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WEAK CONVERGENCE IN NON-COMMUTATIVE SYMMETRIC SPACES

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1. INTRODUCTION

Let E be a separable symmetric sequece space, let C_E be the norm ideal of compact operators associated with E, i.e. C_E is the Banach space of all compact operators x in l_2 such that $s(x) \in E$, where $s(x) = (s_n(x))$ is the sequence of s-numbers of x. J. Arazy showed in [1] that if $x_n, x \in C_E$, then $\|x_n - x\|_{C_E} \to 0$ if and only if $x_n \to x$ weakly and $\|s(x_n) - s(x)\|_E \to 0$.

Let M be a von Neumann algebra, let μ be a faithful normal semifinite trace on M. The purpose of this paper is to obtain the same result for the symmetric space $E(M,\mu)$ associated with a separable symmetric space E of measurable functions on $[0,\infty)$. $E(M,\mu)$ is the Banach space of all μ-measurable operators x affiliated with M such that $\mu_t(x) \in E$, where $\mu_t(x)$ is the rearrangement (generalized s-numbers) of x, with the norm $\|x\|_{E(M,\mu)} = \|\mu_t(x)\|_E$.

Non-commutative symmetric spaces of μ-measurable operators associated with a von Neumann algebra, which differed from a factor of type I_∞, were considered for the first time by V. I. Ovchinnikov [10, 11]. Some properties of these spaces were examined in [3, 4, 8, 14, 15, 18, 19]. In the case of I_∞–factor the class of non-commutative symmetric spaces coincides with the class of norm ideals of compact operators. The most important examples of non-commutative symmetric spaces are non-commutative L_p-spaces, Orlicz, Lorentz and Marcinkiewicz spaces.

The main rezult of this paper is the following.

THEOREM. *Let E be a separable symmetric space of measurable functions on $[0,\infty)$, x_n, $x \in E(M,\mu)$. Then $\|x_n - x\|_{E(M,\mu)} \to 0$ if and only if $x_n \to x$ weakly* *and* $\|\mu_t(x_n) - \mu_t(x)\|_E \to 0$.

This paper consists of six paragraphs. In Section 1, the necessary information about μ-measurable operators is cited. In particular, some properties of rearrangements of such operators are established. In Section 2, symmetric spaces of μ-measurable operators are introduced and the construction of such spaces with the help of functional spaces is defined. Besides, the class of regular non-commutative symmetric spaces is distinguished, and some properties of these spaces are established. In Section 3, some useful criteria for describing convergent sequences in regular non-commutative symmetric spaces using convergence in measure are obtained. In Section 4, the main theorem of this paper is proved for non-commutative Lorentz spaces and then, in Section 5, it is established for any non-commutative symmetric space associated with separable symmetric space of measurable functions on $[0,\infty)$. In the last paragraph, the variant of J. Arazy's theorem for norm ideals C_E of compact operators, acting in an arbitrary (not necessarily separable) Hilbert space is obtained making use of the main theorem.

We use the terminology and notation of the theory of von Neumann algebras from [16], the theory of non-commutative integration from [9, 4] and the theory of symmetric spaces from [10, 11, 19, 7].

1. PRELIMINARIES

Let M be a semifinite von Neumann algebra acting in a Hilbert space H, let μ be a faithful normal semifinite trace on M, let $P(M)$ be the lattice of all projections in M. A densely-defined closed operator x, affiliated with M, is called *μ-measurable* [9], if for each $\varepsilon > 0$ there exists $p \in \mathcal{P}(M)$ such that $p(H) \subset \mathcal{D}(x)$ and $\mu(p^\perp) < \varepsilon$, where $p^\perp = 1 - p$, 1 is the unit in M. The set $\mathcal{K}(M,\mu)$ of all μ-measurable operators is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [9]. For any subset $E \subset \mathcal{K}(M,\mu)$ we shall denote by E_h (respectively by E_+) the set of all self-adjoint (respectively positive self-adjoint) elements from E. We shall denote by "$\leqslant$" the partial order in $\mathcal{K}_h(M,\mu)$ generated by the proper cone $\mathcal{K}_+(M,\mu)$. The rearrangement $\mu_t(x)$ of an operator $x \in \mathcal{K}(M,\mu)$ is the function defined by

$$\mu_t(x) = \inf\{\|xp\|_M : p \in \mathcal{P}(M),\ \mu(p^\perp) \leqslant t\},\ t \geqslant 0,$$

where $\|\cdot\|_M$ is the C^*-norm on M. The function $t \to \mu_t(x)$ from $(0,\infty)$ to $[0,\infty)$ is non-increasing, continuous from the right and $\lim_{t\downarrow 0}\mu_t(x) = \|x\|_M$ for $x \in M$. It is known (see for example [4]), that $\mu_t(x) = \inf\{s \geqslant 0 : \mu(|x| > s) \leqslant t\}$ where

$\{|x| > s\}$ is the spectral projection of $|x| = (x^*x)^{\frac{1}{2}}$ corresponding to the interval (s, ∞). In the commutative case, when $M = L_\infty(\Omega, \Sigma, \mu)$ and $\mu(f) = \int\limits_\Omega f d\mu$, where (Ω, Σ, μ) is a localizable measure space, the algebra $\mathcal{K}(M, \mu)$ coincides with the algebra of all measurable complex functions f on (Ω, Σ, μ) which are bounded everywhere, excepting a set of finite measure. In addition, the rearrangement $\mu_t(f)$ coincides with the non-increasing rearrangement $\tilde{f}$ of function f. The detailed account of many properties of the rearrangements may be found in [4]. We cite below two useful additional properties of rearrangements which will be necessary in the sequel.

PROPOSITION 1.1. *If M is non-atomic von Neumann algebra, x, $y \in \mathcal{K}(M, \mu)$, then*

$$\int\limits_0^t \mu_\tau(xy) d\tau \leqslant \int\limits_0^t \mu_\tau(x)\mu_\tau(y) d\tau$$

for all $t > 0$.

Proof. If $\int\limits_0^t \mu_\tau(x)\mu_\tau(y)d\tau = \infty$, then the statement of the proposition is obvious. Let $\int\limits_0^t \mu_\tau(x)\mu_\tau(y)d\tau < \infty$, $p \in \mathcal{K}(M)$, $\mu(p) \leqslant t$ and let $xy = u|xy|$ be the polar decomposition of xy. Since $|x^*up| \in \mathcal{K}(pMp, \mu)$, then the function $\mu_t(x^*up)$ is identically equal to zero outside the interval $(0, \mu(p))$, i.e. $\mu_\tau(x^*up) = \mu_\tau(x^*up)\chi_{(0,t)}$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0, t)$. Therefore, $\mu_\tau(pu^*x) = \mu_\tau(pu^*x)\chi_{(0,t)}$ and since $\|pu^*\|_M \leqslant 1$, we have

$$\int\limits_0^\infty \mu_\tau(pu^*x)\mu_\tau(y)d\tau = \int\limits_0^t \mu_\tau(pu^*x)\mu_\tau(y)d\tau \leqslant \int\limits_0^t \mu_\tau(x)\mu_\tau(y)d\tau < \infty.$$

By Theorem 3.3 [18] we have

$$\mu(p|xy|) \leqslant \int\limits_0^\infty \mu_\tau(pu^*x)\mu_\tau(y)d\tau \leqslant \int\limits_0^t \mu_\tau(x)\mu_\tau(y)d\tau.$$

It follows from ([4], Lemma 4.1) that if M has no atoms then

$$\int\limits_0^t \mu_\tau(x)d\tau = \sup\{\mu(p|x|p) : p \in \mathcal{P}(M),\ \mu(p) \leqslant t\}$$

for all $x \in \mathcal{K}(M,\mu)$, $t > 0$. Therefore,

$$\int_0^t \mu_t(xy)\mathrm{d}\tau = \sup\{\mu(p|xy|p) : p \in \mathcal{P}(M),\ \mu(p) \leqslant t\} \leqslant \int_0^t \mu_\tau(x)\mu_t(y)\mathrm{d}\tau. \qquad \blacksquare$$

PROPOSITION 1.2. *Let $x \in \mathcal{K}_h(M,\mu)$, $y \in \mathcal{K}_+(M,\mu)$, and let $-y \leqslant x \leqslant y$. Then $\int_0^t \mu_\tau(x)\mathrm{d}\tau \leqslant \int_0^t \mu_\tau(y)\mathrm{d}\tau$ for all $t > 0$.*

Proof. If $\int_0^t \mu_\tau(y)\mathrm{d}\tau = \infty$ for some $t > 0$ then, since $\mu_\tau(y)$ is non-increasing, $\int_0^t \mu_\tau(y)\mathrm{d}\tau = \infty$ for all $t > 0$, and the statement of the proposition is evident in this case. So we assume, that $\int_0^t \mu_\tau(y)\mathrm{d}\tau < \infty$ for all $t > 0$. Since $-y \leqslant x \leqslant y$, then $0 \leqslant x + y \leqslant 2y$ and therefore [4], $\int_0^t \mu_\tau(x)\mathrm{d}\tau \leqslant \int_0^t \mu_\tau(x+y)\mathrm{d}\tau + \int_0^t \mu_t(y)\mathrm{d}\tau \leqslant$ $\leqslant 3\int_0^t \mu_\tau(y)\mathrm{d}\tau < \infty$ for all $t > 0$. Let $e(x_+)$, $e(x_-)$ be the support projections of positive x_+ and negative x_- parts of x respectively. Denote by $\{x\}'$ the commutant of the element x in $\mathcal{K}(M,\mu)$. Since a projection $p \in \{x\}'$ commutes also with $e(x_+)$ and $e(x_-)$, then $q = pe(x_+) \in \mathcal{P}(M)$, $r = pe(x_-) \in \mathcal{P}(M)$. As $-y \leqslant x \leqslant y$, we have $\mu(qx) = \mu(qxq) \leqslant \mu(qyq) = \mu(qy)$, and analogously $-\mu(rx) \leqslant \mu(ry)$.

Hence

$$\mu(p|x|) = \mu(qx) - \mu(rx) \leqslant \mu((q+r)y) \leqslant \mu(py). \tag{1}$$

It follows from the proof of Lemma 4.1. [4] that if M is a non-atomic von Neumann algebra then

$$\int_0^t \mu_\tau(x)\mathrm{d}\tau = \sup\{\mu(p|x|) : p \in \mathcal{P}(M) \cap \{x\}',\ \mu(p) \leqslant t\}.$$

From this and from (1) the statement of proposition follows immediately for a non-atomic algebra M. Let now M be not a non-atomic algebra. Consider the commutative W^*-algebra $L_\infty(0,1)$ with the trace $\nu(f) = \int_0^1 f\mathrm{d}m$, where m is the

Lebesgue measure on [0,1]. Suppose that N acts in $F = L_2(0,1)$. Let $A = M\overline{\otimes}N$ be the tensor product of von Neumann algebras M and N, let $\lambda = \mu \otimes \nu$ be the tensor product of traces μ and ν. It is clear that the algebra A has no atoms. Let $a \in \mathcal{K}(M,\mu)$ and $\mathcal{D}$ be a linear subspace in $H\overline{\otimes}F$ generated by all vectors of the form $\xi\overline{\otimes}\eta$, $\xi \in \mathcal{D}(a)$, $\eta \in F$. For each $\zeta = \sum_{i=1}^{n}\xi_i \otimes \eta_i \in \mathcal{D}$ we put $(a\otimes \mathbf{1})\,(\zeta) = \sum_{i=1}^{n} a\xi_i \otimes \eta_i$. The linear operator $a \otimes \mathbf{1}$ with the domain of definition $\mathcal{D}$ is a preclosed operator and its closure $a\overline{\otimes}\mathbf{1}$ belongs to $\mathcal{K}(A,\lambda)$ [13], in particular $x\overline{\otimes}\mathbf{1}$, $y\overline{\otimes}\mathbf{1} \in \mathcal{K}(A,\lambda)$. In addition, $-y\otimes\mathbf{1} \leqslant x\otimes\mathbf{1} \leqslant y\otimes\mathbf{1}$ and $\mu_t(x) = \lambda_t(x\overline{\otimes}\mathbf{1})$ for all $t > 0$, where $\lambda_t(x\overline{\otimes}\mathbf{1})$ is the rearrangement of operator $x\overline{\otimes}\mathbf{1}$ calculated with respect to the trace λ. Therefore, the required inequality follows from the first part of the proof.

For each $x \in \mathcal{K}(M,\mu)$ we put $\mu_\infty(x) = \lim_{t\to\infty}\mu_t(x)$. The set $\mathcal{K}_0(M,\mu) = \{x \in \mathcal{K}(M,\mu) : \mu_\infty(x) = 0\}$ is a $*$-subalgebra in $\mathcal{K}(M,\mu)$ and it coincides with the closure in the measure topology of the two-sided ideal of elementary operators in M (i.e. $x \in M$ such that $\mu(l(x)) < \infty$, where $l(x)$ is the left support of x). The measure topology τ is determined by fundamental system $\{U(\varepsilon,\delta) : \varepsilon,\delta > 0\}$ of neighborhoods of zero [9] where

$$U(\varepsilon,\delta) = \{x \in \mathcal{K}(M,\mu) : \|xp\|_M < \varepsilon,\ \mu(p^\perp) \leqslant \delta \text{ for some } p \in \mathcal{P}(M)\}.$$

Note that $(\mathcal{K}(M,\mu),\tau)$ is a complete topological $*$-algebra, moreover, M is dense in $\mathcal{K}(M,\mu)$ [9]. We shall denote by $x_n \xrightarrow{\mu} x$ the convergence of the sequence $\{x_n\}$ to x in measure topology generated by the trace μ.

Denote by $(L_1(M,\mu), \|\cdot\|_{L_1(M,\mu)})$ the Banach space of all μ-integrable operators from $\mathcal{K}(M,\mu)$. The spaces $(L_1(M,\mu)\cap M, \|\cdot\|_\cap)$ and $(L_1(M,\mu)+M, \|\cdot\|_+)$, where $\|x\|_\cap = \max\{\|x\|_{L_1(M,\mu)}, \|x\|_M\}$, $\|x\|_+ = \int_0^1 \mu_t(x)\mathrm{d}t$, are Banach spaces. Besides, $x \in L_1(M,\mu)\cap M$ (respectively $x \in L_1(M,\mu)+M$) if and only if $\mu_t(x) \in L_1(0,\infty)\cap L_\infty(0,\infty)$ (respectively $\mu_t(x) \in L_1(0,\infty) + L(0,\infty)$), where $L_1(0,\infty)$ is the space of all integrable and $L_\infty(0,\infty)$ is the space of all bounded measurable functions on $((0,\infty),m)$, where m is Lebesgue measure on $(0,\infty)$.

Let $x,y \in \mathcal{K}(M,\mu)$. The notation $x \prec y$ means $\int_0^t \mu_\tau(x)\mathrm{d}\tau \leqslant \int_0^t \mu_\tau(y)\mathrm{d}\tau$ for all $t > 0$. If $x,y \in L_1(M,\mu)+M$ then $\mu_t(x) = \mu_t(y)$ for all $t > 0$ if and only if $x \prec y$ and $y \prec x$. It is clear that $x \prec y$ is equivalent to $\mu_t(x) \prec \mu_t(y)$. It was established in [3] that $\mu_t(x) - \mu_t(y) \prec \mu_t(x-y)$ for any $x,y \in \mathcal{K}(M,\mu)$.

