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PREDUALS OF SOME FINITE DIMENSIONAL ALGEBRAS

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1. INTRODUCTION

In [2] J. Erdos studied operators on $L^p[0,1]$ which have as eigenvectors the set $\Phi = \{\varphi_a = \chi_{[a,1]} : 0 \leqslant a < 1\}$, and left as a conjecture that all such operators are reflexive. The objective of this paper is to establish properties of finite dimensional algebras which can be used to prove this conjecture for $p = 2$. To this end let T be a linear operator acting on $\mathbb{C}^n$ with Euclidean norm, and let $\mathcal{A}_T$ be the algebra generated by T and the identity. We consider the problem of describing the predual Q_T of $\mathcal{A}_T$ which consists of the set of (weak*-continuous) linear functionals on $\mathcal{A}_T$. It is well-know $\mathcal{A}_T$ has property $(\mathbb{A}_1)$, that is, if φ belongs to Q_T then there exist vectors x and y in $\mathbb{C}^n$ such that $\varphi(A) = (Ax, y)$ for $A \in \mathcal{A}_T$. It is customary to denote φ by $[x \otimes y]$. In a finite dimensional setting an operator T has property $(\mathbb{A}_1(\mathbf{1}))$ if an arbitrary φ acting on $\mathcal{A}_T$ can be written $\varphi = [x \otimes y]$ with $\|x\|^2 = \|y\|^2 = \|\varphi\|$. See [3] for a discussion of the properties $(\mathbb{A}_1)$ and $(\mathbb{A}_1(r))$. Very little is known about property $(\mathbb{A}_1(\mathbf{1}))$ for operators on $\mathbb{C}^n$. It is shown in [4] that for $n \geqslant 4$ not all operators enjoy this property. It is easy to verify that normal operators have property $(\mathbb{A}_1(\mathbf{1}))$ and it is shown in [7] that unweighted shifts also have property $(\mathbb{A}_1(\mathbf{1}))$. Below we introduce another class of operators with property $(\mathbb{A}_1(\mathbf{1}))$ and use results obtained to prove Erdos' conjecture.

2. GENERALITIES

It is assumed throughout this section that T is an operator on $\mathbb{C}^n$ with minimal polynomial of degree n. This assumption guarantees that Q_T is n dimensional.

It is well-known that any functional on $\mathcal{A}_T$ is of the form $\varphi = \sum_{k=1}^{n}[x_k \otimes y_k]$ with $\sum_{k=1}^{n}\|x_k\|^2 = \sum_{k=1}^{n}\|y_k\|^2 = \|\varphi\|$. It easily follows that T has property $(\mathbb{A}_1(1))$ if, and only if, the set

$$\mathcal{C}_1 = \{[x \otimes y] : \|x\| \leqslant 1, \|y\| \leqslant 1\} \text{ is convex.}$$

Define a map $p : \mathcal{C}_1 \to \mathbb{C}^n$ by $p([x \otimes y]) = ((x, y), (Tx, y), \dots, (T^{n-1}x, y))$ and let $\mathcal{C}_2$ be the image of $\mathcal{C}_1$ under this map. Since p extends to a linear map on Q_T it follows that $\mathcal{C}_2$ is convex if, and only if, $\mathcal{C}_1$ is convex. Notice that if $v \in \mathcal{C}_2$ then so does tv if $|t| \leqslant 1$ and that $\mathcal{C}_2$ contains an open ball centered at the origin in Q_T. It will be more convenient to view $\mathcal{C}_2$ as a subset of $\mathbb{R}^{2n}$ in the obvious way. Let L be a linear functional on $\mathbb{R}^{2n}$ and k a real number . Then we call the hyperplane $L(x) = k$ a support plane for $\mathcal{C}_2$ if it has non-empty intersection with $\mathcal{C}_2$ and

(i) for $l > k$ the hyperplane $L(x) = l$ has empty intersection with $\mathcal{C}_2$ or

(ii) for $l < k$ the hyperplane $L(x) = l$ has empty intersection with $\mathcal{C}_2$.

Notice that since $\mathcal{C}_2$ contains the origin there is no loss in assuming each support plane is of the form $L(x) = k$ with $k \geqslant 0$, and that (i) holds.

LEMMA 1. *The set $\mathcal{C}_2$ is convex if, and only if, the intersection of each support plane for $\mathcal{C}_2$ with $\mathcal{C}_2$ is convex.*

Proof. Necessity is obvious. For sufficiency it is enought to show $v = p\left(\frac{1}{2}[x \otimes y] + \frac{1}{2}[u \otimes v]\right)$ belongs to $\mathcal{C}_2$ whenever $[x \otimes y]$ and $[u \otimes v]$ belong to $\mathcal{C}_2$. Choose t so that tv belongs to the boundary of the convex hull of $\mathcal{C}_2$. Then tv for some $t \geqslant 1$ belongs to a support plane π for $\mathcal{C}_2$. It follows that tv is a convex combination of vectors in $\pi \cap \mathcal{C}_2$. Therefore tv, and hence v, belongs to $\mathcal{C}_2$. ■

Next we give a description of support planes. As noted above a given support plane may be written in the form $L(x) = m$, where

$$m = \max\{l > 0 : (L(x) = l) \cap \mathcal{C}_2 \neq \emptyset\}$$

Now, $L(x) = l$ may be written

$$\sum_{k=0}^{n-1} a_k \mathrm{Re}(T^k x, y) + b_k \mathrm{Im}(T^k x, y) = l, \quad \text{or Re} \left(\left(\sum_{k=0}^{n-1} (a_k - ib_k) T^k \right) x, y \right) = l.$$

It is easy to see that l is maximum when x is a unit maximizing vector for $\sum_{k=0}^{n-1}(a_k - ib_k)T^k$ and $y = \sum_{k=0}^{n-1}(a_k - ib_k)T^k x$, normalized. Let $A = \sum_{k=0}^{n-1}(a_k - ib_k)T^k$ and suppose, without loss of generality, that $||A|| = 1$. It follows that if π is a support plane for $\mathcal{C}_2$ then there exists A in $\mathcal{A}_T$ such that

$$\pi \cap \mathcal{C}_2 = \{p([x \otimes Ax]) : x \text{ a unit maximizing vector for } A\} =$$
$$= \{p([x \otimes Ax]) : ||x|| = 1 \text{ and } x \text{ belongs to the eigenspace corresponding the largest eigenvalue of } A^*A\}.$$

Of particular interest is the case where $A = I$. In this case $\pi \cap \mathcal{C}_2$ is the image under p of the set $\mathcal{C}_3 = \{[x \otimes x] : ||x|| = 1\}$. It is easy to see that if $\mathcal{C}_1$ is convex then so is $\mathcal{C}_3$. In light of what is known, the following seems reasonable.

