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APPROXIMATION BY NORMAL ELEMENTS WITH FINITE SPECTRA IN SIMPLE AF-ALGEBRAS

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1. INTRODUCTION

In [3, 2.6], a C^*-algebra A is said to have (FN) if every normal element in A is a norm limit of elements with the form $\sum_{k=1}^{n}\lambda_k p_k$, where each λ_k is a complex number in the spectrum of x and each p_k is mutually orthogonal projection in A. It is known that every von Neumann algebra and every AW*-algebra have (FN). It is recently proved that the corona algebras of finite matroid algebras have (FN) ([16]). In [17], N.C. Phillips constructs two separable simple C^*-algebras have (FN) which are not AF. In this note we will consider the following question:

$\mathbf{Q}_1$: Does every AF-algebra have (FN)?

The question was mentioned in [17, 3.11]. But it may be raised before ([3]).

Recall that a C^*-algebra A is called an AF-algebra if for any $\varepsilon > 0$ and finitely many elements $a_1, a_2, \ldots, a_n \in A$, there are a finite dimensional C^*-subalgebra B of A and elements $b_1, b_2, \ldots, b_n \in B$ such that

$$\|a_i - b_i\| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

AF-algebras have been intensively studied, in particular, those simple separable AF-algebras such as matroid algebras and UHF-algebras (see [6], [10], [11], [12] etc. It is almost impossible to give a complete list.). AF-algebras appear to be most understandable C^*-algebras. Nevertheless, the answer to question $\mathbf{Q}_1$ is still not known.

If x is a self adjoint element in an AF-algebra, then for any $\varepsilon > 0$, there are a finite dimensional C^*-subalgebra B and an element $y \in B$ such that

$$\|x - y\| < \varepsilon.$$

Set $y_1 = \frac{1}{2}(y + y^*)$, then $\|x - y_1\| < \varepsilon$. Since $y_1 \in B$, y_1 has finite spectrum. So the question $\mathbf{Q}_1$ has an affirmative answer for selfadjoint elements. If x is a unitary, then the unitary part of the polar decomposition of y (in B) is close to x in norm, provided that ε is small enough. So the question $\mathbf{Q}_1$ has an affirmative answer for unitaries too. In general, if x is a normal element, one hopes that y is close to a normal element (in B). In fact,

$$\|yy^* - y^*y\| < \varepsilon.$$

Therefore one can give an affirmative answer to question $\mathbf{Q}_1$ if one can prove the following:

$\mathbf{Q}_2$: For any $\varepsilon > 0$, there is a $\delta > 0$ so that whenever B is a finite dimensional C^*-algebra and y is an element in B such that

$$\|y^*y - yy^*\| < \delta \text{ and } \|y\| = 1,$$

then there is a normal element $z \in B$ satisfying

$$\|y - z\| < \varepsilon.$$

But $\mathbf{Q}_2$ is an old problem in linear algebra (see [20], [21] and [9]) and, unfortunately, remains open.

However, without making any effort to solve the problem $\mathbf{Q}_2$, we would like to shed some light on the question $\mathbf{Q}_1$. We will show that the answer to $\mathbf{Q}_1$ is affirmative, if A is a matroid algebra (in particular, A is a UHF-algebra). In fact, we show that for a more general class of simple AF-algebras, the answer to the question $\mathbf{Q}_1$ is affirmative. The techniques used in this note come from [2], [18] and [14].

Let A be a separable simple AF-algebra. Fix a nonzero projection $e \in A$, let T be the set of those (lower semi-continuous and semi-finite) traces τ such that $\tau(e) = 1$. With weak*-topology, T is a compact convex set. If $\tau \in T$ is an extreme point of T, then we say that τ is an extremal trace. Let p be another nonzero projection in A and let T' be the set of those (lower semi-comtinuous and semi finite) traces τ such that $\tau(p) = 1$. If T has only countably many extreme points so does T' (see [13, 6.17]). The main result of this note is the following:

THEOREM A. *Let A be a separable simple* AF-*algebra. Suppose that T has only countably many extremal traces. Then A has* (FN).

It is known ([19, 3.1.8]) that T is a metrizable Choquet simplex. By [1, I.49], every point in T is a barycenter of measure concentrated on its extreme points. If T has only countably many extremal traces $\{\tau_n\}$ then for any $\tau \in T$, there is a sequence of nonnegative numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n = 1$ and

$$\tau = \sum_{n=1}^{\infty} a_n \tau_n.$$

We have the following:

COROLLARY B. *Let A be a unital separable simple* AF-*algebra. Suppose that there is a countable subset $\{\tau_n\}$ of the normalized traces such that for any normalized trace τ there is a sequence of nonnegative numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n = 1$ and*

$$\tau = \sum_{n=1}^{\infty} a_n \tau_n.$$

Then A has (FN).

