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GAUGE INVARIANT STATES OF $\mathcal{O}_\infty$

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1. INTRODUCTION

As is customary, $\mathcal{O}_\infty$ will denote the universal C^*-algebra genarated by an infinite sequence of isometries, $\{v_j : j = 1, 2, \ldots\}$, with mutually orthogonal ranges, as introduced by Cuntz in [2]. For each $\lambda \in \mathbf{T}$ the correspondence $\gamma_\lambda : v_j \mapsto \lambda v_j$, $j = 1, 2, \ldots$ extends uniquely to an automorphism of $\mathcal{O}_\infty$, also indicated by γ_λ. The fixed point algebra of this action, $\mathcal{F}_\infty$, coincides with the C^*-subalgebra generated by the products of an equal number of isometries as of their adjoints. There exists a unique conditional expectation Φ from $\mathcal{O}_\infty$ onto $\mathcal{F}_\infty$, corresponding to the process of averaging over the gauge orbits. This gives rise to a method for extending states from $\mathcal{F}_\infty$ to states of $\mathcal{O}_\infty$, analogous to the inducing construction in a C^*-dynamical system. If ω is a state of $\mathcal{F}_\infty$ we denote by $\overline{\omega}$ its gauge invariant extension $\omega \circ \Phi$ to $\mathcal{O}_\infty$. Such state extensions of product states of $\mathcal{F}_n$ were first studied by Evans in [3] and later by Araki, Carey and Evans in [1]. We consider here the relation between states of $\mathcal{F}_\infty$ and their gauge invariant extensions to $\mathcal{O}_\infty$, induced via the conditional expectation Φ. The present study arose from trying to understand better the pairing construction of [8] concerning the definition of an abstract index for continuous parameters of endomorphisms of $\mathcal{B}(\mathcal{H})$. In particular, it became important to find a good source of examples of ergodic endomorphisms with infinite index and to develop criteria to decide when two endomorphisms are conjugate a task which was started in [4]. As it turns out, the representation theory of the Cuntz algebra $\mathcal{O}_\infty$ is at the heart of the matter, and the path goes both ways for one can use the endomorphisms to gain insight on those representations. The results presented here hold, with the obvious modifications, for all the C^*-algebras $\mathcal{O}_n$. For finite n some of the claims made here are trivially verified, while some others have been obtained by different methods (for product states) in [3] and [1] and, more recently, in [9] for $n = 2$. We have thus concentrated our study to $\mathcal{O}_\infty$. The main result appears in Section 4 where we characterize the pure states of $\mathcal{F}_\infty$ which induce pure states on $\mathcal{O}_\infty$, hence which give ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$. The same analysis, based on the spectral subspaces of the gauge action, yields a necessary and sufficient condition for two factor states of $\mathcal{F}_\infty$ to induce quasi-equivalent states on $\mathcal{O}_\infty$. In Section 5 those results are interpreted when the states being extended are of product type, in which case one can give computable criteria for pureness and quasi equivalence of the corresponding gauge invariant states.

2. FROM $\mathcal{F}_\infty$ TO $\mathcal{O}_\infty$ AND BACK

For each $m \in \mathbf{Z}$, $\mathcal{G}_m$ will denote the spectral subspace $\{x \in \mathcal{O}_\infty : \gamma_\lambda(x) = = \lambda^m x \text{ for } \lambda \in \mathbf{T}\}$ of the gauge action of the circle group $\mathbf{T}$ on $\mathcal{O}_\infty$. It follows from [2] that $\mathcal{G}_m$ is the closed subspace generated by products of the form $v_{i_1}\cdots v_{i_k}\, v_{j_1}^*\cdots v_{j_l}^*$ with $k, l \geqslant 0$ and $k - l = m$. It is then easy to see that $\mathcal{G}_m^* = \mathcal{G}_{-m}$, and that $\mathcal{G}_m\mathcal{G}_n$ spans a dense subspace of $\mathcal{G}_{m+n}$. Moreover $\mathcal{G}_0 = \mathcal{F}_\infty$ and each $\mathcal{G}_m$ is an $\mathcal{F}_\infty$ module.

Suppose now ω is a state of $\mathcal{F}_\infty$ and let $\overline{\omega} = \omega \circ \Phi$ be its gauge invariant extension. Let π denote the associated GNS representation of $\mathcal{O}_\infty$ on the Hilbert space $\mathcal{H}$ with cyclic unit vector Ω, so that $\overline{\omega}(x) = (\pi(x)\Omega, \Omega)$. In order to understand the relation between ω and $\overline{\omega}$ it is convenient to study the canonical decomposition that the spectral subspaces induce on the restriction of π to the subalgebra $\mathcal{F}_\infty$. If we let $\mathcal{H}_m = \overline{\pi(\mathcal{G}_m)\Omega}$ for $m \in \mathbf{Z}$, then the $\mathcal{H}_m$'s are a doubly infinite sequence of $\pi(\mathcal{F}_\infty)$--invariant subspaces of $\mathcal{H}$ which are mutually orthogonal and span the whole space. That they are mutually orthogonal follows from the fact that if $x \in \mathcal{G}_m$, $y \in \mathcal{G}_n$ and $m \neq n$, then $\langle\pi(x)\Omega,\ \pi(y)\Omega\rangle = \overline{\omega}(y^*x) = 0$ because $y^*x \in \mathcal{G}_{m-n}$. Cuntz showed in [2] that elements of the form

$$x = \sum_{j=-m}^{-1} v_1^{*|j|} f_j + f_0 + \sum_{j=1}^{m} f_j v_1^j \quad m = 0, 1, 2, \ldots;\ f_j \in \mathcal{F}_\infty \text{ for } |j| \leqslant m,$$

are dense in $\mathcal{O}_\infty$, thus the corresponding set of vectors, $\{\pi(x)\Omega\}$, is dense in $\mathcal{H}$. It is easy to see that $v_1^{*|j|}f_j \in \mathcal{G}_{-|j|}$ and that $f_j v_1^j \in \mathcal{G}_j$, from which it follows that $\bigoplus_{m\in\mathbf{Z}} \mathcal{H}_m = \mathcal{H}$.

For each integer m, P_m will denote the projection onto $\mathcal{H}_m$ and π_m will denote the associated subrepresentation of $\pi\lceil\mathcal{F}_\infty$. By the above disscusion, $P_m \in \pi(\mathcal{F}_\infty)'$

for all $m \in \mathbb{Z}$ and $\sum_{m\in\mathbb{Z}} P_m = I$, yielding a decomposition of $\pi\restriction\mathcal{F}_\infty$:

$$\pi\restriction\mathcal{F}_\infty = \bigoplus_{m\in\mathbb{Z}} \pi_m \quad \text{on } \mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m. \tag{2.1}$$

Since Ω is cyclic for $\pi_0(\mathcal{F}_\infty)$ on $\mathcal{H}_0$, we see that π_0 is (unitarily equivalent to) the GNS representation of $\mathcal{F}_\infty$ associated with ω. It is not clear that the remaining P_m are nontrivial, and in fact, a quick look shows that for the Fock representation of [3], $P_m = 0$ for all $m < 0$. To avoid such trivialities we will restrict our attention to the *essential* states defined in [4], equivalently, to representations π satisfying $\sum_j \pi(v_j v_j^*) = = I$.

