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POLYNOMIALLY SUBNORMAL OPERATOR TUPLES

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SECTION 0

We begin with several definitions.

DEFINITION. A *commuting d-tuple of operators* S is a d-tuple of bounded operators $(S_1, ..., S_d)$ acting on a complex Hilbert space $\mathcal{H}$ such that

$$S_iS_j = S_jS_i \quad 1 \leqslant i, j \leqslant d.$$

Now let p be a polynomial in d variables $z = (z_1, ..., z_d)$ written

$$p(z) = \sum_{\alpha} a_\alpha z^\alpha \tag{0.1}$$

where $\alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index, $a_\alpha \in \mathbb{C}$, and $z^\alpha = z_1^\alpha z_2^\alpha \cdots z_d^\alpha$ is the usual notation for a monomial. If S is a commuting d-tuple of operators then one can unambiguously define $p(S)$ where p is as in (0.1) by setting

$$p(S) = \sum_{\alpha} a_\alpha S^\alpha$$

where $S^\alpha = S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_d^{\alpha_d}$.

We are interested in studying the following type of commuting d-tuples of operators.

DEFINITION. *A commuting d-tuple of subnormal operators* S is a commuting d-tuple of operators $(S_1, ..., S_d)$ such that S_i is subnormal for $1 \leqslant i \leqslant d$.

DEFINITION. *A subnormal d-tuple* S is a commmuting d-tuple of subnormal operators for which there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a commuting d-tuple N of commuting normal operators on K such that

(i) $N_i\mathcal{H} \subseteq \mathcal{H} \quad 1 \leqslant i \leqslant d$, and

(ii) $N_i|_{\mathcal{H}} = S_i \quad 1 \leqslant i \leqslant d$.

For a subnormal d-tuple S we shall call a d-tuple N and Hilbert space $\mathcal{K}$ as above a *joint normal extension*. A natural question to ask is whether a commuting d-tuple of subnormal operators always has a joint normal extension. This question was settled negatively by counterexamples constructed by Abrahamse and Lubin in [1][8] and [9]. Criteria for determining when a set of commuting subnormal operators has a joint normal extension have been studied by a number of authors including [1], [5], [7], [10] and [12]. Ito generalizes work of Halmos and Bram to give necessary and sufficient conditions for a commutative semi-group of subnormal operators to extend to a commutative semi-group of normal operators. Most other work has been for pairs (S, T) of commuting subnormals. Such a pair has a joint normal extension if either S or T is subnormal [5], if either S or T is cyclic [12], if either S or T is an isometry [11], and under various hypotheses regarding the spectra of S and T [11], [1].

If S is a subnormal d-tuple acting on a complex Hilbert space $\mathcal{H}$, N is a joint normal extension of S, and p is a polynomial in d-variables then

$$p(S) = p(N)|_{\mathcal{H}}.$$

Therefore since $p(N)$ is normal $p(S)$ is subnormal. Thus a necessary condition for a d-tuple of commuting subnormal operators S to have a joint normal extension is that $p(S)$ be subnormal for all polynomials p id d variables. This leads to the following definition.

DEFINITION. *A polynomially subnormal d-tuple S* is a d-tuple of commuting subnormal operators with the property that $p(S)$ is subnormal for all polynomials p in d variables.

With this definition we state the following positive result which will be the main result of this paper.

THEOREM 0.2. *If S is a polynomial subnormal d-tuple then S is a subnormal d-tuple.*

Section 1 uses the work of Agler in *Hypercontractions and Subnormality* [4], to show that Theorem 0.2 is equivalent to a concrete several variables approximation theorem. Section 2 will contain the proof of the approximation theorem via an explicit construction. The author wishes to thank Jim Agler for for suggesting the problem.

SECTION 1

To show that the operator theory question of the existence of a joint normal extension for a polynomially subnormal d-tuple is equivalent to an approximation theorem, Theorem 1.7, we shall pass from operator theory to function theory via the hereditary functional calculus as developed in [3]. Accordingly we have the following definition.

