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L_p-REGULARITY OF SUBELLIPTIC OPERATORS ON LIE GROUPS

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1. INTRODUCTION

In an earlier paper, [6] Theorem 5.3.I, it was established that the C^∞-structure of each continuous representation of a Lie group coincides with the C^∞-structure for each strongly elliptic, or subcoercive, operator, i.e., the C^∞-elements of the representation are precisely the C^∞-elements with respect to the subcoercive operator. It is known, however, that the differential structures, i.e., the C^n-elements, do differ for certain representations such as the left regular representation in $L_1(\mathbf{R}^2)$ or $L_\infty(\mathbf{R}^2)$ (see, for example, [14] and [12]). Nevertheless, in many particular classes of representations the differential structures are the same. For strongly elliptic operators this equality was established for unitary representations in [17], Example II.5.10, for Lipschitz representations in [17], Theorem II.5.8, and for principal series representations in [4] Theorem 6. Moreover, for subcoercive operators the coincidence was proven for unitary representations in [6], Theorem 6.3.II, and for second-order operators with real symmetric coefficents and Lipschitz spaces a comparable conclusion was reached in [5], Theorem 5.1.III. (The last result extends to general subcoercive operators although the proof is not explicitly given in [5].) In the present paper we prove that the differential structure for the left regular representation on the L_p-spaces with respect to the left-, or right-, Haar measure on the Lie group G coincides with the differential structure of each subcoercive operator if $p \in \langle 1, \infty \rangle$.

It is perhaps worthwhile mentioning in this context that the analytic structure of a continuous representation coincides with the analytic structure for each strongly elliptic operator, [17] Theorem II.3.1, but there are subcoercive operators for which these structures differ, even in the case of a unitary representation, [5] Example 8.2.

The comparison of the differential structures is related to the Lie group version of the boundedness of the Riesz transforms. If H is the sublaplacian associated with the left derivatives $A_1, \ldots, A_{d'}$ then we establish that the operators $X_n(\nu) = M_n(\nu I + H)^{-n/2}$, with M_n an n-th order monomial in the A_i, are bounded on the L_p-spaces, $p \in \langle 1, \infty \rangle$, whenever $\nu > 0$. The operators $A_i H^{-1/2}$ are the analogoes of the Riesz transforms and correspond to the X_1 in the limiting case $\nu = 0$. It should be stressed that one cannot expect the $X_n(0)$ to be bounded for all groups, all sublaplacians and all n. Gaudry, Qian and Sjögren [9] have shown that for the $(ax + b)$-group, which is a non-unimodular group of exponential growth, there is an algebraic subbasis such that the operators $A_i H^{-1/2}$ are bounded on L_p, $p \in \langle 1, \infty \rangle$, but the $A_i A_j H^{-1}$ are not bounded on any of the L_p-spaces. Nevertheless, boundedness is restored if H is replaced by $\nu I + H$ with $\nu > 0$. The parameter ν introduces an exponential decrease in the kernels of the operators $X_n(\nu)$ and hence boundedness of these operators becomes a local problem, albeit a problem which has to be handled uniformly over the group. Seemingly stronger results can be obtained if one considers special classes of groups. Lohoué [13] established boundedness of the Riesz transforms for non-amenable unimodular groups and an algebraic basis of left derivatives but since the non-amenability is simply used to deduce an exponential decrease of the operator kernel his results follow from our arguments, even for non-unimodular groups. Folland, [8] Corollary 4.13, established boundedness of the Riesz transforms for stratified groups with H the canonical sublaplacian but this is a simple corollary of our results and a rescaling which removes the factor ν. Other results in this direction have been given by Saloff-Coste [18] who proved boundedness of first-order transforms $A_i H^{-1/2}$ on polynomial groups and by Anker, [1], who established a similar result on noncompact symmetric spaces obtained by the quotient of a semisimple group G by a maximal compact subgroup K. We emphasize that all our results hold for general Lie groups, which need not be unimodular.

In the sequel we adopt the general notation used in [17] and [6] but now we consider two connected Lie groups G and G_1 with $G \subseteq G_1$ and the continuous representation U of G is identified with the left translations L acting on the spaces $L_p(G_1; dg)$ and $L_p(G_1; d\hat{g})$ where dg and $d\hat{g}$ denote the left and right Haar measure, respectively. We use the abbreviated notation $L_p(G_1)$ and $L_{\hat{p}}(G_1)$ and let Δ denote the modular function over G_1. In fact, the group G need not be connected since all analysis takes part on the connected component of the identity of G.

Let $a_1, \ldots, a_{d'}$ be elements of the Lie algebra $\mathfrak{g}$ of G and let A_i, for all $i \in \{1, \ldots, d'\}$, denote the infinitesimal generator of the one-parameter group $t \mapsto L(\exp(ta_i))$ from $\mathbf{R}$ into $L_p(G_1)$ or $L_{\hat{p}}(G_1)$. It will be clear from the context on which space the A_i act. We also denote by $A_i\varphi$ the pointwise left derivative in the direction a_i of a function $\varphi : G \to \mathbb{C}$. The constant $(A_i\Delta)(e)$ is denoted by b_i. We use multi-index notation for products of the generators A or for products of the b_i. For $n \in \mathbb{N}_0$ let

$$J_n(d') = \bigcup_{k=0}^{n} \{1, \ldots, d'\}^k.$$

If $\alpha = (i_1, \ldots, i_k) \in \{1, \ldots, d'\}^k$, we define $|\alpha| = k$, $A^\alpha = A_{i_1} \ldots A_{i_k}$ and $b^\alpha = b_{i_1} \ldots b_{i_k}$. Let $J(d') = \bigcup\limits_{n=1}^{\infty} J_n(d')$. Then for each $n \in \mathbb{N}_0$ we denote the subspace $\bigcap\limits_{\alpha \in J_n(d')} D(A^\alpha)$ in $L_p(G_1)$, or $L_{\hat{p}}(G_1)$, by $L'_{p;n}(G_1)$, or $L'_{\hat{p};n}(G_1)$, respectively. We define a norm and a seminorm on $L'_{p;n}(G_1)$ by setting

$$\|\varphi\|'_{p;n} = \sup_{\alpha \in J_n(d')} \|A^\alpha \varphi\|_p, \quad N'_{p;n}(\varphi) = \sup_{|\alpha|=n} \|A^\alpha \varphi\|_p,$$

for each $\varphi \in L'_{p;n}(G_1)$, and $\|\cdot\|'_{\hat{p};n}$, $N'_{\hat{p};n}$, are defined analogously on $L'_{\hat{p};n}(G_1)$. Let $L'_{p,\infty}(G_1) = \bigcap\limits_{n=1}^{\infty} L'_{p;n}(G_1)$ and $L'_{\hat{p};\infty}(G_1) = \bigcap\limits_{n=1}^{\infty} L'_{\hat{p};n}(G_1)$. We also adopt the corresponding notation $\mathcal{X}'_n$ and $\mathcal{X}'_\infty$ for the subspaces $\bigcap\limits_{\alpha \in J_n(d')} D(A^\alpha)$ and $\bigcap\limits_{\alpha \in J(d')} D(A^\alpha)$ associated with the generators of a continuous representation of G in a Banach space $\mathcal{X}$.

In the absence of a statement to the contrary we assume that $a_1, \ldots, a_{d'}$ is an algebraic basis for $\mathfrak{g}$, i.e., a finite sequence of linearly independent elements of $\mathfrak{g}$ which generate $\mathfrak{g}$. Thus there is an integer r such that $a_1, \ldots, a_{d'}$ together with all commutators $(\operatorname{ad} a_{i_1}) \ldots (\operatorname{ad} a_{i_{n-1}})(a_{i_n})$, $i_j = 1, \ldots, d'$, where $n \leqslant r$, span the vector space $\mathfrak{g}$. The smallest integer r with this property is referred to as the rank of the subbasis and a vector space basis is defined to have rank one. Moreover, the algebraic basis determines in a canonical fashion (see, [17] Section IV.4c) a modulus function $g \mapsto |g|'$ on the group. This function in turn determines a unique local dimension D' such that the ball $B'_\rho = \{g \in G : |g|' < \rho\}$ has measure $|B'_\rho|$, with respect to Haar measure on G, satisfying bounds $c_1 \rho^{D'} \leqslant |B'_\rho| \leqslant c_2 \rho^{D'}$ for all $\rho \in \langle 0, 1]$.

An m-th order form is a function $C : J_m(d') \to \mathbb{C}$ such that $C(\alpha) \neq 0$ for some $\alpha \in J_m(d')$ with $|\alpha| = m$. The principal part P of C is the form with $P(\alpha) = C(\alpha)$ if $|\alpha| = m$ and $P(\alpha) = 0$ if $|\alpha| < m$. The formal adjoint $C^\dagger$ of C is the function $C^\dagger : J_m(d') \to \mathbb{C}$ defined by $C^\dagger(\alpha) = (-1)^{|\alpha|}\overline{C(\alpha_*)}$ where $\alpha_* = (i_n, \ldots, i_1)$ whenever $\alpha = (i_1, \ldots, i_n)$. We consider the operator

$$dL(C) = \sum_{\alpha \in J_m(d')} c_\alpha A^\alpha$$

with domain $L'_{p;m}(H)$ or $L'_{\hat{p};m}(H)$.