2. SYMMETRIC SPACES ON A VON NEUMANN ALGEBRA.

A linear subspace E in $\mathcal{K}(M,\mu)$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric space* on (M,μ) if it follows from $x \in E$, $y \in \mathcal{K}(M,\mu)$ and $\mu_t(y) \leqslant \mu_t(x)$ for all $t > 0$ that $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. A norm $\|\cdot\|_E$ on E is called a *symmetric norm*, if it follows from $x, y \in E$, $y \prec x$ that $\|y\|_E \leqslant \|x\|_E$. All the noncommutative L_p-spaces, Orlicz, Lorentz and Marcinkiewich spaces are symmetric spaces with fully symmetric norms. Any symmetric spaces E on (M,μ) is continuously embedded into $(\mathcal{K}(M,\mu),\tau)$ [3, 11], i.e. from $x_n,\ x \in E$, $\|x_n - x\|_E \to 0$ it follows that $x_n \xrightarrow{\mu} x$. Besides, for any $x \in E$, $a \in M$ we have x^*, $ax \in E$ and $\|x\|_E = \|x^*\|_E$, $\|ax\|_E \leqslant \|a\|_M \|x\|_E$. If M is a non-atomic von Neumann algebra, then for E the following embeddings are continuous [10, 11].

$$(L_1(M,\mu) \cap M, \|\cdot\|_\cap) \subset (E, \|\cdot\|_E) \subset (L_1(M,\mu) + M, \|\cdot\|_+).$$

In the commutative case when $M = L_\infty(\Omega,\Sigma,\mu)$ and $\mu(f) = \int_\Omega f d\mu$, the symmetric space E on (M,μ) and its self-adjoint part E_h are symmetric spaces of complex and real mesurable function on (Ω,Σ,μ) respectively. In particular, if $M = \ell_\infty = L_\infty(\mathbf{N},\Sigma,\mu)$, where $\mathbf{N}$ is the set of all natural numbers, Σ is the σ-algebra of all subsets in $\mathbf{N}$, $\mu(n) = 1$, $n = 1, 2, \ldots$, then the notion of symmetric space on (ℓ_∞, μ) coicides with the notion of symmetric spaces of complex sequences.

The norm $\|\cdot\|_E$ on a symmetric space E is called *order continuous* if for any sequence $\{x_n\}$ in E_+ decreasing to zero we have $\|x_n\|_E \to 0$ when $n \to \infty$. A symmetric space with order continuous norm is called a *regular space*.

PROPOSITION 2.1. *If $(E, \|\cdot\|_E)$ is a regular symmetric space on (M,μ), then $E \subset \mathcal{K}_0(M,\mu)$.*

Proof. It is enough to consider the case, when $\mu(\mathbf{1}) = \infty$. If $\mu_\infty(x) > 0$ for some $x \in E$, then $\mathbf{1} \in E$. Choosing a sequence of projections $p_n \downarrow 0$ such that $\mu(p_n) = \infty$, we get $\|p_n\|_E = \|p_1\|_E$ for all n which contradicts with regularity of E. ∎

Consider the half-interval $[0,\alpha)$, $\alpha = \mu(\mathbf{1})$ with the Lebesgue measure m. For a symmetric space E on (M,μ) denote by $E(0,\alpha)$ the set of all real measurable functions f on $[0,\alpha)$ for which there exists $x_f \in E$ such that $\mu_t(x_f) = \tilde{f}(t)$, $t > 0$. Put $\|f\|_{E(0,\alpha)} = \|x_f\|_E$.

PROPOSITION 2.2. *Let M be a non-atomic von Neumann algebra. Then $(E(0,\alpha), \|\cdot\|_{E(0,\alpha)})$ is a symmetric function space. Moreover, if E is regular, then $E(0,\alpha)$ is regular too.*

The proof uses the folowing two lemmas.

LEMMA 2.1. *Let M be a non-atomic von Neumann algebra, $x \in \mathcal{K}_+(M,\mu)$, $p = \{x > \mu_\infty(x)\}$, $N = pMp$. Then there exists in N a non-atomic abelian von Neumannn subalgebra $\mathcal{U}$ containing all the projections $\{x > \lambda\}$, $\lambda > \mu_\infty(x)$, and the restriction of μ onto $\mathcal{U}$ is σ-finite (i.e. $p = \sup p_n$, $p_n \in \mathcal{P}(\mathcal{U})$, $\mu(p_n) < \infty$, $n = 1, 2, \ldots$).*

Proof. Since $p = \sup\{\{x > \lambda\}, \lambda > \mu_\infty(x)\}$ and $\mu(\{x > \lambda\}) < \infty$ if $\lambda > \mu_\infty(x)$, we may take as $\mathcal{U}$ any maximal abelian $*$-subalgebra in N, containing all the projections $\{x > \lambda\}$, $\lambda > \mu_\infty(x)$. ■

We shall identify $\mathcal{U}$ with the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a measure space with a complete non-atomic σ-finite measure. The space $L_1(\Omega, \Sigma, \mu) + L_\infty(\Omega, \Sigma, \mu)$ we shall write in the form $L_1(\Omega) + L_\infty(\Omega)$. Let $x \in L_1(\Omega) + L_\infty(\Omega)$ and $E_1(x) = \{w \in \Omega : |x(w)| > \tilde{x}(\infty)\}$, $E_2(x) = \{w \in \Omega : |x(w)| = \tilde{x}(\infty)\}$. Put $E(x) = E_1(x)$ if $\mu(E_1(x)) = \infty$ and $E(x) = E_1(x) \cup E_2(x)$ if $\mu(E_1(x)) < \infty$. It is known ([2], p.49), that in case $\mu(\Omega) < \infty$, for every $x \in L_1(\Omega)$ there exists a measure preserving transformation ϕ from Ω onto $[0, \mu(\Omega)]$ such that $|x(w)| = \tilde{x}(\phi(w))$, $w \in \Omega$.

Using this result one can easily establish the following:

LEMMA 2.2. *If $\mu(\Omega) = \infty$, then for any $x \in L_1(\Omega) + L_\infty(\Omega)$ with $\mu(E(x)) = \infty$ there exists a measure preserving transformation ϕ from $E(x)$ onto $[0, \infty)$, for which $|x(\omega)| = \tilde{x}(\phi(\omega))$ for all $w \in E(x)$.*

Pass now to the proof of Proposition 2.2.

Let $\mathcal{U}$ be the commutative von Neumannn algebra from Lemma 2.1., constructed for $x \in E$ such that $x > \mu_\infty(x) \neq 0$ and $\mu(E(x)) = \alpha$. Identify the algebra $\mathcal{U}$ with $L_\infty(\Omega, \Sigma, \mu)$ and consider the measure-preserving map ϕ of $E(x) = \Omega$ onto $[0, \alpha)$ (see Lemma 2.2.). For every $f \in E(0, \alpha)$ put $x_f = f(\phi(w))$, $w \in \Omega$. Then $x_f \in L_1(\Omega) + L_\infty(\Omega) \subset \mathcal{K}(M, \mu)$ and $\mu_t(x_f) = \tilde{f}(t)$; whence $x_f \in E$. It is clear that $x_{\beta f + g} = \beta x_f + x_g$ for all functions $f, g \in E$ and all numbers β. Hence $E(0, \alpha)$ is a subspace of $L_1(0, \alpha) + L_\infty(0, \alpha)$.

It is similarly established that $\|\cdot\|_{E(0,\alpha)}$ is the norm on $E(0, \alpha)$ and in addition, if $f \in E(0, \alpha)$, $g \in L_1(0, \alpha) + L_\infty(0, \alpha)$, $\tilde{g}(t) \leqslant \tilde{f}(t)$, $t \geqslant 0$, then $g \in E(0, \alpha)$ and $\|g\|_{E(0,\alpha)} \leqslant \|f\|_{E(0,\alpha)}$. Let $\{f_n\}$ be an increasing Cauchy sequence of non-negative functions from $(E(0, \alpha), \|\cdot\|_{E(0,\alpha)})$. Then f_n converges in the measure topology to the measurable function $f = \sup f_n$ (see [6], p.139), and therefore $\tilde{f}_n \to \tilde{f}$ almost everywhere ([4], p.93). Since $\{x_{f_n}\}$ is a Cauchy sequence in $(E, \|\cdot\|_E)$, there exists $x \in E$ such that $\|x_{f_n} - x\|_E \to 0$ as $n \to \infty$. Hence $x_n = x_{f_n}$ converges to x in measure [11], and therefore $\mu_t(x_n) \to \mu_t(x)$ almost everywhere (see [4], Lemma 3.4). Using the equality $\mu_t(x_n) = f_n$, $n = 1, 2, \ldots$, we obtain that $\mu_t(x) = \tilde{f}$ almost everywhere. This means that $f \in E(0,\alpha)$, and therefore $\|\cdot\|_{E(0,\alpha)}$ is the Banach norm (see [6], p. 378). Thus $E(0,\alpha)$ is a symmetric functional space on $(0,\alpha)$.

Suppose now that E is regular, $f_n \in E(0,\alpha)$ and $f_n \downarrow 0$. Using Proposition 2.1 we obtain that the sequence f_n converges to zero in measure, and so $\tilde{f}_n \to 0$ almost everywhere. The sequence $\{x_{f_n}\}$ decreases to some element $x \in E$, and since the norm $\|\cdot\|_E$ is order continuous, $\|x_{f_n} - x\|_E \to 0$. Repeating the previous arguments, we obtain, that $x = 0$, i.e. $\|f_n\|_{E(0,\alpha)} = \|x_{f_n}\|_E \to 0$ as $n \to \infty$. ∎

Let $(F, \|\cdot\|_F)$ be a symmetric functional space on $(0,\alpha)$. Denote by $F(M,\mu)$ the set of all $x \in \mathcal{K}(M,\mu)$, for which $\mu_t(x) \in F$. Put $\|x\|_{F(M,\mu)} = \|\mu_t(x)\|_F$. It is clear that if M is a non-atomic von Neumann algebra and $F = E(0,\alpha)$, then $F(M,\mu) = E$ and $\|x\|_E = \|x\|_{F(M,\mu)}$ for all $x \in E$.

Proposition 2.3. (cf [3], Theorems 4.2 and 4.5). *Let $(F, \|\cdot\|_F)$ be a symmetric functional space on $[0,\alpha)$ with a symmetric norm. Then $(F(M,\mu), \|\cdot\|_{F(M,\mu)})$ is a symmetric space on (M,μ) with a symmetric norm. In addition, if F is regular, then $F(M,\mu)$ is also regular.*

Proof. Since $\|\cdot\|_F$ is a symmetric norm we get using the properties of rearrangements that $F(M,\mu)$ is a subspace in $\mathcal{K}(M,\mu)$ and $\|\cdot\|_{F(M,\mu)}$ is a symmetric norm on $F(M,\mu)$. Thus, it is necessary to show only that the norm $\|\cdot\|_{F(M,\mu)}$ is Banach. Let $\{x_n\}$ be an increasing Cauchy sequence from $F_+(M,\mu)$. Then $\{x_n\}$ is a Cauchy sequence in the measure topology [3, 11], and since $\mathcal{K}(M,\mu)$ is complete in this topology, there exists $x \in \mathcal{K}(M,\mu)$ such that $x_n \uparrow x$, $x_n \xrightarrow{\mu} x$ and so $\mu_t(x_n) \to \mu_t(x)$ almost everywhere on $[0,\alpha)$. On the other hand, $\mu_t(x_n) - \mu_t(x_k) \prec \mu_t(x_n - x_k)$ [3]. Hence $\|\mu_t(x_n) - \mu_t(x_k)\|_F \leqslant \|x_n - x_k\|_{F(M,\mu)}$, i.e. the sequence $\{\mu_t(x_n)\}$ is Cauchy in F, and so $\mu_t(x_n)$ converges in F to the function $f(t) = \sup\limits_n \mu_t(x_n)$. Thus $f(t) = \mu_t(x)$ almost everywhere, whence $x \in F(M,\mu)$ and so there exists $\sup\limits_n x_n = x$ in $F_h(M,\mu)$. Repeating the proof of Theorem 2 from ([6], p. 378) we obtain that $F_h(M,\mu)$ is complete, and since $\|x\|_{F(M,\mu)} = \|x^*\|_{F(M,\mu)}$ for all $x \in F(M,\mu)$, the space $F(M,\mu)$ is complete too.

Suppose now that the space F is regular. Let $x_n \in F(M,\mu)$ and $x_n \downarrow 0$. For arbitrary $\varepsilon > 0$ put $q = \{x_1 > \varepsilon\}$, $p = 1 - q$. It is clear that $\|px_np\|_M \leqslant \varepsilon$ for all n. Since $x_1 \in \mathcal{K}_0(M,\mu)$, then $\mu(q) < \infty$ and so $qx_nq \xrightarrow{\mu} 0$. Hence $\mu_t(qx_nq) \to 0$ almost everywhere. Using the inequality $x_n \leqslant 2(px_np + qx_nq)$ and properties of the rearrangements we get

$$\mu_t(x_n) \leqslant 2\mu_t(px_np + qx_nq) \leqslant 2(\mu_{t/2}(px_np) + \mu_{t/2}(qx_nq)) \leqslant 2\varepsilon + 2\mu_{t/2}(qx_nq).$$

Then we have $\mu_t(x_n) \downarrow 0$ and so $\|x_n\|_{F(M,\mu)} = \|\mu_t(x_n)\|_F \to 0$. ∎

REMARK 2.1. If, under the hypothesis of Proposition 2.3, a von Neumann algebra is non-atomic, then it follows from the proof of Proposition 2.2 that $F(M,\mu)(0,\alpha) = F$.

REMARK 2.2. If $(F, \|\cdot\|_F)$ is a separable symmetric functional space on $[0,\alpha)$, then F is regular and its norm $\|\cdot\|_F$ is symmetric (see [7], p. 142). By virtue of Proposition 2.3 the space $F(M,\mu)$ is a symmetric space with order continuous norm.

Let M be a non-atomic von Neumann algebra, let E be a symmetric space on (M,μ). We shall denote by $\phi_E(t)$ the fundamental function of E [10, 11], which is defined by $\phi_E(t) = \|p\|_E$, where $p \in \mathcal{P}(M)$, $\mu(p) = t$. It is evident that $\phi_E(t) = \phi_{E(0,\alpha)}(t)$.

