J. OPERATOR THEORY
31(1994), 23–33

FUNCTIONS VANISHING MODULO SINGULAR INNER FUNCTIONS

A. R. LUBIN

1.

Let H^∞ and H^2 denote the standard Hardy spaces of functions analytic in the unit disc D that are bounded and have square summable power series coefficients respectively. Let σ be a positive Borel measure on the unit circle T, which we identify with $[0, 2\pi]$, singular with respect to Lebesgue measure and let $s = s_\sigma$ be the associated singular inner function

$$s(z) = \exp\left[-\frac{1}{2\pi}\int_T \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\sigma(\theta)\right].$$

For a given $s(z), S_s = S$ will denote the restricted shift defined on $(sH^2)^\perp$ by $Sf = Pzf$, where P denotes the orthogonal projection of H^2 onto $(sH^2)^\perp$. We will assume much of the basic theory of Hardy spaces and inner functions which can be found for example in [3, 4, 13].

In this paper we are concerned with solutions φ of the system of equations

$$(*) \qquad \varphi(z) = (z-\xi)h_\xi(z) + s(z)k_\xi(z) \quad \text{for all } \xi \in Z$$

subject to the condition $(\|h_\xi\|_\infty + \|k_\xi\|_\infty) \leqslant M < \infty$, i.e. we require the bound M to be uniform for $\xi \in Z$, where $Z \subset T$ is some fixed set such that $\sigma(Z) = \sigma(T)$. We say such a solution φ vanishes on Z modulo $s(z)$. Clearly, any such φ must be bounded, in fact $\|\varphi\|_\infty \leqslant 2M$, and any $\varphi \in sH^\infty$ is always a trivial solution. C. Foiaş proved [2] that if $Z = T$ and $\omega(t) = O(t \log t^{-1})$, then every solution of $(*)$ must be trivial; here ω denotes the modulus of continuity $\omega_\sigma(t) = \sup\{\sigma(a,b) : (b-a) \leqslant t\}$.

In section 2, we extend the result of Foiaş first to the case of an arbitrary Z of full σ-measure and then to inner functions corresponding to a larger class of measures, which we refer to as almost almost smooth (a.a.s.) as explained below. We show that these a.a.s. measures must give zero measure to every Carleson set $C \subset T$.

In section 3, as our main result, we show that if σ gives positive measure to some Carleson set, then $(*)$ does have non-trivial solutions. We interpret this in section 4 as a multiplicity condition and apply it to give an unusual approximation result.

It would be of interest to know whether the condition $\sigma(C) > 0$ for some Carleson set is also necessary for $(*)$ to have non-trivial solutions, but we have been unable to establish this.

2.

Suppose now that σ is so smooth that $\omega_\sigma(t) = O(t \log t^{-1})$; this condition is equivalent to the existence of constants C, N such that

$$|s(z)| \geqslant C(1-|z|)^N \quad \text{for all } z \in D$$

where s is the inner function corresponding to σ [15]. Let φ vanish on Z modulo s where $\sigma(Z) = \sigma(T)$, so $\sigma(E) = \sigma(Z \cap E)$ for all $E \subset T$, and let d be the greatest commoninner divisor of φ and s, i.e. $\varphi = \psi d$ and $s = pd$ with ψ and p having no common divisor in H^∞. As in [2], we infer from $(*)$ that for all $\xi \in Z$, there exist $h_\xi(z)$ and $k_\xi(z)$ with $(\|h_\xi\|_\infty + \|k_\xi\|_\infty) \leqslant M_1$ such that

$$\psi(z) = (z-\xi)h_\xi(z) + p(z)k_\xi(z), \quad \xi \in Z.$$

Since clearly $|p(z)| \geqslant C(1-|z|)^N$, we have

$$\psi^N(z) = (z-\xi)^N H_\xi(z) + p(z)K_\xi(z), \quad \xi \in Z$$

with $(\|H_\xi\|_\infty + \|K_\xi\|_\infty) \leqslant M_2$ and for $z = |z|\xi$ with $\xi \in Z$,

$$|\psi^N(z)| \leqslant M_2((1-|z|)^N + |p(z)|) \leqslant M_2(C^{-1}|p(z)| + |p(z)|) = M|p(z)|, \text{ i.e.}$$

$$\left|\frac{\psi^N(z)}{p(z)}\right| \leqslant M \quad \text{for all } z = |z|\xi,\ \xi \in Z,\ |z| < 1.$$