Since every matroid algebra has only one trace (up to the scalar multiples), we immediately have the following:

COROLLARY C. *Every matroid algebra has* (FN).

COROLLARY D. *Every* UHF-*algebra has* (FN).

We also have the following result for non-simple AF-algebras:

THEOREM E. *Let A be a separable unital* AF-*algebra. Suppose that $\tau(1) < \infty$ for every (lower semi-continuous and semi-finite) trace τ. Let T be the set of (lower semi-continuous and semi-finite) traces τ such that $\tau(1) = 1$. If there is a countable subset $\{\tau_n\} \subset T$ such that for every trace $\tau \in T$, there is a sequence of nonnegative numbers $\{\alpha_n\}$ such that $\sum_{n=1}^{n} \alpha_n = 1$ and*

$$\tau = \sum_{n=1}^{\infty} \alpha_n \tau_n.$$

Then A has (FN).

The following are some terminologies which will be used later.

Let p and q be two projections in a C^*-algebra A. We say p is equivalent to q, if there is a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* = q$.

We use the notation $[p]$ for the equivalence class of projections containing p.

We write $[p] > [q]$, if there is a partial isometry $u \in A$ such that $u^*u = q$, $uu^* \leqslant p$ and $uu^* \neq p$.

Let $p \in A^{**}$ be an open projection, where A^{**} is the enveloping von-Neumann algebra of A. We use the notation $\mathrm{Her}(p)$ for the hereditary C^*-subalgebra $pA^{**}p \cap A$.

2. PROOF OF THE RESULTS

LEMMA 1. *Let X be a compact subset of the plane and let $f \in C(X)$. For any $\varepsilon > 0$, there is $\delta > 0$, for any C^*-algebra A and normal elements $x, y \in A$ with $\mathrm{sp}(x), \mathrm{sp}(y) \subset X$, if $\|x - y\| < \delta$, then*

$$\|f(x) - f(y)\| < \varepsilon.$$

Proof. The proof is similar to that of Lemma 2 in [8]. By the Stone-Weierstrass theorem, for any $f \in C(X)$, there is a polynomial p (of two variables) such that

$$\|f(z) - p(z, \bar{z})\| < \frac{\varepsilon}{2} \quad \text{for all} \quad z \in X.$$

Set

$$d = \sup \{|\lambda| : \lambda \in X\}.$$

Then, as in [8, Lemma 2], one has

$$\|x^n(x^m)^* - y^n(y^m)^*\| \leqslant n \cdot m \cdot d \cdot \delta.$$

The rest of the proof is exactly the same as in [8, 2]. ■

LEMMA 2. *Let A be a unital C^*-algebra and x be a normal element in A. For any $\varepsilon > 0$, there is $\delta > 0$ such that if*

(1) *$\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathrm{sp}(x)$ and $|\lambda_i - \lambda_j| \geqslant \delta, i \neq j$;*

(2) *$S_k = \{\lambda : |\lambda - \lambda_k| < \delta\}$;*

(3) *q_k is the spectral projection of x in A^{**} corresponding to the open set S_k;*

(4) *p_k is a projection in $\mathrm{Her}(q_k)$;*

(5) $y = \left(1 - \sum_{i=1}^{n} p_i\right) x \left(1 - \sum_{i=1}^{n} p_k\right)$,

then

$$\left\| x - \left(y + \sum_{k=1}^{n} \lambda_k p_k\right)\right\| < \varepsilon,$$

$$\left\|\left(1 - \sum_{i=1}^{n} p_i\right) x - x \left(1 - \sum_{i=1}^{n} p_i\right)\right\| < \varepsilon.$$

Proof. Notice that $p_k \leqslant q_k$ and q_k's are mutually orthogonal. We have

$$\left\| x\left(\sum_{k=1}^{n} p_k\right) - \sum_{k=1}^{n} \lambda_k p_k \right\| = \left\| x\left(\sum_{k=1}^{n} q_k\right)\left(\sum_{k=1}^{n} p_k\right) - \left(\sum_{k=1}^{n} \lambda_k p_k\right) \right\| =$$

$$= \left\| \sum_{k=1}^{n} q_k \left(x - \lambda_k q_k\right) \right\| < \delta.$$

Similarly,

$$\left\| \left(\sum_{k=1}^{n} p_k\right) x - \sum_{k=1}^{n} \lambda_k p_k \right\| < \delta.$$