Next we want to introduce the concept of subcoercive form of step s, with $s \in \mathbf{N}$. Let $\mathfrak{g}(d', s)$ denote the nilpotent Lie algebra with d' generators which is free of step s, i.e., the quotient of the free Lie algebra with d' generators $\tilde{a}_1, \ldots, \tilde{a}_{d'}$ by the ideal generated by the commutators of order at least $s+1$. Further let $\tilde{G} = G(d', s)$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}(d', s)$ and $L_{\tilde{G}}$ left translations on $L_2(\tilde{G}; dg)$, where dg denotes left Haar measure on $\tilde{G}$. We say that C is an m-th order subcoercive form (of step s) if m is even and there exists $\mu > 0$ such that

$$\mathrm{Re}\,(dL_{\tilde{G}}(P)\varphi, \varphi) \geqslant \mu(N'_{2;m/2}(\varphi))^2$$

for all $\varphi \in L_{2;\infty}(\tilde{G}; dg)$. The largest such μ is called the *ellipticity constant of* C.

The main result of this paper is that if C is a subcoercive form of order m and step r, where r is the rank of the algebraic basis of the Lie algebra $\mathfrak{g}$ of the group G, and if $H = dL(C)$, with L acting on $L_p(G_1)$, then

$$L'_{p;n}(G_1) = D((\nu I + \overline{H})^{n/m}), \tag{1}$$

with equivalent norms, for all $n \in \mathbf{N}$, all large ν and all $p \in \langle 1, \infty \rangle$. A similar identification is valid on the $L_{\hat{p}}(G_1)$-spaces.

Finally note that if $\nu_0 \in \mathbf{R}$ is such that $\nu_0 I + \overline{H}$ generates a bounded semigroup and if (1) is valid for some $\nu \geqslant \nu_0$ then it is automatically valid for all $\nu \geqslant \nu_0$. This follows because $D((\nu I + \overline{H})^{n/m})$ is independent of the value of ν for all $\nu \geqslant \nu_0$, by [17] Lemma II.3.2. Moreover, the identity (1) for one $\nu \geqslant \nu_0$ implies the $L_p(G_1)$-boundedness of the operators $M_n(\nu I + H)^{-n/m}$, with M_n an n-th order monomial in the subelliptic derivatives A_i, for all $\nu > \nu_0$. But the analysis of the $(ax+b)$-group in [9] gives an example of a second-order operator which generates a contraction semigroup for which (1) is valid for $n = 1$ and $\nu \geqslant 0$ but $M_2(\nu I + H)^{-1}$ is not bounded for the critical value $\nu = 0$. Therefore the boundedness properties are more delicate.

2. REGULARITY OF THE LEFT REGULAR REPRESENTATION

In this section we prove that domains of the fractional powers of subcoercive operators associated with left translations of the group G acting on the $L_p(G_1)$-spaces, $p \in \langle 1, \infty \rangle$, of the larger group G_1 coincide with the corresponding C^n-vectors. We begin by observing that it suffices to establish this coincidence for the left differential operators on the space $L_{\hat{p}}(G_1)$.

First let C be a subcoercive form of order m and for $p \in \langle 1, \infty \rangle$ define the m-th order forms $C_{\pm p}$ by

$$C_{\pm p} = \sum_{\alpha \in J_m(d')} \sum_{\substack{\gamma \in J_m(d') \\ (\beta,\gamma) \in \mathrm{Lb}(\alpha)}} c_\alpha (\pm p)^{-|\gamma|} b^\gamma$$

where $\mathrm{Lb}(\alpha)$ is the set of all $(\beta, \gamma) \in J_m(d')^2$ such that β is a multi-index obtained from α by omission of some indices and γ is the multi-index formed by the omitted indices, i.e., the (β, γ) occurring are the pairs of multi-indices in the Leibniz formula for the multi-derivative A^α of a product. Then the principal parts of $C_\pm$ equal the principal part of C, so $C_{\pm p}$ are also subcoercive. In addition the map $C \mapsto C_p$ is invertible and $C = (C_p)_{-p}$. Since $\Delta^{-1/p} A_i \Delta^{1/p} = A_i + p^{-1} b_i I$ it follows that

$$\mathrm{d}L(C_p)\Delta^{-1/p}\varphi = \Delta^{-1/p}\mathrm{d}L(C)\varphi$$

for all $\varphi \in C_c^\infty(G_1)$. Thus if $H = \mathrm{d}L(C)$ and $H_p = \mathrm{d}L(C_p)$ on $L_p(G_1)$ one formally has the relation

$$H_p = \Delta^{-1/p} H \Delta^{1/p}$$

and this is the key to the first result.

LEMMA 2.1. *Let C be a subcoercive form of order m and step r, and $H = \mathrm{d}L(C)$ and $H_p = \mathrm{d}L(C_p)$ the corresponding operators associated with left translations L by the group G acting on the spaces $L_p(G_1)$ and $L_{\hat{p}}(G_1)$ with $p \in \langle 1, \infty \rangle$. Further let $n \in \mathbf{N}$. The following conditions are equivalent.*

I. *The spaces $L'_{p;n}(G_1)$ and $D((\nu I + \overline{H}_p)^{n/m})$ are equal, with equivalent norms, for some large $\nu > 0$.*

II. *The spaces $L'_{\hat{p};n}(G_1)$ and $D((\nu I + \overline{H})^{n/m})$ are equal, with equivalent norms, for some large $\nu > 0$.*

Proof. We only prove I⇒II since the proof of the other implication is almost identical but the map $C \to C_p$ is replaced by its inverse. Moreover, we assume that the real part of the zero-order coefficient of C is large and then we may take $\nu = 0$. We begin by proving that $D(\overline{H}^{n/m})$ is continuously embedded in $L'_{\hat{p};n}(G_1)$.

Let S and K denote the semigroup and kernel corresponding to H acting on $L_p(G_1)$ and S^p and K^p the pair corresponding to H_p. Arguing as in the proof of Corollary 3.5 of [7] it follows that $K_t(g) = \Delta^{-1/p}(g) K_t^p(g)$ for all $t > 0$ and $g \in G$. So $S_t\varphi = \Delta^{1/p} S_t^p \Delta^{-1/p}\varphi$ for all $t > 0$ and $\varphi \in C_c^\infty(G_1)$. Since

$$\overline{H}^{n/m}\varphi = c \int_c^\infty dt\, t^{-1-n/m} (I - S_t)^n \varphi$$

for all $\varphi \in D^\infty(\overline{H})$ (see, for example, [11]), where

$$c^{-1} = \int_0^\infty dt\, t^{-1-n/m}(I - e^{-t})^n,$$

with a similar expression for $\overline{H}_p^{n/m}$, it follows that

$$\Delta^{1/p}\overline{H}_p^{n/m}\Delta^{-1/p}\varphi = \overline{H}^{n/m}\varphi$$

for all $\varphi \in C_c^\infty(G_1)$

Finally, by assumption, one has bounds $\|\varphi\|'_{p;n} \leqslant c\|\overline{H}_p^{n/m}\varphi\|_p$ for $\varphi \in C_c^\infty(G_1)$ and hence

$$\begin{aligned}\|\varphi\|'_{\hat{p};n} \leqslant c'\|\Delta^{-1/p}\varphi\|'_{p;n} \leqslant cc'\|\overline{H}_p^{n/m}\Delta^{-1/p}\varphi\|_p \leqslant \\ \leqslant cc'\|\Delta^{-1/p}\overline{H}^{n/m}\varphi\|_p = cc'\|\overline{H}^{n/m}\varphi\|_{\hat{p}},\end{aligned}$$

for some $c' > 0$ and all $\varphi \in C_c^\infty(G_1)$. Since $C_c^\infty(G_1)$ is dense $L'_{\hat{p};\infty}(G_1)$ by [16] Theorem 1.3, it is a core for $\overline{H}^{n/m}$ and it follows that $D(\overline{H}^{n/m})$ is continuously embedded in $L'_{\hat{p};n}(G_1)$.

