists $\varepsilon' > 0$ having the following property: whenever $\eta_1, \ldots, \eta_n \in H$ satisfy $|(\eta_i, \eta_j) - (\xi_i, \xi_j)| \leqslant \varepsilon'$ for every $1 \leqslant i, j \leqslant n$, then there is a unitary operator U such that $\|U\eta_i - \xi_i\| \leqslant \varepsilon$ for each $1 \leqslant i \leqslant n$.*

LEMMA 3.4 (CF. [1;4.7]). *Suppose that $\varphi = V^*\pi V$ is the Stinespring decomposition of some unital completely positive map of $C^*(T)$ into M_n. If the restriction of φ to the algebra of rational functions in T is a homomorphism and is in the BW-closure of all maps of the form $X \mapsto W^*XW$, where W is an isometry and $X \in C^*(T)$, then there exists a sequence of isometries $W_k : \mathbb{C}^n \to H$ such that $\|W_k^* r(T) W_k - r(\varphi(T))\| \to 0$ for every rational function r with poles off of $\sigma(T)$.*

Proof: There is no loss in generality in assuming that the space H_π on which $\pi(C^*(T))$ acts is a subspace of H. Let $\{r_i\}_i$ be a dense subset of $\mathrm{Rat}\sigma(T)$ with $r_1(z) = 1$, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Select k; we will construct an isometry W_k with the property that $\|W_k^* r_i(T) W_k e_l - r(\varphi(T))e_l\| \leqslant (\|r_i(T)\| + 1)$ for every $1 \leqslant i \leqslant k$ and every $1 \leqslant l \leqslant n$. With $\varepsilon = k^{-1}$ and the nk vectors $r_i(T)Ve_j$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, Lemma 3.3 provides us with an $\varepsilon' > 0$ having the properties stated there. Using this ε', the assumption on φ is that there exists an isometry $V_0 : \mathbb{C}^n \to H$ such that $\|\varphi(r_i(T)) - V_0^* r_i(T) V_0\| < \varepsilon'$ for all $1 \leqslant i \leqslant k$. Because $\varphi(r_i(T)) = V^*\pi(r_i(T))V$, the inequality of the preceding sentence implies that

$$|(\pi(r_i(T))Ve_l, Ve_m) - (r_i(T)V_0e_l, V_0e_m)| < \varepsilon'$$

for all $1 \leqslant i \leqslant k$ and $1 \leqslant l, m \leqslant n$. Therefore, by Lemma 3.3, there is a unitary operator U_k such that for all $1 \leqslant i \leqslant k$, $1 \leqslant l \leqslant n$,

$$\|U_k r_i(T) V_o e_l - \pi(r_i(T))Ve_l\| \leqslant \frac{1}{k}.$$

In particular, it follows from $r_1(z) = 1$ that $\|U_k V_0 e_l - V e_l\| < k^{-1}$ for each l. Define $W_k = U_k^* V$. For every $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant n$,

$$\|W_k^* r_i(T) W_k e_l - r(\varphi(T)) e_l\| \leqslant$$

$$\leqslant \|V^*\| \, \|U_k r_i(T) U_k^* V e_l - \pi(r_i(T)) V e_l\| \leqslant$$

$$\leqslant \|U_k r_i(T) U_k^* V e_l - U_k r_i(T) V_0 e_l\| + \|U_k r_i(T) V_0 e_l - \pi(r_i(T)) V e_l\| \leqslant$$

$$\leqslant \frac{1}{k}(\|r_i(T)\| + 1).$$

In M_n, the strong and the norm topologies coincide and so the above inequality proves that $\lim_k \|W_k^* r_i(T) W_k - \varphi(r_i(T))\| = 0$ for each i. Because $r_1, r_2, \ldots,$ are dense in $\mathrm{Rat}\sigma(T)$, the assertion is proved. ∎

LEMMA 3.5. *Suppose that $T, T_i \in B(H)$ with $T_i \to T$, and suppose that φ_i is a unital completely positive map of $B(H)$ into M_n that is a homomorphism when restricted to the algebra of rational functions in T_i. Suppose further that $\varphi_i(T_i) \to$ $\to A$. Then there is a unital completely positive map $\varphi : C^*(T) \to M_n$ that is a homomorphism on the algebra of rational functions in T and is such that $A = \varphi(T)$.*