DEFINITION. *A hereditary polynomial* is a polynomial over $\mathbf{C}$ in two non-commuting d-tuples of pairwise commuting variables $z = (z_i)$ and $w = (w_i)$ of the form

$$p(z,w) = \sum_{|(\alpha,\beta)|=0}^{n} a_{\alpha\beta} w^{\beta} z^{\alpha}. \tag{1.1}$$

Here (α,β) is th ebi-multi-index $\alpha_1,\ldots,\alpha_d,\beta_1,\ldots,\beta_d)$ and w^β and z^α are the usual notations for nomomials. The set of hereditary polynomials will be denoted $\mathcal{P}_d$. If S is a d-tuple of commuting operators and $p \in \mathcal{P}_d$ is as in (1.1) set

$$p(S) = \sum_{|(\alpha,\beta)|=0}^{n} a_{\alpha\beta} (S^*)^{\beta} S^{\alpha}$$

$$\left((S^*)^\beta = S_1^{*\beta_1} S_2^{*\beta_2} \cdots S_d^{*\beta_d} \text{ and } S^\alpha = S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_d^{\alpha_d}\right).$$

For p as in (1.1) $\overline{p(\overline{w},\overline{z})} = \sum \overline{a_{\alpha\beta}} w^\alpha z^\beta$. Thus $\overline{p(\overline{w},\overline{z})}S = p(S)^*$.

We now recall two theorems from [4] (Theorem 3.1 and Theorem 3.2) about single operators and subnormality.

THEOREM 1.2 [Sz. Nagy, Embry]. *$(1 - zw)^n(S) \geqslant 0$ for all n if and only if $\|S\| \leqslant 1$ and S is subnormal.*

THEOREM 1.3 [Agler]. *S is subnormal with $\|S\| \leqslant 1$ if and only if $p(S) \geqslant 0$ whenever $p \in \mathcal{P}_1$ and $p(z,\overline{z}) \geqslant 0$ for all $z \in \mathbf{D}$.*

These two theorems give necessary and sufficient conditions for an operator to be subnormal in terms of the positivity of two different sets of hereditary functions of the operator. The set of functions in the second theorem is clearly larger than the first, still the fact that both sets determine subnormality shows that in some sense they are the same. Indeed these two theorems about operators enable one to prove the following approximation theorem also from [4], (Theorem 3.4).

THEOREM 1.4 [Agler]. *Let $A^2 = \{f \in C(\mathbf{D}^- \times \mathbf{D}^-) : f$ is analytic on $\mathbf{D} \times \mathbf{D}\}$ with the uniform topology. Let $M = \{f \in A^2 : f(z,\overline{z}) \geqslant 0$ for all $z \in \mathbf{D}\}$, and let C be the* *convex hull of the set functions f in A^2 of the form*

$$f(z,w)=\overline{p(\overline{w})}(1-zw)^n p(z)$$

where n is a nonnegative integer and p is a polynomial in one variable. Then $cc^- = M$.

Clearly if $(1-zw)^n(S) \geqslant 0$ then $h(z,w)(S) \geqslant 0$ for any $h \in C$. Thus with attention to continuity one easily sees that Theorem 1.4 implies Theorem 1.2 given Theorem 1.3. Using the Hahn Banach Theorem and a Gelfand-Naimark-Segal type of construction one can see the reverse implication as well.

Now, by doing barely more than adding indices, we will use these same arguments to show that the question of a joint normal extension for polynomially subnormal d-tuple is equivalent to an approximation theorem. We begin by letting S be a commuting d-tuple of operators and setting

$$\Delta = \mathbf{D}^d$$

the d-dimensional unit polydisc. We now state theorems analogous to Theorem 1.2 and Theorem 1.3.

