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THE C^* PROJECTIVE LENGTH OF n-HOMOGENEOUS C^*-ALGEBRAS

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INTRODUCTION

The quantities C^* projective length $\mathrm{cpl}(A)$ and C^* projective rank $\mathrm{cpr}(A)$ were introduced in [10], to do for projections what the C^* exponential length [14] and C^* exponential rank [8] do for unitaries. For a unital C^*-algebra A, let $P(A)$ denote the set of projections in A. Then $\mathrm{cpl}(A)$ is the supremum of the rectifiable diameters of the connected components of $P(A)$. Also, $\mathrm{cpr}(A)$ is the smallest element of the set $\{0, 1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$ such that any two homotopic projections in A are unitarily equivalent via a product of at most that many $*$-symmetries (selfadjoint unitaries). Here, we say that p and q are unitarily equivalent via a product of $n+\varepsilon$ $*$-symmetries if for every $\varepsilon > 0$ there is a product s of n $*$-symmetries such that $\|sps^* - q\| < \varepsilon$.

The purpose of this paper is to give bounds on the C^* projective length and rank of $C(X) \otimes M_n$. The main results are as follows. If $\dim(X) \leqslant d$ then $\mathrm{cpl}(C(X)\otimes M_n)$ and $\mathrm{cpr}(C(X) \otimes M_n)$ are finite for all n, for $n \geqslant 2d+4$ we have

$$(*) \qquad \mathrm{cpl}(C(X) \otimes M_n) \leqslant 3\pi \ \text{ and } \ \mathrm{cpr}(C(X) \otimes M_n) \leqslant 6 + \varepsilon,$$

and for $n \geqslant 5d+3$ we have

$$(**) \qquad \mathrm{cpl}(C(X) \otimes M_n) \leqslant 2\pi \ \text{ and } \ \mathrm{cpr}(C(X) \otimes M_n) \leqslant 4 + \varepsilon.$$

On the other hand, if B_m is the closed unit ball in $\mathbf{R}^m$, then

$$(***) \qquad \mathrm{cpl}(C(B_{6l+2}) \otimes M_2) > (2l-2)\pi \ \text{ and } \ \mathrm{cpr}(C(B_{6l+2}) \otimes M_2) \geqslant (4l-3)$$

We will obtain the lower bounds $(***)$ as a corollary of a general relationship involving, in its simplest form, $\mathrm{cer}(B \otimes M_2)$, $\mathrm{cpr}(B \otimes M_2)$, and $\mathrm{cpr}(B)$. The basic idea behind this relationship first appeared in [15], and was restated as Proposition 2.10 of [10]. We introduce here a refinement which allows us to obtain, for certain C^*-algebras B, upper bounds on $\mathrm{cer}(B \otimes M_n)$ for large n which actually decrease as $n \to \infty$. The conditions we need on B are $\mathrm{tsr}(B) < \infty$ ($\mathrm{tsr}(B)$ being the topological stable rank of B [12]), $\mathrm{cer}(B \otimes M_k) < \infty$ for some $k \geqslant \mathrm{tsr}(B)$, and $\sup_{n \geqslant k} \mathrm{cpr}(B \otimes M_n) < \infty$. It seems to be often easier to verify all these conditions than it is to estimate $\mathrm{cer}(B \otimes M_n)$ directly. We will use this result, together with the upper bounds (*) and (**), to shed some new light on the still rather poorly understood behavior of $\mathrm{cer}(C(X) \otimes M_n)$.

This paper consists of three sections. The first section contains various preliminaries needed for the proofs of (*) and (**). The second section contains the proofs of (*) and (**), and also a somewhat better estimate which holds if X is a contractible finite complex. Since algebras of sections of locally trivial M_n-bundles (with trivial Dixmier-Douady class) force their way into our proofs anyway, we find it convenient to state our results for this slightly more general sort of algebra. Section 3 is devoted to the relationship between the exponential and projective ranks discussed above, its analog for the exponential and projective lengths, and some consequences. In particular, it contains the proof of (***) and some new upper bounds on $\mathrm{cer}(C(X) \otimes M_n)$.

The first two sections of this paper are a revised and improved version of Section 3 of the preliminary version of [10]. Section 3 is an improved and expanded version of several results which appeared in Sections 2 and 3 of the same preliminary version.

I am grateful to Peter Gilkey, who helped me greatly improve Lemma 1.5 (3). The original conditions for (*) and (**) were $n \geqslant 7(d+2)$ and $n \geqslant \frac{5d}{2}$ for large d, respectively. It is the improvement in Lemma 1.5 (3) that enabled me to reduce them to $n \geqslant 5d+3$ and $n \geqslant 2d+4$.

1. PRELIMINARIES

In this section, we prove, or merely state for convenient reference, several results that will be used in the proofs of the upper bound results of the next section.

Our basic approach to obtaining upper bounds on $\mathrm{cpl}(C(X) \otimes M_n)$ is as follows. Let $p, q \in C(X) \otimes M_n$ be homotopic projections. Let u be a unitary with $upu^* = q$. Assume, as we may, that p and q have rank at most $\frac{n}{2}$. Choose a subprojection p_1 of p, with a certain rank depending on n and $\dim(X)$. Set $q_1 = up_1u^*$. By a small perturbation, we will arrange to have $\mathrm{range}(p_1(x)) \cap \mathrm{range}(q_1(x)) = \{0\}$ for each x. This will enable us to find a projection e_1 such that $p_1, q_1 \leqslant e_1$ and $\mathrm{rank}(e_1(x)) = 2\,\mathrm{rank}(p_1(x))$. From this, we will obtain a projection $f_1 \leqslant 1 - e_1$ which is unitarily equivalent to p_1 and q_1. Then f_1 is also orthogonal to p_1 and q_1, so we can find $*$-symmetries v_1 and w_1 which conjugate p_1 to f_1 and f_1 to q_1. We now repeat this process, with $(1-p_1)[C(X)\otimes M_n](1-p_1)$ replacing $C(X)\otimes M_n$ (and therefore $\overline{n}=\mathrm{rank}(1-p_1)$ replacing n), and with $\overline{p}=p-p_1$ and $\overline{q}=v_1w_1qw_1v_1-p_1$ replacing p and q. Note that we now have $\mathrm{rank}(\overline{p})\leqslant \frac{\overline{n}}{2}-\frac{\mathrm{rank}(p_1)}{2}$. If n sufficiently large compared to $\dim(X)$, this process will terminate after, say, l steps, and we will have $\mathrm{cpl}(C(X)\otimes M_n)\leqslant 2l+\varepsilon$.