LEMMA 2.1. *If $x \in \mathcal{G}_p$ then $\pi(x)P_m = P_{m+p}\pi(x)$ for all $m \in \mathbb{N}$.*

Proof. If $y \in \mathcal{G}_m$, then $\pi(x)(\pi(y)\Omega) = \pi(xy)\Omega \in \mathcal{H}_{m+p}$, and since vectors of the form $\pi(y)\Omega$ are dense in $\mathcal{H}_m$, we conclude that

$$\pi(x)P_m = P_{m+p}\pi(x)P_m \quad m \in \mathbb{Z}. \tag{2.2}$$

Now notice that $x^* \in \mathcal{G}_{-p}$ and apply the above argument to x^* in place of x and $\mathcal{H}_{m+p}$ in place of $\mathcal{H}_m$ to obtain $\pi(x^*)P_{m+p} = P_{m+p-p}\pi(x^*)P_{m+p}$ for $m \in \mathbb{Z}$. By taking adjoints,

$$P_{m+p}\pi(x) = P_{m+p}\pi(x)P_m \quad m \in \mathbb{Z}$$

which together with (2.2) completes the proof. ∎

Denote by α the endomorphism of $\mathcal{B}(\mathcal{H})$ defined by $\alpha(A) = \sum_{j=1}^{\infty} \pi(v_j)A\pi(v_j)^*$. It follows that $\alpha(P_m) = \sum_{j=1}^{\infty} \pi(v_j)P_m\pi(v_j)^* = \sum_{j=1}^{\infty} P_{m+1}\pi(v_j)\pi(v_j)^* = P_{m+1}$ for $m \in \mathbb{Z}$. Thus, for any $S \in \mathcal{B}(\mathcal{H})$ we have $\alpha(P_i S P_j) = P_{i+1}\alpha(S)P_{j+1}$ and, identifying the space $\mathcal{B}(\mathcal{H}_j;\mathcal{H}_i)$ of bounded linear operators from $\mathcal{H}_j$ into $\mathcal{H}_i$ with the subspace $P_i\mathcal{B}(\mathcal{H})P_j$ of $\mathcal{B}(\mathcal{H})$, we can say that α sends $\mathcal{B}(\mathcal{H}_j;\mathcal{H}_i)$ into $\mathcal{B}(\mathcal{H}_{j+1};\mathcal{H}_{i+1})$.

It was proved in [4] that $\pi(\mathcal{F}_\infty)' = \bigcap_{k>0} \alpha^k(\mathcal{B}(\mathcal{H}))$ so the restriction of α to $\pi(\mathcal{F}_\infty)'$ is an automorphism. Thus, whenever $A \in \pi(\mathcal{F}_\infty)'$ and $k \in \mathbb{Z}$, the element $\alpha^k(A) \in \pi(\mathcal{F}_\infty)'$ exists and is uniquely determined.

We may now describe $\pi(\mathcal{F}_\infty)'$ in terms of the decomposition obtained above. Let $\mathcal{I}(\pi_i, \pi_j)$ denote the Banach space of intertwining operators between the representations π_i and π_j, i.e. $\mathcal{I}(\pi_i,\pi_j) = \{T \in \mathcal{B}(\mathcal{H}_j;\mathcal{H}_i) : \pi_i(x)T = T\pi_j(x)(\forall)x \in \mathcal{F}_\infty\}$. Then

$$\pi(\mathcal{F}_\infty)' = \{S \in \mathcal{B}(\mathcal{H}) : P_i S P_j \in \mathcal{I}(\pi_i,\pi_j) \quad (\forall)i,\ j \in \mathbb{Z}\}. \tag{2.3}$$

Notice that if $S \in \mathcal{I}(\pi_{i_0}, \pi_{j_0})$ is extended to an operator (also denoted by S) on $\mathcal{H}$ by letting it vanish on the orthogonal complement of $\mathcal{H}_{j_0}$, then $S \in \pi(\mathcal{F}_\infty)'$.

PROPOSITION 2.2. *For each pair of integers i, j, the endomorphism α induces an isomorphism of Banach spaces from $\mathcal{I}(\pi_i, \pi_j)$ onto $\mathcal{I}(\pi_{i+1}, \pi_{j+1})$.*

Proof. Linearity and boundedness pose no problem since we are dealing with the restriction of a bounded linear map. The point is to prove that α is a bijection between the two subspaces of $\mathcal{B}(\mathcal{H})$.

Suppose $T_0 \in \mathcal{I}(\pi_i, \pi_j)$ and view T_0 as an operator on $\mathcal{H}$ by the identification mentioned above, so $T \in \pi(\mathcal{F}_\infty)'$ by (2.3). There exist T_{-1} and T_1 in $\pi(\mathcal{F}_\infty)'$ such that $\alpha(T_{-1}) = T_0$ and $\alpha^{-1}(T_1) = T_0$. Necessarily $T_{-1} = P_{i-1}T_{-1}P_{j-1}$ and $T_1 = P_{i+1}T_1P_{j+1}$, so that $T_1 \in \mathcal{I}(\pi_{i+1}, \pi_{j+1})$ and $T_{-1} \in \mathcal{I}(\pi_{i-1}, \pi_{j-1})$. We have proved that $\alpha(\mathcal{I}(\pi_{i-1}, \pi_{j-1})) \supseteq \mathcal{I}(\pi_i, \pi_j)$ and that $\alpha(\mathcal{I}(\pi_i, \pi_j)) \subseteq \mathcal{I}(\pi_{i+1}, \pi_{j+1})$, for arbitrary $i, j \in \mathbf{Z}$. It follows that $\alpha(\mathcal{I}(\pi_i, \pi_j)) \supseteq \mathcal{I}(\pi_{i+1}, \pi_{j+1})$, which completes the proof. ∎

Setting $i = j$ it follows that the restriction of α^j to $\pi_0(\mathcal{F}_\infty)'$ establishes an isomorphism between $\pi_0(\mathcal{F}_\infty)'$ and $\pi_j(\mathcal{F}_\infty)'$ for each $j \in \mathbf{Z}$. If the state ω of $\mathcal{F}_\infty$ being extended is pure then π_0 is irreducible, so the above implies that all the representations appearing in the decomposition (2.1) are irreducible.

In order to extract some information about the π_k's in the case when ω is a factor state, we analyze the effect of α on the centers of the π_k's. Although $\alpha(\pi(\mathcal{F}_\infty)'')$ may be strictly contained in $\pi(\mathcal{F}_\infty)''$, α behaves well on the centers. Let $\mathcal{C}$ denote the center of $\pi(\mathcal{F}_\infty)'$ in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{C}_m$ denote the center of $\pi_m(\mathcal{F}_\infty)'$ in $\mathcal{B}(H_m)$.