Since ψ^N is also relatively prime to p and the singular measure corresponding to $p(z)$ also has its full measure on Z, Lemma 2.2 below implies $p(z)$ is constant and hence $\varphi \in sH^\infty$. We have thus proved the following generalization of Foiaş' theorem [2].

PROPOSITION 2.1. *If $\omega_\sigma(t) = O(t \log t^{-1})$ and φ vanishes on Z modulo s, where $\sigma(Z) = \sigma(T)$, then φ is trivial, i.e. $\varphi \in sH^\infty$.*

LEMMA 2.2. *Let $h(z) = f(z)s(z)^{-1}$ with $f \in H^\infty$ and s inner having associated measure σ_1 carried on some $Z_1 \subset T$. Suppose s relatively prime to f and there exists $M < \infty$ such that for all $\xi \in Z_1$ and $|z| < 1$, $|M(|z|\xi)| \leqslant M_1$. Then $s(z)$ is constant.*

Proof. By the standard representation for functions in H^∞, which would yield the result immediately if $Z_1 = T$, we have

$$h(z) = \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right),$$

where $d\mu(\theta) = k(\theta)d\theta + d\sigma_1(\theta) - d\sigma_2(\theta)$ with $k \in L^\infty(T)$ and σ_1 and σ_2 mutually singular positive measures each singular with respect to Lebesgue measure. Hence, we may assume σ_i is carried on Z_i of Lebesgue measure 0, i.e. $|Z_i| = 0$ and $Z_1 \cap Z_2 = \varphi$, $i = 1, 2, \ldots$.

We suppose $\sigma_1 \neq 0$. By a classical theorem of Fatou, it suffices to show $\mu'(t) = = +\infty$ for some $t \in Z_1$ to contradict the boundedness hypothesis of the Lemma [4, p.17]. Although this seems intuitively obvious, we require the following argument which essentially appears in [13, Theorem. 8.10].

Let $V \subset T$ be an open set with $Z_1 \subset V$ and $|V| < \varepsilon$, where $\varepsilon > 0$ is arbitrary, and let K be a compact subset of $E_n = \{t \in Z_1 :$ the lower derivative $\underline{\mu}'(t) < n\}$. For each $t \in K$, there exists an interval $I_t \subset V$ with $t \in I_t$ and $\mu(I_t) \leqslant n|I_t|$. Taking a finite subcover of $\{I_t\}$ such that each $t \in K$ is contained in at most two elements, we have

$$\mu(K) \leqslant \mu\left(\bigcup_j I_{t_j}\right) \leqslant \sum \mu(I_{t_j}) \leqslant n \sum |I_{t_j}| \leqslant 2n \left|\bigcup I_{t_j}\right| \leqslant 2n|V| \leqslant 2n\varepsilon.$$

Thus, $\mu(K) \leqslant 0$, which by regularity implies $\mu(E_n) \leqslant 0$ for all n. Since $\mu = \sigma_1$ on Z_1, we have $\mu'(t) = +\infty$ a.e. $[\sigma_1]$ on Z_1 and the lemma is proved. ■

DEFINITION 2.3. We say a positive singular measure σ is almost almost smooth (a.a.s.) if and only if there exists a sequence $\{\sigma_n\}$ of positive singular measures such that

(i) $(\sigma_{n+1} - \sigma_n) \geqslant 0$ for all n,
(ii) $\omega_{\sigma_n}(t) = O(t \log t^{-1})$,
(iii) $\sigma(E) = \lim \sigma_n(E)$ for all Borel $E \subset T$.