Proof: The hypothesis is that $A_i = \varphi_i(T_i) \in \sigma^n(T_i)$ for each i; because $\sigma(\cdot)$ is an upper-semicontinuous function, it follows via the spectral hierarchy that $\sigma(A) \subset \sigma(T)$. If $r \in \mathrm{Rat}\sigma(T)$ is given, then there is an open set $\mathcal{V} \subset \mathbb{C}$ containing $\sigma(T)$ that excludes the (finitely many) singularities of r, and by the upper-semicontinuity of the spectrum there is an integer N such that $r(T_i)$ is defined whenever $i \geqslant N$. Therefore, when we refer to $\lim_i r(T_i)$, we mean that we begin with the sequence $r(T_N), r(T_{N+1}), \ldots,$ and similarly for $\lim_i r(A_i)$.

The map $\psi : r(T) \mapsto r(A)$ is a well-defined homomorphism. Moreover, this map is a complete contraction, for if $[r_{i,j}(T)]_{i,j=1}^k \in \mathcal{R}(T) \otimes M_k$, then

$$\|[r_{i,j}(A)]\| = \lim_m \|[r_{i,j}(A_m)]\| = \lim_m \|[r_{i,j}(\varphi_m(T_m))]\| =$$

$$= \lim_m \|[\varphi_m(r_{i,j}(T_m))]\| \leqslant \lim_m \|[r_{i,j}(T_m)]\| = \|[r_{i,j}(T)]\|.$$

(Observe that we have used that each φ_m is completely contracitve.) Thus, the map $\Psi : \mathcal{R}(T) + \mathcal{R}(T)^* \to M_n$ sending $A + B^*$ to $\psi(A) + \psi(B)^*$ is completely positive [10;3.4] and extends, therefore, to a completely positive map $\varphi : C^*(T) \to M_n$ that is plainly multiplicative on $\mathcal{R}(T)$. ∎

LEMMA 3.6 *Suppose that $T, T_i \in B(H)$, $A_i \in \sigma_s^n(T_i)$, and that $T_i \to T$, $A_i \to A$. Then $A \in \sigma_s^n(T)$.*

Proof: For each i there exist a sequence $\{\varphi_j^i\}_j$ of unital spatial maps $\varphi_j^i(\cdot) = (V_j^i)^*(\cdot)(V_j^i)$ on the separable C^*-algebra generated by $\{T, T_i\}_i$ such that

$$r(\Lambda_i) = \lim_j \varphi_j^i(r(T_i)) \text{ for every } r \in \mathrm{Rat}\sigma(T_i).$$

For each i, let φ_i be the limit of some BW-convergent subsequence of $\{\varphi_j^i\}_j$. By hypothesis,

$$\varphi_i(r(T_i)) = r(\Lambda_i) \to r(\Lambda) = \varphi(r(T)) \text{ as } i \to \infty,$$

for every $r \in \mathrm{Rat}\sigma(T)$, where the meaning of $\varphi_i(r(T_i))$ and the definition of φ are taken as in Lemma 3.5. Thus, $\varphi|_{\mathcal{R}(T)}$ is a homomorphism and is in the BW-closure of $\{\varphi_i|_{\mathcal{R}(T)}\}$, and so $\varphi|_{\mathcal{R}(T)}$ is approximated by the maps $\{\varphi_j^i|_{\mathcal{R}(T)}\}_{i,j}$ also. Lemma 3.4 asserts that there is a sequence of isometries W_k such that $\varphi(r(T)) = \lim_k W_k^* r(T) W_k$ for every $r \in \mathrm{Rat}\sigma(T)$. Hence, $\Lambda \in \sigma_s^n(T)$. ■

Proof of Theorem 3.2: If the assertion of the theorem is false, then there exists a sequence $\{T_j\}$ in $B(H)$ converging to T but such that $\sigma_s^n(T_j) \not\subset \mathcal{U}$ for every j. So, by choosing $\Lambda_j \in \sigma_s^n(T_j) \backslash \mathcal{U}$ and selecting any limit point Λ of the sequence $\{\Lambda_j\}$, we have that both $\Lambda \notin \mathcal{U}$ and, by Lemma 3.6, $\Lambda \in \sigma_s^n(T) \subset \mathcal{U}$, which is an absurdity. The assertion of the theorem is, therefore, true. ■

4. COMPUTING THE MATRICIAL SPECTRUM: SOME EXAMPLES

Having spent some effort investigating properties of the matricial spectrum, it is appropriate to turn now to some examples. A natural question one could ask is "what is the spatial matricial spectrum of a normal operator?". In this section some computations are made for a number of classes of operators.