Since we have to consider fairly arbitrary corners in $C(X)\otimes M_n$, we may as well consider such algebras to begin with.

NOTATION 1.1. Let X be a compact metric space, and let V be a vector bundle over X. Then $L(V)$ denotes the locally trivial bundle of matrix algebras whose fiber over $x\in X$ is $L(V_x)$, the C^*-algebra of bounded operators on the fiber V_x. If E is a locally trivial bundle of matrix algebras over X, then $\Gamma(E)$ denotes the C^*-algebra of continuous sections of E. If $p\in\Gamma(L(V))$ is a projection, the pV denotes the subbundle of V whose fiber over x is $p(x)(V_x)$.

LEMMA 1.2. ([9], Proposition 4.2) *Let X be a compact Hausdorff space, and let E be a locally trivial M_n-bundle over X. Then the following are equivalent:*

(1) *$E\cong L(V)$ for some n-dimensional vector bundle V.*

(2) *$\Gamma(E)\cong p(C(X)\otimes K)p$ for some rank n projection $p\in C(X)\otimes K$, where K is the algebra of compact operators on a separable infinite dimensional Hilbert space.*

(3) *The Dixmier-Douady invariant of E is trivial.*

LEMMA 1.3. *Let X be a compact Hausdorff space, and let V be a vector bundle over X.*

(1) *The assignment $p\mapsto pV$ is a bijection from projections in $\Gamma(L(V))$ to subbundles of V.*

(2) *Projections $p,q\in\Gamma(L(V))$ are unitarily equivalent if and only if $pV\cong qV$ and $(1-p)V\cong(1-q)V$ as vector bundles.*

Proof. Immediate. ■

LEMMA 1.4. ([10], Lemma 3.4) *Let A be a unital C^*-algebra, and let $p,q\in A$ be unitarily equivalent orthogonal projections. Then there exists a $*$-symmetry v such that $vpv=q$.*

I am grateful to Peter Gilkey for suppling part (3) of the next lemma. (Also see Theorem 2.5 of [4].) The symbol $\langle x\rangle$ denotes the least integer n such that $x\leqslant n$.

LEMMA 1.5. *Let X be a finite simplicial complex of dimension at most d. Then:*

(1) *Every vector bundle over X of dimension $k\geqslant\left\langle\frac{d-1}{2}\right\rangle$ has a trivial direct*

summand of dimension $k - \left\langle \frac{d-1}{2} \right\rangle$.

(2) *Two stably isomorphic vector bundles of dimension at least* $\left\langle \frac{d}{2} \right\rangle$ *are isomorphic.*

(3) *If* E *and* F *are vector bundles over* X *with* $\dim(E) - \dim(F) \geqslant \left\langle \frac{d-1}{2} \right\rangle$, *then* F *is isomorphic to a direct summand of* E.

Proof.

(1) This is Theorem 8.1.2 of [7].

(2) This is Theorem 8.1.5 of [7].

(3) If $\dim(E) = \left\langle \frac{d-1}{2} \right\rangle$ then $F = 0$, and the result is trivial. Therefore we may assume $\dim(E) \geqslant \left\langle \frac{d}{2} \right\rangle$. Let W be a vector bundle such that $F \oplus W$ is trivial. Use (1) to write $E \oplus W \cong V \oplus (X \times \mathbf{C}^n)$ with $\dim(V) = \left\langle \frac{d-1}{2} \right\rangle$. Then

$$n = \dim(E) + \dim(W) - \left\langle \frac{d-1}{2} \right\rangle \geqslant \dim(E) + \dim(W).$$

Set $l = n - \dim(F) - \dim(W)$. Then $W \oplus F \cong X \times \mathbf{C}^{n-l}$. Therefore

$$E \oplus (X \times \mathbf{C}^{n-l}) \cong E \oplus W \oplus F \cong V \oplus (X \times \mathbf{C}^n) \oplus F.$$

Using (2) and $\dim(E) \geqslant \left\langle \frac{d}{2} \right\rangle$, we obtain

$$E \cong V \oplus (X \times \mathbf{C}^l) \oplus F,$$

showing that F is isomorphic to a direct summand in E. ■

LEMMA 1.6. *Let* $r, s, n \in \mathbf{N}$ *with* $r + s \leqslant n$, *and let* $p \in M_n$ *be a projection of rank* r. *Then the set* S *of projections* $q \in M_n$ *of rank* s *such that* $q(\mathbf{C}^n) \cap p(\mathbf{C}^n) \neq \{0\}$ *is the union of finitely many submanifolds of* $P(M_n)$, *each of (real) dimension at most* $2(r-1) + 2(s-1)(n-s)$.

Proof. For $1 \leqslant k \leqslant \min(r, s)$, let S_k be the set of projections $q \in M_n$ such that $q(\mathbf{C}^n) \cap p(\mathbf{C}^n)$ has (complex) dimension exactly k. Then S is the disjoint union of the sets S_k. We will now prove that S_k is a submanifold of $P(M_n)$ of dimension $2k(r-k) + 2(s-k)(n-s)$.

Let G_1 be the set of projections q_1 of rank k such that $q_1 \leqslant p$. Then G_1 is essentially the set of subspaces of $p(\mathbf{C}^n) \cong \mathbf{C}^r$ of complex dimension k, which is a Grassmannian manifold of real dimension $2k(r-k)$. For each $q_1 \in G_1$, let $G_2(q_1)$

be the set of projections q_2 of rank $s-k$ such that $q_2 \leqslant 1-q_1$. Then $G_2(q_1)$ is a Grassmannian of real dimension $2(s-k)(n-s)$, since $(1-q_1)(\mathbf{C}^n) \cong \mathbf{C}^{n-k}$. The set $G = \bigcup_{q_1 \in G_1} \{q_1\} \times G_2(q_1)$ is a locally trivial smooth fiber bundle over G_1, and therefore a manifold of dimension $\dim(G_1) + \dim(G_2(q_1))$ (for any $q_1 \in G_1$). The formula $f(q) = (q_1, q-q_1)$, where q_1 is the projection onto $q(\mathbf{C}^n) \cap p(\mathbf{C}^n)$, defines a diffeomorphism from S_k onto an open subset of G. (This subset is open because its complement $\{(q_1,q_2) \in G : \dim(q_2(\mathbf{C}^n) \cap p(\mathbf{C}^n)) \geqslant 1\}$ is closed.) Therefore S_k is a manifold of the required dimension.