We note that the apparent redundancy of the nomenclature is motivated by the terminology of [18, p. 42] in which measures with $\omega(t) = O(t \log t^{-1})$ are called almost smooth.

By a standard continuity we now establish.

THEOREM 2.4. *Let σ be almost almost smooth and $s(z)$ be the corresponding inner function. If φ vanishes on Z modulo s, where $\sigma(Z) = \sigma(T)$, then $\varphi \in sH^\infty$.*

Proof. Let $\{\sigma_n\}$ be as in 2.3 with associated inner functions $\{s_n\}$, and φ be a solution of $(*)$. For $\xi \in Z$,

$$\varphi(z) = (z-\xi)h_\xi(z) + s(z)k_\xi(z) = (z-\xi)h_\xi(z) + s_n(z)(s_n(z)^{-1}s(z)k_\xi(z)).$$

Since $\sigma_n(Z) = \sigma_n(T)$ and $\|s_n^{-1}s\|_\infty = 1$, we have $\varphi \in s_nH^\infty$ by Theorem 2.1. Since s is the least common multiple of $\{s_n\}$, it follows that $\varphi \in sH^\infty$. ∎

By a similar continuity argument, we can replace the boundedness condition in $(*)$ by a technical measurability condition as follows.

COROLLARY 2.5. *Let σ be almost almost smooth, $s(z)$ the corresponding inner function, and $\sigma(Z) = \sigma(T)$. Suppose $\varphi(z) = (z-\xi)h_\xi(z) + s(z)k_\xi(z)$ for all $\xi \in Z$ where $h_\xi, k_\xi \in H^\infty$ for all $\xi \in Z$ and there exists $n_j \to \infty$ such that $E_j = \{\xi \in Z : (\|h_\xi\|_\infty + \|k_\xi\|_\infty) \leqslant n_j\}$ is Borel measurable for all j. Then $\varphi \in sH^\infty$.*

Almost smooth measures have been relevant to the study of cyclic inner functions in Bergman spaces and some related function-theoretic problems, ([5], [6], [7], [12], [15], [17]). A characterization of cyclic inner functions by B. Korenblum and J. Roberts uses the concept of a Carleson set.

DEFINITION 2.6. A subset $F \subset T$ is called a Carleson set if and only if
(i) F is closed,
(ii) F has Lebesgue measure 0, i.e. $|F| = 0$,
(iii) $\sum_j |I_j| \log |I_j|^{-1} < \infty$ where $\{I_j\}$ are the complementary arcs of F.

Korenblum and Roberts have shown that for H the standard Bergman space of functions analytic in D that are square-integrable with respect to area measure, i.e.

$$H = \{f : f \text{ analytic in } D\} \cap L^2(dxdy),$$

an inner function s is cyclic, i.e. $\{p(z)s(z) : p \text{ a polynomial }\}$ is dense in H, if and only if σ, the measure of s, gives zero mass to every Carleson set ([7], [12], [17]).

Also, V. V. Kapustin has recently shown that the condition $\sigma(C) = 0$ for every Carleson set is necessary and sufficient for the restricted shift S associated with σ to be reflexive [19].

If σ is almost almost smooth, it follows immediately from the definition that $\sigma(C) = 0$ for all C such that $h(C) = 0$, where h is the Hausdorff measure corresponding to the $O(t\log t^{-1})$ condition. Since it is well-known that $h(C) = 0$ for every Carleson set C, we have

COROLLARY 2.7. *If σ is almost almost smooth, then $\sigma(C) = 0$ for every Carleson set.*

The converse of 2.7 is false. We give a counterexample and describe a.a.s. measures in terms of Hausdorff measure in [10].

3.

Suppose σ is a positive singular Borel measure supported on a Carleson set $F \subset T$. It is well-known that the Carleson sets coincide with the null sets for the classes $A^{(n)} = \{f : f^{(n)} \in A\}$, where A denotes the disc algebra $(H^\infty \cap C(\overline{D}))$, and further there exists an outer function $\varphi(z)$ such that $F = \{\xi \in T : \varphi^{(n)}(\xi) = 0,\ n \geqslant 0\}$ and for each $n \geqslant 0$, $|\varphi(\xi)| = O(\rho(\xi)^n)$ as $\xi \to F$ where $\rho(\xi)$ is the distance from ξ to F [5 or 17].