Moreover,

$$\left\| \left(1 - \sum_{k=1}^{n} p_k\right) x \left(1 - \sum_{k=1}^{n} p_k\right) - x\left(1 - \sum_{k=1}^{n} p_k\right) \right\| = \left\| \sum_{k=1}^{n} p_k x \left(1 - \sum_{k=1}^{n} p_k\right) \right\| =$$

$$= \left\| \left[\left(\sum_{k=1}^{n} p_k\right) x - \sum_{k=1}^{n} \lambda_k p_k\right] \left(1 - \sum_{k=1}^{n} p_k\right) \right\| < \delta.$$

Similarly,

$$\left\| \left(1 - \sum_{k=1}^{n} p_k\right) x - \left(1 - \sum_{k=1}^{n} p_k\right) x \left(1 - \sum_{k=1}^{n} p_k\right) \right\| < \delta.$$

Set

$$y = \left(1 - \sum_{k=1}^{n} p_k\right) x \left(1 - \sum_{k=1}^{n} p_k\right).$$

Then

$$\left\| x - \left(y + \sum_{k=1}^{n} \lambda_k p_k\right) \right\| < 2\delta$$

and

$$\left\| \left(1 - \sum_{i=1}^{n} p_i\right) x - x\left(1 - \sum_{i=1}^{n} p_i\right) \right\| < 2\delta.$$

So take $\delta = \varepsilon/2$. ■

LEMMA 3. *Let A be a separable simple unital AF-algebra satisfying the condition in Corollary B, and let x be a normal element in A. For any $\varepsilon > 0$ and positive integer K there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathrm{sp}(x)$ and mutually orthogonal projections $p_1, p_2, \ldots, p_n \in A$ such that*

$$\left\| x - \left(y + \sum_{i=1}^{n} \lambda_i p_i\right) \right\| < \varepsilon,$$

where $y = \left(1 - \sum_{i=1}^{n} p_i\right) x \left(1 - \sum_{i=1}^{n} p_i\right)$,

$$\left\| \left(1 - \sum_{i=1}^{n} p_i\right) x - x \left(1 - \sum_{i=1}^{n} p_i\right) \right\| < \varepsilon$$

and

$$[p_k] > K \left[1 - \sum_{i=1}^{n} p_i\right]$$

for $k = 1, 2, \ldots, n$.

Proof. Without loss of generality, we may assume that $\|x\| \leqslant 1$. Denote by B the C^*-subalgebra generated by x and 1. Then $B \cong C(X)$, where $X = \mathrm{sp}(x)$. Let τ be a trace on A with $\tau(1) = 1$. The restriction of τ on B gives a state on B. By the Riesz representation theorem, the state defines a normalized Borel measure μ_τ on X. Let D denote the unit disk. Then $X \subset D$. For any open subset $O \subset D$, let q_O be the spectral projection of x in A^{**} corresponding to the open subset $O \cap X$. The projection q_O is an open projection in A^{**}. Suppose that $h \in B (\cong C(X))$ such that $1 \geqslant h(t) > 0$ for all $t \in O \cap X$ and $h(t) = 0$ for all $t \in X \setminus O$. Then $\left\{h^{\frac{1}{n}}\right\}$ forms an approximate identity for $\mathrm{Her}(q_O)$. It is clear that

$$\mu_\tau(O \cap X) = \lim_{n \to \infty} \tau(h^{\frac{1}{n}}).$$

Let $\{e_n^O\}$ be an approximate identity for $\mathrm{Her}(q_O)$ consisting of projections. Then

$$\mu_\tau(O \cap X) = \sup \left\{\tau(e_n^O)\right\}.$$

In fact, we have

$$\tau(h^{\frac{1}{k}} e_n^O) = \tau(h^{\frac{1}{2k}} e_n^O h^{\frac{1}{2k}}) \leqslant \tau(h^{\frac{1}{k}}) \leqslant \mu_\tau(O \cap X)$$

and

$$\tau(h^{\frac{1}{k}} e_n^O) = \tau(e_n^O h^{\frac{1}{k}} e_n^O) \leqslant \tau(e_n^O)$$

for all k and n. Since $h^{\frac{1}{k}} e_n^O \to e_n^O$, if $k \to \infty$, and $h^{\frac{1}{k}} e_n^O \to h^{\frac{1}{k}}$ if $n \to \infty$, from above equalities and inequalities, we conclude that

$$\mu_\tau(O \cap X) = \sup \left\{\tau(e_n^O)\right\}.$$

Let T_0 denote the countable subset $\{\tau_n\}$. For the simplicity, we use the notation μ_i for the measure μ_{τ_i}.