PROPOSITION 2.3. *With the above notation, $\alpha(\mathcal{C}) = \mathcal{C}$ and $\alpha(\mathcal{C}_m) = \mathcal{C}_{m+1}$ for $m \in \mathbf{Z}$ so that α restricts to an automorphism of $\mathcal{C}$ and to an isomorphism from $\mathcal{C}_m$ onto $\mathcal{C}_{m+1}$.*

Proof. Since α restricts to an automorphism of $\pi(\mathcal{F}_\infty)'$ it further restricts to an automorphism of $\mathcal{C}$.

To prove $\alpha(\mathcal{C}_m) = \mathcal{C}_{m+1}$ take $Q \in \mathcal{C}_m$, then $\alpha(Q) \in \pi_{m+1}(\mathcal{F}_\infty)'$ and $\alpha^{-1}(Q) \in \pi_{m-1}(\mathcal{F}_\infty)'$. If $T \in \pi_{m\pm1}(\mathcal{F}_\infty)'$ then $\alpha^{\pm}(T) \in \pi_m(\mathcal{F}_\infty)'$ so that $\alpha^{\mp1}(T)Q = Q\alpha^{\mp1}(T)$. Applying $\alpha^{\pm1}$ to both sides it follows that

$$T\alpha^{\pm1}(Q) = \alpha^{\pm}(Q)T \qquad \text{for } T \in \pi_{m\pm1}(\mathcal{F}_\infty)'.$$

Thus $\alpha(Q) \in \mathcal{C}_{m+1}$ and $\alpha^{-1}(Q)\mathcal{C}_{m-1}$, and since m is arbitrary, $\alpha(\mathcal{C}_m) = \mathcal{C}_{m+1}$. ∎

COROLLARY 2.4. *If ω is a factor state, then π_m is a factor representation for each $m \in \mathbf{Z}$.*

Proof. The center of π_0 is trivial because ω is a factor state so the above implies that all the $\mathcal{C}_m$ are trivial. ∎

Finally we give a description of the commutant of $\pi(\mathcal{O}_\infty)$ in terms of the decomposition of $\mathcal{H}$ into invariant subspaces $\mathcal{H}_m$. Let $S_{ij} = P_i S P_j$ be the $(i,j)^{th}$ entry in the matrix of S corresponding to the decomposition of $\mathcal{H}$. Then

$$\pi(\mathcal{O}_\infty)' = \{S \in \mathcal{B}(\mathcal{H}) : \ \alpha(S_{ij}) = S_{i+1\ j+1}\ (\forall) i,j \in \mathbb{Z}\}. \tag{2.4}$$

The justification is as follows: $S \in \pi(\mathcal{O}_\infty)'$ if and only if $\alpha(S) = S$ by Proposition 3.1 in [4]. Since $\alpha(S_{ij}) = (\alpha(S))_{i+1\ j+1}$, this is the case if and only if $\alpha(S_{ij}) = S_{i+1\ j+1}$. Note that $S_{ij} \in \mathcal{I}(\pi_i, \pi_j)$ because $\pi(\mathcal{O}_\infty)' \subseteq \pi(\mathcal{F}_\infty)'$.

3. SHIFTING STATES

Whenever ω is a state of $\mathcal{F}_\infty$, $\alpha^*\omega$ will be the positive linear functional defined by $\alpha^*\omega(x) = \sum_j \omega(v_j x v_j^*)$. This map α^* was introduced and studied in [4] to sidestep the absence of an endomorphism at level of the C^*-algebra $\mathcal{F}_\infty$. There it was shown that ω is singular if and only if $\|\alpha^{*p}\omega\| \to 0$ as $p \to \infty$, and that it is essential if and only if $\alpha^{*p}\omega$ is a state, i.e. $\|\alpha^{*p}\omega\| = 1$, for all $p \geqslant 0$. These definitions also make sense if ω is assumed to be a state of $\mathcal{O}_\infty$, and the corresponding characterizations of essential and singular states hold true. It is convenient to include here some notation associated with the structure of $\mathcal{O}_\infty$. The orthogonality of the ranges of the isometries makes it possible to define an inner product on their closed linear span $\mathcal{E} = \overline{\operatorname{span}}\{v_j\}_{j=1}^\infty$ via $\langle x, y\rangle I = y^*x$, where x and y are in $\mathcal{E}$. Since the Hilbert space norm and the operator norm coincide, $\mathcal{E}$ becomes a Hilbert space inside $\mathcal{O}_\infty$. The isometries in $\mathcal{E}$ are the unit vectors and $\{v_j\}_{j=1}^\infty$ is an orthonormal base.

Along the same lines, let $\mathcal{W}_k$ denote the set of products of k isometries among the generators $\{v_j\}_{j=1}^\infty$ and let $\mathcal{E}^k$ be the closed linear span of $\mathcal{W}_k$, then a similar argument shows that $\mathcal{E}^k$ is a Hilbert space and $\mathcal{W}_k$ is an orthonormal basis. Moreover, the mapping $e_1 \otimes \cdots \otimes e_k \mapsto e_1 \cdots e_k$ extends to a unitary operator from $\mathcal{E}^{\otimes k}$ onto $\mathcal{E}^k$. The fixed point algebra $\mathcal{F}_\infty$ can then be seen as the unital C^*-subalgebra of the infinite tensor product $(\mathcal{K}(\mathcal{E}) + \mathbb{C}I)^{\otimes\infty}$ generated by elementary tensors of the type $K_1 \otimes \cdots \otimes K_n \otimes I \otimes I \cdots$ where $K_j \in \mathcal{K}(\mathcal{E})$ and $n = 1, 2, \ldots$; see [3].

In order to study the representations appearing in the decomposition $\pi\restriction\mathcal{F}_\infty = \bigoplus_{m\in\mathbb{Z}} \pi_m$ up to unitary and quasi-equivalence, we need to examine the orbits of the state ω under powers of α^*. Although α^{*k} has been not defined for $k < 0$, because there are different states of $\mathcal{F}_\infty$ which have the same image under α^*, one can define a "quasi-inverse" for α^* in the following way: If ω is a state of $\mathcal{F}_\infty$ (or of $\mathcal{O}_\infty$), let

$$\beta^*\omega(x) = \omega(v_1^* x v_1) \quad x \in \mathcal{F}_\infty \text{ (or } \mathcal{O}_\infty). \tag{3.1}$$

The particular choice of v_1 is not essential to our purposes for one readily verifies that for each i, the unitary operator $U_{i1} = v_1 v_i^* + v_i v_1^* + \sum_{j\neq 1,i} v_j v_j^*$ establishes the unitary equivalence of $\omega(v_1^* \cdot v_1)$ and $\omega(v_i^* \cdot v_i)$.

The map β^* shifts states in the opposite direction that α^*, by tensoring on the left with the pure state of $\mathcal{K}(\mathcal{E})$ corresponding to the unit vector $v_1 \in \mathcal{E}$.