For fixed $\xi \in F$, we set

$$h_\xi(z) = \varphi(z)(z-\xi)^{-1}.$$

By the mean value theorem, $|h_\xi(z)| = |\varphi'(\omega)|$ for some $\omega \in \overline{D}$, and hence $h_\xi \in H^\infty$ and $\|h_\xi\|_\infty \leqslant \|\varphi'\|_\infty$ uniformly in $\xi \in F$. Setting $k_\xi(z) \equiv 0$, we have

PROPOSITION 3.1. *Suppose σ is supported on a Carleson set $F \subset T$ and s is the corresponding inner function. There exists a non-trivial φ vanishing on F modulo $s(z)$.*

For $\xi \notin F, (S-\xi)$ has a bounded inverse given by $(S-\xi)^{-1} = a_\xi(S)$ where

$$a_\xi(z) = \begin{cases} (s(\xi)-s(z))[s(\xi)(z-\xi)]^{-1}, & z \neq \xi \\ s'(\xi)s(\xi)^{-1}, & z = \xi \end{cases}$$

and $a_\xi(S)f = \mathrm{P}a_\xi f$. We note that $a_\xi \in H^\infty$ since $s(z)$ has analytic continuation across all $\xi \in T \setminus F$.

LEMMA 3.2. *There exists $M < \infty$ such that $\|\varphi a_\xi\|_\infty \leqslant M$ for all $\xi \in T \setminus F$.*

Proof. For $\xi \in T\setminus F$ fixed, $\varphi a_\xi \in H^\infty$ and hence $\|\varphi a_\xi\|_\infty = \|\varphi(\omega)a_\xi(\omega)\|_\infty$ where ω denotes a point on T and the latter norm is in $L^\infty(T)$. Since $|\varphi(\omega)| = O(\rho(\omega)^2)$, it follows that $\varphi(\omega)s(\omega) \in C^1(T)$ and $\varphi(\omega)s'(\omega)$ extends to an element of $C(T)$. Since $|s(\xi)| = 1$, we have that for $\omega \neq \xi$ there exist $\omega_1, \omega_2 \in T$ such that

$$|\varphi(\omega)a_\xi(\omega)| \leqslant |(\varphi(\omega)s(\omega) - \varphi(\xi)s(\xi))(\omega-\xi)^{-1}| + |(\varphi(\xi)-\varphi(\omega))(\omega-\xi)^{-1}| =$$

$$= |(\varphi s)'(\omega_1)| + |\varphi'(\omega_2)| \leqslant \|(\varphi s)'(\omega)\|_\infty + \|\varphi'(\omega)\|_\infty.$$

Since $|\varphi(\xi)a_\xi(\xi)| = |\varphi s'(\xi)| \leqslant \|\varphi s'\|_\infty$, the lemma follows. ∎

PROPOSITION 3.3. *If σ is supported on a Carleson set F, there exists a non-trivial φ vanishing on T modulo $s(z)$.*

Proof. For $\xi \in F$, take h_ξ and k_ξ as in 3.1; for $\xi \in T \setminus F$, let $h_\xi = \varphi a_\xi$. Since

$$P\varphi = (S-\xi)a_\xi(S)P\varphi = P(z-\xi)a_\xi(z)\varphi(z),$$

we have

$$\varphi(z) = (z-\xi)h_\xi(z) + s(z)k_\xi(z)$$

for some $k_\xi \in H^2$. Since $\varphi \in H^\infty$ and $\|h_\xi\|_\infty \leqslant M$, we have $\{\|k_\xi\|_\infty\}$ are uniformly bounded and the theorem follows. ∎