CONJECTURE. *If $\mathcal{C}_3$ is convex, then so is $\mathcal{C}_1$.*

To facilitate the study of $\mathcal{C}_3$, define a map $q : \mathcal{C}_3 \to \mathbb{C}^{n-1}$ by $q([x \otimes x]) = ((Tx, x), (T^2x, x), \ldots, (T^{n-1}x, x))$ and let $\mathcal{C}_4$ denote the image of $\mathcal{C}_3$ under q. We will regard $\mathcal{C}_4$ as a subset of $\mathbb{R}^{2n-2}$ in the obvious way. A necessary condition for $\mathcal{C}_4$ to be convex is that each support plane for $\mathcal{C}_4$ intersected with $\mathcal{C}_4$ is convex. However this alone is not sufficient since we must exclude the possibility that $\mathcal{C}_4$ has "holes" in it. We can give a description of support planes for $\mathcal{C}_4$. Suppose $\sum_{k=1}^{n-1} a_k \mathrm{Re}(Tx, x) + b_k \mathrm{Im}(Tx, x) = m$ is a support plane for $\mathcal{C}_4$.

Then this plane may be written $\mathrm{Re}(Ax, x) = m$ where

$$A = \sum_{k=1}^{n=1} a_k T^i - \mathrm{i} b_k T^i. \tag{1}$$

Therefore we have $\mathrm{Re}(Ax, x) + \mathrm{Re}(x, A^*x) = 2m$ or $((A + A^*)x, x) = 2m$. It follows that $2m$ is either the least or greatest eigenvalue of $A + A^*$. Since we can replace A by $-A$ there is no loss in assuming $2k$ is the greatest eigenvalue of $A + A^*$. We conclude that a support plane for $\mathcal{C}_4$ is of the form $\{q[x \otimes x]) : ||x|| = 1$ and x belongs to the eigenspace corresponding to the largest eigenvalue of $A + A^*$ for some A of the form given in (1)$\}$.

We will need the following lemma concerning maximizing vectors.

LEMMA 2. *Suppose A is an operator on $\mathbb{C}^n$. Then*

(i) *The set of maximizing vectors for A together with 0 is a subspace of $\mathbb{C}^n$, which we call the maximizing subspace of A.*

(ii) *The maximizing subspace of A coincides with the eigenspace of the largest eigenvalue of A^*A.*

(iii) *If x is maximizing for A then Ax is maximizing for A^*.*

(iv) *The maximizing subspace of A and the maximizing subspace of A^* have the same dimension.*

3. OPERATORS WITH PROPERTY $(\mathfrak{A}_1(1))$

THEOREM 3. *The operator T acting on $\mathbb{C}^n$ represented by the matrix below has property $(\mathfrak{A}_1(1))$.*

$$(2)\qquad P=\begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ \lambda_1-\lambda_2 & \lambda_2 & 0 & \cdot & \cdot & 0 & 0 \\ \lambda_1-\lambda_2 & \lambda_2-\lambda_3 & \lambda_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \lambda_{n-1} & 0 \\ \lambda_1-\lambda_2 & \lambda_2-\lambda_3 & \cdot & \cdot & \cdot & \lambda_{n-1}-\lambda_n & \lambda_n \end{bmatrix}$$

Moreover an operator represented by the Schur product of P with a matrix of the form given below also has property $(\mathfrak{A}_1(1))$.

$$(3)\qquad Q=\begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ a_1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ a_1a_2 & a_2 & 1 & 0 & \cdot & \cdot & 0 \\ a_1a_2a_3 & a_2a_3 & a_3 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ a_1a_2\ldots a_{n-1} & a_2\ldots a_{n-1} & \cdot & \cdot & a_{n-2}a_{n-1} & a_{n-1} & 1 \end{bmatrix}$$

REMARKS 1. It is routine to verify that an operator T is of the form described in Theorem 3 if, and only if, it is unitarily equivalent to an operator having eigenvectors given by the columns of Q, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. It follows that each operator in $\mathcal{A}_T$ is representable by a matrix of the form the Schur product of P with Q.

2. It is enought to prove Theorem 3 assuming the λ_i are distinct: Property $(\mathfrak{A}_1(1))$ is a property of $\mathcal{A}_T$ rather than T and if S has repeated eigenvalues then $\mathcal{A}_S \subset \mathcal{A}_T$ for some T with distinct eigenvalues. It follows $\mathcal{A}_S$ has property $(\mathfrak{A}_1(1))$ since any subalgebra of an algebra with property $(\mathfrak{A}_1(1))$ also has property $(\mathfrak{A}_1(1))$.

3. There is no loss of generality in assuming the a_i are all non-zero: If any a_i is zero then $\mathcal{A}_T$ is reducible and direct sums of algebras with property $(\mathfrak{A}_1(1))$ also have property $(\mathfrak{A}_1(1))$.