An approximate eigenvalue λ of an operator T is said to be normal if $\|(T - \lambda 1)^* x_j\| \to 0$ whenever $\|(T - \lambda 1) x_j\| \to 0$. It is clear that every spectral point of a normal operator, or more generally every approximate eigenvalue of a hyponormal operator, is a normal approximate eigenvalue. As well, it follows from the work of Hildebrandt [8; Satz2(ii)] that if an operator has its approximate point of spectrum contained entirely on the boundary of its numerical range, then each of its approximate eigenvalues is normal. The spatial component of the left matricial spectra of these operators is described in full bellow.

EXAMPLE 1. *If every approximate eigenvalue of T is normal, then*

(1) $\Pi_s^n(T) = R_s^n(T)$ *for every n and every matrix in $\Pi_s^n(T)$ is a normal matrix. If a sequence of isometric maps $V_j : \mathbb{C}^n \to H$ satisfies $\|TV_j - V_j\Lambda\| \to 0$ for some $\Lambda \in M_n$, then $\|T^* V_j - V_j \Lambda^*\| \to 0$ as well.*

(2) *If $\lambda_1, \ldots, \lambda_q$ are isolated eigenvalues of finite multiplicity $\nu_1, \ldots, \nu_q$, and if $\lambda_{q+1}, \ldots, \lambda_p \in \Pi_e(T)$, then*

$$\lambda_1 1_{\mu_1} \oplus \cdots \oplus \lambda_p 1_{\mu_p} \in \Pi_s^n(T),$$

where $1 \leqslant \mu_j \leqslant \nu_j$ for $j = 1, \ldots, q$ and $\sum_{j=1}^{p} \mu_j = n$. Conversely, every matrix $\Lambda \in \Pi_s^n(T)$ is unitarily equivalent to a matrix constructed in this way.

Proof: (1) (a) Assume that $\Lambda \in \Pi_s^n(T)$ and $\|TV_j - V_j\Lambda\| \to 0$ for some isometries $V_j : \mathbb{C}^n \to H$. If $\|T^*V_j - V_j\Lambda^*\| \to 0$ as well, then plainly $\Pi_s^n(T) = R_s^n(T)$. We begin by proving that Λ must be normal.

Since Λ acts in finite dimensions, it is normal if and only if $\Lambda^* z = \lambda^* z$ whenever $\Lambda z = \lambda z$. Assume that latter equality holds for a unit vector z and an eigenvalue λ. For every j,

$$\begin{aligned} \|\lambda^* z - \Lambda^* z\| &= \|\lambda^* V_j^* V_j z - \Lambda^* V_j^* V_j z\| \leqslant \\ &\leqslant \|\lambda^* V_j^* V_j z - V_j^* T^* V_j z\| + \|V_j^* T^* V_j z - \Lambda^* V_j^* V_j z\| \leqslant \\ &\leqslant \|V_j^*\| \, \|\lambda^* V_j z - T^* V_j z\| + \|V_j z\| \, \|V_j^* T^* - \Lambda^* V_j^*\| = \\ &= \|T^*(V_j z) - \lambda^*(V_j z)\| + \|TV_j - V_j \Lambda\|. \end{aligned}$$

Now because $\|TV_j z - \lambda V_j z\| \to 0$ and because λ is a normal approximate eigenvalue of T, it follows that $\|T^* V_j z - \lambda^* V_j z\| \to 0$. Hence, the preceding system of inequalities yield

$$\|\lambda^* z - \Lambda^* z\| \leqslant \lim_{j \to \infty} (\|T^* V_j z - \lambda^* V_j z\| + \|TV_j - V_j \Lambda\|) = 0,$$

thereby establishing that Λ is normal.

