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COMPACT OPERATORS OF COMPOSITION ON SOME LOCALLY CONVEX FUNCTION SPACES

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INTRODUCTION

One of the several methods of creating a new function from given two functions S and T is given by their composition $S \circ T$ whenever it is posible. If S varies in a linear space of functions with pointwise linear operations for which composition $S \circ T$ is meaningful, then the correspondence taking S to $S \circ T$ gives rise to a linear transformation which we denote by C_T and designate as a (right) composition transformation or substitution correspondence induced by T. Similarly, if S is a vector space homomorphism and T is allowed to vary in a linear space of functions such that $S \circ T$ makes sense, then the mapping $T \to S \circ T$ is a linear transformation which we denote by ${}_SC$ and designate as a (left) composition transformation induced by S. In case the linear spaces of functions are topological vector spaces and C_T and ${}_SC$ are continuous, they are known as composition operators induced by T and S respectively. These transformations have been the subject matter of thorough and systematic study for the last several decades on different function spaces, viz. spaces of analytic functions, spaces of continuous functions, L^p-spaces, etc. In case we have a function algebra with composition as multiplication, then every element of the function algebra gives rise to a (left) composition operator. For results on composition operators on these function spaces and their applications, we refer to Cowen [3], Kamovitz [5], Mayer [8], McCoy and Ibula [9], Nordgren [11], Rajagopalan and Jamison [13], and Singh [14].

Our interest in this note centers around the study of compact composition operators on the weighted locally convex space $\mathrm{CV}_b(X, E)$ (or $\mathrm{CV}_0(X, E)$) of vector-valued continuous functions on X and on the locally convex space $\mathrm{CL}_b(E)$ of continuous vector space endomorphisms on E. For further details of composition operators on weighted spaces, we refer to Singh and Summers [17] and Singh and Manhas [15].

The study in this note is divided into four sections. In the first section we define the spaces $CV_b(X, E)$, $CV_0(X, E)$ and $CL_b(E)$, and some results regarding composition operators C_T on weighted spaces $CV_b(X)$ and $CV_0(X)$ are reported in Section 2. The characterization of compact composition operators C_T is given in Section 3. In the fourth section we present a characterization of operators on $CL_b(E)$ which are composition operators of the type ${}_TC$.

1. PRELIMINARIES

If X is a completely regular Hausdorff space and E is a locally convex Hausdorff topological vector space over $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$, then by $C(X, E)$ we denote the set of all continuous functions from X into E and by $cs(E)$ the collection of all continuous seminorms on E. A function $f : X \to E$ is said to vanish at infinity if for each $\varepsilon > 0$ and $p \in cs(E)$, there exists a compact subset K of X such that $p(f(x)) < \varepsilon$ for every $x \in X \setminus K$, or equivalently, if the set $\{x \in X : p(f(x)) \geqslant \varepsilon\}$ is relatively compact for every $\varepsilon > 0$ and $p \in cs(E)$. A subset M of E is said to be bounded if for every $p \in cs(E)$, $\sup\{p(x) : x \in M\}$ is finite, or equivalently, if for each neighbourhood N of origin there exists a number $\alpha > 0$ such that $M \subseteq \alpha N$.

Let V be a set of weights on X, where by a *weight* on X we mean a nonnegative upper semi-continuous function on X. If for every $v_1, v_2 \in V$ and $\alpha > 0$, there exists $v \in V$ such that $\alpha v_i(x) \leqslant v(x)$ for each $x \in X$ and $i = 1, 2$, then we say that V is *directed upward*. We say that $V > 0$ if for every $x \in X$ there is some $v \in V$ such that $v(x) > 0$. By a system of weights on X we mean a directed upward set V of weights on X such that $V > 0$. If V and W are two systems of weights on X and if for every $v \in V$ there is a $w \in W$ such that $v(x) \leqslant w(x)$ for each $x \in X$, then we write $V \leqslant W$. For a system V of weights on X, we now define the following weighted spaces of E-valued continuous functions on X:

$$CV_b(X, E) = \{f \in C(X, E) : vf(X) \text{ is bounded in } E \text{ for each } v \in V\}$$

and

$$CV_0(X, E) = \{f \in C(X, E) : vf \text{ vanishes at infinity on } X \text{ for each } v \in V\}.$$

