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SELFADJOINT COMMUTATORS AND INVARIANT SUBSPACES ON THE TORUS

KEIJI IZUCHI and SHÛICHI OHNO

1. INTRODUCTION

It is well known the Beurling characterization of invariant subspaces of $L^2(\mathbf{T})$ on the unit circle $\mathbf{T}$. It is very difficult to describe all invariant subspaces of $L^2(\mathbf{T}^2)$ on the torus $\mathbf{T}^2$ completely. Here a nonzero closed subspace M of $L^2(\mathbf{T}^2)$ is called *invariant* if

$$zM \subset M \quad \text{and} \quad wM \subset M,$$

where $z = \mathrm{e}^{\mathrm{i}\theta}$ and $w = \mathrm{e}^{\mathrm{i}\psi}$. We denote by V_z and V_w the multiplication operator of an invariant subspace M by the functions z and w respectively. Let A be the commutator of the operator V_w and the adjoint operator V_z^* on M;

$$A = V_w V_z^* - V_z^* V_w \quad \text{on } M.$$

Then

$$A^* = V_z V_w^* - V_w^* V_z \quad \text{on } M.$$

$A = 0$ means that V_w and V_z^* commute on M. In [3], Mandrekar showed that if M is an invariant subspace with $M \subset H^2 = H^2(\mathbf{T}^2)$, then $M = qH^2$ for some inner function q if and only if $A = 0$ on M. This is a nice characterization of Beurling type invariant subspaces of H^2. In [2, 6], Ghatage-Mandrekar and Nakazi gave a characterization of general invariant subspaces M such that $A = 0$ on M (see Theorem A). In [6], Nakazi conjectured that if $A = A^*$ on M then $A = 0$ on M. The purpose of this paper is to give a counterexample for this conjecture (in Section 2) and give a characterization of invariant subspaces M such that $A = A^*$ and $A \neq 0$ on M.

We use the following notations and definitions. Let $L^2 = L^2(\mathbf{T}^2)$ be the usual Lebesgue space with respect to the normalized Lebesgue measure m on the torus $\mathbf{T}^2$. We denote by M_h the multiplication operator on L^2 by a bounded measurable function h. For $f, g \in L^2$, the inner product is given by $\langle f, g\rangle = \int_{\mathbf{T}^2} f\overline{g}dm$, where $\overline{g}$ is the complex conjugate of g. If $f = \sum_{n,k=0}^{\infty} a_{n,k} z^n w^k$, the norm of f is given by $\|f\| = \left(\sum_{n,k=0}^{\infty} |a_{n,k}|^2\right)^{1/2}$. If $\langle f, g\rangle = 0$, we write $f \perp g$. For two subspaces M and N of L^2, we write $M \perp N$ if $f \perp g$ for every $f \in M$ and $g \in N$. $M \oplus N$ means that $M \perp N$ and $M \oplus N = \{f + g; f \in M,\ g \in N\}$. When $N \subset M$, $M \ominus N$ denotes the orthogonal complement. For a subset $\mathcal{F}$ of L^2, we denote by $[\mathcal{F}]$ the closed subspace of L^2 generated by functions in $\mathcal{F}$. We denote by χ_E a characteristic function of a measurable subset E of $\mathbf{T}^2$.

Let $\mathbf{Z}$ be the set of integers and $\mathbf{Z}_+ = \{n \in \mathbf{Z}; n \geqslant 0\}$. The Hardy space H^2 is the space of functions f in L^2 such that

$$\int_{\mathbf{T}^2} f(z,w)\overline{z}^n \overline{w}^k dm = 0 \quad \text{for } (n,k) \in \mathbf{Z}^2 \setminus (\mathbf{Z}_+)^2.$$

A function F in L^2 is called *unimodular* if $|F| = 1$ a.e. on $\mathbf{T}^2$. Moreover if $F \in H^2$ then F is called inner. Let $H_z^2 = \left[\bigcup \overline{z}^n H^2; n \in \mathbf{Z}_+\right]$ and $L_z^2 = [z^n; n \in \mathbf{Z}]$. By the same way, we can define H_w^2 and L_w^2. Then $H_z^2 = \sum_{n=0}^{\infty} \oplus w^n L_z^2$ and $H_w^2 = \sum_{n=0}^{\infty} \oplus z^n L_w^2$.

The following theorem gives a characterization of invariant subspaces M such that $A = 0$ on M (see [2, Theorem 2] and [6, Theorem 4]).