THEOREM 1.5. *$(1-p(z)\overline{p(\overline{w})})^n(S) \geqslant 0$ for all n and all polynomials in d-variables such that $p\colon \Delta \to \mathbf{D}$ in and only if $\|S_i\| \leqslant 1$ for $1 \leqslant i \leqslant d$, and S is a polynomially subnormal d-tuple.*

THEOREM 1.6. *S has a joint normal extension with $\|S_i\| \leqslant 1$, $1 \leqslant i \leqslant d$, if and only if $p(S) \geqslant 0$ whenever $p \in \mathcal{P}_d$ and $p(z,\overline{z}) \geqslant 0$ for all $z \in \Delta$.*

Theorem 1.5 is immediate from Theorem 1.2 since $(1-p(z)\overline{p(\overline{w})})^n(S) = (1-zw)^n(p(S))$. Theorem 1.6 is proved using Stinesprings Theorem in exactly the same way as Theorem 1.3 is proven in [4].

The following theorem states the desired equivalence.

THEOREM 1.7. *Let $A^{2d} = \{f \in C(\Delta^- \times \Delta^-)\colon f$ is analytic on $\Delta \times \Delta\}$ with the uniform topology. Let $M = \{f \in A^{2d}\colon f(z,\overline{z}) \geqslant 0$ for all $z \in \Delta\}$, and let cc be the convex hull of the set of functions $f \in A^{2d}$ of the form $f(z,w) = \overline{q(\overline{w})}(1-p(z)\overline{p(\overline{w})})^n q(z)$ where n is a nonnegative integer, p and q are polynomials in d variables, and $p\colon \Delta \to \mathbf{D}$. Then*

$$C^- = M$$

if and only if a polynomially subnormal d-tuple is a subnormal d-tuple.

Theorem 1.5 shows that if S is a polynomially subnormal d-tuple with $\|S_i\| \leqslant 1$ $1 \leqslant i \leqslant d$ then

$$h(z,w)(S) \geqslant 0$$

whenever $h(z,w) \in \mathcal{C}$. Therefore Theorem 1.6 shows that if $\mathcal{C}^- = M$ then a polynomially subnormal d-tuple S is a subnormal d-tuple. Conversely if every polynomially subnormal d-tuple S is a subnormal d-tuple then by applying the Hahn Banach Theorem and using a Gelfand-Naimark-Segal type of construction one can see that $\mathcal{C}^- = M$. Thus we have shown that Theroem 0.2 is equivalent to showing that $\mathcal{C}^- = M$, a fact which will be established in Section 2.

SECTION 2

In this section we shall prove the following approximation theorem.

THEOREM 2.1. *For $\mathcal{C}$ and M as in Theorem 1.7*

$$\mathcal{C}^- = M.$$

Note that it is immediate from the definition of $\mathcal{C}$ that $\mathcal{C}^- \subset M$. To see the reverse inclusion we will construct a sequence of entire functions, $\{G_k(z,w)\}_{k=1}^{\infty}$ in2d variables which will have many properties in common with an approximate identity. $G_k(\lambda - z, \overline{\lambda} - w)$ will be in $\mathcal{C}^-$ for all k and all λ in $\mathbf{C}^d$. For h in M, $\rho < 1, dA(\lambda)$ area measure in $\mathbf{C}^d$, (i.e. the product of d area measures over $\mathbf{C}$) and $h_\rho(z,w) = h(\rho z, \rho v)$ the sequence

$$\left\{ \int_{\frac{1}{\rho}\Delta} h_\rho(\lambda, \overline{\lambda}) G_k(\lambda - z, \overline{\lambda} - w) dA(\lambda) \right\}_{k=0}^{\infty}$$

will be contained in $\mathcal{C}^-$ and converge to h_ρ. Thus h_ρ will be in $\mathcal{C}^-$ and, since $h_\rho \to h$ as $\rho \to 1$, $M \subset \mathcal{C}^-$.