Next, decompose Λ according to its spectral structure: $\Lambda = \sum_{i=1}^{p} \zeta_i E_i$, where $\zeta_1, \ldots, \zeta_p$ are the distinct eigenvalues of Λ and $E_1, \ldots, E_p$ are the corresponding spectral projections. For every $z \in \mathbb{C}^n$,

$$\begin{aligned} \|T^* V_j z - V_j \Lambda^* z\| &= \left\| T^* V_j \left(\sum_{i=1}^{p} E_i z \right) - V_j \left(\sum_{i=1}^{p} \zeta_i^* E_i z \right) \right\| = \\ &= \left\| \sum_{i=1}^{p} (T^* - \zeta_i^* 1) V_j E_i z \right\| \leqslant \sum_{i=1}^{p} \|(T^* - \zeta_i^* 1) V_j E_i z\|. \end{aligned}$$

If $E_i z \neq 0$, then $E_i z$ is an eigenvector of Λ corresponding to the eigenvalue ζ_i, and therefore, for each $1 \leqslant i \leqslant p$, $\{V_j(E_i z)\}_j$ is a sequence of zero vectors, or is a bounded sequence of approximate eigenvectors of T corresponding to the approximate eigenvalue ζ_i of T (we are making use of the hierarchical property). But by hypothesis, these approximate eigenvalues are normal approximate eigenvalues, so $\|(T^* - \zeta_i^* 1)V_j E_i z\| \to 0$ for each i. Hence,

$$0 \leqslant \lim_{j\to\infty} \|T^* V_j z - V_j \Lambda^* z\| \leqslant \lim_{j\to\infty} \sum_{i=1}^{p} \|(T^* - \zeta_i^* 1)V_j E_i z\| = 0.$$

This proves that $\lim_j \|T^* V_j - V_j \Lambda^*\| = 0$, as desired.

(2) Because every approximate eigenvalue is normal, $\Pi_e(T)$ is precisely $R_e^1(T)$, and so the results of Salinas [13] on the reducing essential spectrum apply here.

Since $1 \leqslant \mu_j \leqslant \nu_j$ for all $j = 1, \ldots, q$, there exist orthonormal vectors $x_1^j, \ldots, x_{\mu_j}^j$ in $\ker(T - \lambda_j 1)$ such that the matrices $[(Tx_s^j, x_r^j)]$ are scalar matrices with eigenvalue λ_j for all $1 \leqslant j \leqslant q$. Since the eigenspaces of T are mutually orthogonal, the direct sum Λ_0 of these matrices is an element of $\Pi_0^{\mu}(T)$, where $\mu = \sum_{j=1}^{q} \mu_j$. Let H_0 denote the reducing subspace of T on which the action of T is given by Λ_0, and let V_0 map $\mathbb{C}^{\mu}$ isometrically onto H_0 so that $TV_0 = V_0 \Lambda_0$.

For the elements $\lambda_{q+1}, \ldots, \lambda_p$, given $\varepsilon > 0$, there exist mutually orthogonal projections $Q_{q+1}, \ldots, Q_p$ of infinite rank and nullity such that each $(T - \lambda_i 1)Q_i$ is a compact operator of norm no greater than ε, and the range of each Q_i is orthogonal to H_0 [13;4.1,4.2]. Therefore, there exist isometries $V_i : \mathbb{C}^{\mu_i} \to Q_i(H)$ such that $\|TV_i - V_i(\lambda_i 1)\| < \varepsilon$ for every $i = (q+1), \ldots, p$; hence, $\|TV_\varepsilon - V_\varepsilon \Lambda\| < p^{\frac{1}{2}}\varepsilon$, where $V_\varepsilon = V_0 \oplus V_{q+1} \oplus \cdots \oplus V_p$ and $\Lambda = \Lambda_0 \oplus \lambda_{q+1} 1_{\mu_{q+1}} \oplus \cdots \oplus \lambda_p 1_{\mu_p}$. This shows that $\Lambda \in \Pi_s^n(T)$.