Clearly $CV_b(X, E)$ and $CV_0(X, E)$ are vector spaces over $\mathbf{K}$ with pointwise linear operations while the upper semi-continuity of the weights implies that $CV_0(X, E) \subseteq$ $\subseteq CV_b(X, E)$. For $v \in V$, $p \in cs(E)$ and $f \in C(X, E)$, let us put

$$\|f\|_{v,p} = \sup\{v(x)p(f(x)) : x \in X\}.$$

Then $\|\cdot\|_{v,p}$ is a seminorm on $CV_b(X, E)$ and on $CV_0(X, E)$ and the family $\{\|\cdot\|_{v,p} : v \in V,\ p \in cs(E)\}$ of seminorms generates locally convex Hausdorff topologies on $CV_b(X,E)$ and $CV_0(X,E)$. These spaces with the corresponding topology are known as the weighted spaces of vector-valued continuous functions on X. In case $E = \mathbf{K}$, we shall omit E from our notation and write $CV_b(X)$ and $CV_0(X)$ in place of $CV_b(X,E)$ and $CV_0(X,E)$ respectively. We also then put $\|\cdot\|_{v,p} = \|\cdot\|_v$ for every $v \in V$, where $p(z) = |z|$, $z \in \mathbf{K}$.

The spaces $CV_b(X)$ and $CV_0(X)$ were first introduced by Nachbin [10] and the corresponding vector-valued analogues were subsequently studied in detail by Bierstedt [1,2] and Prolla [12]. The assumption that for each $x \in X$, there exists an $f_x \in CV_0(X)$ such that $f_x(x) \neq 0$ would be in force throughout this note.

The symbol $CL(E)$ stands for the vector space of all continuous vector space endomorphisms on E whereas the symbol $\mathcal{B}$ stands for the family of all bounded subsets of E. For each $M \in \mathcal{B}$ and $p \in cs(E)$, we define the seminorm $\|\cdot\|_{M,p}$ on $CL(E)$ as

$$\|A\|_{M,p} = \sup\{p(A(s)) : s \in M\}.$$

The family $\{\|\cdot\|_{M,p} : M \in \mathcal{B},\ p \in cs(E)\}$ of seminorms defines a locally convex Hausdorff topology on $CL(E)$ and the vector space $CL(E)$ endowed with this topology becomes a locally convex space of continuous endomorphisms on E. We denote this space by $CL_b(E)$. The convergence in this topology is the uniform convergence on bounded subsets of E. For details, we refer to Grothendieck [4] and Köthe [7].

2. COMPOSITION OPERATORS ON THE WEIGHTED SPACES

If $T : X \to X$ is a continuous function and V is a system of weights on X, then

$$V(T) = \{v \circ T : v \in V\}$$

is also a system of weights on X. It has been proved in [17] by Singh and Summers that a function $T : X \to X$ induces a composition operator C_T on $CV_b(X)$ if and only if T is continuous and $V \leqslant V(T)$. In case of $CV_0(X)$, a function $T : X \to X$ induces a composition operator C_T on $CV_0(X)$ if and only if T is continuous, $V \leqslant V(T)$, and for each $v \in V$, $\varepsilon > 0$ and compact subset K of X, $T^{-1}(K) \cap \{x \in X : v(x) \geqslant \varepsilon\}$ is compact. If T is a homeomorphism, then the condition that $V \leqslant V(T)$ is sufficient for T to induce a composition operator C_T on $CV_0(X)$. Singh and Manhas [15] have given similar type of results characterizing composition operators on $CV_b(X,E)$ and $CV_0(X,E)$.

The inducibility of composition operators on the weighted spaces is very much affeccted by the system of weights. Sometimes even the nice functions like homeomorphisms and constant functions fail to induce composition operators. For example, if $X = E = \mathbf{R}$ and $V = \{\alpha v : \alpha > 0\}$, where

$$v(x) = \begin{cases} e^{-x}, & x \geqslant 0 \\ e^{-x^2}, & x < 0 \end{cases}$$

then positive translations induce composition operators whereas all negative translations fail to do so. The conditions under which constant functions induce composition operators on the weighted spaces are given in the following propositions.