We begin the construction of $\{G_k\}$ with some notation. Set

$$\|z\| = \left(\sum_{j=1}^{d} |z_j|^2 \right)^{1/2},$$

and

$$I^n = [0,1]^n.$$

For $t = (t_j)$ in I^d and $z \in \mathbf{C}^d$ define a polynomial $p(t,z)$ by

$$p(t,z) = \sum_{j=1}^{d} t_j z_j^{2d+1}. \tag{2.2}$$

For $k \geqslant 1$ set

$$G_k(z,w) = \frac{\int\limits_{I^d} e^{-kp(t,w)p(t,z)}dt_1 dt_2 \cdots dt_d}{\int\limits_{\Delta}\int\limits_{I^d} e^{-kp(t,\overline{\lambda})p(t,\lambda)}dt_1 dt_2 \cdots dt_d dA(\lambda)}. \tag{2.3}$$

Note that the integrand in the numerator of (2.3) is bounded for t in I^d and (z,w) in any compact subset of $\mathbf{C}^{2d}$, and the denominator is a positive number depending on k. Thus an applicatin of Morera's Theorem and Fubini's Theorem shows that G_k is entire for all k. Also note the following facts which are immediate from the definition of $\{G_k\}$.

$$G_k(\lambda,\overline{\lambda}) \geqslant 0 \text{ for all } k \text{ and all } \lambda \text{ in } \mathbf{C}^d. \tag{2.4}$$

$$G_k(\lambda,\overline{\lambda}) = G_k(\sigma(\lambda),\sigma(\overline{\lambda})) \text{ where } \sigma \text{ is a permutation of the variables.} \tag{2.5}$$

$$\int\limits_{\Delta} G_k(\lambda,\overline{\lambda})dA(\lambda) = 1. \tag{2.6}$$

A less obvious fact about $\{G_k\}$ we shall require is contained in the following proposition.

PROPOSITION 2.7.

$$\lim_{k\to\infty} G_k(\lambda,\overline{\lambda}) = 0, \text{ for all } \lambda \text{ in } \mathbf{C}^d\backslash\{0\}.$$

Proof. We begin by using the definition of $p(t,z)$, (2.2), to make an estimate of the denominator in (2.3), the expression which defines G_k.

$$\int\limits_{\Delta}\int\limits_{I^d} e^{-kp(t,\overline{\lambda})p(t,\lambda)}dt_1 dt_2 \cdots dt_d dA(\lambda)$$

$$> \int\limits_{k^{-\frac{1}{2(2d+1)}}\Delta}\int\limits_{I^d} e^{-k\sup\limits_{(t,\lambda)}|p(t,\lambda)|^2} dt_1 dt_2 \cdots dt_d dA(\lambda) \tag{2.8}$$

$$= \int\limits_{k^{-\frac{1}{2(2d+1)}}\Delta}\int\limits_{I^d} e^{-kd^2(k^{-\frac{1}{2(2d+1)}})^{2(2d+1)}} dt_1, dt_2 \cdots dt_d dA(\lambda) = \pi^d e^{-d^2} k^{-d/(2d+1)}.$$

Next fix $\lambda \in \mathbf{C}^d \backslash \{0\}$. By (2.5) we may assume that $\lambda_1 \neq 0$. Set

$$q(t,\lambda) = \sum_{j=2}^{d} t_j \lambda_j^{2d+1}$$

and

$$\eta = \left(\frac{\lambda_1}{|\lambda_1|}\right)^{2d+1}.$$

By (2.2), (2.3) and the estimate in (2.8) one obtains that

$$G_k(\lambda,\overline{\lambda})$$

$$< \pi^{-} d e^{d^2} k^{d/(2d+1)} \int_{I^{d-1}} \int_0^1 e^{-k[t_1\overline{\eta}|\lambda_1|^{2d+1}+q(t,\overline{\lambda}))(t_1\eta|\lambda_1|^{2d+1}+q(t,\lambda))]} dt_1 dt_2 \cdots dt_d$$

$$\leqslant \pi^{-d} e^{d^2} k^{d/(2d+1)} \int_{I^{d-1}} \int_0^1 e^{-k|t_1|\lambda_1|^{2d+1}} + Re(\eta q(t,\lambda)|^2 dt_1 dt_2 \cdots dt_d.$$