To prove the lemma, it remains to prove that the largest value of this dimension occurs when $k=1$. A calculation shows that

$$\dim(S_k) - \dim(S_{k+1}) = n - (s+r) + 2k + 1 > 0.$$

■

LEMMA 1.7. *Let X be a finite simplicial complex of dimension d, let V be an n-dimensional vector bundle over X, let $p_0, q_0 \in \Gamma(L(V))$ be projections with constant ranks r and s respectively, and let $\varepsilon > 0$. Assume that $d < 2(n-(r+s)+1)$. Then there exist projections $p, q \in \Gamma(L(V))$ such that $\|p-p_0\|$, $\|q-q_0\| < \varepsilon$ and, for every $x \in X$, $p(x)(V_x) \cap q(x)(V_x) = \{0\}$. For any such p and q, there is a projection $e \in \Gamma(L(V))$ of constant rank $r+s$ such that $e \geqslant p$ and $e \geqslant q$.*

Proof. We will construct the perturbation first on the 0-skeleton, then the 1-skeleton, etc., finishing with the d-skeleton. Using the method of proof of Lemma 2.5 of [8], we reduce to the case of constructing the perturbations on a given k-cell, given that the required properties are already satisfied on its boundary. Since a k-cell is contractible, V is trivial over it, and so we can reduce to the case $\Gamma(L(V)) = C(X) \otimes M_n$. Applying a homeomorphism, we can assume we are given p_0, q_0 on the closed unit ball $B_k \subset \mathbf{R}^k$, with $k \leqslant d$, and that $p_0(x)(\mathbf{C}^n) \cap q_0(x)(\mathbf{C}^n) = \{0\}$ for every $x \in \partial B_k = S^{k-1}$. Using contractibility again, p_0 is unitarily equivalent to a constant projection, and we may therefore assume p_0 is a constant projection, $p_0(x) = \overline{p}$ for all x. Now $T = \{\overline{q} \in M_n : \overline{q}\mathbf{C}^n \cap \overline{p}\mathbf{C}^n = \{0\}\}$ is an open subset of $P(M_n)$, so we can assume $q_0(x) \in T$ for $\|x\| \geqslant 1-3\delta$ for some $\delta > 0$. We can furthermore approximate q_0 arbitrarily closely by a projection which agrees with q_0 for $\|x\| \geqslant 1-\delta$, still is in T for $\|x\| \geqslant 1-3\delta$, and is smooth for $\|x\| \geqslant 1-2\delta$. Using the proof of the Transversality Homotopy Theorem ([5], page 70), we can, by a further arbitrarily small perturbation on $\{x \in B_k : \|x\| < 1-2\delta\}$, find a projection q which is transverse to each of the finitely many submanifolds of the previous lemma, using $\overline{p}$ in place of p. One checks that, for each of these submanifolds M, one has $\dim(B_k) + \dim(M)$ strictly less than the dimension $2s(n-s)$ of the space of rank s projections in M_n, using the previous lemma and the inequalities $k \leqslant d$ and $d < 2(n-(r+s)+1)$. Transversality therefore implies that the range of q does not interest any of these manifolds. Therefore $p = p_0$ and q form the required perturbation.

It remains only to prove the existence of e. Let $W_x = \text{span}(p(x)V_x \cup q(x)V_x)$. Since $p(x)V_x \cap q(x)V_x = \{0\}$ for all x, the obvious vector space homomorphism $a(x) : : p(x)V_x \oplus q(x)V_x \to W_x$ is bijective for all x. Therefore $x \mapsto W_x$ is a vector bundle, isomorphic to $pV \oplus qV$. It is a subbundle of V, and we can simply let $e(x)$ be the orthogonal projection from V_x onto W_x. ■

LEMMA 1.8. *Let X be a compact metric space of dimension at most d in the sense of [6], and let V be an n-dimensional vector bundle over X. Then there exist finite simplicial complexes X_k of dimension at most d, n-dimensional vector bundles V_k over X_k, and maps $\varphi_k : \Gamma(L(V_k)) \to \Gamma(L(V_{k+1}))$ such that $\Gamma(L(V)) \cong \varinjlim \Gamma(L(V_k))$.*

Proof. We note that by Theorem 1.7.7 of [2], all three of the usual definitions of dimension agree for compact metric spaces. By Theorem 1.13.5 of [2] there exist X_k as in the statement and maps $X_{k+1} \to X_k$ such that $X \cong \varprojlim X_k$. (I am grateful to Dusan Repovs for supplying this reference.) Then $C(X) \otimes K \cong \varinjlim C(X_k) \otimes K$. By Lemma 1.2 there is a rank n projection $p \in C(X) \otimes K$ such that $p[C(X) \otimes K]p \cong \cong \Gamma(L(V))$. Standard methods produce l and a projection $q_l \in C(X_k) \otimes K$ whose image $q \in C(X) \otimes K$ satisfies $\|q - p\| < \frac{1}{2}$. Then q is unitarily equivalent to p in $(C(X) \otimes K)^+$. In particular, q also has rank n, and $q[C(X) \otimes K]q \cong p[C(X) \otimes K]p \cong \cong \Gamma(L(V))$.

Dropping the initial terms of the sequence, we may assume that $l = 1$. We may furthermore clearly replace each X_k by the union of the connected components of X_k which intersect the image of X in X_k, and restrict q_1 appropriately. It is now easily seen that q_1 has constant rank n. Let q_k be the image of q_1 in $C(X_k) \otimes K$, and use Lemma 1.2 to produce an n-dimensional vector bundle V_k such that $\Gamma(L(V_k)) \cong \cong q_k[C(X_k) \otimes K]q_k$. The vector bundles V_k and the maps $\varphi_k : q_k[C(X) \otimes K]q_k \to \to q_{k+1}[C(X_{k+1}) \otimes K]q_{k+1}$ clearly satisfy the conclusions of the lemma. ■

2. UPPER BOUNDS ON THE PROJECTIVE LENGTH OF $C(X) \otimes M_n$

We prove in this section that $\text{cpl}(C(X) \otimes M_n) \leqslant 2\pi$ if $n \geqslant 5d + 3$, and that $\text{cpl}(C(X) \otimes M_n) \leqslant 3\pi$ if $n \geqslant 2d+4$. We obtain better results if X is a contractible finite complex. We actually state and prove our results for n-homogeneous C^*-algebras with trivial Dixmier-Douady class (compare Lemma 1.2), since the method of proof forces us to consider such algebras anyway.

These results are analogs for projective length and rank of results in Section 3 and 4 of [9]. Note that, unlike [9], they give explicit estimates on how large n must be, and they give smaller upper bounds than those implied by [9].