For any $\varepsilon > 0$, there is a finite subsets $\{\zeta_1, \zeta_2, \ldots, \zeta_m\}$ of D such that for any $\zeta \in D$, there is an integer i such that

$$|\zeta_i - \zeta| < \frac{\varepsilon}{32}$$

and for any i, there is $j \neq i$ such that

$$|\zeta_i - \zeta_j| < \frac{\varepsilon}{16}.$$

For each i set

$$D_i = \left\{\zeta : \frac{\varepsilon}{32} \leqslant |\zeta_i - \zeta| \leqslant \frac{\varepsilon}{16}\right\}.$$

Fix i, for each $\frac{\varepsilon}{32} \leqslant r \leqslant \frac{\varepsilon}{16}$, set

$$S_r = \{\zeta : |\zeta - \zeta_i| = r\}.$$

Since $\mu_k(D_i \cap X) \leqslant 1$ and $S_r \cap S_{r'} = \emptyset$, if $r \neq r'$, there are only countably many r's in $(\frac{\varepsilon}{32}, \frac{\varepsilon}{16})$ such that

$$\mu_k(S_r \cap X) > 0.$$

Since the union of countably many countable sets is still countable, we conclude that for each i, there is $r_i \in (\frac{\varepsilon}{32}, \frac{\varepsilon}{16})$ such that

$$\mu_k(S_{r_i} \cap X) = 0$$

for $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots$.

Now $D \setminus \bigcup S_{r_i}$, is a disjoint union of finitely many open sets $O_1, O_2, \ldots, O_N$ such that the diameter of each O_i is $< \frac{\varepsilon}{8}$ and

$$\mu_k\left(\left(\bigcup S_{r_i}\right) \cap X\right) = 0$$

for all k.

Let $\left\{e_n^{(i)}\right\}$ be an approximate identity for B_{O_i}. Then

$$\tau_j(e_n^{(i)}) \nearrow \mu_j(O_i \cap X)$$

$j = 1, 2, \ldots$ and $i = 1, 2, \ldots, N$. Since $\mu_j\left(X \setminus \bigcup_{i=1}^{N} O_i \cap X\right) = 0$,

$$\tau_j\left(\sum_{i=1}^{N} e_n^{(i)}\right) \nearrow 1,$$

as $n \to \infty$, $j = 1, 2, \ldots$. Since every $\tau \in T$ has the form

$$\tau = \sum_{j=1}^{\infty} \alpha_j \tau_j,$$

where $\alpha_j \geqslant 0$ and $\sum_{j=1}^{\infty} \alpha_j = 1$, we conclude that

$$\tau\left(\sum_{i=1}^{\infty} e_n^{(i)}\right) \nearrow 1, \quad \text{as} \quad n \to \infty$$

for all $\tau \in T$. Since T is compact, by Dini's theorem, the continuous functions $\sum_{i=1}^{N} e_n^{(i)}(\tau)$ defined on T converges to the constant function 1 uniformly on T, as $n \to \infty$. Hence we have projections $p_i \in B_{O_i}$ such that

$$\tau(p_i) > K\tau\left(1 - \sum_{i=1}^{N} p_i\right)$$

for all i and $\tau \in T$. It follows from [4, Prop. 4.1] that

$$[p_i] > K\left[1 - \sum_{i=1}^{N} p_i\right].$$

The rest of proof follows from Lemma 2. ■

LEMMA 4. *Let A be a unital C^*-algebra and let x be a normal element in A. Suppose that there is a projection $p \in A$ such that*

$$\|px - xp\| < \frac{\varepsilon}{2}$$

and there is an element $y \in pAp$ such that

$$\|y - pxp\| < \frac{\varepsilon}{2}.$$

Then (in pAp)

(1) $\mathrm{sp}(y) \subset \{\lambda : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \varepsilon\}$ *and*

(2) $\|(\lambda p - y)^{-1}\| < [\mathrm{dist}(\lambda, \mathrm{sp}(x)) - \varepsilon]^{-1}$

for those λ such that $\mathrm{dist}(\lambda, \mathrm{sp}(x)) \geqslant \varepsilon$.

Proof. Suppose that $\mathrm{dist}(\lambda, \mathrm{sp}(x)) \geqslant \varepsilon$. Then

$$\|p - p(\lambda - x)^{-1}(\lambda p - y)\| \leqslant \|p(\lambda - x)^{-1}[(\lambda - y) - (\lambda - x)p]\| \leqslant$$

$$\leqslant \|(\lambda - x)^{-1}\|(\|y - pxp\| + \|pxp - xp\|) < \frac{\varepsilon}{\operatorname{dist}(\lambda, \operatorname{sp}(x))} \leqslant 1.$$

Similarly,

$$\|p - (\lambda p - y)(\lambda - x)^{-1}p\| < 1.$$

Therefore $\lambda p - y$ is invertible in pAp. This proves (1).