PROPOSITION 2.4. (cf. [7], Theorem 4.8, p. 170). *A symmetric space E on a continuous von Neumann algebra is regular if and only if the fundamental function $\phi_E(t)$ is non-atomic at zero and $L_1(M,\mu) \cap M$ is dense in $(E, \|\cdot\|_E)$.*

Proof. Obviously, continuity of $\phi_E(t)$ at zero follows from regularity of E. For every $x \in E$ and $t > s > \mu_\infty(x)$ the operator $|x|\{s < |x| < t\}$ belongs to $L_1(M,\mu)\cap M$, where $\{s < |x| < t\}$ is the spectral projection of $|x|$, corresponding to the interval (s,t). So, if E is regular, the subspace $L_1(M,\mu) \cap M$ is dense in E. Conversely, let $\phi_E(t)$ be continuous at zero and let $L_1(M,\mu) \cap M$ be dense in E. Denote by F the closure of $(L_1(0,\alpha) \cap L_\infty(0,\alpha))_h$ in $(E(0,\alpha), \|\cdot\|_{E(0,\alpha)})$. Since the fundamental function $\phi_E(t) = \phi_{E(0,\alpha)}(t) = \phi_F(t)$ is continuous at zero, then $(F, \|\cdot\|_{E(0,\alpha)})$ is a regular symmetric functional space on $[0,\alpha)$ ([7], p. 140). By Remark 2.2, $G = F(M,\mu)$ is a regular symmetric space on (M,μ). In addition, $L_1(M,\mu) \cap M \subset G \subset E$, $\|x\|_G = \|x\|_E$ for all $x \in G$. Since $L_1(M,\mu) \cap M$ is dense in E, we have $G = E$. ∎

COROLLARY 2.1. *Let E be a symmetric space on a non-atomic von Neumann algebra (M,μ), let $\phi_E(t)$ be continuous at zero, let E_0 be the closure of $L_1(M,\mu)\cap M$ in E and let F be the closure of $(L_1(0,\alpha) \cap L(0,\alpha))_h$ in $E(0,\alpha)$. Then $(E_0, \|\cdot\|_E)$ is a regular symmetric space on (M,μ) and $E_0 = F(M,\mu)$.*

COROLLARY 2.2. *Let E be a symmetric space on a non-atomic algebra (M,μ) with $\mu(\mathbf{1}) < \infty$ and let $E \neq M$. Then E is regular if and only if M is dense in E.*

Proof. Since $E \neq M$ and $\mu(\mathbf{1}) < \infty$, there exist $x \in E_+$ and a sequence $\{p_n\}$ of non-zero projections from M such that $x \geqslant xp_n = p_n x \geqslant np_n \neq 0$, $n = 1, 2, \ldots$, and $\mu(p_n) \to 0$. As $\|p_n\|_E \leqslant n^{-1}\|x\|_E \to 0$, then $\phi_E(t)$ is continuous at zero. Thus, the statement of Corollary 2.2 follows directly from Proposition 2.4. ∎

One more useful property of regular symmetric spaces now follows.

PROPOSITION 2.5. *Let $(E, \|\cdot\|_E)$ be a regular symmetric space on a non-atomic von Neumann algebra (M, μ), let $p_n \in \mathcal{P}(M)$, $p_n \downarrow 0$. Then $\|xp_n\|_E \to 0$ as $n \to 0$.*

We shall use in the proof of this proposition the following lemma.

LEMMA 2.3. *If E is a regular symmetric space on a non-atomic von Neumann algebra (M, μ), $x, y \in \mathcal{K}(M, \mu)$ and $x^*x, y^*y \in E$, then $x^*y \in E$ and $\|x^*y\|_E \leqslant \leqslant \|x^*x\|_E^{1/2}\|y^*y\|^{1/2}$.*

Proof. We may assume that $\|x^*x\|_E = \|y^*y\|_E = 1$. By Proposition 1.2. we have $\mu_t(x^*y) \prec \mu_t(x^*)\mu_t(y) = \mu_t(x)\mu_t(y) \leqslant 2^{-1}(\mu_t(x^*x) + \mu_t(y^*y))$.

The space $(E(0,\alpha), \|\cdot\|_{E(0,\alpha)})$ is regular (see Propostion 2.2), so it is an interpolation space between $L_1(0,\alpha)$ and $L_\infty(0,\alpha)$ with the interpolation constant one [7]. Therefore the function $\mu_t(x^*y)$ belongs to $E(0,\alpha)$ and

$$\|\mu_t(x^*y)\|_{E(0,\alpha)} \leqslant 2^{-1}(\|\mu_t(x^*x)\|_{E(0,\alpha)} + \|\mu_t(y^*y)\|_{E(0,\alpha)}) = 1.$$

Thus, $x^*y \in E$ and $\|x^*y\|_E \leqslant 1$.

Proof of Proposition 2.5. It is sufficient to show, that $\|xp_n\|_E \to 0$ for every $x \in E_+$. Put $y_n = p_n x^{1/2}$. We have $y_n y_n^* = p_n x p_n \in E$, $n = 1, 2, \ldots$ Since $\mu_t(y_n y_n^*) = \mu_t(y_n^* y_n)$ [18], we have $x^{1/2}px^{1/2} = y_n^* y_n \in E$. Since $x^{1/2}p_n x^{1/2} \downarrow 0$, we get $\|p_n x p_n\|_E = \|x^{1/2}p_n x^{1/2}\|_E \to 0$ as $n \to \infty$. Owing to Lemma 2.3, we obtain that $\|xp_n\|_E = \|x^{1/2}(x^{1/2}p_n)\|_E \leqslant \|x\|_E^{1/2}\|p_n x p_n\|_E^{1/2} \to 0$, as $n \to \infty$. ∎

COROLLARY 2.3. *Let $(E, \|\cdot\|_E)$ be a symmetric space on a non-atomic von Neumann algebra (M, μ). The following conditions are equivalent:*

1. *E is regular;*
2. *$\|xp_n\|_E \to 0$ for every $x \in E$ and any sequence $\{p_n\} \subset \mathcal{P}(M)$ decreasing to zero;*
3. *$\|p_n x p_n\|_E \to 0$ for every $x \in E$ and any sequence $\{p_n\} \subset \mathcal{P}(M)$ decreasing to zero.*

Proof. The implication $1 \Rightarrow 2$ is obtained in Proposition 2.5. The part $2 \Rightarrow 3$ is obvious. Let us establish the statement $3 \Rightarrow 1$. Assume that the condition 3 holds. Using the method of the proof of Proposition 1.2, we obtain that $\|fx_{A_n}\|_{E(0,\alpha)} \to 0$ for any $f \in E(0,\alpha)$ and for any decreasing sequence $\{A_n\}$ of mesurable sets from $[0,\alpha)$ with $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \emptyset$. This means that the functional space $(E, \|\cdot\|_{E(0,\alpha)})$ is regular. Hence, the space $(E, \|\cdot\|_E)$ is regular too (see Proposition 2.3 and Remark 2.2). ∎

A symmetric space E on (M, μ) is said to be *fully symmetric*, if from $y \prec x$, $x \in \in E$, $y \in \mathcal{K}(M, \mu)$ it follows that $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. It is clear that if under the conditions of Proposition 2.3 $(F, \|\cdot\|_F)$ is a fully symmetric functional space on $[0, \alpha)$ (i.e. F is the interpolation space between $L_1(0,\infty)$ and $L_\infty(0,\infty)$ with interpolation constant one), then $F(M, \mu)$ is a fully symmetric space. If M is a non-atomic von Neumann algebra, then the symmetric space E on (M, μ) is fully symmetric if and only if $E(0, \alpha)$ is fully symmetric.

PROPOSITION 2.6. *A regular symmetric space on a non-atomic von Neumann algebra is fully symmetric.*

The proof immediately follows from Proposition 2.2 and Theorem 4.10 (cf. [7], p. 142). ■

3. THE CRITERION OF CONVERGENCE IN REGULAR SYMMETRIC SPACES

We shall say, that a sequence $\{x_n\}$ from a symmetric space E on (M, μ) has absolutely equicontinuous norms (a.e.n.) if $\overline{\lim\limits_{n\to\infty}} \sup\limits_{m\geqslant 1} \|x_m p_n\|_E = 0$ for any sequence of projections $\{p_n\}$ from M, which decreases to zero.

PROPOSITION 3.1 *Let E be a regular symmetric space on a non-atomic von Neumann algebra (M, μ); let $x_n,\ x \in E,\ n = 1, 2, \ldots$. The following conditions are equivalent:*

1. $\|x_n - x\|_E \to 0$;
2. $x_n \xrightarrow{\mu} x$ *and* $\{x_n\}$ *has* a.e.n.

Proof. 1⇒ 2. Let $\|x_n - x\|_E \to 0,\ p_n \in \mathcal{P}(M),\ p_n \downarrow 0,\ \varepsilon > 0$. Then $x_n \xrightarrow{\mu} x$ [3,11] and $\|x_n - x\|_E \leqslant \varepsilon,\ n \geqslant n_0$ for some n_0. By virtue of Proposition 2.5 there exists n_1 such that $\|xp_n\| \leqslant \varepsilon,\ \|x_m p_n\|_E \leqslant \varepsilon$ for $n \geqslant n_1$ and $m = 1, 2, \ldots, n_0$. Since

$$\|x_m p_n\|_E \leqslant \|(x_m - x)p_n\|_E + \|xp_n\| \leqslant 2\varepsilon$$

for $m \geqslant n_0,\ n \geqslant n_1$, we have $\overline{\lim\limits_{n\to\infty}} \sup\limits_{m\geqslant 1} \|x_m p_n\|_E = 0$, i.e. x_n has a.e.n.

2⇒ 1. Let $x_n \xrightarrow{\mu} x$ and $\{x_n\}$ has a.e.n. Using Proposition 2.5 and the continuity of the modulus in the measure topology [17], we may assume that $x_n \xrightarrow{\mu} 0$ and $x_n \in E_+$, $n = 1, 2, \ldots$. Put $e = \sup\limits_{n\geqslant 1} \sup\limits_{\lambda > 0} \{x_n > \lambda\}$. It is clear, that $x_n e = x_n$ for all n. Owing to Proposition 2.1, $\mu(\{x_n > \lambda\}) < \infty$ for all λ and $n = 1, 2, \ldots$. So the restriction of μ onto eMe is σ-finite. Choose a sequence of projections $f_n \uparrow e$ such that $\mu(f_n) < \infty$, $n = 1, 2, \ldots$. Fix an arbitrary $\varepsilon > 0$. Since $\{x_n\}$ has a.e.n., there exists n_0 such that $\|(e - f_{n_0})x_m\|_E = \|x_m(e - f_{n_0})\|_E < \varepsilon$ for all $m = 1, 2, \ldots$. Thus we get $\|x_m\|_E \leqslant \|(e - f_{n_0})x_m\|_E + \|f_{n_0} x_m\|_E < \varepsilon + \|f_{n_0} x_m\|_E$.

Since $y_m = f_{n_0}x_m \xrightarrow{\mu} 0$, we may suppose, passing to a subsequence, that for some sequence of projections $\{q_m\}$ we have $y_m q_m \in M$, $\|y_m q_m\|_M < 2^{-m}$, $\mu(1-q_m) < 2^{-m}$. Put $p_m = \inf_{i \geqslant m} q_i$. Since $\mu(1-p_m) < 2^{-m+1}$, we have $p_m \uparrow 1$. It is clear also that $\|y_m p_n\|_M \to 0$ as $m \to \infty$ for any fixed n. Further, $y_m p_n \in L_1(M,\mu)$ and

$$\|y_m p_n\|_{L_1(M,\mu)} = \|f_{n_0} y_m p_n\|_{L_1(M,\mu)} \leqslant \|f_{n_0}\|_{L_1(M,\mu)} \|y_m p_n\|_M = \mu(f_{n_0})\|y_m p_n\|_M \to 0$$

as $m \to \infty$, for fixed n. This means that $\|y_m p_n\|_\cap \to 0$, where $\| \ \|_\cap$ is the norm of the space $L_1(M,\mu) \cap M$. By virtue of the continuity of the embedding of this space into E, we get $\|y_m p_n\|_E \to 0$ as $m \to \infty$, n is fixed. Using the property a.e.n. for the sequence $\{x_m\}$, choose the number n_1 so that $\|y_m(1-p_{n_1})\|_E < \varepsilon$ for all $m = 1, 2, \ldots$. Thus, we have

$$\overline{\lim}\|x_m\|_E \leqslant \varepsilon + \overline{\lim}\|y_m\|_E \leqslant \varepsilon + \overline{\lim}\|y_m p_{n_1}\|_E + \overline{\lim}\|y_m(1-p_{n_1})\|_E \leqslant 2\varepsilon. \quad \blacksquare$$

Corollary 3.1. *If, under the hypothesis of Proposition 3.1., $\|x_n - x\|_E \to 0$, then $\| |x_n| - |x| \|_E \to 0$.*

Proof. Owing to Proposition 3.1, we have $x_n \xrightarrow{\mu} x$ and x_n has a.e.n. So $|x_n| \xrightarrow{\mu} |x|$ [17] and, using the polar decomposition $x_n = u_n|x_n|$, we have that $|x_n|$ has a.e.n. too. Using proposition 3.1 again we get $\| |x_n| - |x| \|_E \to 0$. ∎

We shall find now a conection between the convergence of a sequence $\{x_n\}$ from a symmetric space and the convergence of the rearrangements $\mu_t(x_n)$.

Proposition 3.2. *Let E be a regular symmetric space on a non-atomic von Neumann algebra (M,μ), let $x_n,\ x \in E,\ n = 1, 2, \ldots$.*

The following conditions are equivalent:

1. $\|x_n - x\|_E \to 0$;
2. $x_n \xrightarrow{\mu} x$ *and* $\|\mu_t(x_n) - \mu_t(x)\|_{E(0,\alpha)} \to 0$.

Proof. The implication $1 \Rightarrow 2$ follows from [11] and from the relation $\mu_t(x_n) - -\mu_t(x) \prec \mu_t(x_n - x)$ [3].

$2 \Rightarrow 1$. Let $x_n \xrightarrow{\mu} x$ and $\|\mu_t(x_n) - \mu_t(x)\|_{E(0,\alpha)} \to 0$. Choose an arbitrary $\varepsilon > 0$ and put $g_r = \chi_{[r,\infty)}$. Since $E(0,\alpha)$ is regular (see Proposition 3.1), there exists a number $r > 0$ such that $\|\mu_t(x_n)g_r\|_{E(0,\alpha)} < \varepsilon$ for all $n = 0, 1, \ldots$ (we set $x_0 = x$). If $\mu_t(x) = 0$ for all $t > 0$, then the implication $2 \Rightarrow 1$ is evident. So we shall assume that $x_n \neq 0$, $n = 0, 1, \ldots$. Let $p_n = \{|x_n| > 0\}$, $N_n = p_n M p_n$ and let $\mathcal{U}_n$ be a non-atomic abelian von Neumann subalgebra of N_n containing all the projections $\{|x_n| > \lambda\}$, $\lambda > 0$ and such that the restriction μ_n of μ onto $\mathcal{U}_n$ is σ-finite (see Lemma 2.1). The algebra $\mathcal{U}_n$ is identified with the algebra $L_\infty(\Omega_n, \Sigma_n, \mu_n)$ and the space $L_1(\mathcal{U}_n, \mu_n) + \mathcal{U}_n$ is identified with the space $L_1(\Omega_n) + L_\infty(\Omega_n)$. Obviously, $|x_n| \in L_1(\mathcal{U}_n, \mu_n) + \mathcal{U}_n$. By Lemma 2.2, there exists a measure-preserving map ϕ_n from $E(x_n) = \Omega_n$ onto $[0, \alpha_n)$, where $\alpha_n = \mu_n(\Omega_n)$, such that $|x_n|(\omega) = \mu_{\phi_n(\omega)}(x_n)$, $\omega \in \Omega_n$, $n = 0, 1, \ldots$. Put $f_n(\omega) = g_r(\phi_n(\omega))$, $\omega \in \Omega_n$. Then f_n is a projection from $\mathcal{U}_n$ and $(|x_n| f_n)(\omega) = \mu_{\phi_n(\omega)}(x_n) g_r(\phi_n(\omega))$ for all $\omega \in \Omega_n$, $n = 0, 1, \ldots$. Hence $\| \, |x_n| f_n \|_E = \|\mu_t(x_n) g_r\|_{E(0,\alpha)} < \varepsilon$.