THEOREM A. *Let M be an invariant subspace of L^2 such that $A = 0$ on M. Then one and only one of the following ocurs:*

(i) $M = F(\chi_{E_1} H_z^2 \oplus \chi_{E_2} L^2)$, *where* $\chi_{E_1} \in L_z^2$, $\chi_{E_1}\chi_{E_2} = 0$ a.e., *and F is unimodular.*

(ii) $M = F(\chi_{E_1} H_w^2 \oplus \chi_{E_2} L^2)$, *where* $\chi_{E_1} \in L_w^2$, $\chi_{E_1}\chi_{E_2} = 0$ a.e., *and F is unimodular.*

(iii) $M = FH^2$, *where F is unimodular.*

A closed subspace M is called *doubly invariant* if $zM = wM = M$, and in this case we have that $A = 0$ on M. The following is the main theorem of this paper.

THEOREM 1. *Let M be an invariant subspace of L^2. Then M satisfies both conditions $A = A^*$ and $A \neq 0$ on M if and only if M has the following form*

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n [\overline{w}\lambda(z\overline{w})]\right)\right) \quad \textit{or} \quad M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus w^n [\overline{z}\lambda(\overline{z}w)]\right)\right),$$

where F is a unimodular function and $\lambda(\zeta) = \dfrac{1}{1-b\zeta}$ *for some real number b with* $0 < |b| < 1$.

We note that $[\overline{w}\lambda(z\overline{w})] = \{c\overline{w}\lambda(z\overline{w}); c$ is a complex number$\}$. Theorems A and 1 give a characterization of invariant subspaces M such that $A = A^*$ on M. The sufficiency of Theorem 1 gives a counterexample for Nazaki's conjecture, and we prove this in Section 2 (when $F = 1$). The proof of the necessity of Theorem 1 is given in Section 4. In Section 3, we give some lemmas which are used in Section 4.

2. A COUNTEREXAMPLE

In this section, we prove the following.

THEOREM 2. *Let*

$$M = H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n[\overline{w}\lambda(z\overline{w})]\right) \quad \text{or} \quad M = H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus w^n[\overline{z}\lambda(\overline{z}w)]\right),$$

where $\lambda(\zeta) = \dfrac{1}{1-b\zeta}$ *for some real number b with $0 < |b| < 1$. Then M satisfies $A = A^*$ and $A \neq 0$ on M.*

Proof. Let b be a real number such that $0 < |b| < 1$ and let

$$\lambda(\zeta) = \frac{1}{1-b\zeta} = \sum_{n=0}^{\infty} b^n \zeta^n \quad \text{for } \zeta \in \mathbb{T};$$

$$N = [\overline{w}\lambda(z\overline{w})].$$

Then $z^n N \perp H^2$ and $z^n N \perp z^k N$ for $n \neq k$ with $n, k \in \mathbb{Z}_+$. Hence the following closed subspace M is well defined;

$$M = H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n N\right). \tag{1}$$

It is easy to see that $zM \subset M$. By an easy calculation, we have

$$w(z^n\overline{w}\lambda(z\overline{w})) = z^n + bz^{n+1}\overline{w}\lambda(z\overline{w}) \in z^nH^2 \oplus z^{n+1}N \subset M \tag{2}$$

for every $n \in \mathbb{Z}_+$, so that $wM \subset M$. Therefore M is an invariant subspace and

$$M \ominus zM = [w^k; k \in \mathbb{Z}_+] \oplus N; \tag{3}$$

$$zM = zH^2 \oplus \left(\sum_{n=1}^{\infty} \oplus z^n N \right). \tag{4}$$

Let P be the orthogonal projection of L^2 onto M. Then the adjoint operator V_z^* has the form $V_z^* f = P(\overline{z} f)$, $f \in M$, because

$$\langle V_z^* f, h \rangle = \langle f, V_z h \rangle = \langle f, zh \rangle = \langle \overline{z} f, h \rangle = \langle P(\overline{z} f), h \rangle$$

for every $f, h \in M$. Hence

$$V_z^*(zf) = f \quad \text{for } f \in M; \tag{5}$$

$$\operatorname{Ker} V_z^* = M \ominus zM. \tag{6}$$

By the same way, we have that

$$V_w^* f = P(\overline{w} f) \ \text{ and } \ V_w^*(wf) = f \quad \text{for } f \in M; \tag{7}$$

$$\operatorname{Ker} V_w^* = M \ominus wM. \tag{8}$$

By our definition of the operator A,

$$A = V_w V_z^* - V_z^* V_w; \tag{9}$$

$$A^* = V_z V_w^* - V_w^* V_z. \tag{10}$$

First we study the operator A on M. By (5),

$$V_z^* = M_{\overline{z}} \quad \text{on } zM. \tag{11}$$

By (9) and (11),

$$A = 0 \quad \text{on } zM. \tag{12}$$

By the form of M in (1), it is easy to see that $A = 0$ on $[w^k; k \in \mathbf{Z}_+]$. Hence by (4) and (12), we have

$$A = 0 \quad \text{on } H^2 \oplus \left(\sum_{n=1}^{\infty} \oplus z^n N \right). \tag{13}$$