THEOREM 2.1. *Let X be a compact metric space of dimension at most d, and let V be a vector bundle over X of dimension $n \geqslant 5d+3$. Then*

$$\mathrm{cpl}(\Gamma(L(V))) \leqslant 2\pi \ \text{ and } \ \mathrm{cpr}(\Gamma(L(V))) \leqslant 4+\varepsilon.$$

Proof. By Proposition 2.11 of [10] and Lemma 1.8, we may assume X is a finite simplicial complex of dimension at most d. We may obviously further assume that X is connected.

Let $r = \left[\frac{n}{2}\right]$, the greatest integer less than or equal to $\frac{n}{2}$. Let r_1 be the integer closest to $\frac{3n}{10}$; round down if $\frac{3n}{10}$ is halfway between two integers. This gives $\frac{3n-5}{10} \leqslant$ $\leqslant r_1 \leqslant \frac{3n+4}{10}$. Let $r_2 = r - r_1$.

We claim that the following seven inequalities are satisfied:

$$r_1 \geqslant \left\langle \frac{d-1}{2} \right\rangle. \tag{1}$$

$$2(n-2r_1+1) > d. \tag{2}$$

$$n-3r_1 \geqslant \left\langle \frac{d-1}{2} \right\rangle. \tag{3}$$

$$n-r_1 \geqslant \left\langle \frac{d}{2} \right\rangle. \tag{4}$$

$$2(n-r_1-2r_2+1) > d. \tag{5}$$

$$n-r_1-3r_2 \geqslant \left\langle \frac{d-1}{2} \right\rangle. \tag{6}$$

$$n-r_1-r_2 \geqslant \left\langle \frac{d}{2} \right\rangle. \tag{7}$$

To verify (1), note that

$$r_1 \geqslant \frac{3n-5}{10} \geqslant \frac{15d+4}{10} \geqslant \frac{3d}{2} \geqslant \frac{d}{2} \geqslant \left\langle \frac{d-1}{2} \right\rangle.$$

For (3),

$$n - 3r_1 \geqslant n - \frac{3(3n+4)}{10} = \frac{n-12}{10} \geqslant \frac{d-1}{2} - \frac{2}{5}.$$

Now $n-3r_1$ is an integer, and there are no integers in the interval $\left[\frac{d-1}{2} - \frac{2}{5}, \frac{d-1}{2}\right)$. Therefore $n - 3r_1 \geqslant \frac{d-1}{2}$, and, again because $n - 3r_1$ is an integer, it follows that $n - 3r_1 \geqslant \left\langle \frac{d-1}{2} \right\rangle$. For (6),

$$n - r_1 - 3r_2 = n + 2r_1 - 3\left[\frac{n}{2}\right] \geqslant n + \frac{2(3n-5)}{10} - \frac{3n}{2} = \frac{n-10}{10}.$$

We have already proved that $\frac{n-12}{10} \geqslant \left\langle \frac{d-1}{2} \right\rangle$, so (6) follows.

For the remaining inequalities, note that

$$r_1 \geqslant \frac{3n-5}{10} \geqslant \frac{4}{10},$$

since $d \geqslant 0$. Therefore $r_1 \geqslant 1$, so (4) follows from (3). Also,

$$n - 2r_1 + 1 \geqslant \left\langle \frac{d-1}{2} \right\rangle + 1 > \frac{1}{2},$$

using (3), and (2) follows. Similarly, (5) follows from (6), and (7) follows from (6) if $r_2 \geqslant 1$ and from (4) if $r_2 = 0$.

Now let p and q be homotopic projections in $\Gamma(L(V))$. Since X is connected, they have the same constant rank, say s. Replacing p and q by $1-p$ and $1-q$ if necessary, we can assume $s \leqslant \frac{n}{2}$. Since s is an integer, this gives $s \leqslant r$. Using Lemma 1.5 (1), Lemma 1.3 (1), and inequality (1), we can write $p = p_1 + p_2$ where p_1 and p_2 are orthogonal projections with $s_i = \operatorname{rank}(p_i) \leqslant r_i$ and with p_2V trivial. The homotopy from p to q yields $w \in U_0(\Gamma(L(V)))$ such that $wpw^* = q$. Set $q_i = wp_iw^*$, and note that p_i is homotopic to q_i.

We will now find, for arbitrary $\varepsilon > 0$, a unitary u_1 with $\operatorname{cel}(u_1) < \pi + \frac{\varepsilon}{2}$ and $u_1p_1u_1^* = q_1$. By Lemma 1.7 and inequality (2), there are arbitrarily small perturbations $\overline{p}_1$ and $\overline{q}_1$ of p_1 and q_1 such that $\overline{p}_1(x)V_x \cap \overline{q}_1(x)V_x = \{0\}$ for all x. Then there is, also by the same lemma, a projection $e_1 \in \Gamma(L(V))$ of constant rank $2r_1$ such that $\overline{p}_1, \overline{q}_1 \leqslant e_1$. According to Lemma 1.5 (3) and inequality (2), there is a subprojection f_1 of $1 - e_1$ such that $f_1V \cong p_1V$. Note that $(1-\overline{p}_1)V$, $(1-f_1)V$, and $(1-\overline{q}_1)V$ are stably isomorphic vector bundles of dimension at least $n - r_1$, and so they are isomorphic by Lemma 1.5 (3) and inequality (4). Therefore, by Lemma 1.3 (2), we have f_1 unitarily equivalent to $\overline{p}_1$ and $\overline{q}_1$. By Lemma 1.4 there are $*$-symmetries v_1, v_2 with $v_1\overline{p}_1v_1 = f_1$ and $v_2f_1v_2 = \overline{q}_1$. There are also unitaries v_3 and v_4 close to 1 such that $v_3 p_1 v_3^* = \overline{p}_1$ and $v_4 \overline{q}_1 v_4^* = q_1$. Then for $\overline{p}_1, \overline{q}_1$ close enough to p_1, q_1, the unitary $u_1 = v_4 v_2 v_1 v_3$ satisfies $u_1 p_1 u_1 = q_1$ and $\mathrm{cel}(u_1) < \pi + \frac{\varepsilon}{2}$.

To finish the proof, we now find a unitary u_2 in

$$A = (1 - q_1)\Gamma(L(V))(1 - q_1) = \Gamma(L((1 - q_1)V))$$

such that $\mathrm{cel}(u_2) < \pi + \frac{\varepsilon}{2}$ and $u_2(u_1 p_2 u_1^*)u_2^* = q_2$. Then $u = [u_2 + (1 - q_1)]u_1$ will be a unitary in $\Gamma(L(V))$ such that $\mathrm{cel}(u) \leqslant 2\pi + \varepsilon$ and $upu^* = q$. The relation $\mathrm{cpl}(\Gamma(L(V))) \leqslant 2\pi$ will follow from Theorem 1.9 of [10], and the relation $\mathrm{cpr}(\Gamma(L(V)) \leqslant 4 + \varepsilon$ will follow from Theorem 2.4 (1) of [10].