LEMMA 3.1. *If ω is essential state of $\mathcal{F}_\infty$, then:*
i) $\alpha^*\beta^*\omega = \omega$, *and*
ii) $\beta^*\alpha^*\omega \stackrel{q}{\sim} \omega$.
The same is true for an essential state of $\mathcal{O}_\infty$.

Proof. For i) we only need to calculate,

$$\alpha^*\beta^*\omega(x) = \sum_{j=1}^{\infty} \beta^*\omega(v_j x v_j^*) = \sum_{j=1}^{\infty} \omega(v_1^* v_j x v_j^* v_1) = \omega(v_1^* v_1 x v_1^* v_1) = \omega(x)$$

whenever $x \in \mathcal{F}_\infty$. For ii), let π_ω be the GNS representation associated with ω, then

$$\beta^*\alpha^*\omega(x) = \alpha^*(v_1^* x v_1) = \sum_{j=1}^{\infty} \omega(v_j v_1^* x v_1 v_j^*) = \sum_{j=1}^{\infty} (\pi_\omega(x)\pi_\omega(v_1 v_j^*)\Omega, \pi_\omega(v_1, v_j^*)\Omega).$$

This yields $\beta^*\alpha^*\omega \prec\!\prec \omega$ at once, for it shows that $\beta^*\alpha^*\omega$ is a normal state in π_ω. Since ω is assumed to be essential,

$$\sum_{j=1}^{\infty} \pi_\omega(v_j v_1^*)\pi_\omega(v_1 v_j^*)\Omega = \sum_{j=1}^{\infty} \pi_\omega(v_j v_j^*)\Omega = \Omega,$$

and since Ω is cyclic it follows that the set $\{\pi_\omega(v_1 v_j^*)\Omega : j = 1, 2, \ldots\}$ is generating for π_ω. The proof is then finished with an application of the following lemma, which is an analogue of the essential uniqueness of the GNS construction. ∎

LEMMA 3.2. *Suppose π is a representation of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and let ρ be the state of $\mathcal{A}$ determined by a positive, trace-one operator $T \in \mathcal{B}(\mathcal{H})$ Assume further that the range of T is generating for $\pi(\mathcal{A})$; then π is quasi-equivalent to the GNS of ρ.*

Proof. Let π_ρ denote the GNS representation associated with ρ, it is immediate that $\pi_\rho \prec\!\prec \pi$, because ρ is normal in π. We must show that there is no nontrivial subrepresentation of π disjoint from π_ρ, for which it suffices to show that there is no such *central* subrepresentation. Let ξ_i be an orthogonal set of vectors diagonalizing T, corresponding to the nonzero eigenvalues of T, so that the range of T is spanned by the ξ_i's, and $\rho(x) = \mathrm{Tr}(T\pi(x)) = \sum_i \langle \pi(x)\xi_i, \xi_i\rangle$ for $x \in \mathcal{A}$. If P_1 is a central projection in $\pi(\mathcal{A})'$ and its associated subrepresentation, π_1, is disjoint from π_ρ, then

$$\rho(x) = \sum_i \langle (P_1 + P_1^\perp)\pi(x)(P_1 + P_1^\perp)\xi_i, \xi_i\rangle = \sum_i \langle \pi(x)P_1\xi_i, P_1\xi_i\rangle +$$

$$+ \sum_i \langle \pi(x)P_1^\perp\xi_i, P_1^\perp\xi_i\rangle = \rho_1(x) + \rho_1^\perp(x).$$

The positive linear functional ρ_1 is clearly normal in $\pi_1 = P_1\pi$, hence π_{ρ_1} is absolutely continuous with respect to π_1. On the other hand, $\rho_1 \leqslant \rho$ so that $\pi_{\rho_1} \prec\!\prec \pi_\rho$. Since π_1 and π_ρ were assumed to be disjoint, we must have $\rho_1 = 0$ hence $P_1\xi_i = 0$ for all i. This concludes the proof because P_1 is in the commutant of $\pi(\mathcal{A})$ and the set $\{\xi_i\}$, assumed to be generating for $\pi(\mathcal{A})$, is separating for $\pi(\mathcal{A})'$, which yields $P_1 = 0$. ∎

DEFINITION 3.1. A state ω of $\mathcal{F}_\infty$ is *periodic* if there is a positive integer p for which $\alpha^{*p}\omega$ is quasi-equivalent to ω. The smallest such integer is the *period*. If no such integer exists, ω is said to be *aperiodic*.

We define the *quasi-orbit* of ω (under both α^* and its "inverse" β^*) to be the set of quasi-equivalence classes corresponding to the states $\alpha^{*k}\omega$ and $\beta^{*k}\omega$ for $k \geqslant 0$. By Lemma 3.1 the periodic states of period p are those with exactly p points in their quasi-orbits, and the aperiodic ones are those with infinite quasi-orbits. For states of $\mathcal{O}_\infty$ the concept of periodicity is uninteresting: all essential states of $\mathcal{O}_\infty$ are quasi-invariant. (They have period 1.)

PROPOSITION 3.3. *For every essential state ω of $\mathcal{O}_\infty$ we have $\alpha^*\omega \stackrel{q}{\sim} \omega \stackrel{u}{\sim} \beta^*\omega$.*

Proof. Let $(\pi, \mathcal{H}, \Omega)$ be the GNS triple associated with ω, then

$$\beta^*\omega(x) = \langle \pi(x)\pi(v_1)\Omega,\ \pi(v_1)\Omega\rangle.$$

Since $\pi(v_1)\Omega$ is cyclic for $\pi(\mathcal{O}_\infty)$, it follows that π is unitarily equivalent to the GNS representation of $\beta^*\omega$. Applying the above argument to $\alpha^*\omega$ in place of ω we obtain $\alpha^*\omega \stackrel{u}{\sim} \beta^*\alpha^*\omega$, and since $\beta^*\alpha^*\omega \stackrel{q}{\sim} \omega$ by Lemma 3.1 ii), the proof is finished. ∎

The following proposition characterizes the quasi-equivalence classes of the summands appearing in (2.1) as those in the quasi-orbit of ω.