More generally, suppose σ is a positive singular Borel measure on T and there exists a Carleson set F with $\sigma(F) > 0$. Then $\sigma = \sigma_1 + \sigma_2$ where $0 \neq \sigma_1$ is supported on F and hence the corresponding inner functions satify $s = s_1 s_2$. By 3.3, there exists $\varphi_1 \notin s_1 H^\infty$ such that

$$\varphi_1(z) = (z-\xi)h_\xi(z) + s_1(z)k_\xi(z) \quad \text{for all } \xi \in T$$

with $(\|h_\xi\|_\infty + \|k_\xi\|_\infty) \leqslant M$. Letting $\varphi = s_2\varphi_1$, we finally have

THEOREM 3.4. *If $\sigma(F) > 0$ for some Carleson set $F \subset T$, there exists $\varphi \notin (sH^\infty)$ such that φ vanishes on T modulo $s(z)$.*

COROLLARY 3.5. *If $\sigma(F) > 0$ for some Carleson set $F \subset T$ there exists $\varphi \notin sH^\infty$ such that*

$$\varphi(z) = (z-\xi)^n h_{\xi,n}(z) + s(z)k_{\xi,n}(z) \quad \text{for all } \xi \in T,$$

where $(\|h_{\xi,n}\|_\infty + \|k_{\xi,n}\|_\infty) \leqslant M_n$.

The proof of the corollary is analogous to the theorem since $\varphi^{(n)}(\xi) = 0$ on F and $|\varphi(\xi)| = O(\rho(\xi)^n)$ for all n.

4.

Our applications are based on the heuristic interpretation of condition (*) as a multiplicity condition on the zero set of $s(z)$, or perhaps alternatively as a spectral multiplicity condition on the associated restricted shift S. Note that if the inner function $s(z)$ is a Blaschke product with zero set Z, then

$$\varphi(z) = (z-\xi)h_\xi(z) + s(z)k_\xi(z) \quad \text{for all } \xi \in Z$$

has a non-trivial solution if and only if s has some multiple zero and the multiplicity of each zero corresponds to the multiplicity of the corresponding generalized eigenspace. Since the density points of the measure σ form the "zero set" of the singular function $s(z)$, our result seems to say that the zeros of s are simple if and only if the measure σ is almost almost smooth. However, we clearly have $s(z) = s_n(z)^n$, where s_n, the inner function corresponding to the measure $\frac{1}{n}\sigma$, has the same zero set as s, so in another sense the zero set of s always has infinite multiplicity. Our results below give two other interpretations in which inner functions whose measures give mass to Carleson sets have infinite multiplicity.

For an interval $(a,b) \subset T$ and σ non-atomic, we denote

$$s_{(a,b)}(z) = \exp\left[-\frac{1}{2\pi}\int_a^b \frac{e^{i\theta}+z}{e^{i\theta}-z}d\sigma(\theta)\right].$$

We have $(s_{(a,b)}H^2)^\perp \subset (sH^2)^\perp$ is an invariant subspace for S^*, where S is the restricted shift corresponding to $s = s_T$.

LEMMA 4.1. *Let $\xi = e^{i\theta_0} \in T$ and $\varepsilon > 0$. There exists $\delta > 0$ such that if $a < \xi < b$ and $(b-a) < \delta$,*

$$\left\|(S^* - \bar{\xi}I)|_{(s_{(a,b)}H^2)^\perp}\right\| < \varepsilon.$$

Proof. We first consider the special case where $\xi = 1$ and $I = (0,\delta)$. By [1,8], S^* is unitarily equivalent to $(M-K)$ acting on $L^2(\sigma)$ where

$$(\mathrm{Mc})(t) = e^{-it}c(t),$$

$$(\mathrm{Kc})(t) = 2e^{-it}e^{\sigma(0,t)}\int_t^{2\pi} e^{-\sigma(0,\lambda)}c(\lambda)d\sigma(\lambda),$$

and $L^2(\sigma|_{(0,\lambda)})$ corresponds to $(s_{(0,\lambda)}H^2)^\perp$. For $c \in L^2$ supported on $(0,\delta)$, i.e. $c(t) = 0$ a.e. if $t > \delta$,