The following results are needed to prove Theorem 3. Assume the λ_i are distinct and that the a_i are non-zero. Let $A \in \mathcal{A}_T$. Then A is represented by a matrix of the form

$$(4)\qquad R = \begin{bmatrix} c_1 & 0 & 0 & \cdot & \cdot & \cdot \\ a_1(c_1-c_2) & c_2 & 0 & \cdot & \cdot & \cdot \\ a_1a_2(c_1-c_2) & a_2(c_2-c_3) & c_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & c_{n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & a_{n-1}(c_{n-1}-c_n) & c_n \end{bmatrix}.$$

LEMMA 5. *Suppose A has matrix R as described above and A^* has two linearly independent maximizing vectors. Then A is a multiple of the identity.*

Proof. As noted in Section 1 the maximizing vectors for A^* constitute a subspace of $\mathbb{C}^n$, hence we can find a maximizing vector on the plane $\pi^{(n)}$ with equation

$$z_1 + \overline{a}_1 z_2 + \overline{a}_1\overline{a}_2 z_3 + \cdots + \overline{a}_1\overline{a}_2\ldots\overline{a}_{n-1}z_n = 0$$

where $(z_1, z_2, \ldots, z_n)$ denotes a generic point in $\mathbb{C}^n$.

We prove by induction on n that if A^* has a maximizing vector on π^n then A is a multiple of the identity. Assume first that $n = 2$ and that (z_1, z_2) is maximizing for A^* and $z_1 + \overline{a}_1 z_2 = 0$. Then we get an equation

$$\begin{bmatrix} \overline{c}_1 & \overline{a}_1(\overline{c}_1-\overline{c}_2) \\ 0 & \overline{c}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \overline{c}_1 z_1 + \overline{a}_1(\overline{c}_1-\overline{c}_2)z_2 \\ \overline{c}_2 z_2 \end{bmatrix} = \begin{bmatrix} -\overline{a}_1\overline{c}_2 z_2 \\ \overline{c}_2 z_2 \end{bmatrix} = \begin{bmatrix} \overline{c}_2 z_1 \\ \overline{c}_2 z_2 \end{bmatrix}$$

It follows that $\|A^*\| = |\overline{c}_2|$ and since it is assumed $a_1 \neq 0$ we have $A = c_2 I$.

Next, the inductive step. Let $x = (z_1, z_2, \ldots, z_n)$ be a maximizing vector for A^* on $\pi^{(n)}$. Then the first component of AA^*x is $c_1(\overline{c}_1 z_1 + \overline{a}_1(c_1 - \overline{c}_2)z_1 + \cdots + {}$ ${}+\overline{a}_1\overline{a}_2\ldots\overline{a}_{n-1}(\overline{c}_1-\overline{c}_2)z_n) = c_1(\overline{c}_2 z_1 + (\overline{c}_1 - \overline{c}_2)(z_1 + \overline{a}_1 z_2 \ldots + \overline{a}_1\overline{a}_2\ldots\overline{a}_{n-1}z_n)) =$ $= c_1\overline{c}_2 z_1$. There are now two cases to consider.

CASE 1. $z_1 = 0$. In this case the operator A_1 which is represented by R with the first row and column removed has a maximizing vector on $\pi^{(n-1)}$. It follows from the inductive hypothesis that A^* has matrix

$$(5)\qquad R_2 = \begin{bmatrix} \overline{c}_1 & \overline{a}_1(\overline{c}_1-\overline{c}_2) & \overline{a}_1\overline{a}_2(\overline{c}_1-\overline{c}_2) & \cdots & \overline{a}_1\overline{a}_2\ldots\overline{a}_{n-1}(\overline{c}_1-\overline{c}_2) \\ 0 & \overline{c}_2 & 0 & \cdots & 0 \\ 0 & 0 & \overline{c}_2 & \cdots & 0 \\ \vdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \overline{c}_2 \end{bmatrix}.$$

Moreover since x is maximizing for A^* and the first component of AA^*x is zero it follows that $\|A^*\| = |\bar{c}_2|$. An examination of the upper left hand two by two block of R_2 shows this is not possible unless $c_1 = c_2$.

CASE 2. $z_1 \neq 0$. It follows that $\|AA^*\| = c_1\bar{c}_2$. An examination of the upper left hand two by two block of R shows that if $c_1 \neq c_2$ then $\|A\|$ is strictly greater than the maximum of $|c_1|$ and $|c_2|$. It follows that $c_1 = c_2$ and that A^* is represented by a matrix of the form

$$\begin{bmatrix} \bar{c}_2 & 0 & 0 & \cdots & 0 \\ 0 & \bar{c}_2 & \bar{a}_2(\bar{c}_2 - \bar{c}_3) & \cdots & \bar{a}_2 \ldots \bar{a}_{n-1}(\bar{c}_2 - \bar{c}_3) \\ 0 & 0 & \bar{c}_3 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \bar{c}_n \end{bmatrix}.$$

Furthermore $\|A^*\| = |\bar{c}_2|$, so since A^* contains the block

$$\begin{bmatrix} \bar{c}_2 & \bar{a}_2(\bar{c}_2 - \bar{c}_3) \\ 0 & \bar{c}_3 \end{bmatrix}$$

we get a contraction if $c_2 \neq c_3$ and so on. It follows that A is a multiple of the identity. ∎

The following is immediate from Lemmas 2 and 4.

COROLLARY 5. *If A has matrix of the form described in (3) and $A \neq I$ then the maximizing subspace for A is one dimensional.*

LEMMA 6. *If A has matrix of the form described in (3) and $A \neq I$ then the eigenspace of the greatest eigenvalue, λ, of $A + A^*$ is one dimensional.*

Proof. The proof is similar to the proof of Lemma 4. Assume the Lemma is false. The proof is by induction on n that this leads to a contradiction. Assume first that $n = 2$ and that (z_1, z_2) is a non-zero vector in the eigenspace corresponding to λ and that $z_1 + \bar{a}_1 z_2 = 0$. Then

$$\left(\begin{bmatrix} c_1 & 0 \\ a_1(c_1 - c_2) & c_2 \end{bmatrix} + \begin{bmatrix} \bar{c}_1 & \bar{a}_1(\bar{c}_1 - \bar{c}_2) \\ 0 & \bar{c}_2 \end{bmatrix}\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 z_1 + \bar{c}_1 z_1 + \bar{a}_1(\bar{c}_1 - \bar{c}_2) z_2 \\ a_1(c_1 - c_2) + \bar{c}_2 z_2 \end{bmatrix} =$$

$$= \begin{bmatrix} (c_1 + \bar{c}_2) z_1 \\ a_1(c_1 - c_2) z_1 + \bar{c}_2 z_2 \end{bmatrix}.$$

Since $z_1 \neq 0$ it follows that $\lambda = c_1 + \bar{c}_2$. This shows that $c_1 + \bar{c}_2$ is real and from the discussion in Section 2 that $\max\{\mathrm{Re}(Ax, x) : \|x\| = 1\} = \dfrac{c_1 + \bar{c}_2}{2}$. With $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

we find $\mathrm{Re}(Ax,x) = \mathrm{Re}\, c_1$ and with $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathrm{Re}(Ax,x) = \mathrm{Re}\, c_2$. It follows that $c_1 = c_2$ and $A = c_1 I$.