Conversely, a matrix $\Lambda \in \Pi_s^n(T)$ is normal, by (a), and so its spectrum is contained in $\Pi(T)$; we need only verify that the multiplicities of those eigenvalues ζ of Λ which are isolated eigenvalues of T do not exceed the dimension of $\ker(T - \zeta 1)$. Therefore, suppose that Λ has such an eigenvalue ζ, say of multiplicity μ; let H_ζ denote the kernel of $T - \zeta 1$, and let ν denote its dimension. Suppose, on the contrary, that $\mu > \nu$. Decompose H as a sum of T-invariant subspaces: $H_\zeta \oplus H_\zeta^{\perp}$. Note that $\sigma(T_{|H_\zeta^\perp}) = \sigma(T) \setminus \{\zeta\}$. Since $\zeta 1_\mu \in \Pi_s^\mu(T)$, there exist orthonormal vectors $x_1^\beta, \ldots, x_\mu^\beta \in H$ such that for each β the first ν of these vectors lie in the subspace H_ζ (and the rest lie in $H_\zeta^\perp$), and such that $\lim_\beta \|(T - \zeta 1)x_j^\beta\| = 0$ for all $1 \leqslant j \leqslant \mu$. This means that the sequence of vectors $x_\mu^\beta \in H_\zeta^\perp$ are approximate eigenvectors of $\tilde{T}$, the restriction of T to $H_\zeta^\perp$, corresponding to the approximate eigenvalue ζ of $\tilde{T}$, contrary to the fact that $\zeta \notin \sigma(\tilde{T})$. Hence, we must have $\mu \leqslant \nu$ as desired. ∎

The next two examples illustrate how properties of an operator can be deduced from higher dimensional spectra in a way unavailable with simply the approximate point spectrum. Part (2) of Example 2 can be arrived at in a manner similar to the computation of $\Pi^n(T)$ given in [1] for the Donoghue shift operator, however the proof given below is direct and typical of what is involved in 2×2 spatial spectral problems. Example 3 shows that $\Pi_s^2(\cdot)$ characterizes normality for algebraic operators.

EXAMPLE 2. *Suppose that T is quasi-nilpotent.*
(1) If T is actually nilpotent, and if $\Pi_s^2(T) = \{0\}$, then $T = 0$.
(2) If T is compact and is without eigenvalues, then $\Pi_s^n(T) = \{0\}$.

Proof: (1) If $T \neq 0$, then there is some $m \geqslant 2$ such that $T^{m-1} \neq 0$ and $T^m = 0$. Thus, there exists a vector $x \in H$ for which $T^{m-1}x = 0$; therefore, the subspace spanned by $\{T^{m-2}x, T^{m-1}x\}$ is a 2-dimensional T-invariant subspace. But $\Pi_s^2(T) = \{0\}$, and so $T^{m-1}x = T(T^{m-2}x) = 0$, contrary to $T^{m-1}x \neq 0$. Thus, T must be 0.

(2) Assume that $n \geqslant 2$. Because every element of $\Pi_s^n(T)$ is nilpotent, it is enough to establish that $\Pi_s^2(T) = \{0\}$, for this would imply that every $\Omega \in \Pi_s^n(T)$ is zero (by the spectral hierarchy and part (1)). If $\Lambda = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \in \Pi_s^2(T)$, then there exist unit vectors $x_k, y_k \in H$ with

$$(x_k, y_k) = 0 \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} \|Tx_k\| = \lim_{k\to\infty} \|Ty_k - \lambda x_k\| = 0.$$

Since T is compact, $\{Ty_k\}$ has a convergent subsequence $\{Ty_\beta\}$ converging to a vector $z \in H$. The equation $0 = \lim_\beta \|Ty_\beta - \lambda x_\beta\|$ implies that $z = \lim_\beta \lambda x_\beta$ and $\|z\| = |\lambda|$. Thus,

$$Tz = \lambda \lim_\beta Tx_\beta = \lambda(0) \quad (\text{since } \|Tx_\beta\| \to 0);$$

hence, $z \in \ker T = \{0\}$ and $\lambda = 0$. ∎

EXAMPLE 3. *If T is an algebraic operator, then T is normal if and only if every matrix in $\Pi_s^2(T)$ is normal.*

Proof: The 2×2 left matricial spectrum of a normal operator, algebraic or not, always consists exclusively of normal matrices, so the necessity of the hypothesis is evident.