PROPOSITION 2.1. *Let E be a non-zero locally convex Hausdorff topological vector space and let V be a system of weights on X. Then the following statements are equivalent:*

(a) *Every $v \in V$ is bounded on X.*

(b) *For every $s \in E$, $\mathbf{1}_s \in CV_b(X, E)$ where $\mathbf{1}_s : X \to E$ denotes the constant map given by $\mathbf{1}_s(x) = s$ for each $x \in X$.*

(c) *Every constant selfmap on X induces a composition operator on $CV_b(X, E)$.*

Proof (a) $\Rightarrow$ (b). Suppose every $v \in V$ is bounded on X. Then there exists an $m_v > 0$ such that $v(x) \leqslant m_v$ for every $x \in X$. Let $s \in E$. Then $\mathbf{1}_s \in C(X, E)$ and

$$p((v\mathbf{1}_s)(x)) = p(v(x)\mathbf{1}_s(x)) = v(x)p(s) \leqslant m_v p(s)$$

for every $x \in X$ and $p \in cs(E)$. Thus $v\mathbf{1}_s(X)$ is bounded in E for every $v \in V$. Hence $\mathbf{1}_s \in CV_b(X, E)$ for every $s \in E$.

(b) $\Rightarrow$ (c). Suppose $\mathbf{1}_s \in CV_b(X, E)$ for every $s \in E$. Let $T : X \to X$ be a constant function given by $T(x) = y$ for each $x \in X$, and some $y \in X$. Then, for each $f \in CV_b(X, E)$, we have $C_T f = f \circ T \in C(X, E)$ and

$$\begin{aligned} p((vC_T f)(x)) &= p(v(x)f(T(x))) = p(v(x)f(y)) \\ &= p(v(x)t), \qquad \text{where } t = f(y) \in E, \\ &= p(v(x)\mathbf{1}_t(x)) = p((v\mathbf{1}_t)(x)) \end{aligned}$$

which is finite for each $v \in V$ and $p \in cs(E)$ since $\mathbf{1}_t \in CV_b(X, E)$. This shows that $C_T f \in CV_b(X, E)$. The continuity of C_T is obvious. Thus C_T is a composition operator on $CV_b(X, E)$.

(c) $\Rightarrow$ (a). Suppose $T(x) = y$ for every $x \in X$ and some $y \in X$ and suppose T induces a composition operator C_T on $CV_b(X, E)$. Let $f_y \in CV_b(X)$ be such that $f_y(y) \neq 0$ and for any $s \in E$, let $g_s : X \to E$ be defined as $g_s(x) = f_y(x)s$ for each $x \in X$. Then $g_s \in CV_b(X, E)$. Since C_T is a composition operator on $CV_b(X, E)$, we have $f \circ T \in CV_b(X, E)$ for every $f \in CV_b(X, E)$. In particular, $g_s \circ T \in CV_b(X, E)$. Let $0 \neq s \in E$ and $q \in cs(E)$ such that $q(s) \neq 0$. Then if $v \in V$, there exists an $m > 0$ such that

$$q(v(x)g_s(T(x))) \leqslant m \quad \text{for each } x \in X,$$

which implies that $v(x)|f_y(y)|q(s) \leqslant m$ for each $x \in X$. From this, we conclude that $v(x) \leqslant \dfrac{m}{|f_y(y)|q(s)}$ for each $x \in X$. This shows that v is a bounded function on X. This completes the proof of the proposition. ■

PROPOSITION 2.2. *Let E be a non-zero locally convex Hausdorff topological vector space and let V be a system of weights on X. Then the following statements are equivalent:*

(a) *Every $v \in V$ vanishes at infinity on X*

(b) *For every $s \in E$, $\mathbf{1}_s \in \mathrm{CV}_0(X,E)$.*

(c) *Every constant selfmap on X induces a composition operator on* $\mathrm{CV}_0(X,E)$.

Proof (a) $\Rightarrow$ (b). Suppose each $v \in V$ vanishes at infinity on X. Then the set $\{x \in X : v(x) \geqslant \varepsilon\}$ is compact in X for each $\varepsilon > 0$. Let $s \in E$. Then for every $p \in \mathrm{cs}(E)$ the set $\{x \in X : p((v\mathbf{1}_s)(x)) \geqslant \varepsilon\}$ is a compact subset of X, because it is empty if $p(s) = 0$ and it is equal to $\left\{x \in X : v(x) \leqslant \dfrac{\varepsilon}{p(s)}\right\}$ if $p(s) \neq 0$, and hence $\mathbf{1}_s \in \mathrm{CV}_0(X,E)$.