On the other hand,

$$\begin{aligned} A(\overline{w}\lambda(z\overline{w})) &= -V_z^* V_w(\overline{w}\lambda(z\overline{w})) && \text{by (3), (6) and (9)} \\ &= -V_z^*(1 + bz\overline{w}\lambda(z\overline{w})) && \text{by (2)} \\ &= -b\overline{w}\lambda(z\overline{w}) && \text{by (3), (5) and (6).} \end{aligned}$$

Hence we get

(14) $$A = -bI \neq 0 \quad \text{on } N.$$

Next we study the operator A^* on M. By (7),

(15) $$A^* = 0 \quad \text{on } wH^2.$$

To study A^* on $M \ominus wH^2$, we need to study $P(z^n\overline{w})$ for $n \geqslant 0$ and $P(z^n\overline{w}^2\lambda(z\overline{w}))$ for $n \geqslant 1$. Since $z^n\overline{w} \perp H^2$ and $z^n\overline{w} \perp z^k N$ for $k \neq n$, by the form of M in (1) we see that $P(z^n\overline{w})$ coincides with the orthogonal projection of $z^n\overline{w}$ onto $z^n N = [z^n\overline{w}\lambda(z\overline{w})]$. Since $\|z^n\overline{w}\lambda(z\overline{w})\|^2 = (1 - b^2)^{-1}$, we have

$$P(z^n\overline{w}) = \frac{\langle z^n\overline{w}, z^n\overline{w}\lambda(z\overline{w})\rangle}{\|z^n\overline{w}\lambda(z\overline{w})\|^2} z^n\overline{w}\lambda(z\overline{w}) = (1 - b^2)z^n\overline{w}\lambda(z\overline{w}) \in z^n N.$$

By (10) and the above, we have

$$A^*(z^n) = zP(z^n\overline{w}) - P(z^{n+1}\overline{w}) = 0$$

for every $n \in \mathbf{Z}_+$. Hence by (15), we get

(16) $$A^* = 0 \quad \text{on } H^2.$$

Since $z^n\overline{w}^2\lambda(z\overline{w})$, $n \geqslant 1$, is orthogonal to H^2 and $z^k N$ for $k \neq n-1$, $P(z^n\overline{w}^2\lambda(z\overline{w}))$ coincides with the orthogonal projection of $z^n\overline{w}^2\lambda(z\overline{w})$ onto $z^{n-1}N = [z^{n-1}\overline{w}\lambda(z\overline{w})]$. Then

$$P(z^n\overline{w}^2\lambda(z\overline{w})) = \frac{\langle z^n\overline{w}^2\lambda(z\overline{w}), z^{n-1}\overline{w}\lambda(z\overline{w})\rangle}{\|z^{n-1}\overline{w}\lambda(z\overline{w})\|^2} z^{n-1}\overline{w}\lambda(z\overline{w}),$$

so that easily we have that

(17) $$P(z^n\overline{w}^2\lambda(z\overline{w})) = bz^{n-1}\overline{w}\lambda(z\overline{w}).$$

By (10) and (17),

$$A^*(z^n\overline{w}\lambda(z\overline{w})) = zP(z^n\overline{w}^2\lambda(z\overline{w})) - P(z^{n+1}\overline{w}^2\lambda(z\overline{w})) = 0 \quad \text{for } n \geqslant 1.$$

Hence $A^* = 0$ on $\sum_{n=1}^{\infty} \oplus z^n N$. Therefore by (16), we get

$$A^* = 0 \quad \text{on } H^2 \oplus \left(\sum_{n=1}^{\infty} \oplus z^n N \right). \tag{18}$$

At last, we study A^* on N. By the forms of $N$and M in (1), $N \perp wM$. Then by (8), $V_w^* = 0$ on N, so that we have

$$A^*(\overline{w}\lambda(z\overline{w})) = -P(z\overline{w}^2\lambda(z\overline{w})) = -b\overline{w}\lambda(z\overline{w}) \qquad \text{by (17)}.$$

Hence

$$A^* = -bI \neq 0 \quad \text{on } N. \tag{19}$$

As a consequence of (1), (13), (14), (18) and (19), we have that $A = A^*$ and $A \neq 0$ on M.

By the same way, we can prove that $A=A^*$ and $A\neq 0$ on $H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus w^n [\overline{z}\lambda(\overline{z}w)] \right)$.