For (2), we have the following inequalities:

$$\|(\lambda p - y)^{-1}\| \leqslant \|(\lambda p - y)^{-1} - p(\lambda - x)^{-1}p\| + \|p(\lambda - x)^{-1}p\| \leqslant$$
$$\leqslant \|(\lambda p - y)^{-1}\|\|p - (\lambda p - y)(\lambda - x)^{-1}p\| + \|(\lambda - x)^{-1}\| <$$
$$< \|(\lambda p - y)^{-1}\| \cdot \frac{\varepsilon}{\operatorname{dist}(\lambda, \operatorname{sp}(x))} + \frac{1}{\operatorname{dist}(\lambda, \operatorname{sp}(x))}.$$

So, we have

$$\|(\lambda p - y)^{-1}\| < \frac{1}{[\operatorname{dist}(\lambda, \operatorname{sp}(x)) - \varepsilon]}.$$

■

Let

$$S = \left\{\alpha + i\beta : |\alpha| \leqslant \frac{1}{2},\ |\beta| \leqslant b\right\}$$

be a subset of the plane, where $b > 0$. Suppose that

$$\frac{1}{2} = t_0 < t_1 < \cdots < t_k = \frac{1}{2}$$

is a partition of the interval $[-1, 1]$. Set

$$D_i = S \cap \{\lambda : t_{i-1} \leqslant \operatorname{Re}\lambda \leqslant t_i\},\ i = 1, 2, \ldots, k$$

and

$$R_i = \left\{\alpha + i\beta : t_i - \frac{\delta}{2} \leqslant \alpha t_i + \frac{\delta}{2}, |\beta| \leqslant b\right\},\ i = 1, 2, \ldots, k - 1.$$

The following lemma is a variation of Lemma 5.2 in [2].

LEMMA 5. *Let X be a closed subset of the square S. For any $0 < \delta < \frac{1}{2}(\min\{(t_i - t_{i-1})\})$ and $\eta > 0$ there is $\varepsilon > 0$, for any finite-dimensional C^*-algebras B, if $x \in B$ satisfies*

(1) $\operatorname{sp}(x) = X$,

(2) $\|x^*x - xx^*\|^{\frac{1}{2}} < \varepsilon$,

(3) $\|(\lambda - x)^{-1}\| < [\operatorname{dist}(\lambda, X_\delta) - \varepsilon]^{-1}$, *where* $X_\delta = \left\{\lambda : \operatorname{dist}(\lambda, X) < \frac{\delta}{2}\right\}$,

(4) $\|\operatorname{Re} x\| \leqslant \frac{1}{2}$ *and* $\|\operatorname{Im} x\| \leqslant b$,

then there are normal elements $y_1, y_2, \ldots, y_{k-1} \in B$ *with* $\mathrm{sp}(y_i) \subset R_i$, $i = 1, 2, \ldots, k-1$, *and elements* $x_1, x_2, \ldots, x_k \in B$ *with*

$$\mathrm{sp}(\mathrm{Re}(x_i)) \subset \left[t_{i-1} - \frac{\delta}{2}, t_i + \frac{\delta}{2}\right],$$

$$\mathrm{sp}(\mathrm{Im}(x_i)) \subset \left[-b - \frac{\delta}{2}, b + \frac{\delta}{2}\right]$$

and

$$\mathrm{sp}(x_i) \subset \left\{\lambda : \mathrm{dist}(\lambda, X_i) < \frac{\delta}{2}\right\},$$

where $X_i = [X \cap D_i] \cup R_i$ *and there is a unitary* u *such that*

$$\|x \oplus y_1 \oplus \ldots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \ldots x_k)u\| < \eta.$$

Proof. By applying Lemma 5.2 of [2] repeatedly, we obtain normal elements $y_1, y_2, \ldots, y_{k-1}$ in B, elements $x_1, x_2, \ldots, x_k \in B$ and a unitary u such that $\mathrm{sp}(y_i) \subset R_i$, $i = 1, 2, \ldots, k-1$,

$$\mathrm{sp}(\mathrm{Re}(x_i)) \subset \left[t_{i-1} - \frac{\delta}{2},\ t_i + \frac{\delta}{2}\right]$$

and

$$\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u\| < \eta$$

(if ε is small enough). Furthermore, if $\eta < \frac{\delta}{4}$,

$$\|\mathrm{Im}(x_i)\| \leqslant b + \frac{\delta}{2}.$$

Let

$$x' = x \oplus y_1 \oplus \cdots \oplus y_{k-1}$$

and

$$x'' = u^*(x_1 \oplus x_2 \oplus \cdots \oplus x_k)u.$$

If $\mathrm{dist}\left(\lambda, X \cup \left(\bigcup_{i=1}^{k-1} R_i\right)\right) \geqslant \frac{\delta}{2}$ and η is small enough (so ε is small), then, by (3),

$$\begin{aligned} &\|1 - (\lambda - x')^{-1}(\lambda - x'')\| \leqslant \\ &\leqslant \|(\lambda - x')^{-1}[(\lambda - x'') - (\lambda - x')]\| < \\ &< \frac{\eta}{(\mathrm{dist}(\lambda, X_\delta) - \varepsilon)} < \frac{\eta}{(\frac{\delta}{2} - \varepsilon)} < 1. \end{aligned}$$