To prove that the hypothesis is sufficient for normality, first note that the spectrum of T is finite and each spectral point is an eigenvalue; this follows from T being algebraic. Suppose that $\sigma(T) = \{\lambda\}$. Then $T - \lambda 1$ is nilpotent and $\Pi_s^2(T - \lambda 1) = \Pi_s^2(T) - \{\lambda 1_2\}$ consists entirely of matrices that are both nilpotent and normal; therefore, $\Pi_s^2(T - \lambda 1) = \{0\}$. By (1) of Example 2, this implies that $T = \lambda 1$. For the general case, if $\lambda \in \sigma(T)$ and if $\tilde{T}$ denotes the restriction of T to the generalized eigenspace $\ker(T - \lambda 1)^k$, where k is the algebraic multiplicity of λ, then the preceding arguments apply to $\tilde{T}$; the conclusion is that $\ker(T - \lambda 1)^k = \ker(T - \lambda 1)$. Because H is an algebraic direct sum of the generalized eigenspaces of T, all that remains is to prove that the eigenspaces of T are mutually orthogonal. To this end, if $\lambda, \mu \in \sigma(T)$ are distinct eigenvalues with correponding eigenvectors x, y respectively, then $\mathcal{L} = \text{sp}\{x, y\}$ is a T- invariant subspace and $T_{|\mathcal{L}}$ is unitarily equivalent to an element of $\Pi_s^2(T)$, and therefore, $T_{|\mathcal{L}}$ is normal. Thus, the eigenvectors x, y of $T_{|\mathcal{L}}$ correponding to the distinct eigenvalues λ, μ must be orthogonal. This proves that T is normal. ■

The remainder of the examples concern the spatial component of the full matricial spectrum.

EXAMPLE 4. *If zero is a boundary point of the numerical range of a quasinilpotent operator T, then $\sigma_s^n(T) = \{0\}$.*

Proof: If $\Lambda \in \sigma_s^n(T)$, then Λ must be nilpotent; therefore, to show that $\Lambda = 0$ it is enough to show, by (1) of Example 2, that $\Pi_s^2(\Lambda) = \{0\}$. Thus, suppose that $\Omega = \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \in \Pi_s^2(\Lambda)$. The numerical range of Ω is a disc centred at 0 of radius $\frac{1}{2}|\omega|$ and this disc must lie within the closure of the numerical range of T. However, the hypothesis states that 0 is a boundary point of the numerical range and so the disc $W(\Omega)$ must be degenerate. Hence, $\omega = 0$ and $\Lambda = 0$. ■

The classical Volterra integral operator is one of many quasi-nilpotent operators that satisfy the hypothesis of Example 4, however different techniques are needed to determine whether the matricial spectra of general Volterra integral operators can have non-zero nilpotents.

EXAMPLE 5. *If a contraction T has the closed unit disc for its spectrum, then $\sigma_s^\nu(T)$ is the closed unit ball of $B(\mathbb{C}^\nu)$.*

Proof: The hypothesis implies that the essential numerical range $W_e(T)$ of T is the closed unit disc and so, from $\sigma_e(T) \cap \partial W_e(T) \subset R_e^1(T)$ [13;3.3], we have that $\partial\mathbf{D} \subset R_e^1(T)$. Suppose that Λ is a contraction on $\mathbb{C}^\nu$. By the Sz.-Nagy dilation theorem, there exists a unitary operator U on a separable space L and an isometry $V : \mathbb{C}^\nu \to L$ such that $\Lambda^m = V^* U^m V$ for every $m \in \mathbb{N}$. Because $\sigma(U) \subset \partial\mathbf{D} \subset R_e^1(T)$ and U is normal, we have that $T \sim_a T \oplus U$ by [13;4,9] and, therefore, that $\sigma_s^\nu(T) = \sigma_s^\nu(T \oplus U)$. Hence, it is clear that $\Lambda^m = W^*(T \oplus U)^m W$, for all m, for some isometry $W : \mathbb{C}^\nu \to H \oplus L$. Finally, because rational functions on the closed unit disc can be approximated uniformly by polynomials, it follows that $r(\Lambda) = W^* r(T \oplus U) W$ for every $r \in \text{Rat}\sigma(T)$. ■

As is shown by Example 5, for normal operators T, one cannot expect that $\sigma_s^n(T) = R_s^n(T)$, for $n > 1$, unlike the scalar case. The following example shows some cases where such equality can occur.