Similarly it follows that $L'_{\hat{p};n}(G_1)$ is continuously embedded in $D(\overline{H}^{n/m})$ since $L'_{\hat{p};\infty}(G_1)$ and hence $C_c^\infty(G_1)$ is dense in $L'_{\hat{p};n}(G_1)$ by [7] Lemma 2.4. ■

COROLLARY 2.2. *Let $p \in \langle 1, \infty\rangle$. The following are equivalent.*

I. *For any subcoercive form C of order m and step r, for all $n \in \mathbf{N}$, all large $\nu > 0$ and with $H = dL(C)$ the operator in L_p the spaces $L'_{p;n}$ and $D((\nu I + \overline{H})^{n/m})$ are equal with equivalent norms.*

II. *For any subcoercive form C of order m and step r, for all $n \in \mathbf{N}$, all large $\nu > 0$ and with $H = dL(C)$ the operator in $L_{\hat{p}}$ the spaces $L'_{\hat{p};n}$ and $D((\nu I + \overline{H})^{n/m})$ are equal with equivalent norms.*

The problem is now reduced to the examination of the left differential operators on the $L_{\hat{p}}(G_1)$-spaces. These operators automatically commute with right translations and as the measure is right-invariant this is useful for obtaining uniform estimates.

THEOREM 2.3. *Let $H = dL(C)$ be an m-th order subcoercive operator associated with left translations L by the group G acting on the spaces $L_{\hat{p}}(G_1)$. If $p \in \langle 1, \infty\rangle$ and $n \in \mathbf{N}$ then $D((\nu I + \overline{H})^{n/m}) = L'_{\hat{p};n}(G_1)$ for all large $\nu > 0$ and the spaces have equivalent norms. In particular, the operator H is closed.*

Similar statements are valid on the $L_p(G_1)$-spaces.

Proof. The proof is in several steps.

First we aim to establish that $D((\nu I+\overline{H})^{n/m}) = R((\nu I+\overline{H})^{-n/m})$ is continuously embedded in $L'_{\hat{p};n}(G_1)$ and this requires proving that $A^\alpha(\nu I+\overline{H})^{-n/m})$, with $|\alpha| = n$, is defined as a bounded operator on $L_{\hat{p}}(G_1)$. This is achieved by establishing that the operator and its adjoint are bounded on $L_{\hat{2}}(G_1)$ and are also bounded from $L_{\hat{1}}(G_1)$ to weak-$L_{\hat{1}}(G_1)$. Then the desired result is obtained by interpolation and duality. But the $L_{\hat{2}}(G_1)$-bounds follows from [6], Theorem 6.3.II, and the main onus of the proof is the derivation of the $L_{\hat{1}}(G_1)$-bounds.

The approach to the $L_{\hat{1}}(G_1)$-bounds begins by observing that

$$(\nu I + \overline{H})^{-n/m}\varphi = L(R_{\nu,n/m})\varphi$$

for an appropriate kernel $R_{\nu,n/m}$ over G where

$$L(f) = \int_G dg\, f(g)L(g)$$

with dg left Haar measure over G. But if $\alpha \in J_n(d')$, $|\alpha| = n$, then $A^\alpha R_{\nu,n/m}$ is not locally integrable and the $L_1(G)$-integral is logarithmically divergent at the identity. Therefore the idea is to use singular integration theory to prove the bound from $L_{\hat{1}}(G_1)$ to weak-$L_{\hat{1}}(G_1)$. Now a straightforward adaptation of the singular integration methods would begin by approximating $A^\alpha R_{\nu,n/m}$ with a sequence of functions obtained by excision of a decreasing family of neighbourhoods of the identity. Thus the convolution formally corresponding to the action of $A^\alpha(\nu I + \overline{H})^{-n/m}$ would be replaced by a principal value integral. But the problem with this approach is that it appears difficult to obtain suitable $L_{\hat{2}}(G_1)$-bounds for the sequence of approximating operators. Therefore we adopt a different type of approximation.

Fix $N \in \mathbf{N}$, $N > D'$ and for large $\nu > 0$ and all $j \in \mathbf{N}$ with $j > 2\nu$ consider the operators

$$X_j = j^N(jI + \overline{H})^{-N}(\nu I + \overline{H})^{-n/m}.$$

Then $A^\alpha X_j$ "approaches" $A^\alpha(\nu I + \overline{H})^{-n/m}$ as j tends to infinity. Therefore if the $A^\alpha X_j$ are bounded, uniformly in j, on $L_{\hat{p}}(G_1)$ one deduces that $(\nu I + \overline{H})^{-n/m}$ maps into the domain of A^α and the $A^\alpha(\nu I+\overline{H})^{-n/m}$ are bounded on $L_{\hat{p}}(G_1)$. The uniform bounds on the $A^\alpha X_j$ are obtained by following the above outline. In particular the bounds from $L_{\hat{1}}(G_1)$ to weak-$L_{\hat{2}}(G_1)$ use singular integration theory and as a prerequisite it is necessary to have uniform $L_{\hat{2}}(G_1)$-bounds on the approximate sequence.

First observe that if $j \in \mathbf{N}$ and $\alpha \in J_n(d')$ one can write $A^\alpha X_j = A^\alpha(\nu I + {} + \overline{H})^{-n/m} \circ j^N(jI+\overline{H})^{-N}$. But by Corollary 2.2 and [6] Theorem 6.3.II the operators $A^\alpha(\nu I + \overline{H})^{-n/m}$ are bounded on $L_{\hat{2}}(G_1)$. First they are bounded on $L_2(G_1)$ by [6] Theorem 6.3.II because the representation of G by left translations is unitary.

Then they are bounded on $L_{\hat{2}}(G_1)$ by Corollary 2.2. Moreover, since $\overline{H}$ generates a holomorphic semigroup, the operators $j^N(jI+\overline{H})^{-N}$ are bounded, uniformly for all large j, on $L_{\hat{2}}(G_1)$. Thus the operators $A^\alpha X_j$ are bounded on $L_{\hat{2}}(G_1)$, uniformly for all large j.

Secondly, remark that if $j \in \mathbf{N}$ then

$$j^N(jI+\overline{H})^{-N}\varphi = j^N L(R_{j,N})\varphi$$

and $(\nu I+\overline{H})^{-n/m}\varphi = L(R_{\nu,n/m})\varphi$, where

$$R_{j,N}(g) = \Gamma(N)^{-1}\int_0^\infty dt\, t^{N-1}e^{-jt}K_t(g)$$

with an analogous expression for $R_{\nu,n/m}$. Using the convolution property of the kernel K_t one then obtains

$$j^N(jI+\overline{H})^{-N}(\nu I+\overline{H})^{-n/m}\varphi = L(k_j)\varphi$$

where $k_j : G\setminus\{e\} \to \mathbf{C}$ is defined by

$$k_j(g) = \int_0^\infty dt\, f_j(t)K_t(g)$$

and

$$f_j(t) = j^N(N-1)!^{-1}\Gamma(n/m)^{-1}\int_0^t dx\, x^{N-1}e^{-jx}(t-x)^{n/m-1}e^{-\nu(t-x)}. \tag{2}$$

We need some estimates for $f_j(t)$.

LEMMA 2.4. *There exists an $a>0$ such that*

$$f_j(t) \leqslant a t^{n/m-1}(jt)^\mu e^{-\nu t}$$

uniformly for all $t>0$, $\nu>0$, $j\in\mathbf{N}$ with $j\geqslant 2\nu$ and $\mu\in[0,N]$.

Proof. A substitution $x = ty$ in (2) gives

(3)

$$f_j(t) = j^N(N-1)!^{-1}\Gamma(n/m)^{-1}t^{N+n/m-1}\int_0^1 dy\, y^{N-1}e^{-jty}(1-y)^{n/m-1}e^{-\nu t(1-y)} \leqslant$$

$$\leqslant (N-1)!^{-1}\Gamma(n/m)^{-1}t^{n/m-1}e^{-\nu t}(jt)^N\int_0^1 dy\, y^{N-1}(1-y)^{n/m-1}e^{-2^{-1}jty}.$$

Now define the function $g : [0,\infty\rangle \to \mathbf{R}$ by

$$g(x) = x^N \int_0^1 dy\, y^{N-1}(1-y)^{n/m-1} e^{-2^{-1}xy}.$$

We shall prove that g is bounded. In order to evaluate g we estimate the integral in two parts: over $\langle 0, 2^{-1}\rangle$ and over $\langle 2^{-1}, 1\rangle$. The first can be estimated as follows

$$\int_0^{2^{-1}} dy\, x^N y^{N-1}(1-y)^{n/m-1} e^{-2^{-1}xy} \leqslant \max(2^{1-n/m}, 1) \int_0^{2^{-1}} dy\, x^N y^{N-1} e^{-2^{-1}xy} =$$

$$= 2^N \max(2^{1-n/m}, 1) \int_0^{4^{-1}x} dt\, t^{N-1} e^{-t} \leqslant 2^N N! \max(2^{1-n/m}, 1).$$

Alternatively,

$$\int_{2^{-1}}^1 dy\, x^N y^{N-1}(1-y)^{n/m-1} e^{-2^{-1}xy} \leqslant x^N e^{-4^{-1}x} \int_{2^{-1}}^1 dy (1-y)^{n/m-1} =$$

$$= mn^{-1} 2^{-n/m} x^N e^{-4^{-1}x}.$$

So there exists an $a > 0$, depending only on N, such that

$$f_j(t) \leqslant a t^{n/m-1} e^{-\nu t}$$

uniformly for all $j \in \mathbf{N}$ and $t > 0$ with $j \geqslant 2\nu$. This proves the case $\mu = 0$.