PROPOSITION 3.4. *Suppose that ω is an essential state of $\mathcal{F}_\infty$ and that π_m for $m \in \mathbb{Z}$ are the representations appearing in the decomposition (2.1) of $\pi \restriction \mathcal{F}_\infty$, then*

i) *For $m \geqslant 0$, π_m is unitarily equivalent to the GNS of $\beta^{*m}\omega$, and*

ii) *for $m < 0$, π_m is quasi-equivalent to the GNS of $\alpha^{*|m|}\omega$.*

Proof. Assume $m \geqslant 0$. The vector $\pi(v_1^m)\Omega$ is cyclic for $\pi_m(\mathcal{F}_\infty)$ acting on $\mathcal{H}_m$, and since $\langle \pi_m(x)\pi(v_1^m)\Omega, \pi(v_1^m)\Omega\rangle = \langle \pi(v_1^{*m} x v_1^m)\Omega, \Omega\rangle = \beta^{*m}\omega(x)$, assertion i) follows from the uniqueness of the GNS representation. To prove ii) let $k = |m|$ and observe that if $x \in \mathcal{F}_\infty$ then

$$\alpha^{*k}\omega(x) = \sum_{s\in\mathcal{W}_k} \omega(sxs^*) = \sum_{s\in\mathcal{W}_k} \langle \pi(sxs^*)\Omega, \Omega\rangle = \sum_{s\in\mathcal{W}_k} \langle \pi_k(x)\pi(s^*)\Omega, \pi(s^*)\Omega\rangle,$$

where $\mathcal{W}_k$ denotes the collection of all products of k isometries chosen from the v_j's. Since ω is essential, $\sum_{s\in\mathcal{W}_k} \pi(ss^*) = I$, hence the set $\{\pi(s^*)\Omega : s \in \mathcal{W}_k\}$ is generating for $\pi_m(\mathcal{F}_\infty)$ on $\mathcal{H}_m$. The assumptions of Lemma 3.2 hold and it follows that π_m is quasi-equivalent to the GNS of $\alpha^{*k}\omega$. ∎

COROLLARY 3.5. *If ω is a factor state, then its quasi-orbit consists of (equivalence classes of) factor states of the same type.*

Proof. If ω is a factor state, then π_0 is a factor representation, so by Proposition 2.4 π_m is a factor representation for each $m \in \mathbb{Z}$. This proves that $\alpha^{*k}\omega$ and $\beta^{*k}\omega$ are factor states. They all have the same type because $\pi_m(\mathcal{F}_\infty)' = \alpha^m(\pi_0(\mathcal{F}_\infty)')$ and type is preserved by taking commutant. ∎

COROLLARY 3.6. *If ω_1 and ω_2 are factor states, and $\omega_1 \overset{q}{\sim} \omega_2$, then $\alpha^{*k}\omega_1 \overset{q}{\sim} \alpha^{*k}\omega_2$ and $\beta^{*k}\omega_1 \overset{q}{\sim} \beta^{*k}\omega_2$ for all $k \geqslant 0$.*

Proof. Use the previous corollary and the fact that two factor states are quasi-equivalent if and their average is also a factor state. ∎

Note that the above two corollaries can also be derived from the generalization of Powers criterion for quasi-equivalence obtained in [4].

4. GAUGE INVARIANT STATES

In this section we examine the questions of unitary and quasi-equivalence of gauge invariant states. The main results are a necessary and sufficient condition for quasi-equivalence of the gauge invariant extensions of factor states and a characterization of the gauge invariant states which are pure. This, in turn, allows us to classify the corresponding ergodic endomorphisms of $\mathcal{B}(\mathcal{H})$ up to conjugacy. The first step is to establish that the process of gauge invariant extension is well behaved vis a vis the relations of unitary and quasi-equivalence.

PROPOSITION 4.1. *Unitary equivalence, absolute continuity and quasi-equivalence of states are preserved by gauge-invariant extension.*

Proof. Suppose $\rho \overset{u}{\sim} \omega$ as states of $\mathcal{F}_\infty$ and let $(\pi, \mathcal{H}, \Omega)$ be the GNS triple of $\overline{\omega}$. π_0 is then unitarily equivalent to the GNS of ρ, so there exists a unit vector ξ, cyclic for $\pi_0(\mathcal{F}_\infty)$ on $\mathcal{H}_0$ such that $\rho(x) = \langle \pi_0(x)\xi, \xi\rangle$ when $x \in \mathcal{F}_\infty$.

The formula $\langle \pi(x)\xi, \xi\rangle$, interpreted for $x \in \mathcal{O}_\infty$, defines a state which is gauge invariant and coincides with $\overline{\rho}$ on $\mathcal{F}_\infty$, thus which is equal to $\overline{\rho}$ on $\mathcal{O}_\infty$. Since ξ is cyclic for the action of $\mathcal{O}_\infty$ on $\mathcal{H}$, it follows that $\overline{\omega}$ is unitarily equivalent to $\overline{\rho}$.

To prove that $\rho \prec\!\!\prec \omega$ implies $\overline{\rho} \prec\!\!\prec \overline{\omega}$, let R be the density operator of ρ in π_0. Extend R to all of $\mathcal{H}$ defining it to be zero on the orthogonal complement of $\mathcal{H}_0$. The formula $\mathrm{Tr}\,(R\pi(x))$ defines a gauge invariant state which coincides with $\overline{\rho}$ on $\mathcal{F}_\infty$, hence is equal to $\overline{\rho}$. It follows that $\overline{\rho} \prec\!\!\prec \overline{\omega}$.

By exchanging the roles of ρ and ω we derive the quasi-equivalence result, for quasi-equivalence means mutual absolute continuity. ■

It may come as a surprise at this point that it is not necessary that ρ be quasi-equivalent to ω for $\overline{\rho}$ to be quasi-equivalent to $\overline{\omega}$. This is, however, implicit in Proposition 3.3. Since it is always true that $\overline{\omega} \overset{q}{\sim} \alpha^*\overline{\omega}$, it suffices to consider any state ω of $\mathcal{F}_\infty$ which is not quasi invariant, i.e. not quasi-equivalent to $\rho = \alpha^*\omega$. As established by the following theorem, this is basically all that can happen, at least for factor states.

THEOREM 4.2. *Suppose that ω and ρ are two factor states of $\mathcal{F}_\infty$. The following are equivalent;*

i) *$\overline{\omega}$ and $\overline{\rho}$ are quasi-equivalent.*

ii) *There exist $p \geqslant 0$ such that $\alpha^{*p}\omega \overset{q}{\sim} \rho$ or $\alpha^{*p}\rho \overset{q}{\sim} \omega$.*

iii) *The quasi-orbits of ω and ρ coincide.*

Proof. i) $\Rightarrow$ ii). Since $\pi_{\overline{\omega}}$ and $\pi_{\overline{\rho}}$ are quasi-equivalent representations of $\mathcal{O}_\infty$, so are their restrictions to $\mathcal{F}_\infty$, thus, in the notation of (2.1),

$$\bigoplus_{m\in\mathbb{Z}} \pi_{\overline{\omega},m} \overset{q}{\sim} \bigoplus_{m\in\mathbb{Z}} \pi_{\overline{\rho},m}. \tag{4.1}$$

It follows that

$$\pi_{\overline{\omega},0} \prec\!\!\prec \bigoplus_{m\in\mathbb{Z}} \pi_{\overline{\rho},m},$$

and since $\pi_{\overline{\omega},0}$ and each $\pi_{\overline{\rho},m}$ are factor representations, there exists some $m \in \mathbb{Z}$ for which $\pi_{\overline{\omega},0} \overset{q}{\sim} \pi_{\overline{\rho},m}$. Let $p = |m|$. By Proposition 3.4, if $m < 0$, $\omega \overset{q}{\sim} \alpha^{*p}\rho$; while if $m \geqslant 0$, $\omega \overset{q}{\sim} \beta^{*p}\rho$, in which case $\alpha^{*p}\omega \overset{q}{\sim} \alpha^{*p}\beta^{*p}\rho \overset{q}{\sim} \rho$ so statement ii) follows.