$$||\mathrm{Kc}||^2 = \int_0^{2\pi} |(\mathrm{Kc})(t)|^2 \mathrm{d}\sigma(t) = \int_0^{\delta} 4\mathrm{e}^{2\sigma(0,t)} \left| \int_t^{2\pi} \mathrm{e}^{-\sigma(0,\lambda)} c(\lambda) \mathrm{d}\sigma(\lambda) \right|^2 \mathrm{d}\sigma(t) \leqslant$$

$$\leqslant 4\sigma(T)\mathrm{e}^{2\sigma(T)} \int_0^{\delta} \sigma(0,\lambda)|c(\lambda)|^2 \mathrm{d}\sigma(\lambda) \leqslant M\sigma(0,\delta)||c||^2,$$

and hence $\left\| K|_{L^2(\sigma|_{(0,\delta)})} \right\|^2 \leqslant M\sigma(0,\delta)$, where M is a constant depending only on σ. Thus, when restricted to $(s_{(0,\delta)}H^2)^\perp$,

$$\left\| (S^* - \bar{\xi}I)|_{(s_{(0,\delta)}H^2)^\perp} \right\| \leqslant ||(M - \bar{\xi}I)_{L^2(\sigma|_{(0,\delta)})}|| + ||K|_{(0,\delta)}|| \leqslant$$

$$\leqslant \delta + (M\sigma(0,\delta))^{\frac{1}{2}}.$$

The general case now follows by the same reasoning since by [9], for any $\theta \in T$, s^* is unitarily equivalent to $(M_\theta - K_\theta)$ acting on $L^2(\sigma_\theta)$ where

$$(M_\theta c)(t) = \mathrm{e}^{-i(t+\theta)} c(t)$$

$$(\mathrm{K}_\theta \mathrm{c})(t) = 2\mathrm{e}^{-i(t+\theta)} \mathrm{e}^{\sigma(\theta,\theta+t)} \int_t^{2\pi} \mathrm{e}^{-\sigma(\theta,\theta+\lambda)} c(\lambda) \mathrm{d}\sigma_\theta(\lambda),$$

$$\sigma_\theta(0,\lambda) = \sigma(\theta,\theta+\lambda),$$

and

$$L^2(\sigma_\theta|_{(0,\lambda)}) \text{ corresponds to } (S_{(\theta,\theta+\lambda)}H^2)^\perp.$$

We now assume that σ is a non-atomic measure such that $\sigma(F) > 0$ for some Carleson set $F \subset T$, s is the associated inner function, and $\varphi(z) \in H^\infty \setminus sH^\infty$ vanishes on T modulo $s(z)$ as in Theorem 3.4. Recall that $\varphi(S)f = P\varphi f$ for $f \in (sH^2)^\perp$.

THEOREM 4.2. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\left\| \varphi(S)^* \big|_{(S_{(a,b)} H^2)^\perp} \right\| < \varepsilon$ *if* $(b-a) < \delta$.

Proof. For any $\xi \in T$, $\varphi(S) = (S - \xi) h_\xi(S)$ and $\|h_\xi(S)\| \leqslant \|h_\xi\|_\infty \leqslant M$. By Lemma 4.2., take $(b-a) < \delta$ and for any $\xi \in (a,b)$

$$\left\| \varphi(S)^* \big|_{(s_{(a,b)} H^2)^\perp} \right\| \leqslant \left\| (S^* - \overline{\xi}) \big|_{(s_{(a,b)} H^2)^\perp} \right\| \cdot M$$

so the theorem follows. ∎

As a direct consequence of this theorem and the famous theorem of D. Sarason [14], we have the following "approximation" result.

COROLLARY 4.3. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\|\varphi + s_{(\theta, \theta+\delta)} H^\infty\| < \varepsilon$ *for all* $\theta \in T$, *where the norm is the coset norm in* L^∞ / H^∞.