The case $\mu = N$ follows from (3):

$$f_j(t) \leqslant (N-1)!^{-1} \Gamma(n/m)^{-1} t^{n/m-1} e^{\nu t} (jt)^N \int_0^1 dy\, y^{N-1}(1-y)^{n/m-1}.$$

The general case can be obtained by interpolation. ■

By the "Gaussian" bounds on K_t and its derivatives one deduces that k_j is infinitely differentiable on $G \setminus \{e\}$. Moreover, using Lemma 2.4 with $\mu = 0$ and with $\mathrm{Re}\, c_0$ large enough, then it follows by the argument in Theorem III.6.7 of [17] that for all $\alpha \in J_n(d')$ and $\beta \in J(d')$ there exist $a, b > 0$ such that

$$|(A^\beta A^\alpha k_j)(g)| \leqslant a(|g|')^{-D'-|\beta|} e^{-b\nu^{1/m}|g|'} \tag{4}$$

uniformly for all $g \in G \setminus \{e\}$ and $j \in \mathbf{N}$. Alternatively, using Lemma 2.4 with $\mu = N$, the inequality $N > D'$ and the argument in Theorem III.6.7 of [17], one deduces for large $\mathrm{Re}\, c_0$ that for all $j \in \mathbf{N}$, $j \geqslant 2\nu$, one has bounds

$$|(A^\alpha k_j)(g)| \leqslant c_j e^{-b\nu^{1/m}|g|'} \tag{5}$$

uniformly for $\alpha \in J_n(d')$ and $g \in G \setminus \{e\}$. So $k_j \in L'_{1;n}(G; e^{\rho|g|'}dg) \cap L_\infty(G; dg)$ for each $j \in \mathbf{N}$ with $j \geqslant 2\nu$, if ν is large enough, where $\rho > 0$ is such that $\Delta(g) \leqslant e^{\rho|g|'}$ for all $g \in G$.

Next note that $A^\alpha X_j \varphi = L(A^\alpha k_j)\varphi$ for all $\alpha \in J_n(d')$. So each operator $A^\alpha X_j$ is continuous on each of the $L_{\hat{p}}(G_1)$-spaces. In order to prove that the $A^\alpha X_j$ are uniformly continuous if $p \in \langle 1, \infty \rangle$ we have to consider two cases, $p \leqslant 2$ and $p \geqslant 2$.

Case 1: $p \in \langle 1, 2]$.
Let $\chi \in C_c^\infty(B'_2)$ with $\chi(g) = 1$ for all $g \in B'_1$ and $0 \leqslant \chi \leqslant 1$. Then

$$A^\alpha X_j \varphi = L(\chi A^\alpha k_j)\varphi + L((1-\chi)A^\alpha k_j)\varphi. \tag{6}$$

But

$$\sup_j \int_G dg |(1-\chi)(g)(A^\alpha k_j)(g)| e^{\rho|g|'} < \infty$$

if ν is large enough. Because of the bounds (4) the operators $\varphi \mapsto L((1-\chi)(A^\alpha k_j))\varphi$ are bounded on all the $L_{\hat{p}}(G_1)$-spaces, $p \in [1, \infty]$, uniformly for $j \in \mathbf{N}$ with $j \geqslant 2\nu$. In particular, this is the case for $p = 1$ and $p = 2$.

Next we prove a local weak-$L_{\hat{1}}(G_1)$ estimate for $A^\alpha X_j$, which is uniform in j. Because of the equality (6) it is sufficient to establish a local weak-$L_{\hat{1}}(G_1)$ estimate for the operator $\varphi \mapsto L(\chi A^\alpha k_j)\varphi$ which is uniform in j. We obtain this estimate by application of Theorem III.2.4 of [3] but since $L(\chi A^\alpha k_j)$ acts by convolution with respect to the subgroup G of G_1 some care has to be taken in applying the result.

Let $a_1, \ldots, a_{d'}, \ldots, a_d$ be a vector space basis for the Lie algebra $\mathfrak{g}$ of G obtained by completing the algebraic basis $a_1, \ldots, a_{d'}$. Further let $a_1, \ldots, a_d, \ldots, a_{d_1}$ be a vector space basis for the Lie algebra $\mathfrak{g}_1$ of G_1 obtained by completing the basis of G. Now G_1 and $G \times \mathbf{R}^{d_1 - d}$ are locally isomorphic. More precisely, define $\Phi : G \times \mathbf{R}^{d_1-d} \to$ $\to G_1$ by

$$\Phi(g, \xi_{d+1}, \ldots, \xi_{d_1}) = g \exp(\xi_{d+1} a_{d+1}) \cdots \exp(\xi_{d_1} a_{d_1}).$$

Next let $U \subset G$ and $V \subset \mathbf{R}^{d_1-d}$ be open bounded neighbourhoods of the identity and the origin. One may choose U and V such that Φ restricted to $U \times V$ is an analytic diffeomorphism from $U \times V$ onto an open neighbourhood Ω of the identity of G_1. If

U and V are small enough there exist $\delta, M > 0$ and a C^∞ function $\sigma : U \times V \to [\delta, M]$ such that

$$\int_\Omega d\hat{g}\, \varphi(g) = \int_U d\hat{g} \int_V d\xi\, \varphi(\Phi(g,\xi))\sigma(g,\xi)$$

for all $\varphi \in C_c(\Omega)$. We may assume that $U = B'_4$ and $V = \langle -4, 4\rangle^{d_1-d}$. One can then introduce $\chi_1 \in C_c^\infty(\Omega)$ such that if $\chi_2 = \chi_1 \circ \Phi$ then $\chi_2 = \chi_3 \otimes \chi_4$ for some $\chi_3 \in C_c^\infty(B'_2)$ and $\chi_4 \in C_c^\infty([-2,2])^{d_1-d})$ and, moreover, $0 \leqslant \chi_4 \leqslant 1$ and $\chi_2(g,\xi) = 1$ for all $(g,\xi) \in B'_1 \times [-1,1]^{d_1-d}$. Then for all $\varphi \in L_{\hat{p}}(G_1)$ one has

$$(L(\chi A^\alpha k_j)(\chi_1\varphi))(\Phi(g,\xi)) = \int_G dh(\chi A^\alpha k_j)(h)(L(h)(\chi_1\varphi))(\Phi(g,\xi)) =$$

$$= \int_G d\hat{h}(\chi A^\alpha k_j)(gh^{-1})\chi_3(h)\chi_4(\xi)\varphi(\Phi(h,\xi))$$

for (g,ξ)-almost everywhere in $U \times V$ and for all large j.

In order to prove suitable weak L_1-bounds we first restrict ourselves to the case $G = G_1$. Define $T_j : L_{\hat{p}}(B'_4) \to L_{\hat{p}}(B'_4)$ by

$$(T_j\varphi)(g) = \int_{B'_4} d\hat{h}(\chi A^\alpha k_j)(gh^{-1})\chi_3(h)\varphi(h)$$

for all $j \in \mathbf{N}$ with $j \geqslant 2\nu$. Then T_j has the form

$$(T_j\varphi)(g) = \int d\mu(h)\kappa_j(g,h)\varphi(h)$$

where $\kappa_j(g,h) = (\chi A^\alpha k_j)(gh^{-1})\chi_3(h)$ and μ denotes the restriction to B'_4 of the right Haar measure on G. Alternatively

$$T_j\varphi = (A^\alpha X_j)(\chi_3\varphi) - L((1-\chi)A^\alpha k_j)\chi_3\varphi)$$

for all $\varphi \in L_p(B'_4;\mu)$. Since we have already established that $A^\alpha X_j$ is bounded on $L_2(G; d\hat{g})$, uniformly in j, and since $\|\chi_3\varphi\|_{\hat{2}} \leqslant \|\varphi\|_{\hat{2}}$ it follows from the observation of the previous paragraph that $\sup_j \|T_j\|_{\hat{2}\to\hat{2}} < \infty$. This is the first condition of [3] for the T_j and it is uniform in the j.