As before, Lemma 1.7 and inequality (5) imply that we can perturb $u_1 p_2 u_1^*$ and q_2 by an arbitrarily small amount to get projections $\overline{p}_2, \overline{q}_2 \in A$ such that there is a projection $e_2 \in A$ of rank $2r_2$ with $\overline{p}_2, \overline{q}_2 \leqslant e_2$. It follows from Lemma 1.5 (1) and inequality (6) that there is $f_2 \leqslant (1-q_1)-e_2$ such that $f_2 V$ is trivial of rank s_2, and so isomorphic to both $\overline{p}_2 V$ and $\overline{q}_2 V$. Also the vector bundles $(1-q_1-\overline{p}_1)V$, $(1-q_1-f_2)V$, and $(1 - q_1 - \overline{q}_2)V$ are all stably isomorphic, and so isomorphic by Lemma 1.5 (2) and inequality (7). So, using Lemma 1.3 (2), we see that $\overline{p}_2, f_2$ and $f_2, \overline{q}_2$ are pairs of unitarily equivalent orthogonal projections in A, and the existence of u_2 follows in the same way as above. ■

Corollary 2.2. (Compare [9], Corollary 3.5.) *For each integer $d \geqslant 0$ there are numbers $C_1(d) < \infty$ and $C_2(d) < \infty$ such that for any n and any compact metric space X of dimension at most d,*

$$\mathrm{cpl}(C(X) \otimes M_n) \leqslant C_1(d) \quad \text{and} \quad \mathrm{cpr}(C(X) \otimes M_n) \leqslant C_2(d).$$

Proof. Define

$$C_1(d, n) = \sup\{\mathrm{cpl}(C(X) \otimes M_n) : X \text{ is compact metric, } \dim(X) \leqslant d\}.$$

It is easy to see, by factoring out the determinant, that if $p, q \in C(X) \otimes M_n$ are homotopic projections, then there is a homotopically trivial $u : X \to \mathrm{SU}_n$ such that $upu^* = q$. It follows from Lemma 3.1 of [9] that there is a finite upper bound, depending only on n and $d = \dim(X)$, for $\mathrm{cel}(u)$ for such unitaries u. Theorem 1.9 of [10] therefore implies that $C_1(d, n) < \infty$ for all d and n. For fixed d, the previous theorem implies $\lim\limits_{n\to\infty} \sup C_1(d, n) \leqslant 2\pi$. So $C_1(d) = \sup\limits_n C_1(d, n) < \infty$. We now get $C_2(d)$ from Theorem 2.4 of [10]. ■

The inequalities (1) and (3) of the proof of Theorem 2.1 imply by themselves that n is at least approximately $2d$. By splitting projections into three pieces instead of two, we will see that we can indeed get $\mathrm{cpl}(\Gamma(L(V))) \leqslant 3\pi$ for $n \geqslant 2d+4$. No significant improvement is made to the allowed values of n by allowing more pieces, except in the special case of contractible spaces, dealt with in Theorem 2.4 below. Even in the cases analogous to the previous theorem and the next theorem, we obtain better results for contractible spaces.

THEOREM 2.3. *Let X be a compact metric space of dimension at most d, and let V be a vector bundle over X of dimension $n \geqslant 2d+4$. Then*

$$\mathrm{cpl}(\Gamma(L(V))) \leqslant 3\pi \quad \text{and} \quad \mathrm{cpr}(\Gamma(L(V))) \leqslant 6+\varepsilon.$$

Proof. We only describe how the proof of the previous theorem needs to be modified. As before, we may assume X is a connected finite complex. Let $r = \left[\frac{n}{2}\right]$, and choose integers r_1 and r_2 to satisfy the following inequalities:

$$\frac{n}{4} - \frac{1}{4} \leqslant r_1 \leqslant \frac{n}{4} + \frac{1}{2} \quad \text{and} \quad \frac{n}{8} - \frac{3}{8} \leqslant r_2 \leqslant \frac{n}{8} + \frac{1}{2}.$$

Set $r_3 = r - r_1 - r_2$. If n is even, then we actually have $r_1 \geqslant \frac{n}{4}$ and $r_2 \geqslant \frac{n}{8} - \frac{1}{4}$, so that

$$r_3 \leqslant \frac{n}{2} - \frac{n}{4} - \left(\frac{n}{8} - \frac{1}{4}\right) = \frac{n}{8} + \frac{1}{4},$$

that is

$$r_3 \leqslant \frac{n}{8} + \frac{1}{4}. \tag{$*$}$$

If n is odd, then $r = \frac{n}{2} - \frac{1}{2}$, and an estimate similar to the above gives $r_3 \leqslant \frac{n}{8} + \frac{1}{8}$. Thus, $(*)$ holds in this case also. One can similarly check that $r_3 \geqslant \frac{n-9}{8}$; since $n \geqslant 4$ and r_3 is an integer, this implies $r_3 \geqslant 0$.

We now claim that the inequalities (1)-(7) of the previous proof hold, along with:

$$2(n - r_1 - r_2 - 2r_3 + 1) > d. \tag{8}$$

$$n - r_1 - r_2 - 3r_3 \geqslant \left\langle \frac{d-1}{2} \right\rangle. \tag{9}$$

$$n - r_1 - r_2 - r_3 \geqslant \left\langle \frac{d}{2} \right\rangle. \tag{10}$$

It suffices to prove (1), (3), (6), and (9), along with the inequality $r_1 \geqslant 1$. Indeed, (2) follows from (3) as in the proof of the previous theorem, and similarly (5) follows from (6) and (8) from (9). Also, (4) follows from (3) and $r_1 \geqslant 1$, (7) follows from (6) if $r_2 \geqslant 1$ and from (4) if $r_2 = 0$, and (10) follows from (9) if $r_3 \geqslant 1$ and from (7) if $r_3 = 0$.