Similarly,

$$\|1 - (\lambda - x'')(\lambda - x')^{-1}\| < 1.$$

So, $\lambda \notin \mathrm{sp}(x'')$. In other words,

$$\mathrm{sp}(x'') \subset \left\{\lambda : \mathrm{dist}\left(\lambda, X \cup \left(\bigcup_{i=1}^{k-1} R_i\right)\right) < \frac{\delta}{2}\right\}.$$

This implies that

$$\mathrm{sp}(x_i) \subset \left\{\lambda : \mathrm{dist}(\lambda, [X \cap D_i] \cup R_i) < \frac{\delta}{2}\right\}.$$ ■

Proof of the Theorem A.

We first assume that A is unital.

Step 1. For any $1 > \delta > 0$, since $\mathrm{sp}(x)$ is compact, there are finitely many open balls $B_1, B_2, \ldots, B_n$ with centers in $\mathrm{sp}(x)$ and diameters less then ε such that

$$\mathrm{sp}(x) \subset \bigcup_{i=1}^{n} B_i.$$

Therefore

$$\bigcup_{i=1}^{n} B_i = \bigcup_{k=1}^{m} X_k,$$

where each X_k is a connected component of $\bigcup_{i=1}^{n} B_i$. Since each X_k is a union of some B_i's, we may assume each X_k is homeomorphic to a rectangular region with possibly finitely many rectangular holes. Notice that $X_k \cap X_{k'} = \emptyset$, if $k \neq k'$. So we may write $x = \sum_{k=1}^{m} \oplus x_k$, where each x_k is normal and $\mathrm{sp}(x_k)$ (in a corner of A) is a subset of X_k. Furthermore, for any $\lambda \in X_i$ there is a $\zeta \in \mathrm{sp}(x_i)$ such that

$$\mathrm{dist}(\lambda, \zeta) < \varepsilon.$$

Without loss of generality, we may assume that $\mathrm{sp}(x)$ is a subset of one of those X_i and denote it by X. Let Ω be the rectangular region with k rectangular holes (k could be zero). Let φ be the homeomorphism from X on to Ω. It is enough to show that $\varphi(x)$ is a norm limit of normal elements with the form $\sum_{i=1}^{n} \lambda_i p_i$ where p_i's are mutually orthogonal projections and $\lambda_i \in \Omega$. Therefore, we may assume that $X = \Omega$. To be more precise, we assume that X is the unit disk, if $k = 0$; and

$$X = \left\{\lambda - \frac{1}{2} : \lambda \in X'\right\}$$

where

$$X' = \{z : 0 \leqslant \operatorname{Re} z \leqslant 1, |\operatorname{Im} z| \leqslant b\} \setminus \bigcup_{j=1}^{k} \left\{z : \left|\frac{2j-1}{2k-z}\right| < r\right\},$$

$0 < r < b$ and

$$(b^2 + [(\frac{1}{2}k)^2])^{\frac{1}{2}} < \frac{1}{k} - 2r - \frac{\delta}{4}.$$

Furthermore, we may simply assume that $\operatorname{sp}(x) = X$.

Step 2. Notice now X is fixed. For any $\varepsilon_1 > 0$, by applying Lemma 3, there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, mutually orthogonal projections $p_1, p_2, \ldots, p_n$ such that

$$\left\| x - \left(y + \sum_{i=1}^{n} \lambda_i p_i \right) \right\| < \varepsilon_1,$$

where $y = \left(1 - \sum_{i=1}^{n} p_i\right) x \left(1 - \sum_{i=1}^{n} p_i\right)$,

$$\left\| \left(1 - \sum_{i=1}^{n} p_i\right) x - x \left(1 - \sum_{i=1}^{n} p_i\right) \right\| < \varepsilon_1,$$

$$[p_i] > 2(k+1) \left[1 - \sum_{i=1}^{n} p_i\right]$$

for $i = 1, 2, \ldots, n$ and for any $\zeta \in \operatorname{sp}(x)$, there is λ_i such that $|\zeta - \lambda_i| < \varepsilon_1$. Since A is an AF-algebra, without loss generality, we may assume that $y \in B$ where B is a finite-dimensional C^*-subalgebra of $\left(1 - \sum_{i=1}^{n} p_i\right) A \left(1 - \sum_{i=1}^{n} p_i\right)$.