(b) $\Rightarrow$ (c). Suppose $\mathbf{1}_s \in \mathrm{CV}_0(X,E)$ for every $s \in E$ and let $T(x) = y$ be a constant function on X, where $y \in X$. Then $C_Tf = f \circ T \in C(X,E)$ for every $f \in \mathrm{CV}_0(X,E)$, and for every $v \in V$, $p \in \mathrm{cs}(E)$ and $\varepsilon > 0$, the set

$$\{x \in X : p((vC_Tf)(x)) \geqslant \varepsilon\} = \{x \in X : p((v\mathbf{1}_{f(y)})(x)) \geqslant \varepsilon\}$$

is compact in X since $\mathbf{1}_{f(y)} \in \mathrm{CV}_0(X,E)$. This shows that $C_Tf \in \mathrm{CV}_0(X,E)$. The continuity of C_T is obvious. Thus C_T is a composition operator on $\mathrm{CV}_0(X,E)$..

(c) $\Rightarrow$ (a). Suppose $T : X \to X$ is a constant function on X with the value $y \in X$ and suppose T induces a composition operator C_T on $\mathrm{CV}_0(X,E)$. Let $f_y \in \mathrm{CV}_0(X)$ be such that $f_y \neq 0$, and let $0 \neq s \in E$ and $q \in \mathrm{cs}(E)$ such that $q(s) \neq 0$. Now define $g_s : X \to E$ as

$$g_s(x) = \frac{f_y(x)}{f_y(y)q(s)}s$$

for each $x \in X$. Then $g_s \in \mathrm{CV}_0(X,E)$. Let $v \in V$ and $\varepsilon > 0$ be arbitrary. Then the set

$$\{x \in X : v(x)q(C_Tg_s(x)) \geqslant \varepsilon\} = \{x \in X : v(x)q(g_s(y)) \geqslant \varepsilon\} =$$
$$= \{x \in X : v(x) \geqslant \varepsilon\}$$

is compact in X because $C_Tg_s \in \mathrm{CV}_0(X,E)$. This concludes the proof of the proposition. ■

3. COMPACT COMPOSITION OPERATORS C_T ON THE WEIGHTED SPACES

A linear transformation A from a topological vector space E into itself is said to be compact if the image of every bounded subset of E under A is relatively compact in E. Recall that a subset F of $CV_b(X,E)$ (or $CV_0(X,E)$) is bounded if for each $v \in V$ and $p \in cs(E)$, there exists a constant $m_{v,p} > 0$ such that $\|f\|_{v,p} \leqslant m_{v,p}$ for each $f \in F$. If A is a compact operator on $CV_b(X,E)$ (or $CV_0(X,E)$) and $\{f_n\}$ is a bounded sequence in $CV_b(X,E)$, then there exists a subsequence $\{f_{n_k}\}$ and $g \in CV_b(X,E)$ (or $CV_0(X,E)$) such that $\{Af_{n_k}\}$ converges to g.

The main aim of this section is to study compact composition operators on the weighted spaces. If X is a compact Hausdorff space, V consists of constant weights and $E = \mathbf{K}$, then $CV_b(X) = CV_0(X) = C(X)$ with the topology of uniform covergence on X. In this case compact composition operators have been characterized by Kamovitz [5]. Singh and Summers in [16] generalized the results of Kamovitz to spaces of vector-valued continuous functions on a completely regular Hausdorff space. Now, we shall prove a theorem which characterizes compact composition operators C_T on the weighted spaces of vector-valued continuous functions on a completely regular connected space. This result generalizes a result presented in [6]. First, we prove the following lemma which is used in the proof of the main theorem.

LEMMA 3.1. *Let V be a system of weights on X, E be a non-zero locally convex Hausdorff topological vector space and let $y \in X$. Then there exists an open set containing y on which each $v \in V$ is bounded.*

Proof. Let $y \in X$ and let $f_y \in CV_0(X)$ such that $f_y(y) \neq 0$. Let $0 \neq s \in E$ and $q \in cs(E)$ such that $q(s) \neq 0$. Define a function $g_s : X \to E$ as $g_s(x) = f_y(x)\dfrac{s}{q(s)}$ for each $x \in X$. Then $g_s \in CV_b(X,E)$. Let

$$G_q = \{x \in X : q(g_s(x)) > \frac{1}{2}|f_y(y)|\}.$$

Then G_q is an open set containing y. For $v \in V$ and $p \in cs(E)$, let $m_{v,p} = \|g_s\|_{v,p}$. Then, for every $x \in G_q$, $v(x)q(g_s(x)) \leqslant m_{v,q}$. Thus

$$v(x) \leqslant \frac{m_{v,q}}{q(g_s(x))} \leqslant \frac{2m_{v,q}}{|f_y(y)|} < \infty.$$

This shows that v is bounded on G_q.