We now verify (1), (3), (6), (9), and $r_1 \geqslant 1$. For (1),

$$r_1 \geqslant \frac{n}{4} - \frac{1}{4} \geqslant \frac{d}{2} + \frac{3}{4} \geqslant \frac{d}{2} \geqslant \left\langle \frac{d-1}{2} \right\rangle.$$

For (3),

$$n - 3r_1 \geqslant n - 3\left(\frac{n}{4} + \frac{1}{2}\right) = \frac{n}{4} - \frac{3}{2} \geqslant \frac{d}{2} - \frac{1}{2}.$$

Since $n - 3r_1$ is an integer, we have $n - 3r_1 \geqslant \left\langle \frac{d-1}{2} \right\rangle$, which is (3). For (6),

$$n - r_1 - 3r_2 \geqslant \frac{3n}{8} - 2 \geqslant \frac{3d}{4} - \frac{1}{2} \geqslant \frac{d}{2} - \frac{1}{2}.$$

Now (6) follows as above. For (9),

$$n - r_1 - r_2 - 3r_3 \geqslant n - \left(\frac{n}{4} + \frac{1}{2}\right) - \left(\frac{n}{8} + \frac{1}{2}\right) - 3\left(\frac{n}{8} + \frac{1}{4}\right) = \frac{n-7}{4} \geqslant \frac{d-1}{2} - \frac{1}{4}.$$

The least integer greater than or equal to $\frac{d-1}{2} - \frac{1}{4}$ is the same as the least integer greater than or equal to $\frac{d-1}{2}$, so, since $n - r_1 - r_2 - 3r_3$ is an integer, we obtain (9). Finally, $r_1 \geqslant \frac{2d+3}{4}$; since $d \geqslant 0$ and r_1 is an integer, we do in the fact get $r_1 \geqslant 1$.

Now let $p, q \in \Gamma(L(V))$ be homotopic projections, of constant rank s. As before, we may assume $s \leqslant \frac{n}{2}$. Using Lemma 1.5 (1) and inequality (1) (from the previous proof), we can write $p = p_1 + p_2 + p_3$, a sum of orthogonal projections with rank $(p_i) \leqslant r_i$ and p_2V and p_3V trivial. The rest of the proof is essentially the same as the previous proof, conjugating first p_1, then p_2, and finally p_3 to appropriate subprojections of q. ■

If X is contractible, then all vector bundles are trivial. This makes possible an improvement of the previous results, which we give next. The term $\left(\frac{2^k}{2^k - 2}\right) \cdot \left(2\left(\frac{3}{2}\right)^k - 3\right)$ accounts for the errors in rounding to integers, and can be omitted entirely if n is divisible by $2(2^k - 1)$; see Remark 2.5 below. In any case, it is not excessively large for small values of k and large values of d. Note that our theorem applies in particular to the closed unit ball in $\mathbf{R}^d$.

THEOREM 2.4. *Let X be a contractible finite complex of dimension at most d. Let*

$$n \geqslant \left(\frac{2^k - 1}{2^k - 2}\right) d + \left(\frac{2^k}{2^k - 2}\right)\left(2\left(\frac{3}{2}\right)^k - 3\right),$$

for some integer $k \geqslant 2$. Then

$$\operatorname{cpl}(C(X) \otimes M_n) \leqslant k\pi \quad \text{and} \quad \operatorname{cpr}(C(X) \otimes M_n) \leqslant 2k + \varepsilon.$$

Proof. For $1 \leqslant l \leqslant k$ define, by induction on l, a number r_l to be the largest integer satisfying

$$r_l < \frac{1}{2}\left(n - r_1 - \cdots - r_{l-1} - \frac{d}{2} + 1\right), \tag{1}$$

or $r_l = 0$ if the right-hand side of (1) is nonpositive. Note that, for $l = 1$, the right-hand side is $\frac{1}{2}\left(n - \frac{d}{2} + 1\right)$, enabling the induction to start. We now prove the following four relations by induction on l:

$$r_l \geqslant 2^{-l}\left(n - \frac{d}{2}\right) - \frac{1}{2}\left(\frac{3}{2}\right)^{l-1}. \tag{2}$$

$$r_1 + \cdots + r_l < (1 - 2^{-l})\left(n - \frac{d}{2}\right) + \left(\frac{3}{2}\right)^l - 1. \tag{3}$$

$$r_1 + \cdots + r_l \geqslant (1 - 2^{-l})\left(n - \frac{d}{2}\right) - \left(\frac{3}{2}\right)^l + 1. \tag{4}$$

$$0 \leqslant r_{l+1} \leqslant r_l. \tag{5}$$

First, note that (5) is obvious from (1), given the restriction that $r_l = 0$ if the right-hand side of (1) is nonpositive. To start on the others, note that for $l = 1$ the inequality (2) follows from the fact that r_l is the greatest integer satisfying (1). Inequalities (3) and (4) are just disguised forms of (1) and (2). (Note that the condition on n implies $\frac{1}{2}\left(n - \frac{d}{2} + 1\right) > 0$.)

So assume (2)-(4) hold for l, and the right-hand side of (1) is strictly positive. Then, using (1) and (4),

$$r_{l+1} < \frac{1}{2}\left(n\left[(1 - 2^{-l})\left(n - \frac{d}{2}\right) - \left(\frac{3}{2}\right)^l + 1\right] - \frac{d}{2} + 1\right) =$$

$$= 2^{-(l+1)}\left(n - \frac{d}{2}\right) + \frac{1}{2}\left(\frac{3}{2}\right)^l.$$

Adding this to (3) gives (3) for $l+1$. Also, using (3) and the fact that r_l is the greatest integer satisfying (1),

$$r_{l+1} \geqslant \frac{1}{2}\left(n\left[(1-2^{-l})\left(n-\frac{d}{2}\right)+\left(\frac{3}{2}\right)^l-1\right]-\frac{d}{2}+1\right)-\frac{1}{2}=$$

$$=2^{-(l+1)}\left(n-\frac{d}{2}\right)-\frac{1}{2}\left(\frac{3}{2}\right)^l,$$

which is (2) for $l+1$. Adding it to (4) gives (4) for $l+1$.

If on the other hand the right-hand side of (1) is nonpositive, then r_{l+1} is chosen larger than called for by (1). This does not affect the proof of (2), and hence also of (4), for $l+1$. To get (3), observe that we are adding $r_{l+1}=0$ to the left-hand side, while the right-hand side is a nondecreasing function of l (since $n \geqslant \frac{d}{2}$). So (3) also holds for $l+1$. Thus, we have proved (2)-(4).

We now claim that

$$r_1+\cdots+r_k \geqslant \left[\frac{n}{2}\right], \tag{6}$$

where $\left[\frac{n}{2}\right]$ is the greatest integer with $\left[\frac{n}{2}\right] \leqslant \frac{n}{2}$. Since $r_1+\cdots+r_k$ is an integer, it suffices to prove that $r_1+\cdots+r_k \geqslant \frac{n}{2}-\frac{1}{2}$. Using (4), we see that it is enough to show that

$$(1-2^{-k})\left(n-\frac{d}{2}\right)-\left(\frac{3}{2}\right)^k+1 \geqslant \frac{n}{2}-\frac{1}{2}.$$

Some algebra shows that this inequality is exactly the condition on n in the statement of the theorem.