ii) $\Rightarrow$ iii). If $\alpha^{*p}\omega \overset{q}{\sim} \rho$, then the quasi-orbit of ω contains that of ρ; conversely, since $\omega \overset{q}{\sim} \beta^{*p}\alpha^{*p}\omega \overset{q}{\sim} \beta^{*p}\rho$, the quasi-orbit of ρ contains that of ω.

iii) $\Rightarrow$ i). If the quasi-orbits coincide then ρ is quasi-equivalent to either $\alpha^{*p}\omega$ or $\beta^{*p}\omega$ for some nonnegative p. Thus $\overline{\rho}$ is quasi-equivalent either to $\overline{\alpha^{*p}\omega} = \alpha^{*p}\overline{\omega}$ or to $\overline{\beta^{*p}\omega} = \beta^{*p}\overline{\omega}$. In both cases Proposition 3.3 yields $\overline{\rho} \overset{q}{\sim} \overline{\omega}$. ∎

The next theorem determines exactly which gauge invariant states are pure; it extends results about quasi-free states obtained in [3] for $\dim \mathcal{E} < \infty$.

THEOREM 4.3. *The following conditions are equivalent for a state ω of $\mathcal{F}_\infty$:*

(1) *$\overline{\omega}$ is pure;*

(2) *ω is pure and aperiodic;*

(3) *ω is pure and restriction of the GNS represention of $\overline{\omega}$ to $\mathcal{F}_\infty$ is multiplicity-free;*

(4) *ω is pure and $\overline{\omega}$ is its unique extension.*

Proof. Let $(\pi, \mathcal{H}, \Omega)$ be the GNS triple associated with $\overline{\omega}$. If ω is pure, the decomposition of $\pi\restriction\mathcal{F}_\infty$ consists of irreducible representations, determined by the quasi-orbit of ω under α^* and β^*, as was proved Proposition 3.4. Since ω is aperiodic if and only if all its α^* and β^*-translates are pairwise disjoint, we see that (2) is equivalent to (3).

Suppose (2) holds, then the intertwining operators $\mathcal{I}(\pi_i, \pi_j)$ are trivial (for $i \neq j$ by aperiodicity, and for $i = j$ by irreducibility), in which case the characterization of the commutant of $\mathcal{O}_\infty$ given by (2.4) yields $\pi(\mathcal{O}_\infty)' = \mathbb{C}I$. Conversely, if ω fails to be pure aperiodic, then at least one of the $\mathcal{I}(\pi_i, \pi_j)$ is nontrivial and the same characterization (2.4) shows that $\overline{\omega}$ is not pure. Therefore (1) is equivalent to (2). To show that (1) and (4) are equivalent let $\tilde{\omega}$ be an extension of ω, then

$$\int_0^{2\pi} \tilde{\omega}(\gamma_\lambda(x)) \frac{\mathrm{d}\lambda}{2\pi} = \overline{\omega}(x) \quad x \in \mathcal{O}_\infty.$$

If in addition $\overline{\omega}$ is pure, then the normalized integral over each subinterval of $[0, 2\pi]$ must also equal $\overline{\omega}(x)$, because for each such subinterval one can write $\overline{\omega}$ as a convex linear combination of the corresponding normalized integrals. Hence, the function

$$\lambda \mapsto \tilde{\omega}(\gamma_\lambda(x))$$

is constant for each $x \in \mathcal{O}_\infty$ and $\tilde{\omega} = \overline{\omega}$.

That (4) implies (1) is the well known fact that if a pure state has a unique state extension, then this extension is pure. ∎

We may now apply the preceding results to give a necessarily and sufficient condition for conjugacy of endomorphisms of $\mathcal{B}(\mathcal{H})$. Suppose ω and ρ are aperiodic pure states of $\mathcal{F}_\infty$ so that by Theorem 4.3 their gauge-invariant extensions $\overline{\omega}$ and $\overline{\rho}$ are pure. Then the endomorphisms associated to these states are ergodic by Proposition 3.1 of [4], and they are conjugate if and only if the states $\overline{\omega}$ and $\overline{\rho}$ are *quasi-free equivalent*.

Thus, two ergodic endomorphisms coming from gauge invariant states are conjugate if and only if there exist a positive integer p and a unitary operator U on the Hilbert space $\mathcal{E}$ generated by the isometries $\{v_j\}_{j\geqslant 1}$ such that $\alpha^{*p}\omega$ is quasi-equivalent to $\rho\circ\gamma_U$ or $\alpha^{*p}\rho$ is quasi-equivalent to $\omega\circ\gamma_U$. Equivalently, $\|\alpha^{*n}\omega-\alpha^{*n+p}\rho\circ\gamma_U\|\to 0$ as $n\to\infty$ for some integer p and some unitary U on $\mathcal{E}$.

The following theorem shows that at least part of Theorem 3.5 of [1] also holds for gauge-invariant extensions of essential *factor* states of $\mathcal{O}_\infty$.

THEOREM 4.4. *The gauge invariant extension of an aperiodic essential factor state of $\mathcal{F}_\infty$ is a factor state of $\mathcal{O}_\infty$.*

Proof. Let ω be an aperiodic essential factor state of $\mathcal{F}_\infty$ and let $\pi\restriction\mathcal{F}_\infty = \bigoplus_{m\in\mathbb{Z}}\pi_m$ be the restriction to $\mathcal{F}_\infty$ of the GNS representation of $\overline{\omega}$. By aperiodicity, if $i\neq j$, there are no nontrivial intertwining operators between π_i and π_j, so $\mathcal{I}(\pi_i,\pi_j)=(0)$ and we have that

$$\pi(\mathcal{O}_\infty)' = \{S\in\mathcal{B}(\mathcal{H}):\quad S_{ij}=\delta_{ij}\alpha^i(S_0);\ i,j\in\mathbb{Z},\quad S_0\in\pi_0(\mathcal{F}_\infty)'\}.$$

by the description given in 2.4. If we now consider an element Q in the center, $Q\in\pi(\mathcal{O}_\infty)'\cap\pi(\mathcal{O}_\infty)''$, then Q is diagonal and is determined by its $(0,0)^{th}$ entry. Moreover $Q_{00}=P_0QP_0$ commutes with S_0 for all $S_0\in\pi_0(\mathcal{F}_\infty)'$, hence Q_{00} is in the center of $\pi_0(\mathcal{F}_\infty)'$, hence Q_{00} is in the center of $\pi_0(\mathcal{F}_\infty)'$, which is trivial because ω is a factor state. Therefore Q_{00} is a scalar and so is Q. ■

5. QUASI-FREE STATES

In this section we consider *quasi-free*, i.e. gauge invariant extensions of product states of $\mathcal{F}_\infty$. The underlying idea is that essential product states of $\mathcal{F}_\infty$ behave as the product states of the UHF algebras of pure type n^∞. The results on quasi-equivalence of product states of UHF algebras given in [5], were extended in [4] to essential product states of $\mathcal{F}_\infty$. For convenience we review briefly the basic facts about product states of $\mathcal{F}_\infty$ and a very useful inequality involving the Hilbert-Schmidt and the trace norms of operators on Hilbert space.