Making the change of variables $t' = \sqrt{k}(t_1|\lambda_1|^{2d+1} + Re(\eta q(t,\overline{\lambda})))$ in this latter integral one obtains that

$$G_k(\lambda,\overline{\lambda})$$

$$< \pi^{-d} e^{d^2} k^{d/2d+1} \int_{I^{d-1}} k^{-\frac{1}{2}} |\lambda_1|^{2d+1} \int_{\sqrt{k}Re(\eta q(t,\lambda))}^{\sqrt{k}(|\lambda_1|^{(2d+1)}+Re(\eta q(t,\lambda))} e^{-}(t')^2 dt' \cdots dt_d$$

$$< \pi^{-d} e^{d^2} |\lambda_1|^{-(2d+1)} k^{\left(\frac{d}{2d+1}-\frac{1}{2}\right)} \int_{-\infty}^{\infty} e^{-t^2} dt.$$

Since the exponent of k in this last expression is negative, its limit in k is zero. This completes the proof of Proposition 2.7.

Next fix h in M. For $\rho < 1$ and $k \geqslant 1$ set $h_\rho(z,w) = h(\rho z, \rho w)$, and

$$h_{\rho,k}(z,w) = \int_{\frac{1}{2}\Delta} h_\rho(\lambda,\overline{\lambda}) G_k(\lambda - z, \overline{\lambda} - w) dA(\lambda). \tag{2.9}$$

Observe that $h_\rho \in M$ and that $h_\rho \to h$ as $\rho \to 1$. To prove that $h \in \mathcal{C}^-$ and hence that $M \subset \mathcal{C}^-$ we shall show in Lemma 2.10 that $h_{\rho,k} \in \mathcal{C}^-$ for all k and all $\rho < 1$ and that $h_{\rho,k} \to h_\rho$ as $k \to \infty$ in Lemma 2.12.

LEMMA 2.10. *If $k \geqslant 1$, $\rho < 1$, and $h_{\rho,k}$ is as in (2.9) then*

$$h_{\rho,k} \in \mathcal{C}^-.$$

Proof. The uniform continuity of $h_\rho(\lambda,\overline{\lambda})G_k(\lambda-z,\overline{\lambda}-w)$ shows that there exists a sequence of Riemann sums of the integral in (2.9) which converges in (z,w) to $h_{\rho,k}$. Thus $h_{\rho,k}\in C^-$ if $h_\rho(\lambda,\overline{\lambda})G_k(\lambda-z,\overline{\lambda}-w)\in C^-$ for all λ in $\frac{1}{\rho}\Delta$. The fact that $h_\rho(\lambda,\overline{\lambda})\geqslant 0$ and the uniform continuity of the integrand in (2.3) (which defines G_k) shows that $h_{\rho,k}\in C^-$ if $e^{-kp(t,\overline{\lambda}-w)p(t,\lambda-z)}\in C^-$ for all t in I^d. Now,

$$e^{-kp(t,\overline{\lambda}-w)p(t,\lambda-z)}=\lim_{n\to\infty}\left(1-\frac{kp(t,\overline{\lambda}-w)p(t,\lambda-z)}{n}\right)^n.$$

Hence $e^{-kp(t,\overline{\lambda}-w)p(t,\lambda-z)}$ will be in C^- if

$$\left(1-\frac{kp(t,\overline{\lambda}-w)p(t,\lambda-z)}{n}\right)\in C^- \tag{2.11}$$

for large n. Note that for n large enough $\left|\sqrt{\frac{k}{n}}p(t,\lambda-z)\right|<1$ for all z in Δ, and that $p(t,\overline{\lambda}-w)=\overline{p(t,\lambda-\overline{w})}$. Therefore by the definition of C (2.11) holds. This completes the proof of Lemma 2.10.