Step 3. Suppose that $k = 0$. In this case X is the unit disk. By [9, 4.5], there are normal elements $y' \in B$ and $y'' \in M_2(B)$ such that

$$\|y \oplus y' - y''\| < \frac{\varepsilon}{2}$$

and $\|y'\| \leqslant 1$, if ε_1 is small enough. For any $\varepsilon > 0$, if ε_1 is small enough, since B is finite dimensional and $\operatorname{sp}(y') \subset X$, we may assume that there are mutually orthogonal projections $q_i \in B$, $i = 1, 2, \ldots, s_1$ and mutually orthogonal projections $q'_j \in M_2(B)$ such that

$$\left\| y \oplus \sum_{i=1}^{s_1} \lambda_i q_i - \sum_{j=1}^{s_2} \alpha_j q'_j \right\| < \frac{\varepsilon}{2},$$

where $0 \leqslant s_1 \leqslant n$ and α_j are complex numbers in X. Since $[p_i] > 2\left[1 - \sum_{i=1}^{n} p_i\right]$, we may write

$$p_i = p_i^{(1)} \oplus p_i^{(2)}, \ i = 1, 2, \ldots, s_1$$

such that there are unitaries $v_i \in A$ with the property that $v_i p_i^{(1)} v_i^* = q_i$, $i = 1, 2, \dots, s_1$. Therefore, there is a unitary $v \in M_2(A)$ such that

$$\left\| \left(\sum_{i=1}^{s_1} \lambda_i p_i^{(1)} \oplus y \right) - \left(\sum_{j=1}^{s_2} \alpha_i v^* q_j' v \right) \right\| < \frac{\varepsilon}{2}.$$

Thus we conclude that

$$\left\| x - \left(\sum_{i=s_1+1}^{n} \lambda_i p_i \oplus \sum_{i=1}^{s_1} \lambda_i p_i^{(2)} \oplus \sum_{j=1}^{s_2} \alpha_j v^* q_j' v \right) \right\| < \varepsilon.$$

This completes the proof for $k = 0$.

Step 4. Now we assume that $k > 0$. Fix $0 < \delta \leqslant (\frac{1}{4})r$. Set

$$t_i = -\frac{1}{2} + \frac{i}{k},\ i = 1, 2, \dots, k.$$

For any $\varepsilon > 0$, if ε_1 is small enough, applying Lemma 3 and Lemma 4, we obtain normal elements $y_1, y_2, \dots, y_{k-1} \in B$ with $\mathrm{sp}(y_i) \subset X$ and elements $x_1, x_2, \dots, x_k \in B$ with

$$\mathrm{sp}(x_i) \subset \left\{ \lambda : \mathrm{dist}(\lambda, [X \cap D_i] \cup R_i) < \frac{\delta}{4} \right\},$$

$$\mathrm{sp}(\mathrm{Re}(x_i)) \subset \left[-\frac{1}{2} + \frac{i}{k} - \frac{\delta}{4}, -\frac{1}{2} + \frac{i+1}{k} + \frac{\delta}{4} \right]$$

and

$$\mathrm{sp}(\mathrm{Im}(x_i)) \subset \left[-b - \frac{\delta}{4}, b + \frac{\delta}{4} \right],$$

and a unitary u such that

$$\| y \oplus y_1 \oplus \dots \oplus y_{k-1} - u^*(x_1 \oplus x_2 \oplus \dots \oplus x_k) u \| < \frac{\varepsilon}{16},$$

where D_i and R_i are as Lemma 4, and $[X \cap D_i] \cup R_i$ is homeomorphic to an annulus. For each i, let

$$x_i' = x_i - [\frac{2i-1}{2k} - \frac{1}{2}],$$

then

$$\mathrm{sp}(x_i) \subset \left\{ z : r - \frac{\delta}{4} \leqslant |z| \leqslant d + \frac{\delta}{4} \right\}$$

and $\|x_i'\| \leqslant d + \frac{\delta}{4}$, where $d = \left[b^2 + \left(\frac{1}{2k} \right)^2 \right]^{\frac{1}{2}}$. Notice that x_i' satisfies the hypothesis of Lemma 4. By applying Lemma 4 and using some inequalities in the proof of Lemma