If σ is actually supported on a Carleson set, by partitioning T into $\bigcup I_j$ where I_j are small intervals whose end points are disjoint from F, we can decompose $(sH^2)^\perp$ into the non-orthogonal direct sum

$$(sH^2)^\perp = \sum_j (s_{I_j} H^2)^\perp.$$

COROLLARY 4.4. *With the above notation, for any* $\varepsilon > 0$, *there exists* $\delta > 0$, *such that if* $|I_j| < \delta$ *for all* j,

$$\varphi(S)^* = \sum \varphi(S)^*_{(s_{I_j} H^2)^\perp} \quad \text{with} \quad \left\| \varphi(S)^* \big|_{(s_{I_j} H^2)^\perp} \right\| < \varepsilon \quad \text{for all } j,$$

i.e. we can diagonalize $\varphi(S^*)$ *with diagonal entries all of arbitrarily small norm.*

We also point out that all the above results can be dualized to $(S - \xi)$ and $\varphi(S)$ either directly from [1] and [7] or by using the well-known duality between S_s and $S^*_{\tilde{s}}$, where $\tilde{s}(z) = \overline{s(\overline{z})}$ [3]. Also, Lemma 4.1 clearly gives quantitative results concerning $\delta(\varepsilon)$, which can be improved using the fact that $\varphi(S) = (S - \xi)^n h_{\xi,n}(S)$ for all n. It would be of interest to know if any non-trivial φ could satisfy the conditions of Theorem 4.2. for an almost almost smooth measure. We now give a final multiplicity interpretation which is internal to the Ahern-Clark-Kriete model theory.

DEFINITION 4.5. For μ a non-atomic measure on T and $c \in L^2(\mu)$, we say $\lim\limits_{\lambda \to 0} c(\lambda) = \ell$ if

$$\lim_{\delta \to 0+} \frac{1}{\mu(0,\delta)} \int_0^\delta |c(\lambda) - \ell|^2 = 0.$$

Given $f \in (sH^2)^\perp$ and the measures $\{\sigma_\theta : \theta \in T\}$ as above, let $c_\theta \in L^2(\sigma_\theta)$ be the element corresponding to f under the unitary map V_θ implementing the equivalence between S and $(M_\theta - K_\theta)$. Let

$$\vartheta = \{f \in (sH^2)^\perp : \text{ for a.e. } \theta[\sigma], \lim_{\lambda \to 0} c_\theta(\lambda) = \ell_\theta \text{ exists } \}$$

and

$$\vartheta_0 = \{f \in \mathcal{V} : \ell_\theta = 0 \text{ a.e. } \theta[\sigma]\}$$

Although it is very difficult to compute any c_{θ_1} from another c_{θ_2}, it follows from [9] that the Szegö kernel $(1 - \overline{s}(\omega)s(z))(1 - \overline{\omega}z)^{-1} \in \vartheta$ for all $\omega \in \vartheta$, and also $Pz^n \in \vartheta$ for all $n \geqslant 0$, so ϑ is dense in $(sH^2)^\perp$.

THEOREM 4.6. *If $\sigma(F) > 0$ for some Carleson set, then ϑ_0 is dense in $(sH^2)^\perp$.*

Proof. We show that $\varphi(S)^*\vartheta \subset \vartheta_0$ which suffices since φ is relatively prime to s. Let $f \in \vartheta$ correspond to $c_\theta = c$ for $\theta \in T$ with $\overline{\xi} = e^{i\theta}$. Then

$$\int_0^\delta |[(S^* - \overline{\xi}I)c](\lambda)|^2 d\sigma_\theta(\lambda) = \int_0^\delta |[(S^* - \overline{\xi}I)(c|_{(0,\delta)})(\lambda)]|^2 d\sigma_\theta(\lambda) =$$

$$= \|(S^* - \overline{\xi}I)(c|_{(0,\delta)})\|^2 \leqslant \varepsilon(\delta)\|c|_{(0,\delta)}\|^2 = \varepsilon(\delta) \int_0^\delta |c(\lambda)|^2 d\sigma_\theta(\lambda).$$