Using the polar decomposition, we obtain that $\|x_n f_n\|_E < \varepsilon$ for all $n = 0, 1, \ldots$. Put $e_n = p_n - f_n = (1 - g_r)(\phi_n(\omega))$, $\omega \in \Omega_n$. It is clear that $\mu(e_n) \leqslant r$, $n = 1, 2 \ldots$. Let $l_n = (1 - e_n) \wedge (1 - e_0)$. Using the equality $x p_n = x_n$, we get $\|x_n l_n\|_E = \|x_n(1 - e_n) l_n\|_E = \|x_n f_n l_n\|_E < \varepsilon$, $n = 0, 1, \ldots$. Since $x_n^* \xrightarrow{\mu} x^*$ and $\sup_{n \geqslant 1} \|e_n \vee e_0\|_M = 1$, we have $(e_n \vee e_0)(x_n^* - x^*) \xrightarrow{\mu} 0$. Passing onto a subsequence, using the inequality $\mu(e_n \vee e_0) \leqslant 2r$ and repeating the end of the proof of Proposition 3.1, we find a sequence of projections $q_m \uparrow 1$ such that $\mu(1 - q_m) < 2^{-m}$ and for any fixed m $\|(e_n \vee e_0)(x_n^* - x^*) q_m\|_E \to 0$ as $n \to \infty$. Since $\mu_t(x^*(1 - q_m)) \prec \mu_t(x_n)\mu_t(1 - q_m)$ (see Proposition 1.1), then using the symmetricity of the norm $\|\cdot\|_{E(0,\alpha)}$ (see Remark 2.2) we get

$$\|x_n^*(1 - q_m)\|_E \leqslant \|\mu_t(x_n)\mu_t(1 - q_m)\|_{E(0,\alpha)}.$$

Since $\mu_t(1 - q_m) = \chi_{(0, \mu(1 - q_m))}$ and $\mu(1 - q_m) \downarrow 0$, the a.e.n.-property of the sequence $\{\mu_t(x_n)\}$ implies the existence of m_0 such that $\|\mu_t(x_n)\mu_t(1 - q_{m_0})\|_{E(0,\alpha)} < \varepsilon$ for all $n = 0, 1, \ldots$. Thus, $\|(e_n \vee e_0)(x_n^* - x^*)(1 - q_{m_0})\|_E < 2\varepsilon$ for all $n = 0, 1, \ldots$. Hence, there exists a number $n(\varepsilon)$ such that $\|(x_n - x)(e_n \vee e_0)\|_E = \|(e_n \vee e_0)(x_n^* - x^*)\|_E < 3\varepsilon$ for all $n \geqslant n(\varepsilon)$.

Then

$$\|x_n - x\|_E \leqslant \|(x_n - x)(e_n \vee e_0)\|_E + \|(x_n - x) l_n\|_E < 5\varepsilon$$

for all $n > n(\varepsilon)$. ∎

COROLLARY 3.2. *If, under the hypothesis of Proposition 3.2 $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$, then $\|x_n - x\|_E \to 0$ if and only if $x_n \xrightarrow{\mu} x$.*

4. WEAK CONVERGENCE IN LORENTZ SPACES

Let $\varphi(t)$ be an increasing continuous concave function on $[0, \alpha)$, $\alpha = \mu(1)$ and let $\varphi(0) = 0$. Put

$$\Lambda_\varphi(M, \mu) = \left\{ x \in \mathcal{K}(M, \mu) : \int_0^\alpha \mu_t(x) \mathrm{d}\varphi(t) < \infty \right\}.$$

The set $\Lambda_\varphi(M,\mu)$ with the norm $\|x\|_{\Lambda_\varphi(M,\mu)} = \int\limits_0^\alpha \mu_t(x)\mathrm{d}\varphi(t)$ is fully symmetric space on (M,μ) [11]. If (M,μ) is a non-atomic von Neumann algebra then $\Lambda_\varphi(M,\mu)(0,\alpha)$ coincides with the functional Lorentz space $\Lambda_\varphi(0,\alpha) = \Lambda_\varphi(L_\infty(0,\alpha),m)_h$ of all real measurable functions f on $[0,\alpha)$ for which $\|f\|_{\Lambda_\varphi(0,\alpha)} = \int\limits_0^\alpha \tilde{f}(t)\mathrm{d}\varphi(t)$. If $\alpha < \infty$ then $(\Lambda_\varphi(0,\alpha), \|\cdot\|_{\Lambda_\varphi(0,\alpha)})$ is regular and so $\Lambda_\varphi(M,\mu)$ is regular too (see Proposition 2.3). If $\alpha = \infty$, then $\Lambda_\varphi(M,\mu)$ is regular if and only if $\lim\limits_{t\to\infty}\varphi(t) = \infty$ (see Lemma 5.1 from [7], p. 150 and Proposition 2.2 and 2.3).

REMARK 4.1. $\|xa\|_{\Lambda_\varphi(M,\mu)} = \|\,|x|a\|_{\Lambda_\varphi(M,\mu)}$ for all $x \in \Lambda_\varphi(M,\mu)$, $a \in M$. The proof immediately follows from the polar decomposition $x = u|x|$ for x and from the inequality $\|bx\|_{\Lambda_\varphi(M,\mu)} \leqslant \|b\|_M\|x\|_{\Lambda_\varphi(M,\mu)}$, $b \in M$, $x \in \Lambda_\varphi(M,\mu)$.

Let $\Gamma \subseteq L_1(M,\mu) \cap M = \Lambda$. For any symmetric space E on (M,μ) the notation $x_n \xrightarrow[\Gamma]{} x$, $x_n, x \in E$ means that $\mu(x_n y) \to \mu(xy)$ for all $y \in \Gamma$.

REMARK 4.2. If E is regular symmetric space on a non-atomic von Neumann algebra (M,μ), $x_n, x \in E$, $\sup\limits_{n\geqslant 1}\|x_n\|_E < \infty$, $\mu(x_n p) \to \mu(xp)$ for all $p \in \mathcal{P}(M)$ with $\mu(p) < \infty$, then $x_n \xrightarrow[\Lambda]{} x$.

Proof. By the assumption we have $\mu(x_n y) \to \mu(xy)$ for every $y \in N =$
$= \left\{a = \sum\limits_{i=1}^n \lambda_i p_i : p_i \in \mathcal{P}(M),\ \mu(p_i) < \infty,\ \lambda_i \text{ are complex numbers, } i = 1,2,\dots,n,\ n \text{ is a natural number}\right\}$. Let $a \in \Lambda_+$, $\varepsilon > 0$. Choose $y \in N$ so that $\|a - y\|_\cap < \varepsilon$. Since $(L_1(0,\alpha) \cap L_\infty(0,\alpha))_h$ is continuously embedded into the functional space $E'(0,\alpha)$ associated with $E(0,\alpha)$ (see [7]), we have $\|\mu_t(b)\|_{E'(0,\alpha)} \leqslant c\|b\|_\cap$ for all $b \in \Lambda$ and some $c > 0$. Thus, for some n_0 and for $n \geqslant n_0$ we have

$$|\mu(x_n a) - \mu(xa)| \leqslant |\mu(x_n y) - \mu(xy)| + |\mu(x_n(a-y))| + |\mu(x(a-y))| \leqslant$$
$$\leqslant \varepsilon + \int\limits_0^\infty \mu_t(x_n)\mu_t(a-y)\mathrm{d}t + \int\limits_0^\infty \mu_t(x)\mu(a-y)\mathrm{d}t \leqslant$$
$$\leqslant \varepsilon + (\|\mu_t(x_n)\|_{E(0,\alpha)} + \|\mu_t(x)\|_{E(0,\alpha)})\|\mu_t(a-y)\|_{E'(0,\alpha)} \leqslant \varepsilon + c\varepsilon(\sup_{n\geqslant 1}\|x_n\|_E + \|x\|_E).$$

It means that $\mu(x_n a) \to \mu(xa)$. Using the decomposition of any element from E into the linear combination of four positive elements from E, we obtain the statement of Remark 4.2. ■

PROPOSITION 4.1. *Let E be a regular symmetric space on a non-atomic von Neumann algebra (M,μ), let $x_n, x \in E$, and $x_n \xrightarrow[\Lambda]{} x$. Then*

$$\underline{\lim}\|x_n\|_E \geqslant \|x\|_E.$$

Proof. Let $x = u|x|$ be the polar decomposition of x. Since $x_n \underset{\Lambda}{\rightarrow} x$, we have $y_n = u^* x_n \underset{\Lambda}{\rightarrow} |x|$. In addition, $\underline{\lim}\|x_n\|_E \geqslant \underline{\lim}\|y_n\|_E$. Hence we may suppose that $x \geqslant 0$. Assume first that M is commutative von Neumann algebra and $\mu(1) < \infty$. Identify M with $L_\infty(\Omega, \Sigma, \mu)$ for some measurable space with a complete non-atomic measure, and $L_1(M, \mu)$ identify with $L_1(\Omega, \Sigma, \mu)$. In addition, E has been identified with some regular symmetric space of complex measurable functions on (Ω, Σ, μ).

Since E is regular, the adjoint space E^* coincides with E' and $\|f\|_E = = \sup\left\{\int\limits_\Omega fg \mathrm{d}\mu : g \in E', \|g\|_{E'} \leqslant 1\right\}$. Fix $\varepsilon > 0$ and choose $g \in E_+$ such that $\|g\|_{E'} \leqslant 1$ and $\int\limits_\Omega xg\mathrm{d}\mu > \|x\|_E - \varepsilon$. Put $A_n = \{\omega \in \Omega : g(\omega) < n\}$ and $g_n = g\chi_{A_n}$. There exists n_0 such that $\|x\|_E - \varepsilon < \int\limits_\Omega xg_{n_0}\mathrm{d}\mu = \lim\limits_{n\to\infty} \int\limits_\Omega x_n g_{n_0} \mathrm{d}\mu \leqslant \leqslant \underline{\lim}\|x_n\|_E \|g_{n_0}\|_{E'} \leqslant \underline{\lim}\|x_n\|_E$. Hence $\underline{\lim}\|x_n\|_E \geqslant \|x\|_E$. ■

Let now M be an arbitrary non-atomic von Neumann algebra and μ be a faithful normal semi-finite trace on M. Fix an arbitrary $\varepsilon > 0$. Using Proposition 2.1 and the regularity of E, choose a projection $p = \{x > \lambda\}$, $\lambda > 0$, such that $\mu(p) < \infty$ and $\|x(1-p)\|_E < \varepsilon$. Let N be a maximal commutative $*$-subalgebra of pMp, containing xp. We shall continue to denotate by μ the restriction of μ onto pMp and N. Let T be the conditional expectation from $L_1(pMp, \mu)$ onto $L_1(M, \mu)$. Since

$$y_n = px_np \underset{\Lambda}{\rightarrow} xp,$$

we have $\mu(T(y_n)y) = \mu(y_ny) \rightarrow \mu(xpy)$ for any $y \in N$, i.e. $T(y_n) \underset{\Lambda}{\rightarrow} xp$. It is clear that $E(N, \mu) = E \cap L_1(N, \mu)$ is a regular symmetric space on (N, μ) with the norm induced from E. Since E is fully symmetric (see Proposition 2.6) and $Ta \prec a$ for any $a \in L_1(pMp, \mu)$ [19], we have $Ty_n \in E(N, \mu)$. From this and from the first part of the proof we get

$$\underline{\lim}\|x_n\|_E \geqslant \underline{\lim}\|y_n\|_E \geqslant \underline{\lim}\|Ty_n\|_E \geqslant \|xp\|_E \geqslant \|x\|_E - \varepsilon.$$

We shall say that a symmetric space E on (M, μ) has the property (H) with respect to Γ, if from $x_n, x \in E$, $x_n \underset{\Gamma}{\rightarrow} x$, $\|x_n\|_E \rightarrow \|x\|_E$ it follows, that $\|x_n - x\|_E \rightarrow 0$. It is shown in [12] that if φ is a strictly concave increasing continuous function on $(0, \alpha)$, $\alpha < \infty$, $\varphi(0) = 0$, then $\Lambda_\varphi(0, \alpha)$ has the property (H) with respect to $(L_\infty(0, \alpha))_h$. The next proposition states the analogous result for the self-adjoint part $(\Lambda_\varphi(M, \mu))_h$ of a Lorentz space associated with an arbitrary commutative non-atomic von Neumann algebra M with a faithful normal finite trace μ.

PROPOSITION 4.2. *Let E be a symmetric space on a non-atomic commutative von Neumann algebra (M,μ), $\mu(1)<\infty$. Then E_h has the property (H) with respect to M_h if and only if $E(0,\alpha)$ has the property (H) with respect to $(L_\infty(0,\alpha))$.*

In the proof of this proposition we shall use the following.

LEMMA 4.1. *Let M be a non-atomic commutative von Neumann algebra, let μ be a faithful normal finite trace on M, let $\alpha=\mu(1)$, $x_n\in(L_1(M,\mu))_h$, $n= 1,2,\ldots$. Then there exists a non-atomic commutative von Neumann subalgebra N in M and a positive isometry U from $L_1(M,\mu)$ onto $L_1(0,\alpha)$ such that $x_n\in L_1(M,\mu)$, $n=1,2,\ldots$, $\mu_t(x)=(\tilde{U}x)(t)$ for all $t>0$, $x\in L_1(N,\mu)$, $U(N)=L_\infty(0,\alpha)$ and $U(xy)=U(x)U(y)$, $x\in L_1(N,\mu)$, $y\in N$.*

Proof. Let ∇_0 be a countable Boolean subalgebra in $\mathcal{P}(M)$ which contains all the projections $\{x_n>r\}$, $n=1,2,\ldots$, where r is a rational number. Let ∇ be the closure of ∇_0 in the measure topology τ. Then ∇ is a complete Boolean subalgebra in $\mathcal{P}(M)$, and besides the least upper bound in ∇ for any subset $A\subset\nabla$ coincides with the least upper bound of A in $\mathcal{P}(M)$. Such subalgebras are called regular. Let Δ be the set of all atoms in ∇ and $\Delta\neq 0$. Since $\mathcal{P}(M)$ is a non-atomic Boolean algebra, for every $q\in\Delta$ there exists in $q\nabla$ a non-atomic regular Boolean algebra ∇_q which is separable in the topology τ. Let B be the set of all $e\in\mathcal{P}(M)$ for which $e(1-\sup\Delta)\in\nabla$ and $eq\in\nabla_q$ for any $q\in\Delta$. It is clear that B is complete regular non-atomic and separable (with respect to the topology τ) Boolean subalgebra in $\mathcal{P}(M)$ which contains all the projections $x_n>\lambda$, $n=1,2,\ldots$, where λ is a real number. Hence, there exists an isomorphism ϕ from B on the Boolean algebra $\mathcal{P}(L_\infty(0,a))$ such that $m(\phi(e))=\mu(e)$ for all $e\in B$ [5]. Let us denote by N the non-atomic commutative von Neumann subalgebra in M for which $\mathcal{P}(N)=B$. Then $x_n\in L_1(N,\mu)$ for all $n=1,2,\ldots$. Evidently, the isomorphism ϕ may be extended up to the positive isometry U from $L_1(N,\mu)$ onto $L_1(0,\alpha)$ and, in addition, $U(N)=L_\infty(0,\alpha)$, $\mu_t(x)=(\tilde{U}x)(t)$ for all $t>0$, $x\in L_1(N,\mu)$ and $U(xy)=U(x)U(y)$, $x\in L_1(N,\mu)$, $y\in N$. ■

Proof of Proposition 4.2. Let $E(0,\alpha)$ have the property (H) with respect to $(L_\infty(0,\alpha))_h$, let $x_n,x\in E_h$ and let $x_n\xrightarrow[M_h]{}x$, $\|x_n\|_E\to\|x\|_E$. Let N be a $*$-subalgebra in M, let U be an isometry from $L_1(N,\mu)$ onto $L_1(0,a)$ for which $x,x_n\in E(N,\mu)=L_1(N,\mu)\cap E$ and the statement of Lemma 4.1 is fulfilled. It follows from the equality $\mu_t(y)=(\tilde{U}y)(t)$, $y\in E(N,\mu)$ that $U(E_h(N,\mu))=E(0,\alpha)$ and $\|Ux\|_{E(0,\alpha)}=\|x\|_E$ for all $x\in E_h(N,\mu)$. Put $f_n=Ux_n$, $f=Ux$. Then f_n, $f\in E(0,\alpha)$, $n=1,2,\ldots$, $f_n\xrightarrow[(L_\infty(0,\alpha))_h]{}f$ and $\|f_n\|_{E(0,\alpha)}\to\|f\|_{E(0,\alpha)}$. So $\|x_n-x\|_E=\|f_n-f\|_{E(0,\alpha)}\to 0$. The converse part is proved similarly. ■

Proposition 4.2 and [12] imply the following.