We note that the invariant subspace M given by (1) is singly generated. For by (2), we have

$$w(\overline{w}\lambda(z\overline{w})) = 1 + bz\overline{w}\lambda(z\overline{w}).$$

This implies that the constant function 1 belongs to the invariant subspace generated by $\overline{w}\lambda(z\overline{w})$. Hence H^2 and M are contained in the invariant subspace generated by $\overline{w}\lambda(z\overline{w}) \in M$. Therefore M is a singly generated invariant subspace. ∎

3. LEMMAS

To prove Theorem 1, we need some lemmas. The following is proved in [4, Theorem 6] and [6, Proposition 2] essentially, but there are some differences in the forms (see [4, 6] in detail).

LEMMA 1. *Let M be an invariant subspace and $S_1 = M \ominus zM \neq \{0\}$. Let S be the largest closed subspace of S_1 such that $wS \subset S$. Suppose that $S \neq \{0\}$. Then we have the following:*

(i) *If $wS \neq S$, then there exists a unimodular function F such that*

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n N \right) \right),$$

where N is a closed subspace of $H_w^2 \ominus H^2$, $\overline{w} \notin N$, and $S = F[w^n; n \in \mathbf{Z}_+]$.

(ii) *If $wS = S$, then $S = S_1$ and there exists a unimodular function F such that*

$$M = F(\chi_{E_1} H_w^2 \oplus \chi_{E_2} L^2),$$

where $\chi_{E_1} \in L_w^2$, $\chi_{E_1} \neq 0$, and $\chi_{E_1}\chi_{E_2} = 0$ a.e.

By an easy computation, we have the following (see [6, Lemma 2]).

LEMMA 2. *Let M be an invariant subspace and $S_1 = M \ominus zM$. Then $A = 0$ on M if and only if $wS_1 \subset S_1$.*

LEMMA 3. *Let M and M_1 be invariant subspaces and $M = FM_1$ for a unimodular function F. We denote by A and A_1 the operators $V_w V_z^* - V_z^* V_w$ on M and M_1 respectively. Then $A = A^*$ and $A \neq 0$ on M if and only if $A_1 = A_1^*$ and $A_1 \neq 0$ on M_1.*

Proof. To avoid confusion, we use v_z and v_w for the operators V_z and V_w on M_1. Let $U : M_1 \ni f \to Ff \in M$. Then

$$v_z = U^{-1} V_z U \quad \text{and} \quad v_w = U^{-1} V_w U.$$

Hence

$$v_z^* = U^{-1} V_z^* U \quad \text{and} \quad v_w^* = U^{-1} V_w^* U.$$

Therefore we have our assertion easily.

The following lemma is proved in [5] essentially.

LEMMA 4. *Let S_1 be a nonzero closed subspace of L^2 and $S_n = [zS_{n-1}, wS_{n-1}]$ for $n \geqslant 2$. If $S_n \perp S_k$ for $n \neq k$, then $S_1 = FK$, where F is a unimodular function and K is a closed subspace of $[(\bar{z}w)^n; n \in \mathbf{Z}]$. Moreover suppose that $wS_1 \subset zS_1$. If $wS_1 = zS_1$ then $K = \chi_E[(\bar{z}w)^n; n \in \mathbf{Z}]$ for some $\chi_E \in [(\bar{z}w)^n; n \in \mathbf{Z}]$, and if $wS_1 \neq zS_1$ then $K = [(\bar{z}w)^n; n \in \mathbf{Z}_+]$.*

Since the statement of Lemma 4 is not written explicitly in [5], we give some comments. If $S_n \perp S_k$ for $n \neq k$, then by our definition of S_n, $M = \sum_{n=1}^{\infty} \oplus S_n$ becomes an invariant subspace, and such an M is called a *homogeneous invariant subspace* in [5]. Also by the condition $S_n \perp S_k$ for $n \neq k$, there exists a unimodular function q such that

$$S_1 = qK,$$

where K is a closed subspace of $[(\bar{z}w)^n; n \in \mathbf{Z}]$ (see the proof of [5, Theorem 3]). Moreover suppose that $wS_1 \subset zS_1$, that is, $\bar{z}wS_1 \subset S_1$. By considering $\zeta = \bar{z}w$, we can consider that K is an invariant subspace as a variable ζ. Hence by the

Beurling theorem, if $wS_1 = zS_1$ then $K = \chi_E[(\overline{z}w)^n; n \in \mathbf{Z}]$, and if $wS_1 \neq zS_1$ then $K = q_1[(\overline{z}w)^n; n \in \mathbf{Z}_+]$ for some unimodular function q_1 (see the proof of [5, Proposition 5]). Combining these facts, we can get Lemma 4.