Next, the inductive step. Let $x = (z_1, z_2, \ldots, z_n)$ be an eigenvector corresponding to λ on $\pi^{(n)}$. Then the first component of $(A + A^*)x$ is

$$\begin{aligned} &c_1 z_1 + (\bar{c}_1 z_1 + \bar{a}_1(\bar{c}_1 - \bar{c}_2) z_1 + \cdots + \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{n-1}(\bar{c}_1 - \bar{c}_2) z_n) = \\ &= c_1 z_1 + (\bar{c}_2 z_1 + (\bar{c}_1 - \bar{c}_2)(z_1 + \bar{a}_1 z_2 \cdots + \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{n-1} z_n)) = (c_1 + \bar{c}_2) z_1. \end{aligned} \tag{6}$$

There are now two cases to consider.

CASE 1. $z_1 = 0$. It follows from the inductive hypothesis that A^* has matrix R_2 given in (5) and that $\lambda = c_2 + \bar{c}_2$, that is, $\max\{\mathrm{Re}(Ax,x) : \|x\| = 1\} = \mathrm{Re}\, c_2$. Now consider (Ax,x) for $x = (\sqrt{\varepsilon}\mathrm{sgn}(\overline{a_1(c_1 - c_2)}), \sqrt{1-\varepsilon}, 0, 0, \ldots, 0)$. It is routine to show $\mathrm{Re}(Ax,x) = \mathrm{Re}(c_2 + \varepsilon(c_1 - c_2) + |a_1(c_1 - c_2)|\sqrt{\varepsilon}\sqrt{1-\varepsilon})$, and that this quantity is greater than $\mathrm{Re}\, c_2$ if $c_1 \neq c_2$, and ε is sufficiently small.

CASE 2. $z_1 \neq 0$. It follows that $\max\{\mathrm{Re}(Ax,x) : \|x\| = 1\} = \frac{1}{2}(c_1 + \bar{c}_2)$. Assume $c_1 \neq c_2$. Then it is shown in Case 1 that $\max\{\mathrm{Re}(Ax,x) : \|x\| = 1\}$ is greater than $\mathrm{Re}\, c_2$ and a similar argument shows it also to be greater than $\mathrm{Re}\, c_1$ which is a contradiction. Equation (6) shows $c_1 + \bar{c}_2$ to be real since it is an eigenvector of $A + A^*$ and it follows that $c_1 = c_2$. It follows that $\max\{\mathrm{Re}(Ax,x) : \|x\| = 1\} = \mathrm{Re}\, c_2$. An argument similar the one used in case 1 shows that if any diagonal entry of A does not equal c_2 we get a contradiction. ∎

LEMMA 7. *Let T be of the form described in Theorem 3 with the λ_i distinct and each a_i non-zero. Then $\{[x \otimes x] : \|x\| = 1\}$ is convex.*

Proof. We show that the set C_4 described in Section 2 is convex. Let S be the subset of the unit sphere in $\mathbb{C}^n$ consisting of those $(z_1, z_2, \ldots, z_n)$ for which $z_1 + \bar{a}_1 z_2 + + \bar{a}_1 \bar{a}_2 z_3 + \cdots + \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{n-1} z_n \geqslant 0$. Then S is homeomorphic with a unit hemisphere in $\mathbb{R}^{2n-1}$. Define a map φ from S to C_4 by $\varphi(x) = ((Tx,x), (T^2x,x), \ldots, (T^{n-1}x,x))$. Since $[x \otimes x] = [e^{i\theta}x \otimes e^{i\theta}x]$ for any θ it follows that φ is surjective. Lemma 6 shows that the intersection of each support plane for C_4 with C_4 is a single point. This shows that ∂C_4, the outer boundary of C_4, is homeomorphic with the unit sphere in R^{2n-2}. Lemma 6 also shows that each point in ∂C_4 is the image under φ of a unique point in S. It follows from the continuity of φ that the preimage of ∂C_4 in S is homeomorphic with the unit sphere in R^{2n-2}. It is a consequence of the Browder Fixed Point Theorem, see, for example [5], that C_4 is homeomorphic with the unit ball in R^{2n-2}. Since each support plane for C_4 intersects ∂C_4 in a single point it follows that C_4 is convex. ∎

Proof of Theorem 3. By Lemma 1 it is enought to show that the intersection of each support plane for C_2 with C_2 is convex. It is shown in Section 2 that such an intersection has the form

$$\mathcal{I} = \{[x \otimes Ax] : x \text{ is a unit maximizing vector for } A \text{ and } \|A\| = 1\}.$$

If A is not multiple of the identity then $\mathcal{I}$ is a single point by Lemma 5, and if A is a multiple of the identity it follows from Lemma 7 that $\mathcal{I}$ is convex. ■

4. AN APPLICATION

We now turn attention to the conjecture of Erdos cited in the introduction. He showed that the algebra $\mathcal{A}(\Phi)$ of operators having as eigenvectors the set Φ consists of operators of the form

$$(A_\lambda f)(x) = \lambda(x)f(x) - \int_0^x \lambda'(t)f(t)\mathrm{d}t \tag{7}$$

where λ is a function which is bounded on $[0,1)$, absolutely continuous on $[0,a]$ for each $a < 1$, and for which $\sup_{0<x<1} (1-x)^{\frac{1}{p}} \left\{ \int_0^x |\lambda'(t)|^q \mathrm{d}t \right\}^{\frac{1}{q}} < \infty$. Let Λ denote the set of all such λ.