COROLLARY 4.1. *Let M be a non-atomic commutative von Neumann algebra, let μ be a faithful normal finite trace on M, let φ be a strictly concave increasing continuous function on $[0,\mu(\mathbf{1}))$, $\varphi(0)=0$. Then $(\Lambda_\varphi(M,\mu))_h$ has the property (H) with respect to M_h.*

The basic aim of this section is proving the following theorem.

THEOREM 4.1. *Let M be a non-atomic von Neumann algebra, let μ be a faithful, normal, semifinite trace on M, let φ be a strictly concave increasing continuous function on $(0,\mu(\mathbf{1}))$, and let $\varphi(0)=0$, $\lim_{t\to\infty}\varphi(t)=\infty$ if $\mu(\mathbf{1})=\infty$. Let $x_n, x\in\Lambda_\varphi(M,\mu)$, $x_n \underset{\Lambda}{\to} x$, $\|\mu_t(x_n)-\mu_t(x)\|_{\Lambda_{\varphi(0,\alpha)}}\to 0$. Then $\|x_n-x\|_{\Lambda_\varphi}(M,\mu)\longrightarrow 0$.*

We shall divide the proof of the theorem into several steps.

LEMMA 4.2. *It is sufficient to prove the statement of Theorem 4.1 for the case, when $\mu_t(x_n)=\mu_t(x)$, $n=1,2,\ldots$.*

Proof. Suppose that the statement of Theorem 4.1 is valid under the extra condition $\mu_t(x_n)=\mu_t(x)$, $n=1,2,\ldots$, $t>0$. Let $x_n,x\in\Lambda_\varphi(M,\mu)$, $x_n\underset{\Lambda}{\to}x$ and $\|\mu_t(x_n)-\mu_t(x)\|_{\Lambda_\varphi(0,\alpha)}\to 0$. Suppose that $\mu(\mathbf{1})=\infty$ (in the case $\mu(\mathbf{1})<\infty$ the proof is similar). Denote by J the set of all indices n for which $\mu(s(|x_n|))<\infty$, where $s(|x_n|)$ is the support projection of $|x_n|$. Using Lemmas 2.1 and 2.2 for each $n\in J$, we can construct $y_n\in(\Lambda_\varphi(M,\mu))_+$ such that $s(y_n)(l(x_n)+l(x_n^*))=0, \mu(s(y_n))=\infty$ and $\|y_n\|_{\Lambda_\varphi(M,\mu)}=\mathbf{1}$.

Put $a_n=x_n+n^{-1}y_n$ for all $n\in J$ and $a_n=x_n$ if $n\in J$. Since $|a_n|=|x_n|+$ $+n^{-1}y_n$, we have $\mu(s(|a_n|))=\infty$. Let $\mathcal{U}_n$ be a non-atomic commutative von Neumann subalgebra in $s(|a_n|)Ms(|a_n|)$ containing all the projections $\{|a_n|>\lambda\}$, $\lambda>0$ and such that the restriction μ_n of μ onto $\mathcal{U}_n$ is σ-finite (see Lemma 2.1). We shall identify $\mathcal{U}_n$ with $L_\infty(\Omega_n,\Sigma_n,\mu_n)$ and $L_1(\mathcal{U}_n,\mu_n)+\mathcal{U}_n$ with $L_1(\Omega_n)+L_1(\Omega_n)$. Since $|a_n|\in L_1(\mathcal{U}_n,\mu)+\mathcal{U}_n$, by Lemma 2.2, there exists a measure preserving transformation ϕ_n from Ω_n onto $[0,\infty)$ such that $|a_n|(\omega)=\mu_{\phi_n(\omega)}(a_n)$, $\omega\in\Omega_n$, $n=1,2,\ldots$. Put $b_n(\omega)=\mu_{\phi_n(\omega)}(x)-|a_n|(\omega)$, $\omega\in\Omega_n$. It is clear that $\mu_t(|a_n|+b_n)=\mu_t(x)$, $t>0$. Let $a_n=v_n|a_n|$ be the polar decomposition of a_n. Put $d_n=v_nb_n$, $n=1,2,\ldots$. We have $\|d_n\|_{\Lambda_\varphi(M,\mu)}\leqslant\|\mu_t(x)-\mu_t(a_n)\|_{\Lambda_\varphi(0,\alpha)}\leqslant\|\mu_t(x)-\mu_t(x_n)\|_{\Lambda_\varphi(0,\alpha)}+\|\mu_t(x_n)-\mu_t(a_n)\|_{\Lambda_\varphi(0,\alpha)}$. Since

$$\|\mu_t(x_n)-\mu_t(a_n)\|_{\Lambda_\varphi(0,\alpha)}\leqslant\|\mu_t(x_n-a_n)\|_{\Lambda_\varphi(0,\alpha)}=\|x_n-a_n\|_{\Lambda_\varphi(0,\alpha)}\to 0,$$

we have $\|d_n\|_{\Lambda_\varphi(M,\mu)}\to 0$ and, in particular, $d_n\underset{\Lambda}{\to}0$. From this and from the convergences $x_n\underset{\Lambda}{\to}x$, $\|x_n-a_n\|_{\Lambda_\varphi(M,\mu)}\to 0$ we get $z_n=a_n+d_n\underset{\Lambda}{\to}x$. Since $z_n=$

$= v_n(|a_n| + b_n)$ and $v_n^* v_n = s(|a_n|)$, we have $z_n^* z_n = (|a_n| + b_n)^2$ and, therefore, $\mu_t(z_n) = \mu_t(x)$. It follows from the conditions of the lemma that $\|z_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. Hence $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. ■

LEMMA 4.3. *Let the conditions of Theorem 4.1 be satisfied and, besides,* $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2 \ldots$, $p = \{|x| > \lambda\}$, $\lambda > 0$. *Then*

$$\lim_{n\to\infty} \|x_n p\|_{\Lambda_\varphi(M,\mu)} = \|xp\|_{\Lambda_\varphi(M,\mu)}.$$

Proof. Put $\beta = \mu(p)$. Using Proposition 1.1 and Remark 4.1 we get

$$\|x_n p\|_{\Lambda_\varphi(M,\mu)} = \int_0^\infty \mu_t(x_n p) \mathrm{d}\varphi(t) \leqslant \int_0^\beta \mu_t(x_n) \mathrm{d}\varphi(t) =$$

$$\int_0^\beta \mu_t(x) \mathrm{d}\varphi(t) = \| \, |x| p \|_{\Lambda_\varphi(M,\mu)} = \|xp\|_{\Lambda_\varphi(M,\mu)}.$$

On the other hand, since $x_n \underset{\Lambda}{\to} x$, we have $x_n p \underset{\Lambda}{\to} xp$, and, by Proposition 4.1, $\|xp\|_{\Lambda_\varphi(M,\mu)} \leqslant \underline{\lim} \|x_n p\|_{\Lambda_\varphi(M,\mu)}$. ■

LEMMA 4.4. *Let the conditions of Theorem 4.1 be satisfied and, besides,* $x \in \Lambda_\varphi(M, \mu)$, $p = \{|x| > \lambda\}$, $\lambda > 0$, $q \in \mathcal{P}(M)$, $\mu(q) = \mu(p) = \beta$, $\mu(|p - q|) = \varepsilon$. *Then* $\|xq\|_{\Lambda_\varphi(M,\mu)} \leqslant \|xp\|_{\Lambda_\varphi(M,\mu)} - d$, *where* $d > 0$ *depends on* $\varepsilon, \beta, \mu_t(x)$ *and does not depend on* q.

Proof. By Remark 4.1, we may suppose that $x \geqslant 0$. It follows from Proposition 1.1 that $xq \prec \mu_t(x)\mu_t(q) = \mu_t(x)\chi_{(0,\beta)} = \mu_t(xp)$, i.e. $xq \prec xp$. At first we shall prove, that

$$\int_0^\beta \mu_t(xq) \mathrm{d}t \leqslant \int_0^\beta \mu_t(xp) - \gamma,$$

where γ depends on $\varepsilon, \beta, \mu_t(x)$ and does not depend on q. Using Lemma 2.1, we can find a commutative von Neumann subalgebra $\mathcal{U}$ in M which contains all the spectral projections of x and such that the restriction of μ onto $\mathcal{U}$ is semifinite. Let T be the conditional expectation from $L_1(M, \mu) + M$ onto $L_1(\mathcal{U}, \mu) + \mathcal{U}$. Using the polar decomposition $xq = v|xq|$ of xq, we get

$$\int_0^\beta \mu_t(xq) \mathrm{d}t = \mu(|xq|) = \mu(v^* xq) = \mu(T(xqv^*)) = \mu(xT(qv^*)).$$

Since $(2\beta)^{-1}\|p-q\|^2_{L_1(M,\mu)} \leqslant \|p-q\|^2_{L_2(M,\mu)} = \mu((p-q)^2) = 2\beta - 2\mu(pq)$, we have $0 \leqslant \mu(pq) \leqslant \beta - \rho$, where $\rho = (4\beta)^{-1}\varepsilon^2$.
Hence

$$\|T(pqv^*)\|_{L_1(M,\mu)} \leqslant \|pq\|_{L_1(M,\mu)} \leqslant \|p\|_{L_2(M,\mu)} \cdot \|pq\|_{L_2(M,\mu)} =$$
$$= \beta^{1/2}\mu(qpq)^{1/2} \leqslant \beta(1-\rho\beta^{-1})^{1/2} \leqslant \beta - 2^{-1}\rho.$$

Choose $\rho' \in [2^{-1}\rho, \beta]$ such that $\int\limits_0^\infty \mu_t(T(pqv^*))dt = \|T(pqv^*)\|_{L_1(M,\mu)} = \beta - \rho'$. Then, using the inequality $\mu_t(T(pqv^*)) \leqslant 1$ for all $t > 0$, we get $\mu_t(T(pqv^*)) \prec \chi_{(0,\beta-\rho')}$. Therefore $|\mu(xT(pqv^*))| \leqslant \int\limits_0^\infty \mu_t(x)\mu_t(T(pqv^*))dt \leqslant \int\limits_0^{\beta-\rho'} \mu_t(x)dt$. Further,

$$|\mu(xT((1-p)qv^*))| = |\mu(x(1-p)T((1-p)qv^*))| \leqslant \int\limits_0^\infty \mu_t((1-p))\mu_t(T((1-p)qv^*))dt.$$

Since $\mathcal{U}$ is a commutative algebra, we have

$$\mu(|T((1-p)qv^*|) = \mu((1-p)|T(qv^*)|) = \mu(|T(qv^*)|) - \mu(|T(pqv^*)|) \leqslant$$
$$\leqslant \|q\|_{L_1(M,\mu)} - \|T(pqv^*)\|_{L_1(M,\mu)} = \rho'.$$

From this and from the inequality $\|T((1-p)qv^*)\|_M \leqslant 1$, we get $\mu_t(T((1-p)qv^*)) \prec \prec \chi_{(0,\rho')}$. Therefore

$$|\mu(xT((1-p)qv^*))| \leqslant \int\limits_0^{\rho'} \mu_t(x(1-p))dt = \int\limits_\beta^{\beta+\rho'} \mu_t(x)dt.$$

Thus we have

$$\int\limits_0^\beta \mu_t(xq)dt \leqslant \int\limits_0^{\beta-\rho'} \mu_t(x)dt + \int\limits_\beta^{\beta+\rho'} \mu_t(x)dt = \int\limits_0^\beta \mu_t(x)dt - \gamma_0,$$

where $\gamma_0 = \int\limits_{\beta-\rho'}^\beta \mu_t(x)dt - \int\limits_\beta^{\beta+\rho'} \mu_t(x)dt$. Since $\mu_t(x)$ is non-increasing and $\mu_t(x) > \mu_\beta(x)$ for $t \in (0,\beta)$, we have

$$\gamma_0 \geqslant \int\limits_{\beta-2^{-1}\rho}^\beta \mu_t(x)dt - \int\limits_\beta^{\beta+2^{-1}\rho} \mu_t(x)dt = \gamma > 0.$$

and, in addition, γ depends on $\varepsilon, \beta, \mu_t(x)$ and does not depend on q. Thus we have

$$\int_0^\beta \mu_t(xq)\mathrm{d}t + \gamma \leqslant \int_0^\beta \mu_t(x)\mathrm{d}t.$$

Choose $\beta_1 \in (0, \beta)$ such that $\int_{\beta_1}^\beta \mu_t(x)\mathrm{d}t \leqslant 2^{-1}\gamma$. Then $\int_0^\beta \mu_t(xq)\mathrm{d}t + 2^{-1}\gamma \leqslant \int_0^{\beta_1} \mu_t(x)\mathrm{d}t$. It follows from this inequality and from the relation $xq \prec xp$ that $\mu_t(xq) \prec \mu_t(x)\chi_{(0,\beta_1)}$. Hence

$$\|xq\|_{\Lambda_\varphi(M,\mu)} = \int_0^\beta \mu_t(xq)\varphi'(t)\mathrm{d}t \leqslant \int_0^{\beta_1} \mu_t(x)\varphi'(t)\mathrm{d}t = \|xp\|_{\Lambda_\varphi(M,\mu)} - \int_{\beta_1}^\beta \mu_t(x)\varphi'(t)\mathrm{d}t.$$

Since φ is strictly concave, we have $d = \int_{\beta_1}^\beta \mu_t(x)\varphi'(t)\mathrm{d}t > 0$. From the way we have constructed d we see that this number depends on ε, β, $\mu_t(x)$ and does not depend on q. ∎

LEMMA 4.5. *Let the conditions of Theorem 4.1 be satisfied and, besides,* $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$. *Then* $\| |x_n| - |x| \|_{\Lambda_\varphi(M,\mu)} \to 0$.