4. PROOF OF THEOREM 1

In this section, we prove our theorem. Let M be an invariant subspace and let P be the orthogonal projection of L^2 onto M. As Section 2, we have

$$V_z^* f = P(\overline{z}f) \text{ and } V_z^*(zf) = f \quad \text{for } f \in M; \tag{1}$$

$$\operatorname{Ker} V_z^* = S_1 = M \ominus zM; \tag{2}$$

$$V_w^* f = P(\overline{w}f) \text{ and } V_w^*(wf) = f \quad \text{for } f \in M; \tag{3}$$

$$\operatorname{Ker} V_w^* = M \ominus wM; \tag{4}$$

First suppose that

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n[\overline{w}\lambda(z\overline{w})]\right)\right), \quad \text{or} \quad M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus w^n[\overline{z}\lambda(\overline{z}w)]\right)\right),$$

where F is unimodular and $\lambda(\zeta) = \dfrac{1}{1-b\zeta}$ for some real number b with $0 < |b| < 1$. By Theorem 2 and Lemma 3, we have $A = A^*$ and $A \neq 0$ on M.

Next suppose that $A = A^*$ and $A \neq 0$ on M. If M is doubly invariant, then $A = 0$ on M. Hence M is not doubly invariant. Here we assume that $M \neq zM$. In this case, we shall prove that

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n[\overline{w}\lambda(z\overline{w})]\right)\right)$$

for a unimodular function F and $\lambda(\zeta) = \dfrac{1}{1-b\zeta}$ for some real number b with $0 < |b| <$ < 1. Let

$$S_1 = M \ominus zM \neq \{0\}. \tag{5}$$

Then we have the following Wold decomposition

$$M = \left(\sum_{n=0}^{\infty} \oplus z^n S_1\right) \oplus S_0, \quad S_0 = \bigcap_{k=0}^{\infty} z^k M. \tag{6}$$

There may happen the following three cases.

Case 1. wS_1 is contained in S_1.

Case 2. wS_1 is contained in zS_1.

Case 3. Both Cases 1 and 2 do not happen.

We study the above three cases separately.

Case 1. By Lemma 2, we have that $A = 0$ on M. Hence Case 1 does not happen.

Case 2. We shall also prove that Case 2 does not happen. Suppose that $wS_1 \subset \subset zS_1$. Here we define $S_n = [zS_{n-1}, wS_{n-1}]$ for $n \geqslant 2$. Then we have $z^n S_1 = S_n$, and we can use Lemma 4. Hence there exists a unimodular function F such that

$$S_1 = FK, \tag{7}$$

where K is a closed subspace of $[(\overline{z}w)^n; n \in \mathbb{Z}]$, and if $wS_1 = zS_1$ then K has a form

$$K = \chi_E[(\overline{z}w)^n; n \in \mathbb{Z}], \quad \chi_E \in [(\overline{z}w)^n; n \in \mathbb{Z}], \tag{8}$$

and if $wS_1 \neq zS_1$ then K has a form

$$K = [(\overline{z}w)^n; n \in \mathbb{Z}_+]. \tag{9}$$

First we study when $wS_1 = zS_1$. Then by the form of M in (6), $S_1 \perp wM$. Hence by (7) and (8),

$$F\chi_E \in S_1 \subset M \ominus wM,$$

so that by (4) we have $V_w^*(F\chi_E) = 0$. Since $F\chi_E \in S_1$, by (2) $V_z^*(F\chi_E) = 0$. Here we have

$$A(F\chi_E) = -V_z^* V_w(F\chi_E) = -P(\overline{z}wF\chi_E) = -\overline{z}wF\chi_E \in FK;$$

$$A^*(F\chi_E) = -V_w^* V_z(F\chi_E) = -P(z\overline{w}F\chi_E) = -z\overline{w}F\chi_E \in FK.$$

Since $A = A^*$, we get $\overline{z}wF\chi_E = z\overline{w}F\chi_E$, and then $(\overline{z}w)^2\chi_E = \chi_E$. Therefore $\chi_E = 0$ a.e., so that by (7) and (8) we have $S_1 = \{0\}$. This contradicts (5).

Next we study when $wS_1 \neq zS_1$. Then by (6), (7) and (9),

$$M = F[\overline{z}^k w^n; k \leqslant n,\ n \in \mathbb{Z}_+] \oplus S_0.$$

Since $F[\overline{z}^k w^n;\ k \leqslant n,\ n \in \mathbb{Z}_+] \perp S_0$ and S_0 is an invariant subspace, we have $S_0 \perp L^2$, so that $S_0 = \{0\}$. Hence

$$M = F[\overline{z}^k w^n; k \leqslant n,\ n \in \mathbb{Z}_+].$$

By the above form of M, $F \perp wM$ and $zF \perp wM$, so that by (4) we have $V_w^* F = = V_w^* V_z F = 0$ and $A^* F = 0$. Since $F \in S_1$, by (2) $V_z^* F = 0$. Hence we get $AF =$

$= -V_z^* V_w F = -P(\bar{z}wF) = -\bar{z}wF$. This contradicts $AF = A^*F$. Therefore Case 2 does not happen.