In this paper we restrict attention to the case $p = 2$, although it seems likely that the techiniques could be extended to deal with the general case.

It is shown in [4] that if $\mathcal{A}$ is a reflexive algebra with property $(\mathbb{A}_1)$ then every weakly closed subalgebra of $\mathcal{A}$ is reflexive. Erdos showed that $\mathcal{A}(\Phi)$ is reflexive, so we will be able to prove his conjecture, at least for $p = 2$, by showing that $\mathcal{A}(\Phi)$ has property $(\mathbb{A}_1)$.

Note that an example of Larsen and Wogen [6] shows there exists operators with spanning sets of eigenvectors which are not reflexive.

The following is a standard result. See [1].

LEMMA 8. *Let $\mathcal{A}$ be a weak* closed algebra of operators acting on a Hilbert space $\mathcal{H}$ and let $\varepsilon > 0$ be given. Suppose φ is a weak*-continuous linear functional on $\mathcal{A}$ of unit norm. Then there exist sequences of vectors $\{x_n\}$ and $\{y_n\}$ in $\mathcal{H}$ with $\sum_{i=1}^{\infty} \|x_i\|^2 = \sum_{i=1}^{\infty} \|y_i\|^2 \leqslant 1 + \varepsilon$ such that $\varphi = \sum_{i=1}^{\infty} [x_i \otimes y_i]$.*

LEMMA 9. *Suppose φ is a weak*-continuous linear functional of unit norm acting on $\mathcal{A}(\Phi)$. Let $\varepsilon > 0$ be given. then there exist vectors x and y in the unit ball of $L^2[0,1]$ such that $\|\varphi - [x \otimes y]\| < \varepsilon$.*

Proof. It follows from Lemma 8 that there exist sequences of vectors $\{x_n\}$ and $\{y_n\}$ in $L^2[0,1]$ with $\sum_{i=1}^{n}\|x_i\|^2 = \sum_{i=1}^{n}\|y_i\|^2 \leqslant 1$ such that $\left\|\varphi - \sum_{i=1}^{n}[x_i \otimes y_i]\right\| < \frac{\varepsilon}{2}$. Partition [0,1] into m equal subintervals and let $\{e_k\}$ denote the characteristic functions of these subintervals normalized. For m sufficiently large each x_i and y_i can be approximated arbitrarily closely by linear combinations of $\{e_k\}$. It follows that there exist m and such step functions that $\left\|\varphi - \sum_{i=1}^{n}[f_i \otimes g_i]\right\| < \varepsilon$. Let $\mathcal{H}$ be the subspace of $L^2[0,1]$ with orthonormal basis $\{e_k\}$. Then it is easy to see that each $A \in \mathcal{A}(\Phi)$ leaves $\mathcal{H}$ invariant and that the restriction of A to $\mathcal{H}$ has matrix representation of the form given in (2). It follows from Theorem 3 that there exist f and g in the unit ball of $\mathcal{H}$ such that

$$\sum_{i=1}^{n}(A|\mathcal{H}f_i, g_i)) = (A|\mathcal{H}f, g) \quad \text{for} \quad A \in \mathcal{A}(\Phi),$$

and hence from the invariance of $\mathcal{H}$ that

$$\sum_{i=1}^{n}(Af_i, g_i) = (Af, g) \quad \text{for} \quad A \in \mathcal{A}(\Phi).$$

It follows that $\|\varphi - [f \otimes g]\| < \varepsilon$. ■

LEMMA 10. *Suppose $A_\lambda \in \mathcal{A}(\Phi)$, that $|\lambda(a)| = \|\lambda\|_\infty$ for some $a \in [0,1)$ and that λ is not constant. Then $\|A_\lambda\| > \|\lambda\|_\infty$.*

Proof. Assume first that λ is constant on [a,1). Then as in the proof of Lemma 9 the restriction of A_λ to some invariant subspace has matrix given by (2) with $\lambda_i = \lambda\left(\frac{i-1}{n}\right)$. An examination of the 2 by 2 diagonal blocks shows $\|A_\lambda\| > \|\lambda\|_\infty$.

Now assume λ is not constant on [a,1). If $b > a$ then

$$A_\lambda(\varphi_a - \varphi_b) = \lambda(a)\varphi_a - \lambda(b)\varphi_b = \lambda(a)(\varphi_a - \varphi_b) + (\lambda(a) - \lambda(b))\varphi_b.$$

Since this is an orthogonal sum any choice of b with $\lambda(a) \neq \lambda(b)$ shows $\|A_\lambda\| >$ $> \|\lambda\|_\infty$. ■

DEFINITION 11. For $\lambda \in \Lambda$, define λ_a by $\lambda_a(t) = \begin{cases} \lambda(t), & t < a \\ \lambda(a), & t \geqslant a \end{cases}$. Let Λ_1 be the subset of Λ consisting of those λ for which

(a) $\|A_\lambda\| > \|\lambda\|_\infty$, and

(b) $\lim_{a \to 1} \|A_\lambda - A_{\lambda_a}\| = 0$.

It follows from Lemma 3.1 of [2] that (b) is equivalent to

(b)′ $\lim_{a\to 1}(1-a)^{\frac{1}{2}} \left\{\int_0^a |\lambda'(t)|^2 dt\right\}^{\frac{1}{2}} = 0.$

LEMMA 12. *Suppose $\lambda \in \Lambda_1$. Then A_λ has a maximizing vector.*

Proof. Assume without loss that $\|A_\lambda\| = 1$. Then there exists a sequence $\{f_n\}$ in the unit ball of $L^2[0,1]$ with $\|A_\lambda f_n\| \to 1$. By dropping to a subsequence, if necessary, it can be assumed $\{f_n\}$ converges weakly to some f. Since $\|A_\lambda(f_n - f) + A_\lambda f\| \to 1$, it follows from the facts that $\{f_n - f\}$ converges weakly to 0 and $\|A_\lambda\| = 1$ that

$$\|A_\lambda f\| = \|f\| \text{ and } \{\|A_\lambda(f_n - f)\| - \|f_n - f\|\} \to 0. \tag{8}$$

We will now assume that $f = 0$ and obtain a contradiction.