Proof. For every $\lambda > 0$ we put $p_\lambda(n) = \{|x_n| > \lambda\}$, $p_\lambda\{|x| > \lambda\}$. We shall prove that $\|p_\lambda(n) - p_\lambda\|_{L_1(M,\mu)} \to 0$. If this is not the case then, passing to a subsequence, we get $\|p_\lambda(n) - p_\lambda\|_{L_1(M,\mu)} \geqslant \varepsilon$ for all $n = 1, 2, \ldots$ and some $\varepsilon > 0$. Since $\mu_t(x_n) = \mu_t(x)$, we have $\mu(p_\lambda(n)) = \mu(p_\lambda) = \beta$, $n = 1, 2, \ldots$. By Lemma 4.4, we have $\|x_n p_\lambda\|_{\Lambda_\varphi(M,\mu)} \leqslant \|x_n p_\lambda(n)\|_{\Lambda_\varphi(M,\mu)} - d$, where $d > 0$ depends only on $\varepsilon, \beta, \mu_t(x_n)$. Since $\|x_n p_\lambda(n)\|_{\Lambda_\varphi(M,\mu)} = \| |x_n| p_\lambda(n)\|_{\Lambda_\varphi(M,\mu)} = \int_0^\beta \mu_t(x_n)\mathrm{d}\varphi(t) = \int_0^\beta \mu_t(x)\mathrm{d}\varphi(t) = \|xp_\lambda\|_{\Lambda_\varphi(M,\mu)}$, we have $\|x_n p_\lambda\|_{\Lambda_\varphi(M,\mu)} \leqslant \|xp\|_{\Lambda_\varphi(M,\mu)} - d$, which contradicts Lemma 4.3. Hence $\|p_\lambda(n) - p_\lambda\|_{L_1(M,\mu)} \to 0$. Therefore, $\mu_t(p_\lambda(n) - p_\lambda) \xrightarrow{m} 0$ on $(0, 2\beta)$. Using the Lebesgue dominated convergence theorem we get

$$(1) \quad \|p_\lambda(n) - p_\lambda\|_{\Lambda_\varphi(M,\mu)} = \int_0^\infty \mu_t(p_\lambda(n) - p_\lambda)\mathrm{d}\varphi(t) = \int_0^{2\beta} \mu_t(p_\lambda(n) - p_\lambda)\varphi'(t)\mathrm{d}t \to 0.$$

Fix $\varepsilon > 0$. Since $\Lambda_\varphi(M, \mu)$ is a regular symmetric space on (M, μ), there exist $\lambda'' > \lambda' > 0$ such that for all $n = 1, 2, \ldots$,

$$(2) \qquad \| |x_n| p_{\lambda''}(n)\|_{\Lambda_\varphi(M,\mu)} = \| |x| p_{\lambda''}\|_{\Lambda_\varphi(M,\mu)} \leqslant \varepsilon$$

$$(3)\qquad \| |x_n|(\mathbf{1}-p_{\lambda'}(n))\|_{\Lambda_\varphi(M,\mu)} = \| |x|(\mathbf{1}-p_{\lambda'})\|_{\Lambda_\varphi(M,\mu)} \leqslant \varepsilon$$

(see Proposition 2.5). Choose a partition $\lambda' = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_r = \lambda''$ such that $\|\lambda_{i-1} - \lambda_i\| \leqslant \varepsilon\|p_{\lambda'} - p_{\lambda''}\|^{-1}_{\Lambda_\varphi(M,\mu)}$ for all $i = 1, 2, \ldots, r$. Then

$$(4)\qquad \| |x|(p_{\lambda'} - p_{\lambda''}) - \sum_{i=1}^{r} \lambda_{i-1}(p_{\lambda_{i-1}} - p_{\lambda_i})\|_{\Lambda_\varphi(M,\mu)} \leqslant \varepsilon$$

$$(5)\qquad \| |x_n|(p_{\lambda'} - p_{\lambda''}) - \sum_{i=1}^{r} \lambda_{i-1}(p_{\lambda_{i-1}}(n) - p_{\lambda_i}(n))\|_{\Lambda_\varphi(M,\mu)} \leqslant \varepsilon.$$

Using (1)–(5) and arbitrarness of ε, we get

$$\| |x_n| - |x| \|_{\Lambda_\varphi(M,\mu)} \to 0. \qquad \blacksquare$$

LEMMA 4.6. *Let the conditions of the Theorem 4.1 be satisfied and, besides,* $x \geqslant 0$, $x_n = x_n^*$, $n = 1, 2, \ldots$. *Then* $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$.

Proof. Suppose first that $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$. Fix $\varepsilon > 0$ and choose $p = \{x > \lambda\}$ such that $\|x(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} < \varepsilon$. By Remark 4.1 and Lemma 4.5, there exists n_0 such that for $n \geqslant n_0$

$$(6)\qquad \|(\mathbf{1}-p)x_n\|_{\Lambda_\varphi(M,\mu)} = \|x_n(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} = \| |x_n|(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} < \varepsilon.$$

Since $\mu(p) < \infty$, we have px, $p|x_n|p \in \Lambda_\varphi(pMp, \mu) \subset L_1(M, \mu)$. Therefore, the convergence $\|p|x_n|p - px\|_{\Lambda_\varphi(M,\mu)} \to 0$ (see Lemma 4.5) implies $\mu(p|x_n|p) \to \mu(px)$. Since $x_n \underset{\Lambda}{\to} x$, we have $\mu(px_np) \to \mu(px)$. Hence, $\|p(x_n)_- p\|_{L_1(M,\mu)} = \mu(p(x_n)_- p) = 2^{-1}\mu(p(|x_n| - x)p) \to 0$, where $(x_n)_-$ is the negative part of x_n. Therefore, $|(x_n)_-^{1/2}p|^2 = p(x_n)_- p \xrightarrow{\mu} 0$, and so [17] $(x_n)_-^{1/2}p \xrightarrow{\mu} 0$ from which $p(x_n)_-^{1/2} \xrightarrow{\mu} 0$. Since $\| |x_n| - x\|_{\Lambda_\varphi(M,\mu)} \to 0$, we have $|x_n| \xrightarrow{\mu} x$ and $|x_n|^{1/2} \xrightarrow{\mu} x^{1/2}$. Hence, $p(x_n)_- = p(x_n)_-^{1/2}|x_n|^{1/2} \xrightarrow{\mu} 0$. Since the sequence $\{|x_n|\}$ has a.e.n. (see Proposition 3.1), $\{p(x_n)_-\}$ has a.e.n. too, and so, by Proposition 3.1, $\|p(x_n)_-\|_{\Lambda_\varphi(M,\mu)} \to 0$. Therefore, $\|p(x_n)_+ - px\|_{\Lambda_\varphi(M,\mu)} \to 0$, where $(x_n)_+$ is the positive part of x_n. Thus $\|px_n - px\|_{\Lambda_\varphi(M,\mu)} \to 0$. From this and from (6) we get $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} < 3\varepsilon$ if $n \geqslant n_1$ for some $n_1 \geqslant n_0$.

Suppose now, that the assumption $\mu_t(x_n) = \mu_t(x)$ does not hold, but $\|\mu_t(x_n) - \mu_t(x)\|_{\Lambda_\varphi(M,\mu)} \to 0$. Take, as in the proof of Lemma 4.2, a non-atomic commutative von Neumann subalgebra $\mathcal{U}_n$ containing all the projections $\{(a_n)_+ > \lambda\}$, $(a_n)_- > \lambda\}$, $\lambda > 0$. Repeating the proof of Lemma 4.2 and taking into account, that $v_n =$

$= v_n^* \in \mathcal{U}$, $y_n = y_n^*$ (these notations are from the proof of Lemma 4.2), we get $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. ■

LEMMA 4.7. *Let φ be a strictly concave increasing continuous function on $[0,\alpha)$, $\alpha < \infty$, $h_n, f_n, g_n \in \Lambda_\varphi(0,\alpha)$, $h_n = \tilde{h}_n$, $f_n = \tilde{f}_n$, $g_n = g_n$, $h_n \prec f_n \prec g_n$, $n = 1,2,\dots$, $\|h_n - f\|_{\Lambda_\varphi(0,\alpha)} \to 0$, $\|g_n - f\|_{\Lambda_\varphi(0,\alpha)} \to 0$. Then $\|f_n - f\|_{\Lambda_\varphi(0,\alpha)} \to 0$.*

Proof. Since $\|h_n\|_{\Lambda_\varphi(0,\alpha)} \leqslant \|f_n\|_{\Lambda_\varphi(0,\alpha)} \leqslant \|g_n\|_{\Lambda_\varphi(0,\alpha)}$, we have $\|f_n\|_{\Lambda_\varphi(0,\alpha)} \to \|f\|_{\Lambda_\varphi(0,\alpha)}$. We shall show that $f_n \underset{(L_\infty(0,\alpha))_h}{\longrightarrow} f$. Then it will follow from [12] that $\|f_n - f\|_{\Lambda_\varphi(0,\alpha)} \to 0$. For every $t \in (0,\alpha)$ we have

$$\int_0^t h_n(\tau)\mathrm{d}\tau \leqslant \int_0^t f_n(\tau)\mathrm{d}\tau \leqslant \int_0^t g_n(\tau)\mathrm{d}\tau.$$

It follows from these inequalities and the conditions of the lemma that

$$\int_0^\alpha f_n\chi_{(0,t)}\mathrm{d}\tau \to \int_0^\alpha f\chi_{(0,t)}\mathrm{d}\tau.$$

Hence, $\int_0^\alpha f_n\chi_{(s,t)}\mathrm{d}\tau \to \int_0^\alpha f\chi_{(s,t)}\mathrm{d}\tau$ for all $0 < s < t < \alpha$. Fix $\varepsilon > 0$. Since $\Lambda_\varphi(0,\alpha)$ is continuously imbedded into $L_1(0,\alpha)$, we have $\|g_n - f\|_{L_1(0,\alpha)} \to 0$ and, therefore, there exists $\delta > 0$ such that $\int_0^\delta g_n(t)\mathrm{d}t < \varepsilon$ for all $n = 1,2,\dots$ (see Proposition 3.1) and $\int_0^\delta \tilde{f}(t)\mathrm{d}t < \varepsilon$. For any measurable set $A \subset [0,\alpha)$ choose

$$B = \bigcup_{i=1}^n (s_i, t_i) \subset [0,\alpha)$$

such that $m(A\Delta B) < \delta$. Then $\left|\int_0^\alpha f_n\chi_A\mathrm{d}t - \int_0^\alpha f\chi_A\mathrm{d}t\right| \leqslant \int_0^\alpha f_n\chi_{A\Delta B}\mathrm{d}t + \left|\int_0^\alpha f_n\chi_B\mathrm{d}t - \int_0^\alpha f\chi_B\mathrm{d}t\right| + \int_0^\alpha f\chi_{A\Delta B}\mathrm{d}t \leqslant \int_0^\delta g_n(t)\mathrm{d}t + \varepsilon + \int_0^\delta \tilde{f}(t)\mathrm{d}t < 3\varepsilon$ as $n \geqslant n_0$ for some n_0. Using Remark 4.2, we get $f_n \underset{(L_\infty(0,\alpha))_h}{\longrightarrow} f$. ■

LEMMA 4.8. *Let the conditions of the Theorem 4.1 be satisfied and, besides, $x \geqslant 0$, $\mu_t(x_n) = \mu_t(x)$, $n = 1,2,\dots$. Then $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$.*

Proof. By Lemma 4.5, $\| |x_n| - x \|_{\Lambda_\varphi(M,\mu)} \to 0$, $\| |x_n^*| - x \|_{\Lambda_\varphi(M,\mu)} \to 0$. Therefore, for given $\varepsilon > 0$ there exist $p = \{x > \lambda\}$, $\lambda > 0$ and n_0 such that for $n \geqslant n_0$

$$(7) \qquad \|x(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} < \varepsilon, \quad \|x_n(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} < \varepsilon, \quad \|x_n^*(\mathbf{1}-p)\|_{\Lambda_\varphi(M,\mu)} < \varepsilon.$$

Put $a_n = \operatorname{Re} x_n = 2^{-1}(x_n + x_n^*)$, and let N_n be a commutative von Neumann subalgebra containing all the spectral projections of $|a_n|$ and such that the restriction of μ onto N_n is semifinite, $n = 0, 1, \ldots$, (we put $x_0 = x$). Let T_n be the conditional expectations from $L_1(M,\mu) + M$ onto $L_1(N_n,\mu) + N_n$. Then $T_n(x_n) = a_n + \mathrm{i} T_n(\operatorname{Im} x_n)$, where $\operatorname{Im} x_n = (2\mathrm{i})^{-1}(x_n - x_n^*)$, and, so $|a_n| \leqslant |T_n(x_n)|$, whence, $\mu_t(a_n) \leqslant \mu_t(T_n(x_n)) \prec \mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$. Therefore

$$\mu_t(T_0(pa_np)) \prec \mu_t(pa_np) \leqslant \mu_t(a_n)\mu_t(p) \prec \mu_t(x)\mu_t(p) = \mu_t(xp).$$

Hence, $\|T_0(pa_np)\|_{\Lambda_\varphi(pN_0p,\mu)} \leqslant \|xp\|_{\Lambda_\varphi(M,\mu)}$ (remember that $T_0 a \prec a$ for all $a \in L_1(M,\mu) + M$ [19] and so $T_0(\Lambda_\varphi(M,\mu)) = \Lambda_\varphi(N_0,\mu))$. On the other hand, since $pa_np \xrightarrow[\Lambda]{} xp$, we have $T_0(pa_np) \xrightarrow[N_0]{} T_0(xp) = xp$ and hence (see Proposition 4.1) $\underline{\lim} \|T_0(pa_np)\|_{\Lambda_\varphi(pN_0p,\mu)} \geqslant \|xp\|_{\Lambda_\varphi(pN_0p,\mu)}$. Thus, $\lim \|T_0(pa_np)\|_{\Lambda_\varphi(pN_0p,\mu)} = \|xp\|_{\Lambda_\varphi(pN_0p,\mu)}$. By Corollary 4.1, $\|T_0(pa_np) - xp\|_{\Lambda_\varphi(pN_0p,\mu)} \to 0$, whence $\|\mu_t(T_0(pa_np)) - \mu_t(xp)\|_{\Lambda_\varphi(0,\alpha)} \to 0$ (see Proposition 3.2). Therefore, by Lemma 4.7, we have $\|\mu_t(pa_np) - \mu_t(xp)\|_{\Lambda_\varphi(0,\alpha)} \to 0$. By Lemma 4.6, we get $\|pa_np - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. Thus, $|x_n| \xrightarrow{\mu} x$, $|x_n^*| \xrightarrow{\mu} x$, $\operatorname{Re} x_n \xrightarrow{\mu} x$. Since $2((\operatorname{Re} x_n)^2 + (\operatorname{Im} x_n)^2) = |x_n|^2 + |x_n^*|^2 \xrightarrow{\mu} 2x^2$, we have $|\operatorname{Im} x_n| \xrightarrow{\mu} 0$, whence $\operatorname{Im} x_n \xrightarrow{\mu} 0$. It means that $x_n = \operatorname{Re} x_n + \mathrm{i}\operatorname{Im} x_n \xrightarrow{\mu} x$. Since the sequence $\{|x_n|\}$ has a.e.n. (see Proposition 3.1) we get, by Remark 4.1, that the sequence $\{x_n\}$ has a.e.n. too. Then, by Proposition 3.1, $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. ∎