With the tensor product picture of $\mathcal{F}_\infty$ mentioned in Section 3 in mind, a product state of $\mathcal{F}_\infty$ is determined by a sequence $\{A_j\}$ of positive operators on $\mathcal{E}$ such that

$\operatorname{Tr} A_j \leqslant 1$ for $j \geqslant 1$. The state $\omega_{\{A_j\}}$ is then defined by the formula

$$\omega_{\{A_j\}}(K_1 \oplus \cdots \oplus K_n \oplus I \oplus I \cdots) = \prod_{j=1}^{n} \operatorname{Tr}(K_j A_j)$$

together with $\omega_{\{A_j\}}(I) = 1$. Such a state is singular if and only if $\prod_1^\infty \operatorname{Tr}(A_j) = 0$, and is essential if and only if $\operatorname{Tr}(A_j) = 1$ for every $j \geqslant 1$.

The essential product state corresponding to a sequence of positive trace-one operators $A_j \in \mathcal{K}(\mathcal{E})$ is denoted by $\omega_A = \bigotimes_1^\infty \omega_{A_j}$, and its value the elementary tensor $T_1 \oplus \cdots \oplus T_k \oplus I \oplus \ldots$ is $\prod_1^\infty \operatorname{Tr}(T_j A_j)$. Since α^* corresponds to the right shift in the infinite tensor product it follows that $\alpha^* \omega_A = \bigoplus_{j=1}^\infty \omega_{A_{j+1}}$. This leads to the following theorem, which states that a celebrated result about equivalence of states of UHF algebras from [5] also holds for essential product states of $\mathcal{F}_\infty$.

THEOREM 5.1. *Suppose $\omega_A = \bigotimes_1^\infty \omega_{A_j}$ and $\omega_B = \bigotimes_1^\infty \omega_{B_j}$ are two product states of $\mathcal{F}_\infty$ corresponding to the sequences $\{A_j\}$ and $\{B_j\}$ of positive trace-one operators on $\mathcal{E}$. the following are equivalent:*

(1) *ω_A and ω_B are quasi-equivalent.*

(2) $\prod_m^\infty \operatorname{Tr}(A_j^{\frac{1}{2}} B_j^{\frac{1}{2}}) \neq 0$ *for some $m \in \mathbf{N}$.*

(3) $\sum_1^\infty \|A_j^{\frac{1}{2}} B_j^{\frac{1}{2}}\|_2^2 < \infty$.

Before giving the proof we recall the Powers-Størmer inequality [7, Lemma 4.1]: *If A and B are positive operators on a separable Hilbert space, then*

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2 \leqslant \|A - B\|_1. \tag{5.1}$$

As usual, $\|\cdot\|_2$, indicates the Hilbert-Schimidt norm, and $\|\cdot\|_1$ the trace-class norm. An argument borrowed from the proof of Lemma 4.2 of [7] yields another useful inequality.

LEMMA 5.2. *If A and B are positive operators on a separable Hilbert space, then*

$$\|A - B\|_1 \leqslant (\|A^{\frac{1}{2}}\|_2 + \|B^{\frac{1}{2}}\|_2)\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2. \tag{5.2}$$

Proof. If either A or B is not of trace class then $\|A^{\frac{1}{2}}\|_2 + \|B^{\frac{1}{2}}\|_2 = \infty$ and the inequality is trivially satisfied, so assume A and B are of trace class. In this case, both $X = A^{\frac{1}{2}}$ and $Y = B^{\frac{1}{2}}$ are Hilbert-Schmidt operators with Hilbert-Schmidt norms $\|X\|_2 = \|A\|_1^{\frac{1}{2}}$ and $\|Y\|_2 = \|B\|_1^{\frac{1}{2}}$. With this considerations, the inequality to be proved becomes $\|X^2 - Y^2\|_1 \leqslant (\|X\|_2 + \|Y\|_2)\|X - Y\|_2$. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis diagonalizing Y, so that $Y\xi_i = y_i\xi_i$ for $i \in \mathbb{N}$, where the y_i's are the eigenvalues of Y. Further, let U be the partial isometry in the polar decomposition $X^2 - Y^2 = U|X^2 - Y^2|$. Then

$$\|X^2 - Y^2\|_1 = \operatorname{Tr}|X^2 - Y^2| = \operatorname{Tr} U^*(X^2 - Y^2) = \sum_i \langle U^*(X^2 - Y^2)\xi_i, \xi_i\rangle =$$

$$= \sum_i \langle U^*(X^2 - y_i^2)\xi_i, \xi_i\rangle = \sum_i \langle U^*(X + y_i)(X - y_i)\xi_i, \xi_i\rangle =$$

$$= \sum_i \langle (X - y_i)\xi_i, (X + y_i)U\xi_i\rangle = \sum_i \langle (X - Y)\xi_i, (XU + UY)\xi_i\rangle =$$

$$= \sum_i \sum_j \langle (X - Y)\xi_i, \xi_j\rangle\langle \xi_j, (XU + UY)\xi_i\rangle.$$

The last equality holds because the inner product can be calculated using the coefficients corresponding to any orthonormal basis. We can now apply the Cauchy-Schwarz inequality to this double summation and obtain

$$\|X^2 - Y^2\|_1 \leqslant \left(\sum_{i,j} |\langle (X - Y)\xi_i, \xi_j\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{i,j} |\langle \xi_j, (XU + UY)\xi_i\rangle|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_i \|(X - Y)\xi_i\|^2\right)^{\frac{1}{2}} \left(\sum_i \|(XU + UY)\xi_i\|^2\right)^{\frac{1}{2}} = \|X - Y\|_2\|XU + UY\|_2.$$

Since U is partial isometry, $\|XU + UY\|_2 \leqslant \|X\|_2 + \|Y\|_2$, hence

$$\|X^2 - Y^2\|_1 \leqslant \|X - Y\|_2(\|X\|_2 + \|Y\|_2)$$

as claimed. ∎

If in addition to the hypothesis of the lemmas, $\operatorname{Tr} A = \operatorname{Tr} B = 1$, then the following double inequality holds:

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2 \leqslant \|A - B\|_1 \leqslant 2\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2.$$

Moreover, since $\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2 = \operatorname{Tr}(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 = \operatorname{Tr}(A + B - A^{\frac{1}{2}}B^{\frac{1}{2}} - B^{\frac{1}{2}}A^{\frac{1}{2}}) = = 2(1 - \operatorname{Tr}(A^{\frac{1}{2}}B^{\frac{1}{2}}))$ the double inequality above yields:

$$2(1 - \operatorname{Tr}(A^{\frac{1}{2}}B^{\frac{1}{2}})) \leqslant \|A - B\|_1 \leqslant 2\sqrt{2}(1 - \operatorname{Tr}(A^{\frac{1}{2}}B^{\frac{1}{2}}))^{\frac{1}{2}}. \tag{5.3}$$

With the aid of (5.3) we are ready to give the proof of Theorem 5.1.