Fix a in (0,1) and write $f_n = k_n \oplus h_n$, where $k_n = \chi_{[0,a]} f_n$ and $h_n = \chi_{[a,1]} f_n$.

CLAIM 1. The sequence $\{k_n\} \to 0$ strongly.

Proof. Assume not. then by dropping to a subsequence if necessary we can assume $\|k_n\| \to \delta > 0$. Let K_λ be the operator defined by $(K_\lambda f)x = \int_0^x \lambda'(t)f(t)\mathrm{d}t$. Then $K_\lambda k_n$ tends pointwise to zero and $|(K_\lambda k_n)x| < \left\{\int_0^a |\lambda'(t)|^2\mathrm{d}t\right\}^{\frac{1}{2}} \|k_n\|$ for each x. It follows from the bounded convergence theorem that $\|K_\lambda k_n\| \to 0$. Let M_λ be the operator defined by $(M_\lambda f)x = \lambda(x)f(x)$. Then $A_\lambda(k_n \oplus h_n) = (M_\lambda k_n \oplus A_\lambda h_n) - -K_\lambda k_n$. By assumption $\|M_\lambda\| = \|\lambda\|_\infty < 1$, and hence, given $\varepsilon > 0$, for n sufficiently large $\|A_\lambda(k_n \oplus h_n)\|^2 \leqslant \|M_\lambda k_n\|^2 + \|A_\lambda h_n\|^2 + \varepsilon$, and hence $\lim_{n\to\infty} \|A_\lambda(k_n \oplus h_n)\|^2 \leqslant \leqslant \|\lambda\|_\infty \|k_n\|^2 + \|h_n\|^2$, which contradicts (8) and establishes the claim. The following is now immediate.

CLAIM 2. Without loss of generality given a in (0,1) it can be assumed that for n sufficiently large $f_n = \chi_{[a,1]} f_n$.

Now for each $a \in [0,1)$ it follows that $0 = \lim_{n\to\infty} \|A_\lambda f_n - A_{\lambda_a} f_n\| = \lim_{n\to\infty} \|A_\lambda f_n - -\lambda(a) f_n\| \geqslant 1 - \|\lambda\|_\infty$, and this contradiction completes the proof.

LEMMA 13. *Suppose $A_\lambda \in \mathcal{A}(\Phi)$ has unit norm and a unit maximizing vector with $A_\lambda f = g$. Then for almost every $a \in [0,1)$*

$$f(a) = \overline{\lambda(a)} g(a) - \overline{\lambda'(a)} \int_a^1 g(t)\mathrm{d}t.$$

Proof. Fix $a \in [0,1)$ and write orthogonal sums $f = f_1 \oplus \alpha\varphi_a$ and $g = g_1 \oplus \beta\varphi_a$. Now if γ is any scalar $A_\lambda(f_1 \oplus \alpha\varphi_a + \gamma\varphi_a) = g_1 \oplus \beta\varphi_a + \lambda(a)\gamma\varphi_a$.

By assumption for all γ we must have $\|f_1 \oplus \alpha\varphi_a + \gamma\varphi_a\| \geqslant \|g_1 \oplus \beta\varphi_a + \lambda(a)\gamma\varphi_a\|$ and since f and g have unit norm

$$\|f_1 \oplus \alpha\varphi_a + \gamma\varphi_a\|^2 - \|f_1 \oplus \alpha\varphi_a\|^2 \geqslant \|g_1 \oplus \beta\varphi_a + \lambda(a)\gamma\varphi_a\|^2 - \|g_1 \oplus \beta\varphi_a\|^2$$

which reduces to

$$|\gamma|^2 + \alpha\overline{\gamma} + \overline{\alpha}\gamma \geqslant |\lambda(a)|^2|\gamma|^2 + \beta\overline{\lambda(a)\gamma} + \overline{\beta}\lambda(a)\gamma.$$

And it easily follows that $\alpha = \beta\overline{\lambda(a)}$. In other words

$$\int_a^1 f(t)\mathrm{d}t = \overline{\lambda(a)} \int_a^1 g(t)\mathrm{d}t.$$

Differentiating the sides of this equation now completes the proof. ■

LEMMA 14. *For each n we have $\|A_{t^n}\| = 1$.*

Proof. Since $A_{t^n} = (A_t)^n$ it is enough to prove the lemma for $n = 1$. Assume $\|A_{rt}\| = 1$ for some $r < 1$. Then since $\lambda(t) = rt$ satisfies condition (b′) of Definition 11, it follows that $\lambda(t) = rt$ belongs to Λ_1. By Lemma 12, A_{rt} has a unit maximizing vector f with $A_{rt}f = g$. By (7) $g(x) = rxf(x) - \int_0^x rf(t)\mathrm{d}t$. By Lemma 13, $f(x) = rxg(x) - r\int_x^1 g(s)\mathrm{d}s$. Hence, $g(x) = r^2x^2g(x) - r^2x\int_x^1 g(s)\mathrm{d}s - \int_0^x \left[r^2tg(t) - \int_t^1 r^2g(s)\mathrm{d}s\right]\mathrm{d}t = r^2x^2g(x) - r^2x\int_x^1 g(s)\mathrm{d}s - \int_0^x r^2tg(t) - \int_0^x \left[\int_t^1 r^2g(s)\mathrm{d}s\right]\mathrm{d}t$. Integrating the third term by parts gives

$$\int_0^x r^2tg(t)\mathrm{d}t = -r^2t\int_t^1 g(s)\mathrm{d}s\Big|_0^x + \int_0^x \left[\int_t^1 r^2g(s)\mathrm{d}s\right]\mathrm{d}t = -r^2x\int_x^1 g(s)\mathrm{d}s + \int_0^x \left[\int_t^1 r^2g(s)\mathrm{d}s\right]\mathrm{d}t.$$