Case 3. Suppose that wS_1 is not contained in S_1 and also wS_1 is not contained in zS_1. By (1), $A = 0$ on zM. Since $A = A^*$, we have $V_z V_w^* = V_w^* V_z$ on zM, so that

$$V_z V_w^*(zg) = V_w^*(z^2 g) \quad \text{for } g \in M.$$

Since $V_w^*(zg) \in M$, $V_w^*(z^2 M) \subset zM$. Then by (5), $S_1 \perp V_w^*(z^2 M)$, so that $wS_1 \perp z^2 M$. Since $zS_0 = S_0$,

$$z^2 M = \left(\sum_{n=2}^{\infty} \oplus z^n S_1 \right) \oplus S_0.$$

Therefore we get

$$wS_1 \subset S_1 \oplus zS_1. \tag{10}$$

Let $g \in S_1$. Then we can write wg as

$$wg = g_0 + zg_1 \quad \text{for some } g_0, g_1 \in S_1. \tag{11}$$

By (3), $A^*(wg) = zg - zg = 0$. Since $A = A^*$, $A(wg) = 0$. Hence

$$V_w V_z^*(wg) = V_z^* V_w(wg) \quad \text{for every } g \in S_1, \tag{12}$$

so that by (11) we have

$$V_w V_z^* g_0 = V_w V_z^*(wg) - wg_1 = V_z^* V_w(wg) - wg_1 = V_z^* V_w g_0.$$

Since $g_0 \in S_1$, by (2) $V_z^* g_0 = 0$, therefore we get $V_z^*(wg_0) = 0$, so that $wg_0 \in S_1$. Then by (2) and (12), we have

$$V_z^*(w^2 g_0) = V_z^* V_w(wg_0) = V_w V_z^*(wg_0) = 0.$$

Hence $w^2 g_0 \in S_1$. By repeating the same argument, we can get

$$w^k g_0 \in S_1 \quad \text{for every } k \in \mathbf{Z}_+. \tag{13}$$

Let S be the largest closed subspace of S_1 such that $wS \subset S$. The condition of Case 3 implies that there exists $g \in S_1$ such that $g_0 \neq 0$ in the form of (11). Therefore by (13), we have $S \neq \{0\}$. Here we can rewrite the condition of Case 3 as follows;

$$S \neq \{0\} \text{ and } S \neq S_1. \tag{14}$$

The condition $S \neq \{0\}$ corresponds to the condition that wS_1 is not contained in zS_1, and $S \neq S_1$ corresponds to that wS_1 is not contained in S_1. By Lemma 1 (ii) and (14), $wS = S$ does not happen. By Lemma 1 (i), there exists a unimodular function F and there exists a closed subspace N such that

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n N\right)\right); \tag{15}$$

$$N \subset H^2_w \ominus H^2; \tag{16}$$

$$S = F[w^k; k \in \mathbf{Z}_+];$$

$$S_1 \ominus S = FN;$$

$$\overline{w} \notin N. \tag{17}$$

Now we shall determine the form of N and M. By Lemma 3, we may assume that $F = 1$ in (15), so that for a while we consider that

$$M = H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n N\right); \tag{18}$$

$$S = [w^k; k \in \mathbf{Z}_+]; \tag{19}$$

$$S_1 \ominus S = N \neq \{0\} \qquad \text{by (14)}. \tag{20}$$

Let $f \in N$ and $f \neq 0$. By (20), $f \in S_1$, and by (10)

$$wf = f_0 + zf_1, \qquad f_0, f_1 \in S_1. \tag{21}$$

By (19) and (20)

$$zf_1 \in zS_1 = z(S \oplus N) \subset z(H^2 \oplus N) \subset H^2 \oplus zN.$$

By (13), we have $f_0 \in S$. Then by (19) and (21) we have $wN \subset H^2 \oplus zN$. Therefore by (18),

$$wM \subset H^2 \oplus \left(\sum_{n=1}^{\infty} \oplus z^n N\right).$$

Hence we get

$$N \perp wM.$$

By (4), this implies that $V_w^* = 0$ on N, so that

$$A^* = -V_w^* V_z \quad \text{on } N. \tag{22}$$

Here we shall prove

$$f_1 \in N \tag{23}$$

for a function $f \in N$ of the form (21). Since $f \in N \subset S_1$ and $f_0 \in S_1$, by (1) and (2)

$$Af = V_w V_z^* f - V_z^* V_w f = -V_z^*(f_0 + zf_1) = -f_1. \tag{24}$$

By (22),

$$A^* f = -V_w^* V_z f = -P(z\overline{w}f). \tag{25}$$

Since $f \in N \subset H_w^2 \ominus H^2$, we have $z\overline{w}f \in H_w^2 \ominus H^2$. Then by (18),

$$P(z\overline{w}f) \in \sum_{n=0}^{\infty} \oplus z^n N. \tag{26}$$

Since $A = A^*$, by (24) and (25) $f_1 = P(z\overline{w}f)$. By (19) and (20), $S_1 = N \oplus [w^k; k \in \mathbf{Z}_+]$. Therefore by (16) and (26),

$$f_1 = P(z\overline{w}f) \in \left(\sum_{n=0}^{\infty} \oplus z^n N\right) \cap S_1 = N.$$