Proof. Since $\|\alpha^{*m}(\omega_A - \omega_B\| = \sup_k \|\bigotimes_{j=1}^{k} A_{m+j} - \bigotimes_{j=1}^{k} B_{m+j}\|_1$, it follows from Proposition 3.6 of [4] that the states ω_A and ω_B are quasi-equivalent if and only if $\sup_k \|\bigotimes_{j=1}^{k} A_{m+j} - \bigotimes_{j=1}^{k} B_{m+j}\|_1 \to 0$ as $m \to \infty$. The evaluation of this trace norm in terms of the individual members of the sequences $\{A_j\}$ and $\{B_j\}$ is not easy so one consideres (5.3) with $\bigotimes_{j=1}^{k} A_{m+j}$ in place of A and $\bigotimes_{j=1}^{k} B_{m+j}$ in place of B to see that quasi-equivalence occurs if and only if

$$\inf_k \operatorname{Tr}\left(\bigotimes_{j=1}^{k} A_{m+j}^{\frac{1}{2}} \bigotimes_{j=1}^{k} B_{m+j}^{\frac{1}{2}}\right) \to 1 \qquad \text{as } m \to \infty,$$

i.e., if and only if $\inf_k \prod_{j=m+1}^{m+k} \operatorname{Tr}(A_j^{\frac{1}{2}} B_j^{\frac{1}{2}}) \to 1$ as $m \to \infty$.

In view of the fact that the numbers $\operatorname{Tr}(A_j^{\frac{1}{2}} B_j^{\frac{1}{2}})$ are all between 0 and 1, the above happenes if and only if their infinite product starting at m is not zero for some $m \in \mathbb{N}$. Therefore conditions (1) and (2) of Theorem 5.1 are equivalent . It is easy to see that condition (2) holds if and only if $\sum_j \left(1 - \operatorname{Tr}(A_j^{\frac{1}{2}} B_j^{\frac{1}{2}})\right) < \infty$, which happens if and only if $\sum_1^\infty \|A_j^{\frac{1}{2}} - B_j^{\frac{1}{2}}\|_2^2 < \infty$. Thus conditions (2) and (3) are seen to be equivalent. ■

As a consequence we mention the corresponding result when the states considered are pure, that is, when each A_j and B_j is a rank-one projection. For this it becomes convenient to alter the notation slightly, and considering unit vectors in the ranges of those projections, we label the states with the corresponding sequences of unit vectors (which are determined only up to scalar multiples).

COROLLARY 5.3. *The two pure product states $\bigotimes_{j=1}^{\infty} \omega_{f_j}$ and $\bigotimes_{j=1}^{\infty} \omega_{g_j}$ of $\mathcal{F}_\infty$ corresponding to the sequences of unit vectors $\{f_j\}$ and $\{g_j\}$ in $\mathcal{E}$ are unitarily equivalent if and only if $\sum_{j=1}^{\infty}(1 - |\langle f_j, g_j\rangle|) < \infty$.*

Proof. Since $1 \leqslant 1 + |\langle f_j, g_j\rangle| \leqslant 2$ convergence of the series above is equivalent to convergence of the series $\sum_{j=1}^{\infty}(1 - |\langle f_j, g_j\rangle|^2)$. In view of the formula for the Hilbert-Schmidt distance between two rank-one projections, this is a restatement of condition (3) of Theorem 5.1. The corollary follows because for pure states quasi-equivalence is the same as unitary equivalence. ■

Of course this can be combined with the results of Section 4 to obtained the following two corollaries.

COROLLARY 5.4 (Quasi-equivalence of quasi-free states of $\mathcal{O}_\infty$). *Suppose* $\omega_A = = \bigotimes_{j=1}^{\infty} \omega_A$ *and* $\omega_B = \bigotimes_{j=1}^{\infty} \omega_B$ *are two product states of* $\mathcal{F}_\infty$*; their gauge invariant extensions* $\overline{\omega}_A$ *and* $\overline{\omega}_B$ *are quasi-equivalent if and only if there exists an integer* p *such that* $\sum_{j=|p|}^{\infty} \|A_j^{\frac{1}{2}} - B_{j+p}^{\frac{1}{2}}\|_2^2 < \infty$.

Proof. It follows at once from Theorem 5.1 and Theorem 4.2. ■

If the product states under consideration are pure, the corollary says that $\overline{\omega}_f$ is quasi-equivalent to $\overline{\omega}_g$ if and only if $\sum_{j=|p|}^{\infty}(1 - |\langle f_j, g_{j+p}\rangle|) < \infty$ for some $p \in \mathbf{Z}$. We underline the fact that quasi-equivalence cannot be replaced by unitary equivalence at this point because gauge invariant extensions of pure states need not be pure. In fact, the pure product state $\bigotimes \omega_{f_j}$, induces a pure quasi-free state of $\mathcal{O}_\infty$ if and only if it is aperiodic, that is, if and only if $\sum_{j=|p|}^{\infty}(1 - \langle f_j, g_{j+p}\rangle|) = \infty$ for each $p \in \mathbf{Z}$.

COROLLARY 5.5. (Unitary equivalence of pure quasi-free states). *Suppose* $\overline{\omega_f}$ *and* $\overline{\omega_g}$ *are pure quasi-free states extending the pure aperiodic product states* $\bigotimes \omega_{f_j}$ *and* $\bigotimes \omega_{g_j}$ *respectively. Then they are unitarily equivalent if and only if there exists an integer* $p \in \mathbf{Z}$ *such that* $\sum_{j=|p|}^{\infty}(1 - |\langle f_j, g_{j+p}\rangle|) < \infty$.

Proof. The states are pure because they extend aperiodic states, so we can replace quasi-equivalence by unitary equivalence in the application of Corollary 5.4. ■

Finally, suppose α is the ergodic endomorphism of $\mathcal{B}(\mathcal{H})$ induced by the pure aperiodic state $\bigotimes \omega_{f_j}$, and similarly β is the one induced by $\bigotimes \omega_{g_j}$. Then α is conjugate to β if and only if $\sum_{j=|p|}^{\infty}(1 - |\langle f_j, Ug_{j+p}\rangle|) < \infty$ for some integer p and some unitary U on $\mathcal{E}$. In other words, the sequences of rank-one projections on $\mathcal{E}$ determining the states are such that by shifting one of them and "turning" each of its terms by the same unitary U, they become close in the Hilbert-Schmidt norm, in the sense that $\sum_{j=|p|}^{\infty} \|P_{f_j} - UP_{g_{j+p}}U^*\|_2^2$ converges.

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