Therefore $g(x) = r^2x^2g(x)$ and this contradiction completes the proof. ■

LEMMA 15. *Suppose φ is a weak*-continuous linear functional of unit norm acting on $\mathcal{A}(\Phi)$ such that $|\varphi(I)| < 1$. Then there exist vectors f and g in the unit ball of $L^2[0,1]$ such that $\varphi = [f \otimes g]$.*

Proof. Since the unit ball in $\mathcal{A}(\Phi)$ is weak*-compact there exists A_λ of unit norm in $\mathcal{A}(\Phi)$ such that $\varphi(A_\lambda) = 1$. By Lemma 9 there exist sequences $\{f_n\}$ and $\{g_n\}$ in the unit ball of $L^2[0,1]$ with $\|\varphi - [f_n \otimes g_n]\| \to 0$. By dropping to subsequences, if necessary, it can be assumed these sequences converge weakly to f and g respectively.

Since $\varphi(A_\lambda) = 1$ we have $(A_\lambda f_n, g_n) \to 1$ and hence $\|A_\lambda(f_n - f) + A_\lambda f\| \to 1$. It follows from the facts that $\{f_n - f\}$ converges weakly to 0 and $\|A_\lambda\| = 1$ that

$$\|A_\lambda f\| = \|f\| \text{ and } \{\|A_\lambda(f_n - f)\| - \|f_n - f\|\} \to 0, \tag{9}$$

and hence that

$$A_\lambda f = g \text{ and } \{\|A_\lambda(f_n - f)\| - (g_n - g)\} \to 0. \tag{10}$$

We will now assume that $\{f_n - f\}$ does not converge strongly to zero and obtain a contradiction.

Fix a in $(0,1)$ and write $f_n - f = k_n \oplus h_n$, where $k_n = \chi_{[0,a]}(f_n - f)$ and $h_n = \chi_{[a,1]}(f_n - f)$.

CLAIM 1. The sequence $\{k_n\} \to 0$ strongly.

Proof. This is almost identical to the proof of Claim 1 in Lemma 11. The only difference being that an appeal to Lemma 10 shows the existence of $\eta < 1$ with $\|M_\lambda k_n\| < \eta\|k_n\|$.

The following is an immediate consequence of Claim 1.

CLAIM 2. Without loss of generality given a in $(0,1)$ it can be assumed that for n sufficiently large $f_n - f = \chi_{[a,1]}(f_n - f)$ and hence that $g_n - g = \chi_{[a,1]}(g_n - g)$.

It follows from the assumption that $\{f_n - f\}$ does not converge strongly to zero that by replacing $\{f_n - f\}$ by a subsequence it can be assumed that $\|f_n - f\| \to \delta > 0$. Let $u_n = f_n - f$, normalized and let $v_n = g_n - g$, normalized.

CLAIM 3. The sequence $\{[f \otimes v_n]\}$ converges to 0.

Proof. Fix $a \in (0,1)$. Then given b with $a < b < 1$, for n sufficiently large $[f \otimes v_n] = [\chi_{[0,a]}f \otimes \chi_{[b,1]}v_n] + [\chi_{[a,1]}f \otimes \chi_{[b,1]}v_n]$. If a is sufficiently close to 1 then second term is small for any choice of b. To show that the first term can be made small choose $\lambda \in \Lambda$ with $\|A_\lambda\| = 1$. Then

$$[\chi_{0,a]}f \otimes \chi_{[b,1]}v_n](A_\lambda) = (A_\lambda\chi_{[0,a]}f, \chi_{[b,1]}v_n) = (\chi_{[b,1]}A_\lambda\chi_{[0,a]}f, \chi_{[b,1]}v_n). \tag{11}$$

It follows from (7) that for $x > a$ we have

$$|(A_\lambda\chi_{[0,a]}f)(x)| = \left|-\int_0^a \lambda'(t)f(t)\mathrm{d}t\right| \leqslant \left\{\int_0^a |\lambda'(t)|^2\mathrm{d}t\right\}^{\frac{1}{2}}\left\{\int_0^a |f(t)|^2\mathrm{d}t\right\}^{\frac{1}{2}} \leqslant \left\{\int_0^a |\lambda'(t)|^2\mathrm{d}t\right\}^{\frac{1}{2}}.$$

It follows from Lemma 3.1 of [2] that $(1-a)^{\frac{1}{2}}\left\{\int\limits_0^a|\lambda'(t)|^2\mathrm{d}t\right\}^{\frac{1}{2}} \leqslant 3$. Hence for $x>a$ that $|(A_\lambda\chi_{[0,a]}f)(x)|\leqslant 3(1-a)^{-\frac{1}{2}}$, and so $\|\chi_{[b,1]}A_\lambda\chi_{[0,a]}f\|\leqslant 3(1-a)^{-\frac{1}{2}}(1-b)^{\frac{1}{2}}$. Then from (11) $|[\chi_{[0,a]}f\otimes\chi_{[b,1]}v_n](A_\lambda)|\leqslant 3(1-a)^{-\frac{1}{2}}(1-b)^{\frac{1}{2}}$, and this can be arbitrarily small by an appropriate choice of b.

CLAIM 4. The sequence $\{[u_n\otimes v_n]\}$ converges to a functional μ on $\mathcal{A}(\Phi)$ such that $\|\mu\|=1$, $\mu(A_\lambda)=1$.