Secondly, κ_j has support in $B'_4 \times B'_4$, and in fact $\kappa_j \in L_2(B'_4 \times B'_4; \mu\otimes\mu)$. This follows because $A^\alpha k_j \in L_\infty(G; dg)$ by (5). Finally, for the third and most difficult condition, it suffices to prove that

$$\sup_j \sup_{h,h_0\in B'_4} \int_{\Omega(h,h_0)} d\mu(g)|\kappa_j(g,h) - \kappa_j(g,h_0)| < \infty$$

where $\Omega(h, h_0) = \{g \in B_4' : \mathrm{d}(g, h_0) > 4d(h, h_0)\}$ and d is the subelliptic distance on G, defined by $\mathrm{d}(g, h) = |hg^{-1}|'$. Then by right invariance

$$\sup_j \sup_{h,h_0 \in B_4'} \int_{\Omega(h,h_0)} \mathrm{d}\mu(g)|\kappa_j(g, h) - \kappa_j(g, h_0)| \leqslant$$

$$\leqslant \sup_j \sup_{h,h_0 \in B_4'} \int_{\Omega_1(h,h_0)} \mathrm{d}\mu(g)|\kappa_j(gh_0, h) - \kappa_j(gh_0, h_0)|,$$

where $\Omega_1(h, h_0) = \{g \in B_8' : |g|' > 4|hh_0^{-1}|'\}$ and μ also denotes the restriction to B_8' of the right Haar measure on G. For $a_i \in \mathfrak{g}$ let $\widetilde{X_i}$ be the corresponding *right* invariant vector field on G. So

$$(\widetilde{X_i}\psi)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(\exp(ta_i)g)\Big|_{t=0}$$

for all $\psi \in C_c^\infty(G)$. Now let $h, h_0 \in G$ and choose an absolutely continuous path $\omega : [0, 1] \to G$ from h_0 to h with tangential coordinates in the directions $\widetilde{X_i}$, i.e.,

$$\dot\omega(t) = \sum_{i=1}^{d'} \omega_i(t)\widetilde{X_i}\Big|_{\omega(t)},$$

such that

$$\int_0^1 \mathrm{d}t \left(\sum_{i=1}^{d'} \omega_i(t)^2\right)^{1/2} \leqslant 2d(h, h_0).$$

Then

$$|\kappa_j(gh_0, h) - \kappa_j(gh_0, h_0)| \leqslant \int_0^1 \mathrm{d}t \left|\frac{\mathrm{d}}{\mathrm{d}t}\kappa_j(gh_0, \omega(t))\right|.$$

Now if $\varphi \in C_c^\infty(G)$, $k \in G$ and $\psi(g) = \varphi(kg^{-1})$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\omega(t)) = \sum_{i=1}^{d'} \omega_i(t)(L(k\omega(t)^{-1})A_i L((k\omega(t)^{-1})^{-1})\varphi)(k\omega(t)^{-1}).$$

By [6], Lemma 7.3, there exist functions $c_{i,\beta} : G \to \mathbf{R}$, where $i \in \{1, \ldots, d'\}$ and $\beta \in J_r(d')$, and constants M_1, $\sigma > 0$ such that

$$L(g^{-1})A_i L(g) = \sum_{\substack{\beta \in J_r(d') \\ |\beta| \neq 0}} c_{i,\beta}(g)A^\beta$$

with $|c_{i,\beta}(g)| \leqslant M_1(|g|')^{|\beta|-1}e^{\sigma|g|'}$ for all $g \in G$, $i \in \{1, \ldots, d'\}$ and $\beta \in J_r(d')$, $|\beta| \neq$ $\neq 0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(\omega(t)) = \sum_{i=1}^{d'} \sum_{\substack{\beta \in J_r(d') \\ |\beta| \neq 0}} \omega_i(t)c_{i,\beta}(k\omega(t)^{-1})(A^\beta\varphi)(k\omega(t)^{-1}).$$

Moreover, for all $t \in [0,1]$ and $g \in \Omega_1(h, h_0)$ one has

$$|gh_0\omega(t)^{-1}|' \geqslant |g|' - \mathrm{d}(h_0, \omega(t)) \geqslant |g|' - 2\mathrm{d}(h_0, h) \geqslant 2^{-1}|g|'.$$

Combining these two observations with the bounds (4) one obtains for all $g \in \Omega_1(h, h_0)$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} k_j(gh_0, \omega(t))\right| \leqslant \sum_{i=1}^{d'} |\omega_i(t)| \cdot |(\chi A^\alpha k_j)(gh_0\omega(t)^{-1})(\widetilde{X}_i\chi_3)(\omega(t))| +$$

$$+ \sum_{i=1}^{d'} \sum_{\substack{\beta \in J_r(d') \\ |\beta| \neq 0}} |\omega_i(t)| \cdot |c_{i,\beta}(gh_0\omega(t)^{-1})| \cdot |(A^\beta(\chi A^\alpha k_j))(gh_0\omega(t)^{-1})| \cdot |\chi_3(\omega(t))| \leqslant$$

$$\leqslant \left(\sum_{i=1}^{d'} |\omega_i(t)|\right) \Bigg(a 2^{D'} (|g|')^{-D'} \sum_{i=1}^{d'} \|\widetilde{X}_i\chi_3\|_\infty +$$

$$+ \sum_{\substack{\beta \in J_r(d') \\ |\beta| \neq 0}} \sum_{(\gamma,\delta) \in \mathrm{Lb}(\beta)} M_1 (2|g|')^{|\beta|-1} e^{2\sigma|g|'} a (2^{-1}|g|')^{-D'-|\gamma|} \|A^\delta \chi\|_\infty \|\chi_3\|_\infty \Bigg) \leqslant$$

$$\leqslant M_2 (|g|')^{-D'-1} \sum_{i=1}^{d'} |\omega_i(t)|.$$

Hence

$$|\kappa_j(gh_0, h)) - \kappa_j(gh_0, h_0))| \leqslant (d')^{1/2} M_2 d(h_0, h) (|g|')^{-D'-1}.$$

But if $c = \sup_{t \in \langle 0,8]} t^{-D'}|B_t'|$, $s = d(h, h_0)$ and $N_s \in \mathbb{N}_0$ is such that $2^{N_s - 1} \leqslant s^{-1} \leqslant 2^{N_s}$ then we obtain

$$\int_{B_8' \setminus B_{4s}'} d\hat{g}\, s(|g|')^{-D'-1} \leqslant \sum_{n=0}^{N_s} \int_{B'_{2^{-n+3}} \setminus B'_{2^{-n+2}}} d\hat{g}\, s(|g|')^{-D'-1} \leqslant$$

$$\leqslant \sum_{n=0}^{N_s} cs(2^{-n+2})^{-D'-1}(2^{-n+3})^{D'} = 2^{D'-2} cs(2^{N_s+1} - 1) \leqslant 2^{D'} c.$$

Hence

$$\int_{\Omega_1(h,h_0)} d\hat{g}\, |\kappa_j(gh_0, h)) - \kappa_j(gh_0, h_0))| \leqslant 2^{D'} c (d')^{1/2} M_2,$$

which is the third condition of Theorem III.2.4 of [3], uniform in j.

Now we can use this latter theorem to deduce that there exists $M_3 > 0$, independent of j, such that

$$\mu(\{g \in B_4' : |(T_j\varphi)(g)| > \gamma\}) \leqslant M_3 \gamma^{-1} \|\varphi\|_1$$

for all $\varphi \in L_1(B'_4; d\hat{g}) \cap L_2(B'_4; d\hat{g})$ and $\gamma > 0$.

Next we drop the restriction that $G = G_1$ and extend the last bounds to G_1. Let μ_2 denote the product measure of μ and the Lebesgue measure on $\mathbf{R}^{d_1-d}$. Then with $\varphi^\xi(g) = \varphi(\Phi(g, \xi))$ one has

$$\mu_2(\{(g,\xi) \in U \times V : |((L(\chi A^\alpha k_j))(\chi_3\varphi))(\Phi(g,\xi))| > \gamma\}) =$$
$$= \mu_2(\{(g,\xi) \in U \times V : \chi_4(\xi)|(T_j\varphi^\xi)(g)| > \gamma\}) \leqslant$$
$$\leqslant \int\limits_{[-4,4]^{d_1-d}} d\xi \, \mu(\{g \in B'_4 : |(T_j\varphi^\xi)(g)| > \gamma\}) \leqslant M_3 \int\limits_{[-4,4]^{d_1-d}} d\xi \int\limits_{B'_4} d\hat{g} \, |\varphi^\xi(g)|\gamma^{-1} \leqslant$$
$$\leqslant M_3\delta^{-1} \int\limits_{[-4,4]^{d_1-d}} d\xi \int\limits_{B'_4} d\hat{g} \, |\varphi(\Phi(g,\xi))|\sigma(g,\xi)\gamma^{-1} = M_3\delta^{-1}\|\varphi\|_{\hat{1}}\gamma^{-1}$$

for all $\varphi \in L_1(\Omega; d\hat{g})$. In particular, if $\varphi \in L_1(G_1; \mu_1) \cap L_2(G_1; \mu_1)$, where we now use μ_1 to denote the right Haar measure on G_1, with $\operatorname{supp}\varphi \subset \Omega' = \Phi(B'_1 \times [-1,1]^{d_1-d})$, then there exists $c > 0$ such that

$$\mu_1(\{g \in G_1 : |(L(\chi A^\alpha k_j)\varphi)(g)| > \gamma\}) \leqslant c\|\varphi\|_{\hat{1}}\gamma^{-1}$$

with c independent of j.