Now let $p, q \in C(X) \otimes M_n$ be homotopic projections. Then they have the same rank, say s. Replacing p and q by $1-p$ and $1-q$ if necessary, we may assume $s \leqslant \left[\frac{n}{2}\right]$. Since X is contractible, p is unitarily equivalent to the constant projection $p_0(x) = \mathbf{1}_s \oplus \mathbf{0}_{n-s}$ (where $\mathbf{1}_s \in M_s$ and $\mathbf{0}_{n-s} \in M_{n-s}$). Conjugating p and q by the unitary involved, we may assume $p = p_0$. Also, since p and q are homotopic, there is a unitary u such that $uqu^* = p$.

At this point, we divide into two cases, the first of which is $3r_1 > n$. Combining this inequality with (1), and noting that (1) holds because $r_1 \neq 0$, we get $\frac{2n}{3} > d-2$. Let $s_1 = \min(s, \left[\frac{n}{3}\right])$, and let $p_1 \leqslant p$ be the constant projection $p_1(x) = \mathbf{1}_{s_1} \oplus \mathbf{0}_{n-s_1}$. Let $q_1 = u^* p_1 u$. We have

$$2(n-2s_1+1) \geqslant 2\left(\frac{2n}{3}+1\right) > d,$$

so Lemma 1.4 enables us to perturb p_1 and q_1 slightly so as to get projections $\overline{p}_1$ and $\overline{q}_1$ such that $\overline{p}_1(x)(\mathbb{C}^n) \cap \overline{q}_1(x)(\mathbb{C}^n) = \{0\}$ for all x, and a projection e of rank

$2s_1$ such that $\overline{p}_1, \overline{q}_1 \leqslant e$. Now $(1-e)(X \times \mathbb{C}^n)$ is a trivial vector bundle (since X is contractible), and has rank at least $\frac{n}{3} \geqslant s_1$, so there is a projection $f \leqslant 1-e$ which is unitarily equivalent to both $\overline{p}_1$ and $\overline{q}_1$. It follows, as in the proof of Theorem 3.1, that there is a unitary u_1 such that $\mathrm{cel}(u_1) < \pi + \varepsilon$ and $u_1 q_1 u_1^* = p_1$.

If $s_1 = s$, we have shown that the rectifiable distance $d_r(p,q)$ satisfies $d_r(p,q) < \pi + \varepsilon$. If not, we further split into the two subcases $k = 2$ and $k \geqslant 3$. If $k = 2$, set $s_2 = s - s_1$. Then

$$s_2 \leqslant \left[\frac{n}{2}\right] - \left[\frac{n}{3}\right] \leqslant \left[\frac{n}{6}\right] + \frac{2}{3}.$$

The condition on n is $n \geqslant \frac{3d}{2} + 3$. Therefore

$$2(n - s_1 - 2s_2 + 1) = 2(n - s - s_2 + 1) \geqslant \frac{2n}{3} + \frac{2}{3} > d. \tag{7}$$

Also,

$$n - s_1 - 3s_2 = n - s - 2s_2 \geqslant \frac{n}{6} - \frac{4}{3}.$$

If $n \geqslant 8$, this implies

$$n - s_1 - 3s_2 \geqslant 0. \tag{8}$$

For $n = 5, 6$, or 7, one can check directly that

$$n - s_1 - 3s_2 \geqslant n - \left[\frac{n}{3}\right] - 3\left(\left[\frac{n}{2}\right] - \left[\frac{n}{3}\right]\right) \geqslant 0,$$

thus yielding (8) in these cases also. (If $n \leqslant 4$, then $d = 0$, and the algebra $C(X) \otimes M_n$ has real rank 0 and cancellation. Therefore the conclusion of our theorem follows from Theorem 3.2 of [10]). The inequalities (7) and (8) suffice to be able to apply the argument of the previous paragraph to the rank s_2 projections $p_2 = p - p_1$ and $q_2 = u_1 q_1 u_1^* - p_1$ in $(1-p_1)(C(X) \otimes M_n)(1-p_1) \cong C(X) \otimes M_{n-s_1}$. The result is a unitary $v \in C(X) \otimes M_{n-s_1}$ with $\mathrm{cel}(v) < \pi + \varepsilon$ and $v q_2 v^* = p_2$. Setting $u_2 = p_1 + v$, we get $\mathrm{cel}(u_2 u_1) < 2\pi + 2\varepsilon$ and $(u_2 u_1) q (u_2 u_1)^* = p$. Thus $d_r(p,q) < 2\pi + 2\varepsilon$. Since p, q, and $\varepsilon > 0$ are arbitrary, the theorem is proved in this case.

If $k \geqslant 3$, set $s_2 = \min\left(s - s_1, \left[\frac{n}{6}\right]\right)$. The assumption $3r_1 > n$ and the requirement $r_1 < \frac{1}{2}\left(n - \frac{d}{2} + 1\right)$ imply that $n > \frac{3d}{2} - 3$. Therefore

$$2(n - s_1 - 2s_2 + 1) \geqslant \frac{2n}{3} + 2 > d.$$

It is clear that $n - s_1 - 3s_2 \geqslant 0$. Therefore the method of previous paragraph, applied to the rank s_2 subprojection $p_2 = 0 \oplus 1_{s_2} \oplus 0_{n-s_1-s_2}$ and the corresponding rank s_2 subprojection $q_2 = u_1u^*p_2uu_1^* \leqslant u_1qu_1^* - p_1$, produces a unitary u_2 with $\mathrm{cel}(u_2) < \pi+\varepsilon$ and $u_2u_1q(u_2u_1)^* \geqslant p_1+p_2$. If $s_2 = s - s_1$, we have $d_\Gamma(p,q) < 2\pi+2\varepsilon$, and we are done. Otherwise set $s_3 = s - s_1 - s_2$. Then

$$s_3 \leqslant \left[\frac{n}{2}\right] - \left[\frac{n}{3}\right] - \left[\frac{n}{6}\right] \leqslant \frac{2}{3} + \frac{5}{6}.$$

Since s_3 is an integer, we have $s_3 \leqslant 1$. Therefore

$$2(n - s_1 - s_2 - 2s_3 + 1) = 2(n - s - s_3 + 1) \geqslant 2(n - s) \geqslant n > d$$

(the last step following from the condition on n in the statement of the theorem) and

$$n - s_1 - s_2 - 3s_3 \geqslant n - s - 2 \geqslant \frac{n}{2} - 2 \geqslant 0$$

(since if $k \geqslant 3$ then $n \geqslant 4$). Another step similar to the one at the beginning of this paragraph shows that $d_\Gamma(p,q) < 3\pi + 3\varepsilon$. Since $3 \leqslant k$ and $\varepsilon > 0$ is arbitrary, we are done in this case.