Proof. To show $\{[u_n\otimes v_n]\}$ converges it suffices to show $\{[(f_n-f)\otimes(g_n-g)]\}$ converges. Now, $[(f_n-f)\otimes(g_n-g)]=[f_n\otimes g_n]-[f\otimes(g_n-g)]-[(f_n-f)\otimes g]+[f\otimes g]$. By Claim 2 for $a\in[0,1)$ and n sufficiently large this may be rewritten $[(f_n-f)\otimes(g_n-g)]+[f\otimes\chi_{[a,1]}(g_n-g)]+[\chi_{[a,1]}(f_n-f)\otimes g]+[f\otimes g]=[(f_n-f)\otimes(g_n-g)]+[f\otimes\chi_{[a,1]}(g_n-g)]+[\chi_{[a,1]}(f_n-f)\otimes\chi_{[a,1]}g]+[f\otimes g]$. Since the third term can be made arbitrarily small by choosing a sufficiently close to 1, and by Claim 3 the second term is small for n sufficiently large, the first part of Claim 4 is established. That $\|\mu\|=1$ and $\mu(A_\lambda)=1$ follows from (9) and (10).

To complete the proof we obtain the contradiction that $\{[u_n\otimes v_n]\}$ is not a Cauchy sequence. Consider $[u_n\otimes v_n](A_t m_\lambda]=(A_t m_\lambda u_n,v_n)$. Since $[u_n\otimes v_n](A_\lambda)= =(A_\lambda u_n,v_n)\to 1$ as $n\to\infty$, it follows from Claim 2 that $(A_t m_\lambda u_n,v_n)\to 1$ as $n\to\infty$ for fixed m. On the other hand if n is fixed then $(A_t m_\lambda u_n,v_n)\to 0$ as $m\to\infty$. Since, by Lemma 14, $\|A_t m_\lambda\|=1$ this gives the desired contradiction.

We conclude that $\{f_n-f\}$ converges strongly to 0 and that $\varphi=[f\otimes g]$. ■

It remains to deal with case where φ is a functional of unit norm on $\mathcal{A}(\Phi)$ and $\varphi(I)=1$.

LEMMA 16. *Suppose φ is a functional on $\mathcal{A}(\Phi)$ of unit norm and that $\varphi(I)=1$. Then either*

(a) *φ is a point evaluation. That is there is an in $[0,1)$ such that $\varphi(A_\lambda)=\lambda(a)$*

or

(b) *There is $A_\lambda\in\mathcal{A}(\Phi)$ with λ bounded away from 0 and $\varphi(A_\lambda)=0$.*

Proof. Assume (b) is false and suppose $\lambda\in\Lambda$ is strictly monotonic increasing. Then $\varphi(A_\lambda-\alpha I)=0$ for some $\alpha\in\mathbb{C}$. Since (b) is assumed false it follows that $\lambda(a)=\alpha$ for some $a\in[0,1]$ and that $\varphi(A_\lambda)=\lambda(a)$. Let μ be another strictly monotonic increasing function in Λ. Then $\varphi(A_\mu)=\mu(b)$ for some $b\in[0,1]$. If $b\neq a$ then $\nu=\lambda-\lambda(a)+\mathrm{i}(\mu-\mu(b))$ is bounded away from 0 and $\varphi(A_\nu)=0$. It follows that $\varphi(A_\lambda)=\lambda(a)$ for each $\lambda\in\Lambda$ which is absolutely continuous on $[0,1]$. To show that $a\in[0,1)$ it is enough to show that the densely defined functional $\eta(A_\lambda)=\lambda(1)$

is not weak*-continuous.

Given $\lambda \in \Lambda$ define λ_n by $\lambda_n(x) = \begin{cases} \lambda(x) & \text{if } x \leqslant 1 - \frac{1}{2^n} \\ \lambda\left(1 - \frac{1}{2^n}\right) & \text{if } x > 1 - \frac{1}{2^n} \end{cases}.$

CLAIM. The sequence $\{\lambda_n\}$ converges weak* to λ.

Proof. This is essentialy contained in Lemma 3.5 of [2].

Let μ be the piecewise linear function for which $\mu\left(1 - \frac{1}{2^n}\right) = 1$ for n even and $\mu\left(1 - \frac{1}{2^n}\right) = -1$ for n odd. A routine calculation shows $\mu \in \Lambda$.

It now follows that $\eta(\mu_n) = (-1)^n$, showing that η is not weak*-continuous. Now, it follows that $a < 1$ and from the claim that φ is a point evaluation. ∎

LEMMA 17. *Suppose φ is a functional of unit norm acting on $\mathcal{A}(\Phi)$ such that $\varphi(I) = 1$. Then there exist x any y in $L^2[0,1]$ such that $\varphi = [x \otimes y]$.*

Proof. If φ is point evaluation at a then $\varphi = [\varphi_a \otimes \varphi_a]$ normalized. By Lemma 16 if φ is not point evaluation there is $\mu \in \Lambda$ bounded away from 0 with $\varphi(A_\mu) = 0$. Notice that A_μ is invertible. Define a new functional on $\mathcal{A}(\Phi)$ by $\varphi_\mu(A_\lambda) = \varphi(A_\mu A_\lambda)$. Since $\varphi_\mu(I) = 0$ it follows from Lemma 15 that there exist x and y in $L^2[0,1]$ such that $\varphi_\mu = [x \otimes y]$. Now, $\varphi(A_\lambda) = \varphi(A_\mu A_\mu^{-1} A_\lambda) = \varphi_\mu(A_\mu^{-1} A_\lambda) = (A_\mu^{-1} A_\lambda x, y) = (A_\lambda(A_\mu^{-1} x), y)$. Therefore $\varphi = [A_\mu^{-1} x \otimes y]$. ∎

The following is now immediate from the above.

LEMMA 18. *The algebra $\mathcal{A}(\Phi)$ has property $(\mathbb{A}_1)$ and hence every unital subalgebra of $\mathcal{A}(\Phi)$ is reflexive.*

Finally, a word about maximizing vectors. The proof of Lemma 14 (with $r = 1$) shows that not all operators in $\mathcal{A}(\Phi)$ have maximizing vectors. However it is a consequence of the above that the following is true.

PROPOSITION 19. *Suppose $A_\lambda \in \mathcal{A}(\Phi)$. Then A_λ has a maximizing vector if, and only if, there is a weak*-continuous linear functional φ of unit norm acting on $\mathcal{A}(\Phi)$ with $\varphi(A_\lambda) = 1$.*

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