Next for $j \in \mathbf{N}$, $j \geqslant 2\nu$ define $P_j : L_{\hat{1}}(G_1) \to L_{\hat{1}}(G_1)$ by

$$P_j\varphi = L(\chi A^\alpha k_j)\varphi.$$

Obviously each P_j is continuous by the estimates (5). It follows that

$$\mu_1(\{g \in G_1 : |(P_j\varphi)(g)| > \gamma\}) \leqslant c\gamma^{-1}\|\varphi\|_{\hat{1}}$$

for all $\varphi \in L_1(\Omega'; \mu_1) \cap L_2(\Omega'; \mu_1)$ and $\gamma > 0$.

Moreover, for all $k \in G_1$ and $\varphi \in L_1(G_1; \mu_1)$, one has

$$R(k)L(\chi A^\alpha k_j)\varphi = L(\chi A^\alpha k_j)R(k)\varphi,$$

where R denotes right translations. Therefore

$$\mu_1(\{g \in G_1 : |(P_j\varphi)(g)| > \gamma\}) \leqslant c\|\varphi\|_{\hat{1}}\gamma^{-1} \tag{7}$$

for all $j \in \mathbf{N}$, $\gamma > 0$ and all $\varphi \in L_1(G_1; \mu_1)$ such that $\operatorname{supp}\varphi \subset \Omega' k$ for some $k \in G_1$. It then follows by a finite covering argument that a similar estimate is valid for all $\varphi \in L_1(\Omega; \mu_1)$ and for each bounded open neighbourhood Ω of e. So if G_1 is compact it follows that the P_j satisfy a global weak-$L_{\hat{1}}$ estimate. However, we need a global bound also if G_1 is not compact.

Next we establish that the operators P_j satisfy a global weak-$L_{\hat{1}}$ estimate if G_1 is not compact by use of the following covering lemma, [2] Lemma 3.2.7 (see also [15] page 66).

LEMMA 2.5. *Suppose G_1 is not compact and let B_ε denote the ball $\{g_1 \in \in G_1; |g_1| < \varepsilon\}$ relative to a fixed modulus on G_1. Given $\varepsilon > 0$, there is a sequence $g_1, g_2, \ldots$, of points in G_1 such that*

$$G_1 = \bigcup_{i=1}^{\infty} B_\varepsilon g_i$$

and the additional two properties are valid.

I. *There is an $N_1 \in \mathbf{N}$ such that each $g \in G_1$ lies in at most N_1 balls $B_\varepsilon g_i$.*

II. *Given $\delta > 0$ there is an $N_2 \in \mathbf{N}$ such that each $g \in G_1$ lies in at most N_2 of the balls $B_{\varepsilon+\delta} g_i$.*

Proof. The existence of a covering sequence with the first property has been established by Pier (see [15] page 66). The second property is established as follows.

Fix $g \in G$ and let $\mathcal{I}$ denote the set of indices i such that $B_\varepsilon g_i \subseteq B_{2\varepsilon+\delta} g$. Further let m_j denote the μ_1-measure of the set of $h \in B_{2\varepsilon+\delta}$ such that h lies in exactly j of the balls $B_\varepsilon g_i$ with $i \in \mathcal{I}$. Then

$$\sum_{i \in \mathcal{I}} \mu_1(B_\varepsilon g_i) = \sum_{j=1}^{N_1} j m_j$$

since any point of $B_{2\varepsilon+\delta}$ is contained in at most N_1 of the balls $B_\varepsilon g_i$. But

$$\sum_{j=1}^{N_1} m_j \leqslant \mu_1(B_{2\varepsilon+\delta}).$$

Therefore if k denotes the number of indices in $\mathcal{I}$ then

$$k\mu_1(B_\varepsilon) = \sum_{i \in \mathcal{I}} \mu_1(B_\varepsilon g_i) \leqslant N_1 \mu_1(B_{2\varepsilon+\delta})$$

and k has the g-independent bound

$$k \leqslant N_1 \mu_1(B_{2\varepsilon+\delta}) / \mu_1(B_\varepsilon).$$

Finally suppose that $h \in G$ lies in l balls $B_{\varepsilon+\delta} g_i$ then $B_\varepsilon g_i \subseteq B_{2\varepsilon+\delta} h$ for each such ball. Hence one may choose

$$N_2 = N_1 \mu_1(B_{2\varepsilon+\delta}) / \mu_1(B_\varepsilon)$$

independent of the choice of g. ∎

Now apply the lemma with an $\varepsilon > 0$ such that $B_\varepsilon \subseteq \Omega'$ and $\delta > 0$ such that $\Phi(B'_3 \times [-1,1]^{d_1-d}) \subseteq B_{\varepsilon+\delta}$. Choose a partition of the unity $(\psi_i)_i$ relative to the cover $G_1 = \bigcup_{i=1}^{\infty} B_\varepsilon g_i$, i.e., $\operatorname{supp}\psi_i \subseteq B_\varepsilon g_i$. Then for all $j \in \mathbb{N}$ with $j \geqslant 2\nu$ and $\varphi \in L_1(G_1;\mu_1)$ one has $\varphi = \sum_{i=1}^{\infty} \psi_i\varphi$ in $L_1(G_1;\mu_1)$. Then by the continuity of P_j

$$P_j\varphi = \sum_{i=1}^{\infty} P_j(\psi_i\varphi) = \sum_{i=1}^{\infty} R(g_i^{-1})P_jR(g_i)(\psi_i\varphi).$$

Moreover, $\operatorname{supp} R(g_i)(\psi_i\varphi) \subseteq B_\varepsilon$ and $\operatorname{supp} R(g_i^{-1})P_jR(g_i)(\psi_i\varphi) \subseteq B_{\varepsilon+\delta}g_i$, so each $g \in G_1$ lies in the support of at most N_2 functions $R(g_i^{-1})P_jR(g_i)(\psi_i\varphi)$. Therefore we obtain by (7) that

$$\begin{aligned}
&\mu_1(\{g \in G_1 : |(P_j\varphi)(g)| > \gamma\}) \leqslant \\
&\leqslant \sum_{i=1}^{\infty} \mu_1(\{g \in G_1 : |(R(g_i^{-1})P_jR(g_i)(\psi_i\varphi))(g)| > \gamma N_2^{-1}\}) \leqslant \\
&\leqslant \sum_{i=1}^{\infty} c\gamma^{-1}N_2\|\psi_i\varphi\|_{\hat{1}} = c\gamma^{-1}N_2\|\varphi\|_{\hat{1}}.
\end{aligned}$$

Thus the operators P_j satisfy a global weak-$L_{\hat{1}}$ estimate for any Lie group G_1, uniformly in j. Hence the operators $A^\alpha X_j$ also satisfy a global weak-$L_{\hat{1}}$ estimate, uniformly in j. By interpolation one deduces that the operators $A^\alpha X_j$ are uniformly bounded on the $L_{\hat{p}}(G_1)$-spaces, with $p \in \langle 1,2]$ and $\alpha \in J_n(d')$.

Next we prove by induction that

$$D((\nu I + \overline{H})^{n/m}) \subseteq L'_{\hat{p};k}(G_1)$$

for all $k \in \{0,\ldots,n\}$ and that the inclusion is continuous. The case $k = 0$ is trivial. Let $\alpha \in J_{n-1}(d')$, $i \in \{1,\ldots,d'\}$ and suppose that $D((\nu I + \overline{H})^{n/m})$ is continuously embedded in $L_{\hat{p};|\alpha|}(G_1)$. Then there exists a $c > 0$ such that $\|\varphi\|'_{\hat{p};|\alpha|} \leqslant c\|(\nu I + \overline{H})^{n/m}\varphi\|_{\hat{p}}$ for all $\varphi \in D((\nu I + \overline{H})^{n/m})$. Let $p \in \langle 1,2]$ and $\varphi \in L_{\hat{p}}(G_1)$. Then for all $j \in \mathbb{N}$ with $j \geqslant 2\nu$ one obtains the estimate

$$\begin{aligned}
&\|A^\alpha(\nu I + \overline{H})^{-n/m}\varphi - A^\alpha j^N(jI + \overline{H})^{-N}(\nu I + \overline{H})^{-n/m}\varphi\|_{\hat{p}} \leqslant \\
&\leqslant c\|(\nu I + \overline{H})^{n/m}((\nu I + \overline{H})^{-n/m}\varphi - j^N(jI + \overline{H})^{-N}(\nu I + \overline{H})^{-n/m}\varphi)\|_{\hat{p}} = \\
&= c\|(I - j^N(jI + \overline{H})^{-N})\varphi\|_{\hat{p}}.
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} A^\alpha j^N(jI + \overline{H})^{-N}(\nu I + \overline{H})^{-n/m}\varphi = A^\alpha(\nu I + \overline{H})^{-n/m}\varphi$$

in the $L_{\hat{p}}(G_1)$-sense. Now let $M > 0$ be such that $\|A^\alpha X_j\|_{\hat{p}\to\hat{p}} \leqslant M$ for all $j \in \mathbf{N}$, $j \geqslant 2\nu$, $1 < p \leqslant 2$ and $\alpha \in J_n(d')$. Then for all $\psi \in D(A_i^*) \subseteq L_{\hat{q}}(G_1)$, where q is the conjugate to p, one obtains:

$$|\langle A_i^*\psi, A^\alpha(\nu I + \overline{H})^{-n/m}\varphi\rangle| = \lim_{j\to\infty} |\langle A_i^*\psi, A^\alpha j^N(jI + \overline{H})^{-N}(\nu I + \overline{H})^{-n/m}\varphi\rangle| =$$

$$= \lim_{j\to\infty} |\langle \psi, A_i A^\alpha X_j\varphi\rangle| \leqslant M\|\psi\|_{\hat{q}}\|\varphi\|_{\hat{p}}.$$

Hence $A^\alpha(\nu I+\overline{H})^{-n/m}\varphi \in D((A_i^{**}) = D(A_i)$ and $\|A^i A^\alpha(\nu I+\overline{H})^{-n/m}\varphi\|_{\hat{p}} \leqslant M\|\varphi\|_{\hat{p}}$.