The following theorem shows that the collection of compact composition operators on the weighted spaces is not too large if the underlying topological space X is connected and dimension of E is finite.

THEOREM 3.2. *Let V be a system of weights on a connected completely regular Hausdorff space X satisfying* 2.1 (a) *(or* 2.2 (a)*), and let E be a non-zero finite* *dimensional locally convex Hausdorff topological vector space. Then a composition operator C_T on $\mathrm{CV}_b(X,E)$ (or $\mathrm{CV}_0(X,E)$) is compact if and only if the inducing function T is constant.*

Proof. Let $T : X \to X$ be a map such that C_T is a compact composition operator on $\mathrm{CV}_b(X,E)$. Then we want to show that T is a constant function. Let $x_1, x_2 \in X$ and let $y_1 = T(x_1) \neq T(x_2) = y_2$. Then, by Lemma 3.1, there exists an open set G containing y_1 on which every member v of V is bounded and $y_2 \notin G$. By complete regularity of X, there is an $f \in C(X)$ such that $0 \leqslant f \leqslant 1$, $f(y_1) = 1$ and $f(x) = 0$ for each $x \notin G$. Let $0 \neq s \in E$ and $q \in \mathrm{cs}(E)$ such that $q(s) \neq 0$. Define $f_s : X \to E$ as

$$f_s(x) = f(x)\frac{s}{q(s)}$$

for every $x \in X$. Then $f_s \in \mathrm{CV}_b(X,E)$ with $f_s(y_1) = f(y_1)\dfrac{s}{q(s)} = \dfrac{s}{q(s)}$ and $f_s(y_2) = 0$. Let $g_n(x) = f^{n-1}(x)f_s(x)$ for every $n \in \mathbb{N}$, where f^n is the product of f with itself n-times. Let $F = \{g_n : n \in \mathbb{N}\}$. Then boundedness of every $v \in V$ on G together with the fact that g_n vanishes off G implies that F is a bounded subset of $\mathrm{CV}_b(X,E)$. Since C_T is compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in \mathrm{CV}_b(X,E)$ such that $\{C_T g_{n_k}\}$ converges to g. This implies that for every $v \in V$ and $p \in \mathrm{cs}(E)$, $\{\|g_{n_k} \circ T - g\|_{v,p}\}$ converges to zero, which further implies that $v(x)p(g_{n_k}(T(x)) - g(x))$ converges to zero for each $x \in X$. From this, we conclude that $p(g_{n_k}(T(x)) - g(x))$ conveges to zero for each $x \in X$. In particular, $q \circ g_{n_k}(T(x))$ conveges to $q \circ g(x)$ for each $x \in X$. But this means that $q \circ g$ is a characteristic function with $q \circ g(x_1) = 1$ and $q \circ g(x_2) = 0$, which is a contradiction since $q \circ g$ is continuous and X is connected. Thus $T(x_1) = T(x_2)$ for every $x_1, x_2 \in X$. This shows that T is a constant function. The converse is obvious. This completes the proof of the theorem. ■

REMARK 3.3. In case $\dim E = \infty$, then the constant functions do not always induce compact composition operators.

4. COMPOSITION OPERATORS ${}_TC$ ON $\mathrm{CL}_b(E)$

In this section we shall characterize all those vector space endomorphisms T on E which induce composition operators ${}_TC$ on $\mathrm{CL}_b(E)$ and then identify all operators on $\mathrm{CL}_b(E)$ which are of the type ${}_TC$.

THEOREM 4.1. *Let T be an endomorphism on E. Then ${}_TC$ is a composition operator on $\mathrm{CL}_b(E)$ if and only if T is continuous.*

Proof. Suppose T is continuous. In order to show that ${}_TC$ is a composition operator on $\mathrm{CL}_b(E)$, it is enough to show that ${}_TC$ is continuous at the origin. Let $\{A_\alpha\}$ be a net in $\mathrm{CL}_b(E)$ such that $\{\|A_\alpha\|\}_{M,p}\}$ converges to zero for each $M \in B$ and $p \in \mathrm{cs}(E)$. Then

$$\|{}_TCA_\alpha\|_{M,p} = \|T \circ A_\alpha\|_{M,p} \leqslant \|A_\alpha\|_{M,p \circ T} \to 0.$$