Now we have that

(27) $\qquad f_0$ is a constant function, say $f_0 = a_0$.

For, by (16) and (23) $f_0 = wf - zf_1 \perp [w^k; k \geqslant 1]$. By (13) and (19), $f_0 \in [w^k; k \in \mathbf{Z}_+]$, so that we get (27).

As a consequence of (21), (23), and (27), we have

$$f = a_0\overline{w} + z\overline{w}f_1, \quad f_1 \in N$$

for every $f \in N$. We can also write f_1 as

$$f_1 = a_1\overline{w} + z\overline{w}f_2, \quad f_2 \in N.$$

By repeating this argument, we have a representation of f and f_1 as follows;

$$f = \overline{w} \sum_{n=0}^{\infty} a_n (z\overline{w})^n \quad \text{for every } f \in N; \tag{28}$$

$$f_1 = \overline{w} \sum_{n=1}^{\infty} a_n (z\overline{w})^{n-1}. \tag{29}$$

By (20) and the definition of S, $wf \notin S_1$ for some $f \in N$. Then by (21),

$$f_1 = \overline{w} \sum_{n=1}^{\infty} a_n (z\overline{w})^{n-1} \neq 0 \quad \text{for some } f \in N. \tag{30}$$

Now we look at the above situation in a new light. By (23) and (24),

$$-A : N \ni f \to f_1 \in N$$

is a bounded linear operator on N. Then by (28) and (29),

$$M_w(-A)M_{\overline{w}} : wN \ni wf = \sum_{n=0}^{\infty} a_n(z\overline{w})^n \to \sum_{n=1}^{\infty}(z\overline{w})^{n-1} = wf_1 \in wN. \tag{31}$$

Here putting $\zeta = z\overline{w}$, we identify the space wN with the closed subspace $\mathcal{H}$ of $H^2(\mathbf{T})$ such that

$$\mathcal{H} = \left\{ \sum_{n=0}^{\infty} a_n\zeta^n ; \overline{w}\sum_{n=0}^{\infty} a_n(z\overline{w})^n \in N \right\}. \tag{32}$$

We denote by U_ζ^* the unilateral backward shift operator on $H^2(\mathbf{T})$, that is,

$$U_\zeta^* h = \overline{\zeta}(h(\zeta) - h(0)) \quad \text{for } h \in H^2(\mathbf{T}).$$

Then (31) and (32) say that

$$M_w(-A)M_{\overline{w}} = U_\zeta^* \quad \text{on } \mathcal{H}; \tag{33}$$

$$U_\zeta^*\mathcal{H} \subset \mathcal{H}. \tag{34}$$

Next we study the operator $M_w(V_w^* V_z)M_{\overline{w}}$ on wN. Let

$$L = [(z\overline{w})^n ; n \in \mathbf{Z}].$$

By (28), we have

$$N \subset \overline{w}L \text{ and } \overline{w}L \perp H^2 \oplus \left(\sum_{n=1}^{\infty} \oplus z^n N\right). \tag{35}$$

It is not difficult to see that $M_w P M_{\overline{w}}$ is the orthogonal projection from L^2 onto wM. By (18) and (35), $M_w P M_{\overline{w}|L}$ is the orthogonal projection from L onto wN. We denote this projection by P'. Here we have

$$M_w(V_w^* V_z)M_{\overline{w}} = (M_w P M_{\overline{w}})M_z M_{\overline{w}} \quad \text{on } wN.$$

Then for $f \in N$ of the form (28), we have

$$M_z M_{\overline{w}}(wf) = M_z f = \sum_{n=0}^{\infty} a_n(z\overline{w})^{n+1} \in L.$$

Hence

$$(36)\qquad M_w(V_w^*V_z)M_{\overline{w}} : wN \ni wf = \sum_{n=0}^{\infty} a_n(z\overline{w})^n \to P'\left(\sum_{n=0}^{\infty} a_n(z\overline{w})^{n+1}\right) \in wN.$$