Case 2. $p \in [2,\infty\rangle$.

For all $\alpha \in J_{n-1}(d')$, $i \in \{1,\ldots,d'\}$, $j \in \mathbf{N}$ with $j \geqslant 2\nu$, $\varphi \in L_{\hat{p}}(G_1)$ and $\psi \in D(A_i^*) \subset L_{\hat{q}}(G_1)$, where q is the conjugate to p, one has

$$\langle A_i^*\psi, A^\alpha X_j\varphi\rangle = \langle\psi, A_i A^\alpha X_j\varphi\rangle = \langle\psi, L(A_i A^\alpha k_j)\varphi\rangle = \langle L(\overline{(A_i A^\alpha k_j)\check{}})\psi, \varphi\rangle,$$

where $\tau\check{}(g) = \tau(g^{-1})$. Now $q \in \langle 1, 2]$ since $p \in [2,\infty\rangle$. Because

$$\|(L(\overline{(A_i A^\alpha k_j)\check{}})\psi\|_{\hat{2}} = \|(A_i A^\alpha X_j)^*\psi\|_{\hat{2}} \leqslant \|A_i A^\alpha X_j\|_{\hat{2}\to\hat{2}}\|\psi\|_{\hat{2}}$$

it follows that the operators $\psi \mapsto L(\overline{(A_i A^\alpha k_j)\check{}})\psi$ are bounded on $L_{\hat{2}}(G_1)$ uniformly in j. Moreover, if $\mathrm{Re}\, c_0$ is large, then for all $\beta \in J(d')$ one has bounds

$$|(A^\beta\overline{(A_i A^\alpha k_j)\check{}})(g)| \leqslant a(|g|')^{-D'-|\beta|}e^{-b\nu^{1/m}|g|'}$$

because of the inequalities (4). Therefore, arguing as above, it follows that the operators $(A_i A^\alpha X_j)^*$ are uniformly bounded on $L_{\hat{q}}(G_1)$. Finally by repetition of the foregoing induction argument one deduces that $D((\nu I + \overline{H})^{n/m})$ is continuously embedded in $L_{\hat{p};n}(G_1)$ for all $p \in [2,\infty\rangle$.

The next step in the proof consists of establishing the converse inclusion, $L'_{\hat{p};n}(G_1) \subseteq D((\nu I + \overline{H})^{n/m})$.

First suppose that $n \in \{1,\ldots,m-1\}$. We may assume that the real part of the zero-order coefficient of C is sufficiently large that $\overline{H}$ has a bounded inverse. Let $C_i : J_m(d') \to \mathbf{C}$ be the form defined by

$$C_1 = \sum_{\alpha\in J_m(d')} \sum_{\substack{\gamma\in J_m(d') \\ (\beta,\gamma)\in \mathrm{Lb}(\alpha)}} c_\alpha b^\gamma$$

and let $H_1^\dagger = dL(C_1^\dagger)$. So $\langle\psi, H\varphi\rangle = \langle H_1^\dagger\psi, \varphi\rangle$ for all smooth enough φ and ψ. Next, for all $\alpha \in J_m(d')$ let $\alpha' \in J_{m-n}(d')$ and $\alpha'' \in J_n(d')$ be such that $\alpha = \langle\alpha', \alpha''\rangle$. By the first part of the proof of this theorem there exists $c > 0$ such that

$$\|\psi\|'_{\hat{q};m-n} \leqslant c\|(\overline{H_1^\dagger})^{(m-n)/m}\psi\|_{\hat{q}}$$

for all $\psi \in C_c^\infty(G_1)$. Then for all $\varphi, \psi \in C_c^\infty(G_1)$ one obtains

$$|\langle\psi,\varphi|\rangle = |n\psi, \overline{H}^{-(m-n)/m} H \overline{H}^{-n/m}\varphi\rangle| =$$

$$= \left|\sum_{\alpha\in J_m(d')} c_\alpha \langle (A^{\alpha'})^* (\overline{H_1^\dagger})^{-(m-n)/m}\psi, A^{\alpha''}\overline{H}^{-n/m}\varphi\rangle\right| =$$

$$= \left|\sum_{\alpha\in J_m(d')} c_\alpha(-1)^{|\alpha'|} \sum_{(\beta,\gamma)\in Lb(\alpha')} b^\gamma \langle (A^{\beta_*}(\overline{H_1^\dagger})^{-(m-n)/m}\psi, A^{\alpha''}\overline{H}^{-n/m}\varphi\rangle\right| \leqslant$$

$$\leqslant c \sum_{\alpha\in J_m(d')} |c_\alpha| \sum_{(\beta,\gamma)\in Lb(\alpha')} b^\gamma \|\psi\|_{\hat{q}} \|\overline{H}^{-n/m}\varphi\|'_{\hat{p};n}.$$

Hence $\|\varphi\|_{\hat{p}} \leqslant c'\|\overline{H}^{-n/m}\varphi\|'_{\hat{p};n}$ for all $\varphi \in C_c^\infty(G_1)$ for some $c' > 0$ and, by density, for all $\varphi \in L_{\hat{p}}(G_1)$. So $\|\overline{H}^{n/m}\varphi\|_{\hat{p}} \leqslant c'\|\varphi\|'_{\hat{p};n}$ for all $\varphi \in D(\overline{H}^{n/m})$. Since $L'_{\hat{p};\infty}(G_1)$ and hence $D(\overline{H}^{n/m})$ is dense in $L'_{\hat{p};n}(G_1)$, see [7] Lemma 2.4, it follows that $L'_{\hat{p};n}(G_1)$ is continuously embedded in $D(\overline{H}^{n/m})$.

Finally we consider the case $n \geqslant m$. Write $n = Nm + k$ with $N \in \mathbf{N}$ and $k \in \{0, \ldots, m-1\}$. There exists $c > 0$ such that $\|\overline{H}^{k/m}\varphi\|_{\hat{p}} \leqslant c\|\varphi\|'_{\hat{p};k}$ for all $\varphi \in C_c^\infty(G_1)$. Then

$$\|\overline{H}^{k/m}\varphi\|_{\hat{p}} = \|\overline{H}^{k/m} H^N\varphi\|_{\hat{p}} \leqslant c\|H^N\varphi\|'_{\hat{p};k}$$

for all $\varphi \in C_c^\infty(G_1)$. But H^N is an operator of order Nm. So

$$\|\overline{H}^{n/m}\varphi\|_{\hat{p}} \leqslant c'\|\varphi\|'_{\hat{p};k+Nm} = c'\|\varphi\|'_{\hat{p};n}$$

for all $\varphi \in C_c^\infty(G_1)$. Again, since $C_c^\infty(G_1)$ is dense in $L'_{\hat{p};n}(G_1)$ it follows that $L'_{\hat{p};n}(G_1)$ is continuously embedded in $D(\overline{H}^{n/m})$. This completes the proof of the theorem. ■

One can immediately deduce from the theorem a characterization of the C^n--elements associated with a finite sequence $a_1, \ldots, a_{d'}$ of elements of $\mathfrak{g}$. Let $\mathfrak{g}'$ be the Lie subalgebra of $\mathfrak{g}$ generated by $a_1, \ldots, a_{d'}$. If G' is the connected subgroup of G with Lie algebra $\mathfrak{g}'$ one can apply the theorem with G and G_1 replaced by G' and G, respectively.

Corollary 2.6. *Let $a_1, \ldots, a_{d'}$ be elements of the Lie algebra $\mathfrak{g}$ of a connected Lie group G and $L'_{p;n}(G)$, $L'_{\hat{p};n}(G)$ the corresponding C^n-subspaces. Then*

$$L'_{p;n}(G) = \bigcap_{i=1}^{d'} D(A_i^n)$$

for all $p \in \langle 1, \infty\rangle$ and $n \in \mathbf{N}$. Similar identities are valid for the $L'_{\hat{p};n}(G)$-spaces.