This proves that ${}_TC$ is continuous at the origin. Conversely, if ${}_TC$ is a composition operator on $\mathrm{CL}_b(E)$, then ${}_TCA \in \mathrm{CL}_b(E)$ for each $A \in \mathrm{CL}_b(E)$. Let $A = I$, the identity homomorphism on E. Then $T = {}_TCI \in \mathrm{CL}_b(E)$. Thus T is a continuous endomorphism on E. ■

COROLLARY 4.2. *If E is a Banach space, then $\mathrm{CL}_b(E)$ becomes a Banach space with operator norm. In this case ${}_TC$ is a composition operator if and only if T is bounded. Moreover $\|{}_TC\| = \|T\|$.*

THEOREM 4.3. *Let Φ be an operator on $\mathrm{CL}_b(E)$. Then $\Phi = {}_TC$ for some $T \in \mathrm{CL}_b(E)$ if and only if $\Phi(A \circ B) = \Phi(A) \circ B$ for every $A, B \in \mathrm{CL}_b(E)$.*

Proof. Suppose $\Phi = {}_TC$ for some $T \in \mathrm{CL}_b(E)$, and let $A, B \in \mathrm{CL}_b(E)$. Then

$$\Phi(A \circ B) = {}_TC(A \circ B) = T \circ A \circ B = {}_TC(A) \circ B = \Phi(A) \circ B.$$

Conversely, we assume that $\Phi(A \circ B) = \Phi(A) \circ B$ for each $A, B \in \mathrm{CL}_b(E)$. Taking $A = I$, the identity homomorphism on E we have $\Phi(B) = \Phi(I) \circ B = T \circ B = {}_TC(B)$, where $T = \Phi(I) \in \mathrm{CL}_b(E)$. This shows that $\Phi = {}_TC$ and the proof is complete. ■

We now characterize invertible composition operators ${}_TC$ on $\mathrm{CL}_b(E)$. Recall that a transformation $S \in \mathrm{CL}_b(E)$ is left (or right) inverse of $T \in \mathrm{CL}_b(E)$ if $S \circ T = I$ (or $T \circ S = I$).

THEOREM 4.4. *Let ${}_TC$ be a composition operator on $\mathrm{CL}_b(E)$. Then*
(a) *${}_TC$ is right invertible if and only if $T \in CL(E)$ is right invertible.*
(b) *${}_TC$ is left invertible if $T \in CL(E)$ is left invertible.*

Proof (a). If S is right inverse of T, then ${}_SC$ is right inverse of ${}_TC$. Conversely, if ${}_TC$ is right invertible, then there exists an operator Φ on $\mathrm{CL}_b(E)$ such that ${}_TC \circ \Phi = I$, the identity operator on $\mathrm{CL}_b(E)$, that is, for each $A \in \mathrm{CL}_b(E)$, $T \circ \Phi(A) = A$. Taking $A = I$, the identity homomorphism on E, we see that $\Phi(I)$ is the right inverse of T. Thus T is right invertible.

(b) It is clear that if S is left inverse of T, then ${}_SC$ is left inverse of ${}_TC$. ■

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Added in proof (June 17, 1994).

(1) After this paper was accepted for publication, Professor (Mrs.) A. I. Singh pointed out some errors in the preprint and that some of the results of section 4 can be obtained from B. E. Johnson (*Proc. London Math. Soc.*, **14**(1964), 299–320). The errors have been corrected.

(2) It will be worthwhile to characterize compact composition operators ${}_TC$ on $CL_b(E)$, when E is any locally convex space. In case E is a normed space, it is known that ${}_TC$ is compact if and only if T is compact, details can be seen in F. F. Bonsall and J. Duncan (*Complete Normed Algebras*, Springer-Verlag, New York, 1973).

(3) There has been further progress on the subject since this paper was written. We refer to the following references:

LINDSTROM, M.; LLAVONA, J., Compact and weakly compact homomorphisms between algebras of continuous functions, *J. Math. Anal. Appl.*, **166**(1992), 325–330.

SINGH, R. K.; MANHAS, J. S., *Composition operators on function spaces*, Math. Studies **179**, North-Holland, Amsterdam, 1993.

TAKAGI, H., Compact weighted composition operators on certain subspaces of $C(X, E)$, *Tokyo J. Math.*, **14**(1991), 121–127.

TAKAGI, H.; WADA, J., Weakly compact weighted composition operators on certain subspaces of $C(X, E)$, *Proc. Japan Acad.*, **67**(1991), 304–307.