LEMMA 2.12. *If $h_{\rho,k}$ is defined as in (2.9), then, $h_{\rho,k}\to h_\rho$ in A^{2d}.*

Proof. Set

$$\tilde{h}_{\rho(z,w)}=\begin{cases}h_\rho(z,w) & \text{for } (z,w) \text{ in } \frac{1}{\rho}(\Delta\times\Delta)\\ 0 & \text{for } (z,w)\in \mathbf{C}^{2d}\setminus\frac{1}{\rho}(\Delta\times\Delta)\end{cases} \tag{2.13}$$

and

$$\tilde{h}_{\rho,k}(z,w)=\int_{2\frac{1}{\rho}\Delta}\tilde{h}_\rho(\lambda+z,\overline{\lambda}+w)G_k(\lambda,\overline{\lambda})dA(\lambda). \tag{2.14}$$

Clearly $\tilde{h}_{\rho,k}\in A^{2d}$. Using (2.13) and (2.9) one obtains that

$$h_{\rho,k}(z,w)=\int_{2\frac{1}{\rho}\Delta}\tilde{h}_\rho(\lambda,\overline{\lambda})G_k(\lambda-z,\overline{\lambda}-w)dA(\lambda). \tag{2.15}$$

For $w=\overline{z}$ the change of variables $\lambda'=\lambda-z$ applied to (2.15) shows that $h_{\rho,k}(z,w)=$ $=\tilde{h}_{\rho,k}(z,w)$ whenever $w=\overline{z}$. Since $\{(z,w)\colon(z,w)\in\frac{1}{\rho}(\Delta\times\Delta)$ and $w=\overline{z}\}$ is a set of uniquennes for holomorphic functions in $2d$ variables, [6], one has that $h_{\rho,k}=\tilde{h}_{\rho,k}$ on $\frac{1}{\rho}(\Delta\times\Delta)$. Thus $h_{\rho,k}\to h_\rho$ if

$$\tilde{h}_{\rho,k}\to h_\rho. \tag{2.16}$$

To see (2.16) fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\delta < \frac{1}{\rho} - 1$$

and if (z, w) and (z', w') are in $\frac{1}{\rho}(\Delta \times \Delta)$ with $\|(z, w) - (z', w')\| < \delta$ then

$$|h_\rho(z, w) - h_\rho(z', w')| < \varepsilon.$$

Set

$$B = \sup\left\{|h_\rho(z, w)| : (z, w) \in \frac{1}{\rho}(\Delta \times \Delta)\right\}.$$

(2.6), (2.13), and (2.14) show that for $(z, w) \in \Delta \times \Delta$

$$|h_\rho(z, w) - \tilde{h}_{\rho,k}(z, w)|$$

$$= \left| \int_\Delta h_\rho(z, w) G_k(\lambda, \overline{\lambda}) dA(\lambda) - \int_{2\frac{1}{\rho}\Delta} \tilde{h}_\rho(\lambda + z, \overline{\lambda} + w) G_k(\lambda, \overline{\lambda}) dA(\lambda) \right|$$

$$\leqslant \int_{\|\lambda\| < \frac{\delta}{\sqrt{2}}} |h_\rho(z, w) - \tilde{h}_\rho(\lambda + z, \overline{\lambda} + w)| G_k(\lambda, \overline{\lambda}) dA(\lambda)$$

$$+ \int_{2\frac{1}{\rho}\Delta \setminus \{\lambda \| \lambda \| < \frac{\delta}{\sqrt{2}}\}} 2B G_k(\lambda, \overline{\lambda}) dA(\lambda).$$

By our choice of δ, the definition of $\tilde{h}_\rho$, and (2.6) the first integral above is less than ε. Further using Proposition 2.7 one sees that the limit in k of the second integral is 0. Thus $\tilde{h}_{\rho,k} \to h_\rho$ and the proof of Lemma 2.12 is complete, completing the proof of Theorem 2.1.

REMARKS. These techniques can be modified to directly prove Theorem 3.1 of [4] by simply setting

$$G_k(z, w) = e^{-kzw} \left(\int_{\mathbb{D}} e^{-k|\lambda|^2} dA(\lambda) \right)^{-1}.$$

More importantly we not ethat the hypotheses of Theorem 0.2 can be weakened somewhat. In fact we need only know that $p(S)$ is subnormal for polynomials of degree $2d + 1$. This fact can be seen by examining the proof that $k_{\rho,k} \in C^-$ Lemma 2.10. The number $2d + 1$ is important because of its role in the behaviour of G_k near zero, specifically in the estimate in (2.8).

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