Since $f \in \vartheta$, the integral is $O(\sigma_\theta(0,\delta))$ and thus, since $\varepsilon(\delta) \to 0$, we have $\lim_{\lambda \to 0}[(S^* - \overline{\xi}I)c_\theta](\lambda) = 0$. By a similar argument, $h(S)^*\vartheta_0 \subset \vartheta_0$ for all $h \in H^\infty$ and hence $h_\xi(S)^*(S^* - \overline{\xi}I) = \varphi(S^*)$ maps ϑ into ϑ_0. ■

It would also be of interest to know if ϑ_0 is dense for almost almost smooth measures. More particularily, is it ever the case that $\vartheta_0 = \{0\}$; if $\vartheta_0 = \{0\}$, it is not hard to show that the associated restricted shift is reflexive and that it has a natural "diagonalization" in terms of generalized eigenvalues and rigged Hilbert spaces. Finally, we note that 4.6. translates to

COROLLARY 4.7.*If $\sigma(F) > 0$ for some Carleson set F,*

$$\left\{f \in (sH^2)^\perp : \lim_{\delta \to 0}(\|P_{(s(\theta,\theta+\delta)H^2)^\perp} f\|)(\|P_{cs(\theta,\theta+\delta)H^2)^\perp} l\|)^{-1} = 0 \text{ for all } \theta \in T\right\}$$

is dense in $(sH^2)^\perp$.

REFERENCES

1. AHERN, P. R.; CLARK, D. N., Functions orthogonal to invariant subspaces, *Acta Math.*, **124**(1970), 191–204.
2. FOIAŞ, C., On the scalar part of a decomposable operator, *Rev. Roumanie Math. Pures Appl.*, **17**(1972), 1181–1198.
3. FUHRMANN, P., *Linear Systems and Operators in Hilbert Space*, Mc-Graw Hill, 1981.
4. KOOSIS, P., *Introduction to H_p Spaces*, Cambridge University Press, 1980.
5. KORENBLUM, B., Functions holomorphic in the disc and smooth in its closure, *Soviet Math. Dokl.*, **12**(1971), 1312–1315.
6. KORENBLUM, B., A Beurling-type theorem, *Acta Math.*, **138**(1977), 265–293.
7. KORENBLUM, B., Cyclic elements in some spaces of analytic functions, *Bull. Amer. Math. Soc.*, **5**(1981), 317–318.
8. KRIETE, T. L., A generalized Paley-Wiener theorem, *J. Math. Anal. Appl.*, **36**(1971), 529–555.
9. LUBIN, A. R., Isometries of $*$-invariant subspaces, *Trans. Amer. Math. Soc.*, **190**(1974), 405–415.
10. LUBIN, A. R., Almost almost smooth measures (to appear).
11. RADJAVI, H.; ROSENTHAL P., *Invariant Subspaces*, Springer-Verlag, 1973.
12. ROBERTS, J., Cyclic inner functions in the Bergman spaces and weak outer function in $H^p, 0 < p < 1$, Ilinois *J. Math.*, **29**(1985), 25–38.
13. RUDIN, W., *Real and Complex Analysis*, Mc-Graw Hill, 1974.
14. SARASON, D., Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127**(1967), 179–203.
15. SHAPIRO, H. S., Weakly invertible elements in certain function spaces and generations of ℓ^1, *Michigan Math. J.*, **11**(1964), 161–165.
16. SHAPIRO, H. S., Some remarks on weighted polynomial approximation by holomorphic functions, *Math. USSR - Sb.*, **2**(1967), 285–294.
17. SHAPIRO, J., Cyclic inner functions in Bergman spaces, Unpublished seminar notes.
18. ZYGMUND, A., *Trigonometric Series*, Cambridge University Press, 1959.
19. KAPUSTIN, V. V., Reflexivity criterion for restricted shift operators, LOMI, preprint E-2-91.

A. R. LUBIN
Department of Mathematics,
Illinois Institute of Technology,
Chicago, IL 60616,
U.S.A.

Received September 27, 1991; revised January 29, 1993.