LEMMA 4.9. *Let conditions of the Theorem 4.1 be satisfied and, besides, $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$. Then $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$.*

Proof. Put $f = \mathbf{1} - l(x)$, $e = \mathbf{1} - l(x^*)$. By the projections comparison theorem (see, for example, [16], p. 293) there exists a central projection z in M such that $fz \preceq ez$ and $e(\mathbf{1}-z) \preceq f(\mathbf{1}-z)$. Let v and w be partial isometries from M such that $vv^* = fz$, $v^*v \leqslant ez$, $ww^* = e(\mathbf{1}-z)$, $w^*w \leqslant f(\mathbf{1}-z)$. Put $y = xz + z^*(\mathbf{1}-z)$, $y_n = x_nz + x_n^*(\mathbf{1}-z)$. It is clear that $y_n \xrightarrow[\Lambda]{} y$. Besides, $|y| = |x|z + |x^*|(\mathbf{1}-z)$, hence, using the equality $\mu_t(x^*(\mathbf{1}-z)) = \mu_t(x(\mathbf{1}-z))$ [18], we get $\mu(|y| > \lambda) = \mu(|x|z > \lambda) + \mu(|x^*|(\mathbf{1}-z) > \lambda) = \mu(|x| > \lambda)$, $\lambda > 0$. This means that $\mu_t(y) = \mu_t(x)$, $t > 0$. Similarly we get $\mu_t(y_n) = \mu_t(x_n) = \mu_t(y)$, $n = 1, 2, \ldots$. Let $x = u|x|$ be the polar decomposition of x. Put $a = (v+u)z + (w+u^*)(\mathbf{1}-z)$. Since $uv^* = 0$, $wu = 0$, we have $aa^* = (vv^* + uu^*)z + (ww^* + u^*u)(\mathbf{1}-z) = \mathbf{1}$. Put $b = a^*y$, $b_n = a^*y_n$, $n = 1, 2, \ldots$. Since $b_n^*b_n = y_n^*y_n$, we have $\mu_t(b_n) = \mu_t(y_n) =$

$= \mu_t(y) = \mu_t(b)$, $n = 1, 2, \ldots$. Using the equalities $v^*x = 0$ and $w^*x^* = 0$, we get $b = ((v^* + u^*)z + (w^* + u)(1 - z))(xz + x^*(1 - z)) = u^*xz + ux^*(1 - z) = = |x|z + |x^*|(1 - z) \geqslant 0$. Since $b_n \underset{\Lambda}{\to} b$, we have, by Lemma 4.8, $\|b_n - b\|_{\Lambda_\varphi(M,\mu)} \to 0$. Using now the equality $aa^* = 1$, we get $\|y_n - y\|_{\Lambda_\varphi(M,\mu)} \to 0$. Hence $\|(x_n - x)z\|_{\Lambda_\varphi(M,\mu)} = \|(y_n - y)z\|_{\Lambda_\varphi(M,\mu)} \to 0$ and $\|(x_n - x)(1 - z)\|_{\Lambda_\varphi(M,\mu)} \to 0$. This means that $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. ■

The proof of Theorem 4.1 now follows from Lemmas 4.2 and 4.9.

5. WEAK CONVERGENCE IN REGULAR SYMMETRIC SPACES

The aim of this section is proving the basic results of this paper: Theorems 5.1 and 5.2.

THEOREM 5.1. *Let M be a non-atomic von Neumann algebra, let μ be a faithful, normal, semifinite trace on M, let E be a regular symmetric space on (M, μ), let $x_n, x \in E$, $n = 1, 2, \ldots$. Then the following conditions are equivalent:*

1. $\|x_n - x\|_E \to 0$;
2. $\|\mu_t(x_n) - \mu_t(x)\|_{E(0,\alpha)} \to 0$ *and* $x_n \to x$ *weakly;*
3. $\|\mu_t(x_n) - \mu_t(x)\|_{E(0,\alpha)} \to 0$ *and* $\mu(x_n p) \to \mu(xp)$ *for all* $p \in \mathcal{P}(M)$ *with* $\mu(p) < \infty$.

The proof of Theorem 5.1 essentially uses the statement of Theorem 4.1 and requires some preliminary preparations.

LEMMA 5.1. *Let $(F, \|\cdot\|_F)$ be a regular symmetric function space on $[0, \alpha)$. Then there exists a continuous concave function φ on $[0, \alpha)$ such that $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ if $\alpha = \infty$, $F \subset \Lambda_\varphi(L_\infty(0, \infty), m)$ and $\|f\|_{\Lambda_\varphi(L_\infty(0,\alpha),m)} \leqslant \|f\|_F$, for every $f \in F$.*

Proof. Let $\alpha = \infty$. We shall show that $\tilde{f}(\infty) = 0$ for all $f \in F''$, where F'' is the second associated space of F. If this is not the case, then $\chi_{[0,\infty)} \in F''$ and, therefore,

$$\sup_{n\geqslant 1} \|\chi_{[0,n]}\|_F = \sup_{n\geqslant 1} \|\chi_{[0,n]}\|_{F''} \leqslant \|\chi_{[0,\infty)}\|_{F''} < \infty.$$

It means that the fundamental function $\phi_{(F,\|\cdot\|_F)}(t)$ of the space F is bounded on $[0, \infty)$. Consider another norm $\|\cdot\|'$ on F which is equivalent to the norm $\|\cdot\|_F$ and such that $(F, \|\cdot\|')$ is a symmetric space on $[0, \infty)$ and $\phi(t) = \phi_{(F,\|\cdot\|')}(t)$ is a concave function (see [7], p. 164). Then (see [7], Theorem 5.5, p. 160) $\Lambda_\varphi(0, \infty) \subset F_h$ and, since $\phi(t)$ is bounded on $[0, \infty)$, we have $(L_\infty(0, \infty))_h \subset \Lambda_\varphi(0, \infty) \subset F_h$ which contradicts with the regularity of F (see the Proposition 2.1). Hence, $\tilde{f}(\infty) = 0$

for all $f \in F''$. This means that there exists $g \in F'$ such that $g = \tilde{g}$, $||g||_{F'} = 1$, $g \overline{\in} L_1(0,\infty)$ (otherwise $F' \subset L_1(L_\infty(0,\infty), m)$ and so $L_\infty(0,\infty) \subset F''$). The function $\varphi(t) = \int_0^t g(\tau)\mathrm{d}\tau$ is continuous concave increasing and $\varphi(0) = 0$, $\varphi(\infty) = \infty$. For every $f \in F$ we have

$$\int_0^\infty \tilde{f}(t)\mathrm{d}\varphi(t) = \int_0^\infty \tilde{f}(t)g(t)\mathrm{d}t \leqslant ||f||_F||g||_{F'} = ||f||_{F'}$$

i.e. $f \in \Lambda_\varphi(L_\infty(0,\infty), m)$ and $||f||_{\Lambda_\varphi(L_\infty(0,\infty),m)} \leqslant ||f||_F$. If $\alpha < \infty$, then the function $\varphi(t) = \int_0^t g(\tau)\mathrm{d}\tau$, where $g = \tilde{g} \in F'$, $||g||_{F'} = 1$, does the job. ∎

LEMMA 5.2. *Let E be a regular symmetric space on a non-atomic von Neumann algebra (M,μ). Then there exists a continuous strictly concave increasing function φ on $[0,\alpha)$, $\alpha = \mu(\mathbf{1})$ such that $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ if $\alpha = \infty$, $E \subset \Lambda_\varphi(M,\mu)$ and $||x||_{\Lambda_\varphi(M,\mu)} \leqslant c||x||_E$ for all $x \in E$ and some $c > 0$.*

Proof. By Proposition 2.2, Remark 2.1 and Lemma 5.1, there exists a function φ satisfying all the conditions of Lemma 5.2 except, maybe, the strict concavity of φ. Put $\varphi_0(t) = \operatorname{arctg} t$, and $\varphi_1(t) = \varphi(t) + \varphi_0(t)$, $t \in [0,\infty)$. The function $\varphi_1(t)$ is continuous strictly concave and increasing on $[0,\alpha)$ and, in addition, $\varphi_1(0) = 0$ and $\lim_{t\to\infty} \varphi_1(t) = \infty$ if $\alpha = \infty$. Since $\lim_{t\to\infty} \varphi_0(t) < \infty$, we have $(L_\infty(0,\alpha))_h \subset \Lambda_{\varphi_0}(0,\alpha)$. Besides, it is evident that $L_1(0,\alpha) \subset \Lambda_{\varphi_0}(0,\alpha)$. Therefore $\Lambda_{\varphi_0}(0,\alpha) = L_1(0,\alpha) + L_\infty(0,\alpha)$, and so $\Lambda_{\varphi_0}(M,\mu) = L_1(0,\alpha) + M$. Hence by virtue of the continuity of the embedding of $\Lambda_{\varphi_0}(M,\mu)$ into $L_1(0,\alpha) + M$, we have that the norms $||\cdot||_{\Lambda_\varphi(M,\mu)}$ and $||\cdot||_+$ are equivalent on $L_1(M\mu) + M$. In particular, $||x||_{\Lambda_{\varphi_0}(M,\mu)} \leqslant c_0||x||_+$ for all $x \in L_1(M,\mu) + M$ and some $c_0 > 0$. Since E is continuously embedded into $L_1(M,\mu) + M$ too, we have $||x||_{\Lambda_{\varphi_0}(M,\mu)} \leqslant c_0c_1||x||_E$ for all $x \in E$ and some $c_1 > 0$. We get from embeddings $E \subset \Lambda_{\varphi_0}(M,\mu)$ and $E \subset \Lambda_\varphi(M,\mu)$ that $E \subset \Lambda_{\varphi_1}(M,\mu)$ and, in addition, $||x||_{\Lambda_{\varphi_1}(M,\mu)} = ||x||_{\Lambda_\varphi(M,\mu)} + ||x||_{\Lambda_{\varphi_0}(M,\mu)} \leqslant (1 + c_0c_1)||x||_E$ for all $x \in E$, which concludes the proof of the lemma. ∎

Proof of Theorem 5.1. The implication $1 \Rightarrow 2$ follows from Proposition 3.2. If $p \in \mathcal{P}(M)$ and $\mu(p) < \infty$, then it follows from the inequality $|\mu(xp)| \leqslant ||x||_E \cdot ||\chi_{(0,\mu(p))}||_{E'(0,\alpha)}$, $x \in E$, that $\mu(xp)$ is a continuous linear functional on $(E, ||\cdot||_E)$, and, therefore, the implication $2 \Rightarrow 3$ is obvious.

$3 \Rightarrow 1$. Let $||\mu_t(x_n) - \mu_t(x)||_{E(0,\alpha)} \to 0$ and $\mu(x_np) \to \mu(xp)$ for all $p \in \mathcal{P}(M)$, $\mu(p) < \infty$. Then, by Remark 4.2, we have $x_n \underset{\Lambda}{\to} x$. Let φ be strictly convex function from Lemma 5.2. Since

$$||\mu_t(x_n) - \mu_t(x)||_{\Lambda_\varphi(0,\alpha)} \leqslant c||\mu_t(x_n) - \mu_t(x)||_{E(0,\alpha)},$$

we have, by Theorem 4.1, $\|x_n - x\|_{\Lambda_\varphi(M,\mu)} \to 0$. Hence, $x_n \xrightarrow{\mu} x$ [11]. Using Proposition 3.2 we have $\|x_n - x\|_E \to 0$. ∎

REMARK 5.1. The implications $1 \Rightarrow 2 \Rightarrow 3$ from the statement of Theorem 5.1 are valid for any symmetric space E on a non-atomic von Neumann algebra (M, μ) with fully symmetric norm (see proof of Proposition 3.2).

PROPOSITION 5.1. *If E is a non-regular symmetric space on a non-atomic von Neumann algebra (M, μ), then the statement of Theorem 5.1 is false.*

Proof. We shall prove that the implication $3 \Rightarrow 1$ is incorrect. Suppose at first, that the fundamental function $\phi_E(t)$ is discontinuous at zero, i.e. $\lim_{t\downarrow 0} \phi_E(t) = \varepsilon > 0$. Take a projection p with $\mu(p) < \infty$ and represent it in the form $p = \sup_{n\geqslant 1} p_n$, where $p_n \in \mathcal{P}(M)$, $p_n p_k = 0$, $n \neq k$, $\mu(p) = 2^{-n}\mu(p)$ for all $n = 1, 2, \ldots$. Put $x_n = p - 2p_n$. It is clear that $\mu_t(x_n) = \mu_t(p)$ for all $n = 1, 2, \ldots$. Besides, if $g \in \mathcal{P}(M)$, then

$$\mu(p_n g) = \mu(p_n g p_n) \to 0,$$

and therefore $\mu(x_n g) \to \mu(pg)$. On the other hand, $\|x_n - p\|_E = 2\|p_n\|_E \geqslant 2\varepsilon$, $n = 1, 2, \ldots$, i.e. the implication $3 \Rightarrow 1$ of Theorem 5.1 is false.

Suppose now, that $\phi_E(t)$ is continuous at zero. Then, since E is non-regular, we have $\alpha = \mu(\mathbf{1}) = \infty$ (see the proof of Corollary 2.2). If there exists $x \in E \setminus \mathcal{K}_0(M, \mu)$, then $\mathbf{1} \in E$. Choose $p_n \in \mathcal{P}(M)$ such that $p_n p_k = 0$, $n \neq k$, $\mu(p_n) = 2^{-n}$, and set $x_n = \mathbf{1} - p_n$. Repeating the previous arguments, we obtain again that Theorem 5.1 is false. Let now $E \subset \mathcal{K}_0(M, \mu)$. Using Corollary 2.1, choose $x \in E_+$ such that $\mu_t(x) \overline{\in} F$, where F is the closure of $(L_1(0, \infty) \cap L_\infty(0, \infty))_h$ in $E(0, \infty)$. Suppose first that $\mu_t(x)\chi_{(0,r)} \in F$ for all $r > 0$. Let F'' and $E''(0, \infty)$ be the second associate spaces of F and $E(0, \infty)$ respectively. Since F is regular (see Proposition 2.4), we have $\|f\|_{E''(0,\infty)} = \|f\|_{F''} = \|f\|_F = \|f\|_{E(0,\infty)}$ for all $f \in F$. Hence, if $\|\mu_t(x)\chi_{(n,\infty)}\|_{E''(0,\infty)} \to 0$, then $\mu_t(x)\chi_{(0,n)}$ is a Cauchy sequence in $(F, \|\cdot\|_F)$ and therefore $\mu_t(x) \in F$, but this is not the case. This means that $\inf_{r>0} \|\mu_t(x)\chi_{[r,\infty)}\|_{E''(0,\infty)} = \delta > 0$. The norm $\|\cdot\|_{E''(0,\infty)}$ has the Fatou property, therefore

$$\|\mu_t(x)\chi_{[r,\infty)}\|_{E''(0,\infty)} = \lim_{\gamma\to\infty} \|\mu_t \chi_{[r,\gamma)}\|_{E''(0,\infty)}.$$

Hence, there exists a sequence of positive numbers $r_n \uparrow \infty$ such that $\|\mu_t(x)\chi_{[r_n, r_{n+1})}\|_{E(0,\infty)} > 2^{-1}\delta$ for all $n = 1, 2, \ldots$. Let $\mathcal{U}$ be a non-atomic commutative von Neumann subalgebra in $s(x)Ms(x)$ containing all the projections $\{x > \lambda\}$, $\lambda > 0$ and such that the restriction of μ onto $\mathcal{U}$ is σ-finite (see Lemma 2.1). Identify $\mathcal{U}$ with $L_\infty(\Omega, \Sigma, \mu)$, $L_1(\mathcal{U}, \mu) + \mathcal{U}$ with $L_1(\Omega) + L_\infty(\Omega)$. By Lemma 2.2 there exists a measure preserving trasnsformation ϕ from Ω onto $[0, \infty)$ such that

$x(\omega) = \mu_{\phi(\omega)}(x)$, $\omega \in \Omega$. Put $g_n = \chi_{[r_n, r_{n+1})}(\phi(\omega))$ and $x_n = x - 2xg_n$. It is clear that $\mu_t(x_n) = \mu_t(x)$, $n = 1, 2, \ldots$. Besides, if $g \in \mathcal{P}(M)$, $\mu(g) < \infty$, then since $x \in \mathcal{K}_0(M, \mu)$, we have $|\mu(xg_ng)| \leqslant \|xg_n\|_M \mu(g) \to 0$. Hence, $\mu(x_ng) \to \mu(xg)$. On the other hand $\|x_n - x\|_E = 2\|xg_n\|_E = 2\|\mu_t(x)\chi_{[r_n, r_{n+1})}\|_{E(0,\infty)} > \delta$, $n = 1, 2, \ldots$, i.e. the implication $3 \Rightarrow 1$ from the statement of the Theorem 5.1 is false.