5, if $\varepsilon_1$is small enough, we have $\|(x_i')^{-1}\| \leqslant \left(r - \frac{\delta}{2}\right)^{-1}$. By applying [2, 4.1] (see also [2, 4.2]), if ε_1 is small enough, we have normal elements $z_i' \in B$ and $z_i'' \in M_2(B)$ such that

$$\|u^* x_i' u \oplus z_i' - z_i''\| < \frac{\varepsilon}{16}$$

and $\mathrm{sp}(z_i') \subset \left\{z : r - \frac{\delta}{2} \leqslant |z| \leqslant d + \frac{\delta}{4}\right\}$. Now let

$$y_i' = z_i' + \left[\frac{2i-1}{2k} - \frac{1}{2}\right] \text{ and } y_i'' = z_i'' + \left[\frac{2i-1}{2k} - \frac{1}{2}\right],$$

$$y' = y_1 \oplus y_2 \oplus \ldots \oplus y_{k-1} \oplus y_1' \oplus y_2' \oplus \ldots \oplus y_k',$$

and

$$y'' = y_1'' \oplus y_2'' \oplus \ldots \oplus y_k''.$$

Then

$$\|y \oplus y' - y''\| < \frac{\varepsilon}{8}.$$

Notice that even $\mathrm{sp}(y_i')$ may intersect with $\mathrm{sp}(y_j')$, $\mathrm{sp}(y_i')$ does not intersect the hole $\left\{z : \left|\frac{2j-1}{2k} - z\right| < r\right\}$, $i \neq j$. So there is a region Y such that $\mathrm{sp}(y'), \mathrm{sp}(x) \subset Y$, and there is a retraction r from Y onto $\mathrm{sp}(x)$. It is important to notice that such Y and r do not depend on ε or ε_1 but depend on X and δ. If ε_1 is small enough we can make

$$\left\|x \oplus y' - \sum_{i=1}^{n} \lambda_i p_i \oplus y''\right\| < \frac{\varepsilon}{4}.$$

Furthermore, since y'' is normal, by assuming ε_1 and ε small, we may assume that $\mathrm{sp}(y'') \subset Y$ too. By applying Lemma 1, we may assume that

$$\left[r(x) \oplus r(y') - \sum_{i=1}^{n} r(\lambda_i) p_i \oplus r(y'')\right\| < \frac{\varepsilon}{2}.$$

Since $r(x) = x$ and $r(\lambda_i) = \lambda_i$, we may further assume that $\mathrm{sp}(y') \subset \mathrm{sp}(x)$. We now apply the absorption argument used in the step 3. Notice in this case $k > 0$. However, we have

$$[p_i] > 2(k+1)\left[1 - \sum_{j=1}^{n} p_i\right], \quad i = 1, 2, \ldots, n.$$

This completes the proof for the case that A is unital.

Now we assume that A is not unital.

Then $0 \in \text{sp}(x)$. Let h be a continuous function defined on the unit disk D such that $\|h\| \leqslant 1$, $h(\zeta) = \zeta$ if $|\zeta| > \frac{\varepsilon}{2}$ and $h(\zeta) = 0$ if $|\zeta| < \frac{\varepsilon}{4}$. Then

$$\|h(x) - x\| < \frac{\varepsilon}{2}.$$

Now let p be the spectral projection of x in A^{**} corresponding to the open subset $\left\{\zeta \in D : |\zeta| > \frac{\varepsilon}{16}\right\}$ and q be the spectral projection of x in A^{**} corresponding to the closed subset $\left\{\zeta \in D : |\zeta| \geqslant \frac{\varepsilon}{8}\right\}$. Then p is an open projection in A^{**} and q is closed projection in A^{**}. Moreover, $q \leqslant p$. Suppose that g is a countinuous function defined on D such that $\|g\| \leqslant 1, g(\zeta) = 1$ if $|\zeta| \geqslant \frac{\varepsilon}{8}$ and $g(\zeta) = \emptyset$ if $|\zeta| > \frac{\varepsilon}{16}$. Then $g(x) \in A$ and $g(x) \geqslant q$. So q is compact. It follows from [7] that there is a projection $e \in A$ such that

$$q \leqslant e \leqslant p.$$

Clearly,

$$h(x)q = qh(x) = h(x).$$

So $h(x) \in eAe$. Since eAe is a unital simple AF-algebra and the compact convex space of the normalized traces (of eAe) has only countably many extreme points, from what we have established, there is a normal element $z \in eAe$ with finite spectrum contained in $\text{sp}(x)$ such that

$$\|h(x) - z\| < \frac{\varepsilon}{2}.$$

Therefore

$$\|x - z\| < \varepsilon. \qquad \blacksquare$$

Proof of Corollary B and Theorem E. By [4, Proposition 4.1], we known that if p and q are two projections in A and $\tau(p) > \tau(q)$ for every (lower semi-continuous and semifinite) trace τ on A, then $[p] \geqslant [q]$. From the proof of Lemma 2, we know that Lemma 2 holds for unital separable AF-algebras that satisfy the condition in Theorem E (and Corollary B). So the rest of the proof is exactly the same as that of Theorem A (and Corollary B).

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