Proof. We may assume that $a_1, \ldots, a_{d'}$ are linearly independent. Let C_{2n} be the form such that $\mathrm{d}L(C_{2n}) = (-1)^n \sum_{i=1}^{d'} A_i^{2n}$. Let $\varphi \in \bigcap_{i=1}^{d'} D(A_i^n) \subset L_p(G)$. Let $c_1 = \sum_{i=1}^{d'} \|A_i^n \varphi\|_p + \|\varphi\|_p$. Then for all $\psi \in L'_{q;\infty}(G)$

$$|((\mathrm{d}U(C_{2n}) + I)\psi, \varphi)| = \left|(-1)^n \left(\sum_{i=1}^{d'} A_i^{2n}\psi, \varphi\right) + (\psi, \varphi)\right| =$$

$$= \left|\sum_{i=1}^{d'} (A_i^n \psi, A_i^n \varphi) + (\psi, \varphi)\right| \leqslant c_1 \|\psi\|'_{q;n}.$$

By Theorem 2.3, with G and G_1 replaced by G' and G, respectively, there exists $c_2 > 0$ such that

$$\|\psi\|'_{q;n} \leqslant c_2 \|(\mathrm{d}L(C_{2n}) + I)^{1/2}\psi\|_q$$

for all $\psi \in L'_{q;\infty}(G)$. Since $(\mathrm{d}L(C_{2n}) + I)^{1/2}$ maps $L'_{q;\infty}(G)$ onto $L'_{q;\infty}(G)$ it follows that

$$|((\mathrm{d}L(C_{2n}) + I)^{1/2}\psi, \varphi)| \leqslant c_1 c_2 \|\psi\|_q$$

for all $\psi \in L'_{q;\infty}(G)$ and, by continuity, for all $\psi \in D((\mathrm{d}L(C_{2n}) + I)^{1/2})$. So $\varphi \in$ $\in D(((\mathrm{d}L(C_{2n}) + I)^{1/2})^*) = D((\mathrm{d}L(C_{2n}) + I)^{1/2}) = L'_{p;n}(G)$ by Theorem 2.3 again.

The proof for $L'_{\hat{p};n}(G)$ is nearly the same but a minor complications occurs because of the modular function. This can be handled as before. ■

The theorem and the corollary can be combined to give a variety of other statements. For example, if

$$H = -\sum_{i=1}^{d'} A_i^2$$

is the sublaplacian formed from the left derivatives associated with the general subbasis $a_1, \ldots, a_{d'}$ then

$$D((\nu I + H)^{n/2}) = \bigcap_{i=1}^{d'} D(A_i^n)$$

on each L_p-space with $p \in \langle 1, \infty \rangle$, for all $\nu \geqslant 0$. In particular, if $d' = 1$, and one sets $A_i = A$ and $\nu = 0$, then

$$D(|A|^n) = D(A^n)$$

for all $n \in \mathbf{N}$ where the modulus of A is defined by $|A| = (-A^2)^{1/2}$.

The situation on the $L_{\hat{p}}$-spaces is slightly more complicated. But one finds that

$$D((\nu I + H)^{n/2}) = \bigcap_{i=1}^{d'} D(A_i^n)$$

on each $L_{\hat{p}}$-space with $p \in \langle 1, \infty \rangle$, for all $\nu \geqslant b^2/p^2$ where $b = \left(\sum_{i=1}^{d'} (A_i \Delta)(e)^2 \right)^{1/2}$.

The foregoing argument with G and G' can be used to extend earlier results on unitary representations. One has the direct analogue of the foregoing corollary and theorem.

Corollary 2.7. *Let $(\mathcal{H}, G, U)$ be a unitary representation, $a_1, \ldots, a_{d'}$ elements of the Lie algebra $\mathfrak{g}$ of the Lie group G and $A_i = dU(a_i)$ the corresponding generators. Further let $\mathcal{H}'_n$ denote the C^n-subspaces associated with $A_1, \ldots, A_{d'}$ and set*

$$H = -\sum_{i,j=1}^{d'} c_{ij} A_i A_j + \sum_{i=1}^{d'} c_i A_i$$

where c_{ij}, $c_i \in \mathbf{C}$ and the real part $2^{-1}(C + C^*)$ of the matrix $C = (c_{ij})$ is strictly positive-definite.

Then

$$\mathcal{H}'_n = \bigcap_{i=1}^{d'} D(A_i^n) = D((\nu I + H)^{n/2})$$

for all $n \in \mathbf{N}$ and $\nu \geqslant 0$.

The corollary is a direct consequence of [6], Theorem 6.3, applied to the unitary representation $(\mathcal{H}, G', U')$ where G' is defined as above and $U' = U|_{G'}$. More general statements are possible in terms of higher-order subelliptic operators.

For general representations one has the following extension of [6] Corollary 6.2. If $a_1, \ldots, a_{d'}$ is a basis for the Lie algebra this result reproduces Theorem 1.1 of [10].

Corollary 2.8. *Let $(\mathcal{X}, G, U)$ be a strongly continuous, or weakly*-continuous, representation of G on a Banach space $\mathcal{X}$, $a_1, \ldots, a_{d'}$ elements of the Lie algebra $\mathfrak{g}$ of the Lie group G and $A_i = dU(a_i)$ the corresponding generators. Then*

$$\mathcal{X}'_\infty = \bigcap_{i=1}^{d'} D^\infty(A_i).$$

Next we consider homogeneous spaces for which the subgroup is compact.

Theorem 2.9. *Let K be a compact subgroup of a unimodular connected group G_1 and let μ be a left invariant measure on the homogeneous space G/K. Let G be* *a subgroup of G_1. Let $p \in \langle 1, \infty \rangle$ and let U be the left regular representation of G in $\mathcal{X} = L_p(G_1/K; \mu)$. If $a_1, \ldots, a_{d'}$ is an algebraic basis of the Lie algebra $\mathfrak{g}$ of G and $C : J_m(d') \to \mathbf{C}$ a subcoercive form of order m and step r then for $H = dU(C)$ one has*

$$D((\nu I + \overline{H})^{n/m}) = \mathcal{X}'_n$$

for each $n \in \mathbf{N}$ and all large ν, with equivalent norms.

Proof. Consider the corresponding problem in $L_p(G_1; dg)$. If X_j is the operator on $L_p(G_1, dg)$ as in the proof of Theorem 2.3 and $X_j^\flat$ is the corresponding convolution operator on $\mathcal{X} = L_p(G_1/K; \mu)$, then the $A^\alpha X_j$ satisfy a weak L_1-estimate uniformly in j, so since K is compact it immediately follows that also the $A^\alpha X_j^\flat$ satisfy a weak L_1-estimate on the homogeneous space, uniformly in j. Since U is a unitary representation if $p = 2$, the theorem is valid for $p = 2$ by [6] Theorem 6.3.II. Hence by interpolation and a similar approximation to that used in the proof of Theorem 2.3 the result follows for $p \in \langle 1, 2]$. But the same argument also works for $(A^\alpha X_j^\flat)^*$ and hence the result for $p \in [2, \infty\rangle$ follows by duality.

3. CONCLUSION

The characterization of the differential structure given by Theorem 2.3 is related to the Lie group version of the boundedness of the Riesz transforms. If H is the sublaplacian formed from the left derivatives $A_1, \ldots, A_{d'}$ then we have established that $D(H^{n/2}) = L'_{p;n}$ and one has bounds

$$\|A^\alpha \varphi\|_p \leqslant c_{p,n,\nu} \|(\nu I + H)^{n/2} \varphi\|_p$$

for all α with $|\alpha| = n$, all $\varphi \in L'_{p;n}$ with $p \in \langle 1, \infty \rangle$ and all $\nu > 0$. The limit case $\nu = 0$ corresponds to the Riesz transform problem. Our results do extend to $\nu = 0$ for certain classes of groups, e.g., compact groups.

If G is compact and φ is a constant function then $\varphi \in L_p$ and since $A^\alpha \varphi = 0 = = H\varphi$ the required estimates are obvious. Next let $P\varphi = \int_G dg\, L(g)\varphi$ be the projection of φ on the space of constant functions. Then on the subspace $(I - P)L_p$ of L_p the operator H has a bounded inverse as a direct consequence of spectral properties (see [17] Proposition I.7.1). Therefore it follows straightforwardly from (8) that one has bounds

$$\|A^\alpha \varphi\|_p \leqslant c_{p,n} \|H^{n/2} \varphi\|_p \tag{9}$$

for all α with $|\alpha| = n$ and all $\varphi \in L'_{p;n}$ with $p \in \langle 1, \infty \rangle$. Therefore these estimates are valid on L_p.

If G is non-compact the boundedness of the Riesz transforms is much more delicate and the example of Gaudry, Qian and Sjögren [9] shows that (9) may be valid with $n = 1$ but false for $n = 2$.

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