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A SIMPLE PROOF OF AN OPERATOR INEQUALITY OF JOCIĆ AND KITTANEH

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The arithmetic-geometric mean inequality proved in [1,3] says that for every unitarily invariant norm on Hilbert space operators we have

$$|||A^*XB||| \leqslant \frac{1}{2}|||AA^*X + XBB^*|||. \tag{1}$$

Using this Jocić and Kittaneh [2] have proved the following interesting result.

LEMMA: *Let A, B be self adjoint and X an arbitrary operator. Then for every positive integer n and $j = 1, 2, \ldots, n$*

$$|||A^{n+j}XB^{n-j+1} - A^{n-j+1}XB^{n+j}||| \leqslant |||A^{n+j+1}XB^{n-j} - A^{n-j}XB^{n+j+1}|||. \tag{2}$$

A proof much simpler than the one in [2] is given below.

Using (1) we have for all $n \geqslant 1$

$$\begin{aligned} |||A^{n+1}XB^n - A^nXB^{n+1}||| &= |||A(A^nXB^{n-1} - A^{n-1}XB^n)B||| \leqslant \\ &\leqslant \frac{1}{2}|||A^2(A^nXB^{n-1} - A^{n-1}XB^n) + (A^nXB^{n-1} - A^{n-1}XB^n)B^2||| \leqslant \\ &\leqslant \frac{1}{2}|||A^{n+2}XB^{n-1} - A^{n-1}XB^{n+2}||| + \frac{1}{2}|||A^{n+1}XB^n - A^nXB^{n+1}|||. \end{aligned}$$

Hence

$$|||A^{n+1}XB^n - A^nXB^{n+1}||| \leqslant |||A^{n+2}XB^{n-1} - A^{n-1}XB^{n+2}|||. \tag{3}$$

This proves (2) in the special case $j = 1$. The general case is proved by induction. Suppose (2) has been proved for $j - 1$ in place of j. Then using (1), the triangle inequality and the induction hypothesis we obtain

$$|||A^{n+j}XB^{n-j+1} - A^{n-j+1}XB^{n+j}||| = |||A(A^{n+j-1}XB^{n-j} - A^{n-j}XB^{n+j-1})B||| \leqslant$$

$$\leqslant \frac{1}{2}|||A^2(A^{n+j-1}XB^{n-j} - A^{n-j}XB^{n+j-1}) + (A^{n+j-1}XB^{n-j} - A^{n-j}XB^{n+j-1})B^2||| \leqslant$$

$$\leqslant \frac{1}{2}|||A^{n+j+1}XB^{n-j} - A^{n-j}XB^{n+j+1}||| +$$

$$+\frac{1}{2}|||A^{n+(j-1)}XB^{n-(j-1)+1} - A^{n-(j-1)+1}XB^{n+(j-1)}||| \leqslant$$

$$\leqslant \frac{1}{2}|||A^{n+j+1}XB^{n-j} - A^{n-j}XB^{n+j+1}||| + \frac{1}{2}|||A^{n+j}XB^{n-(j-1)} - A^{n-(j-1)}XB^{n+j}|||.$$

From this the inequality (2) follows. ∎

This Lemma is crucial to the proof of the main Theorem in [2] and its proof there is perhaps the most intricate part of that paper. The simple proof presented above might, therefore, have some interest.

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