By putting $\zeta = z\overline{w}$, we identify the space L with $L^2(\mathbf{T})$. We denote by Q the orthogonal projection of $L^2(\mathbf{T})$ onto $\mathcal{H}$. Then (36) says that

$$(37)\qquad M_w(V_w^*V_z)M_{\overline{w}} = QM_\zeta \quad \text{on } \mathcal{H}.$$

Since $A = A^*$, by (22) we have $-A = V_w^*V_z$ on N. Hence by (33) and (37), we get

$$(38)\qquad U_\zeta^* = QM_\zeta \quad \text{on } \mathcal{H}.$$

By (34), either $\mathcal{H} = H^2(\mathbf{T})$ or $\mathcal{H}$ has the following form

$$(39)\qquad \mathcal{H} = H^2(\mathbf{T}) \ominus \varphi H^2(\mathbf{T})$$

for some inner function φ. If $\mathcal{H} = H^2(\mathbf{T})$ then (38) does not happen. Hence (39) happens. By (17), (23), (30) and (32), $\mathcal{H}$ contains non-constant functions. Therefore $\varphi(\zeta)$ is not a constant function, and also $\varphi(\zeta) \neq c\zeta$ for every constant c with $|c| = 1$. Let

$$\varphi(\zeta) = \sum_{n=0}^{\infty} b_n\zeta^n.$$

Then we have $U_\zeta^*\varphi = \overline{\zeta}(\varphi - b_0)$ and $U_\zeta^*(U_\zeta^*\varphi) = \overline{\zeta}^2(\varphi - b_0 - b_1\zeta)$. Since $Q(\varphi) = 0$,

$$QM_\zeta(U_\zeta^*\varphi) = Q(\varphi - b_0) = -b_0Q(1).$$

By (39), $U_\zeta^*\varphi \in \mathcal{H}$. Then by (38),

$$\overline{\zeta}^2(\varphi - b_0 - b_1\zeta) = -b_0Q(1).$$

Since Q is the orthogonal projection onto $\mathcal{H}$, by (39) it is not difficult to see that $Q(1) = 1 - \overline{b}_0\varphi$ (see [7, p. 34]). Then we have $\overline{\zeta}^2(\varphi - b_0 - b_1\zeta) = -b_0(1 - \overline{b}_0\varphi)$, so that

$$(40)\qquad \varphi = \frac{-b_0\zeta^2 + b_1\zeta + b_0}{1 - |b_0|^2\zeta^2}.$$

Since φ is a non-costant inner function, we have $|b_0| < 1$. If $b_0 = 0$, $\varphi = b_1\zeta$ and $|b_1| = 1$. Since $\varphi(\zeta) \neq c\zeta$, we get

$$0 < |b_0| < 1.$$

Since φ is a non-constant inner function, by (40) the form of φ is given by either

$$\varphi = c\frac{\zeta - a}{1 - \overline{a}\zeta} \tag{41}$$

for some complex numbers a and c with $|c| = 1$and $0 < |a| < 1$, or

$$\varphi = c\frac{\zeta - |b_0|}{1 - |b_0|\zeta}\,\frac{\zeta + |b_0|}{1 + |b_0|\zeta} \tag{42}$$

for a complex number c with $|c| = 1$. By compairing the coefficients of ζ^2 in numerators of (40) and (42), we have $c = -b_0$. Since $|c| = 1$ and $|b_0| < 1$, this is a contradiction. Hence φ has a form in (41). Then

$$\frac{-b_0\zeta^2 + b_1\zeta + b_0}{1 - |b_0|^2\zeta^2} = c\frac{\zeta - a}{1 - \overline{a}\zeta},$$

so that $a = |b_0|$ or $a = -|b_0|$. Therefore

$$\varphi = c\frac{\zeta - b}{1 - b\zeta}$$

for some real number b such that $0 < |b| < 1$. Let

$$\lambda(\zeta) = \frac{1}{1 - b\zeta}.$$

Then by (39), $\mathcal{H} = \{d\lambda(\zeta); d$ is a complex number$\}$. Hence by (32), we have $N =$ $= [\overline{w}\lambda(z\overline{w})]$. Therefore by (18),

$$M = H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus z^n[\overline{w}\lambda(z\overline{w})]\right).$$

Since we assumed that $F = 1$ in (15), M has the desired form.

If we start from $M \neq wM$, we have

$$M = F\left(H^2 \oplus \left(\sum_{n=0}^{\infty} \oplus w^n[\overline{z}\lambda(\overline{z}w)]\right)\right)$$

for a unimodular function F. This completes the proof. ■

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KEIJI IZUCHI
Department of Mathematics
Niigata University
Niigata 950-21,
Japan.

SHÛICHI OHNO
Department of Mathematics
Nippon Institute of Technology,
Miyashiro, Saitama 345,
Japan.

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