EXAMPLE 6. *If T is a normal operator such that every continuous function on $\sigma(T)$ is approximated uniformly by functions in* $\mathrm{Rat}\sigma(T)$, *then* $\sigma_s^\nu(T) = R_s^\nu(T)$.

Proof: By Theorem 2.2, if $A \in \sigma_s^\nu(T)$, then there exists a (normal) operator $S \sim_a T$ on a separable space L and an isometry $V : \mathbb{C}^\nu \to L$ such that $r(A) = V^* r(S) V$ for every $r \in \mathrm{Rat}\sigma(S)$. The hypothesis is that $\mathrm{Rat}\sigma(S)$ is uniformly dense in $C(\sigma(S))$ and so $\mathcal{R}(S)$ is dense in $C^*(S)$ and the map $X \mapsto V^*XV$ is, therefore, a $*$-homomorphism $C^*(S) \to B(\mathbb{C}^\nu)$. Thus, A is a direct summand of S and an approximate direct summand of T.

Conversely, it is clear that $R_s^\nu(T) \subset \sigma_s^\nu(T)$. ∎

EXAMPLE 7. *The following operators T satisfy $\sigma_s^\nu(T) = R_s^\nu(T)$: selfadjoint operators, unitary operators, compact normal operators, and normal operators with spectrum that neither has interior nor disconnects the plane.*

Proof: Lavrentiev's theorem states that for compact planar sets X, all continuous functions on X can be approximated uniformly by polynomials if and only if X is nowhere dense and does not disconnect the plane. The spectra of all of the normal operators mentioned above, with the sole exception of the unitaries, have spectra that fulfil the sufficiency part of Lavrentiev's theorem and, consequently, the result follows from Example 6. In the case of the unitary operators, every continuous function on the unit circle or some compact subset thereof can be approximated uniformly by rational functions and so Example 6 applies once again. ∎

REMARK. If T is a normal operator such that $\sigma(T)$ is nowhere dense and has conected complement, then T is a reductive operator. Actually, T is more special in that T is reductive in an asymptotic sense, meaning that $\|TP_j - P_jT\| \to 0$ whenever P_j are projections satisfying $\|(1-P_j)TP_j\| \to 0$ (see Harrison [7]). What, then, is the spatial matricial spectrum of an arbitrary reductive normal operator ?

In M_n, the set of normal matrices is nowhere dense, and so the left matricial spectrum of a normal operator never has interior. The concluding example, given below, shows in particular that the left matricial spectrum of a normal operator will be much larger than the left matricial spectrum if the essential spectrum has interior.

EXAMPLE 8. *The spatial matricial spectrum has interior whenever the essential spectrum has.*

Proof: Because $\sigma_e(T) = \Pi_e(T) \cup \Pi_e(T^*)^*$, one of the two sets in this union must have interior, if $\sigma_e(T)$ has. Assume, therefore and withoutloss in generality, that $\Pi_e(T)$ contains the closed unit disc. Then there is an operator S approximately equivalent to T such that

$$S = T \oplus \begin{pmatrix} N & * \\ 0 & * \end{pmatrix},$$

where N is normal and has spectrum $\mathbb{D}^-$. Let P be the projection onto the (S-invariant) subspace that N acts upon; let Γ be a smooth contour containing $\sigma(T)$ in its interior. Then for any $r \in \mathrm{Rat}\sigma(S)$ (note: $\sigma(S) = \sigma(T)$),

$$Pr(S)_{|P(H)} = -\frac{1}{2\pi i}\int_\Gamma r(\zeta) P \begin{pmatrix} (T-\zeta 1)^{-1} & 0 & 0 \\ 0 & (N-\zeta 1)^{-1} & * \\ 0 & 0 & * \end{pmatrix}_{|P(H)} d\zeta =$$

$$= -\frac{1}{2\pi i}\int_\Gamma r(\zeta)(N-\zeta 1)^{-1} d\zeta = r(N),$$

whence $N \in \sigma_s^\infty(S) = \sigma_s^\infty(T)$. By the spectral hierarchy it follows that the closed unit ball of $B(\mathbb{C}^\nu)$, which is the spectrum of N by Example 5, is contained within $\sigma_s^\nu(T)$. ■

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