It remains to consider the case $\mu_t(x)\chi_{(0,r)} \overline{\in} F$ for some r. Using the same arguments we obtain that $\inf\limits_{s>0} \|\mu_t(x)\chi_{(0,s)}\|_{E''(0,\infty)} = \delta_1 > 0$. With the help of the Fatou property we can construct a sequence of positive numbers $s_n \downarrow 0$ such that $\|\mu_t(x)\chi_{[s_{n+1}, s_n)}\|_{E(0,\infty)} > 2^{-1}\delta_1$. Repeating previous arguments, we obtain, that in this case the implication $3 \Rightarrow 1$ from Theorem 5.1 is false too. ■

In the next theorem the variant of Theorem 5.1 for an arbitrary semifinite von Neumann algebra M is given.

THEOREM 5.2. *Let M be an arbitrary von Neumann algebra, let μ be a faithful normal semifinite trace on M, let E be a separable symmetric space on $[0, \mu(\mathbf{1}))$, let $x_n, x \in E(M, \mu)$, $n = 1, 2, \ldots$. Then the following conditions are equivalent:*

1. $\|x_n - x\|_{E(M,\mu)} \to 0$;
2. $\|\mu_t(x_n) - \mu_t(x)\|_E \to 0$ *and* $x_n \to x$ *weakly;*
3. $\|\mu_t(x_n) - \mu_t(x)\|_E \to 0$ *and* $\mu(x_np) \to \mu(xp)$ *for all* $p \in \mathcal{P}(M)$, $\mu(p) < \infty$.

Proof. Consider a commutative W^*-algebra $N = L_\infty(0,1)$ with the trace $\nu(f) = \int\limits_0^1 f dm$, acting on the separable Hilbert space $F = L_2(0,1)$. Let $A = M\overline{\otimes}N$ be the tensor product of the von Neumann algebras M and N, $\lambda = \mu \otimes \nu$. It is clear, that A is a non-atomic von Neumann algebra and λ is a faithful, normal, semifinite trace on A. Consider a regular symmetric space $E(A, \lambda)$ on (A, λ) (see Remark 2.2). Since $\mu_t(x) = \lambda_t(x \otimes \mathbf{1})$, $t > 0$ for all $x \in L_1(M, \mu) + M$ (the notations are taken from the proof of Propositin 1.2), then we can identify the subspace $\{x\overline{\otimes}\mathbf{1} : x \in E(M, \mu)\}$ in $E(A, \lambda)$ with $E(M, \mu)$. In addition, $\|x\|_{E(M,\mu)} = \|\mu_t(x)\|_E = \|\lambda_t(x\overline{\otimes}\mathbf{1})\|_E = \|x\overline{\otimes}\mathbf{1}\|_{E(A,\lambda)}$ for all $x \in E(M, \mu)$. This, together with Proposition 3.2, completes the proof of implications $1 \Rightarrow 2$. The implication $2 \Rightarrow 3$ can be proved in just the same way as in Theorem 5.1.

$3 \Rightarrow 1$. Let $\|\lambda_t(x_n\overline{\otimes}\mathbf{1}) - \lambda_t(x\overline{\otimes}\mathbf{1})\|_E = \|\mu_t(x_n) - \mu_t(x)\| \to 0$ and $\mu(x_np) \to \mu(xp)$ for all $p \in \mathcal{P}(M)$, $\mu(p) < \infty$. Repeating the arguments from Remark 4.2, we obtain, that for all $y \in L_1(M, \mu) \cap M$ we have the convergence

$$\lambda(x_n\overline{\otimes}\mathbf{1})(y\overline{\otimes}\mathbf{1}) = \mu(x_ny) \to \lambda((x\overline{\otimes}\mathbf{1})(y\overline{\otimes}\mathbf{1}).$$

Let T be the conditional expectation from $L_1(A, \lambda) + A$ onto $L_1(M\overline{\otimes}\mathbf{1}, \lambda) + M\overline{\otimes}\mathbf{1}$, $p \in \mathcal{P}(A), \lambda(p) < \infty$. Then $Tp = y\overline{\otimes}\mathbf{1}$ for some $y \in L_1(M, \mu) \cap M$ and hence

$$\lambda((x_n\overline{\otimes}\mathbf{1})p) = \lambda((x_n\overline{\otimes}\mathbf{1})Tp) \to \lambda((x\overline{\otimes}\mathbf{1})Tp) = \lambda((x\overline{\otimes}\mathbf{1})p).$$

It remains to use Theorem 5.1, by which $\|x_n - x\|_{E(M,\mu)} = \|x_n\overline{\otimes}\mathbf{1} - x\overline{\otimes}\mathbf{1}\|_{E(A,\lambda)} \to 0$. ■

REMARK 5.2. It follows immediately from Theorem 5.2 the statement of the theorem, formulated in the introduction.

6. WEAK CONVERGENCE IN THE SPACE C_E

Let $(E, \|\cdot\|_E)$ be an arbitrary fully symmetric space of sequences of complex numbers (i.e. E is a fully symmetric space on (ℓ_∞, μ) (see Section 2)). Denote by $E(0,\infty)$ the set of all the functions $f \in L_1(0,\infty) + L_\infty(0,\infty)$, for which $\pi(f) = \left\{ \int_{n-1}^{n} \tilde{f}(t)\mathrm{d}t \right\}_{n=1}^{\infty} \in E$. Put $\|f\|_{E(0,\infty)} = \|\pi(f)\|_E$.

PROPOSITION 6.1. *$(E(0,\infty), \|\cdot\|_{E(0,\infty)})$ is a fully symmetric function space on $[0,\infty)$. In addition, if E is separable, then $E(0,\infty)$ is regular.*

Proof. Using the properties of rearrangements we get for any $f, g \in E(0,\infty)$

$$\sum_{n=1}^{k} \int_{n-1}^{n} (\widetilde{f+g})(t)\mathrm{d}t \leqslant \sum_{n=1}^{k} \int_{n-1}^{n} \tilde{f}(t)\mathrm{d}t + \sum_{n=1}^{j} \int_{n-1}^{n} \tilde{g}(t)\mathrm{d}t,$$

$k = 1, 2, \ldots$, i.e. $\pi(f+g) \prec \pi(f) + \pi(g)$. Since E is fully symmetric, $f + g \in E(0,\infty)$ and $\|f+g\|_{E(0,\infty)} \leqslant \|f\|_{E(0,\infty)} + \|g\|_{E(0,\infty)}$. Hence, $E(0,\infty)$ is a subspace of $L_1(0,\infty) + L_\infty(0,\infty)$ and $\|\cdot\|_{E(0,\infty)}$ is a fully symmetric norm on $E(0,\infty)$. Let $\{f_n\}$ be an increasing Cauchy sequence of negative functions from $E(0,\infty)$. Since ([7], p. 113)

$$\tilde{f}_n(t) - \tilde{f}_m(t) \prec (\widetilde{f_n - f_m})(t), \tag{8}$$

$\pi(f_n) - \pi(f_m) \prec \pi(f_n - f_m)$ and therefore $\{\pi(f_n)\}$ is a Cauchy sequence in E. Hence, there exists $x = \{\beta_n\} \in E$ such that $\|\pi(f_n) - x\|_E \to 0$. Besides, it follows from (8) also, that $\{\tilde{f}_n(t)\}$ is a Cauchy sequence in $E(0,\infty)$, and therefore it is a Cauchy sequence in the measure topology with respect to Lebesgue measure m ([6], p. 139). Consequently, there exists a function h on $[0,\infty)$ such that $f_n \xrightarrow{m} h = \sup_{n\geqslant 1} \tilde{f}_n$. By Levy's theorem we have for any fixed n, $\int_{n-1}^{n} \tilde{f}_k(t)\mathrm{d}t \to \int_{n-1}^{n} h(t)\mathrm{d}t = \beta_n$ as $k \to \infty$.

It means that $h \in E(0,\infty)$. Since $\{f_n\}$ is a Cauchy sequence with respect to the convergence by Lebesgue measure m, there exists a measurable function f on $[0,\infty)$, such that $f_n \xrightarrow{m} f = \sup\limits_{n\geqslant 1} f_n$. In particular, $\tilde{f}_n \to \tilde{f}$ almost everywhere. Hence, $\tilde{f} = h$ almost everywhere, and therefore $f \in E(0,\infty)$. By the Amemiya's theorem (see for example [6], p. 378) $E_h(0,\infty) = \{g \in E(0,\infty) : g = \overline{g}\}$ is complete, and since $\|g\|_{E(0,\infty)} = \|\overline{g}\|_{E(0,\infty)}$ for all $g \in E(0,\infty)$, $E(0,\infty)$ is complete too.

Suppose now, that E is separable. Then the norm $\|\cdot\|_E$ is order continuous, in particular, $\beta_n \to 0$ for any $x = \{\beta_n\} \in E$. Let $f_n \in E(0,\infty)$ and $f_n \downarrow 0$. Since $f_1(\infty) = 0$, we have $f_n \downarrow 0$ ([7], p. 94). Therefore $\pi(f_n) \downarrow 0$ and hence $\|f_n\|_{E(0,\infty)} = \|\pi(f_n)\|_E \to 0$. ■

Let H be a Hilbert space, $B(H)$ be a $*$-algebra of all continuous linear operators acting in H, let tr be the canonical trace on $B(H)$, let F be symmetric space on $(B(H),\mathrm{tr})$. It is clear, that $\mathcal{K}(B(H),\mathrm{tr}) = B(H)$. Besides, since F is a two-sided $*$-ideal in $B(H)$, then, in the case $F \neq B(H)$, the space F is contained in the ideal $C(H)$ of all compact operators in H. In addition, the rearrangement $\mu_t(x)$ for any $x \in F$ has the form

$$\mu_t(x) = \sum_{n=1}^{\infty} s_n(x)\chi_{[n-1,n)}(t),$$

where $\{s_n(x)\}$ is the sequence of s-numbers of x.

Let $(E, \|\cdot\|_E)$ be a fully symmetric space of sequences, $E \neq \ell_\infty$, let $G = E(0,\infty)$ be the fully symmetric space on $[0,\infty)$ associated with E (see Proposition 6.1). Denote by C_E the set of all $x \in C(H)$, for which $\{s_n(x)\} \in E$, and put $\|x\|_{C_E} = \|s_n(x)\|_E$ (separability is not supposed). The following proposition is obvious (see Proposition 2.3).

Proposition 6.2. *$C_E = G(B(H),\mathrm{tr})$, $\|x\|_{C_E} = \|x\|_{G(B(H),\mathrm{tr})}$ for all $x \in C_E$.*

By Propositions 2.3, 6.1 and 6.2 we have.

Corollary 6.1. *$(C_E, \|\cdot\|_{C_E})$ is a fully symmetric space on $(B(H),\mathrm{tr})$. In addition, if E is separable, then C_E is regular.*

Theorem 6.1. *Let E be a separable symmetric sequence space, C_E be the symmetric space of compact operators acting in a Hilbert space H, associated with E, x_k, $x \in C_E$, $k = 1,2,\ldots$. The following conditions are equivalent:*

1. $\|x_k - x\|_{C_E} \to 0$;
2. $\|s_n(x_k) - s_n(x)\|_E \to 0$ *and* $x_k \to x$ *in the weak topology* $\sigma(C_E, C_E^*)$;
3. $\|s_n(x_k) - s_n(x)\|_E \to 0$ *and* $(x_k\xi,\xi) \to (x\xi,\xi)$ *for any* $\xi \in H$.

The proof of Theorem 6.1 follows from Theorem 5.2 and Propositions 6.1, 6.2.

In the case, when H is separable Hilbert space, the statement of Theorem 6.1 was obtained in [1], essentially using the matrix representations of operators from $B(H)$. Our theorem is valid for any Hilbert space.

PROPOSITION 6.3. *If E is a non-separable fully symmetric space, $E \neq \ell_\infty$, then the implication $3 \Rightarrow 1$ from the statement of Theorem 6.1 is false.*

Proof. Let F be the closure in $(E, \|\cdot\|_E)$ of the subspace of all $x = \{\alpha_n\} \in E$ such that $\alpha_n = 0$ for all but finitely many n. Since E is non-separable, $F \neq E$. Choose $x \in (C_E)_+$ such that $a = \{s_n(x)\} \overline{\in} F$. Put $p_n = \{\beta_k\}_{k=1}$, where $\beta_k = 0$ if $1 \leqslant k \leqslant n$ and $\beta_k = 1$ if $k \geqslant n$. In just the same way as in the proof of Proposition 5.1 we can establish that $\inf \|ap_n\|_{E''} = \delta > 0$, where E'' is the second associate space of E. Using the Fatou property for the norm $\|\cdot\|_{E''}$, construct an increasing sequence of indices n_k such that $\|a(p_{n_k} - p_{n_{k+1}})\|_{E''} > 2^{-1}\delta$. Set $x = \sum_{n=1}^{\infty} s_n(x) e_n$, where e_n is a sequence of one-dimensional pairwise orthogonal projections from $B(H)$. Put $x_k = x - 2 \sum_{i=n_k}^{n_{k+1}-1} s_i(x) e_i$. It is clear, that $s_n(x_k) = s_n(x)$, $k = 1, 2, \ldots$, and $(x_k \xi, \xi) \to (x\xi, \xi)$ for all $\xi \in H$. On the other hand

$$\|x_k - x\|_{C_E} = 2 \left\| \sum_{i=n_k}^{n_{k+1}-1} s_i(x) e_i \right\|_{C_E} = 2\|a(p_{n_k} - p_{n_{k+1}})\|_{E''} > \delta,$$

$k = 1, 2, \ldots$, i.e. the implication $3 \Rightarrow 1$ from the statement of Theorem 6.1 is false.

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