It remains to consider the case $3r_1 \leqslant n$. We inductively define $s_1 = \min(s, r_1)$ and $s_j = \min(s - s_1 - \cdots - s_{j-1}, r_j)$. The relations (5), (6), and $s \leqslant \left[\frac{n}{2}\right]$ imply that the s_j are nonincreasing and sum to s. Let the nonzero s_j be $s_1, \ldots, s_l$; then $l \leqslant k$ (by (6)). We now claim that l steps, of the sort used in the argument for $3r_1 > n$, yield a unitary v with $\mathrm{cel}(v) < l(\pi + \varepsilon)$ and $vqv^* = p$. This will show $d_\Gamma(p,q) < l(\pi + \varepsilon) \leqslant k(\pi + \varepsilon)$ for any $\varepsilon > 0$, and prove the theorem. Just as above, the conditions we need for the j step are:

$$2(n - s_1 - \cdots - s_{j-1} - 2s_j + 1) > d \tag{9}$$

and

$$n - s_1 - \cdots - s_{j-1} - 3s_j + 1 \geqslant 0. \tag{10}$$

Relation (9) follows from the relation obtained by substituting r_i for s_i, which is exactly (1) for j. (Since $0 \leqslant s_j \leqslant r_j$ and $s_j \neq 0$, we have $r_j \neq 0$, so (1) does in fact hold.) The relation (10) will follow from (9) provided $s_j \leqslant \frac{d}{2} - 1$. By assumption, $3r_1 \leqslant n$, and we therefore have

$$r_1 \geqslant \frac{1}{2}\left(n - \frac{d}{2} + 1\right) \geqslant \frac{1}{2}\left(3r_1 - \frac{d}{2} + 1\right),$$

from which we obtain:

$$\frac{d}{2} - 1 \geqslant r_1 \geqslant r_j \geqslant s_j,$$

using (5). This proves (10) and completes the proof of the theorem. ■

REMARK 2.5. The extra constants in Theorems 2.1, 2.3, and 2.4 are present to account for the rounding errors that accumulate because the ranks of projections must be integers. Possibly they can be improved by paying more careful attention to the number theory. For example, in Theorem 2.1, if d is odd and at least 3, and $n = 5d - 5$, then the proof goes through using $r_1 = \dfrac{3n}{10}$ and $r_2 = \dfrac{2n}{10}$, yielding $\mathrm{cpl}(C(X) \otimes M_n) \leqslant 2\pi$. Similarly, in Theorem 2.4, if $d = 2(2^k - 2)m$ for some integer m, and if $n = \left(\dfrac{2^k - 1}{2^k - 2}\right) d$, without the extra term, then in the proof of the theorem we can take $r_j = 2^{k-j}m$, and obtain $\mathrm{cpl}(C(X) \otimes M_n) \leqslant k\pi$.

3. A RELATION BETWEEN THE EXPONENTIAL AND PROJECTIVE RANKS

In this section, we will present an improvement and generalization of Proposition 2.10 of [10]. As a consequence we will show that projective length and projective rank can be arbitrarily large. (See Theorem 3.10 and Corollary 3.11.) Our result also sheds some further light on the behavior of $\mathrm{cer}(C(X) \otimes M_n)$ as a function of n (see Corollary 3.9) and provides the first concrete evidence for a connection between exponential rank and topological stable rank [12] (see Theorem 3.8).

The idea we exploit here first appeared in the proof of Theorem 2.2 and Corollary 2.3 of [15]. The version we present here incorporates a slight refinement, which allows us (unlike [15]) to obtain in good cases upper bounds on $\mathrm{cer}(B \otimes M_n)$ which actually decrease with n. (See Theorem 3.8 and Corollary 3.9).

DEFINITION 3.1. Let A be a unital C^*-algebra. Then $U(A)$ is the unitary group of A and $U_0(A)$ is the connected component of $U(A)$ which contains the identity. If $p \in A$ is a projection, then we say that the inclusion $pAp \to A$ is *injective on U/U_0* if the map $U(pAp)/U_0(pAp) \to U(A)/U_0(A)$, induced by $u \mapsto u + (1-p)$, is injective. In matrix notation, with respect to the decomposition $1 = p + (1-p)$, this is the same as saying that if $u \in U(pAp)$ and

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U_0(A),$$

then $u \in U_0(pAp)$.

We will use the following two conditions for injectivity on U/U_0. Here, $\mathrm{tsr}(A)$ is the topological stable rank [12].

PROPOSITION 3.2. (1) *If $m \geqslant n \geqslant \mathrm{tsr}(A)$ then the standard inclusion $A \otimes M_n \to$ $\to A \otimes M_m$ is injective on U/U_0.*

(2) *If $U(pAp)$ is connected then $pAp \to A$ is injective on U/U_0.*

Proof. (1) follows immediately from Theorem 2.10 of [13] and (2) is obvious. ∎

Our main result relating exponential and projective rank and length, Theorem 3.6, is essentially a combination of the next two propositions. It is not quite a corollary, but it follows by joining the proofs together in a fairly trivial manner.

In the following results, $\langle\alpha\rangle$ means the least integer $n \geqslant \alpha$, and $[\alpha]$ means the greatest integer $n \leqslant \alpha$. In arithmetic operations on $\mathrm{cpr}(A)$ and $\mathrm{cer}(A)$, we disregard the ε in values of the form $n+\varepsilon$.

PROPOSITION 3.3. *Let B be a unital C^*-algebra, and assume that the inclusion $B \to M_2(B)$ (in the upper left corner) is injective on U/U_0. Then*

(1) $\mathrm{cel}(B\otimes M_2) \leqslant \mathrm{cpl}(B\otimes M_2) + \dfrac{\mathrm{cel}(B)}{2} + \pi$.

(2) $\mathrm{cer}(B\otimes M_2) \leqslant \left\langle \dfrac{1}{2}\mathrm{cpr}(B\otimes M_2)\right\rangle + \left\langle \dfrac{1}{2}\mathrm{cer}(B)\right\rangle + 1 + \varepsilon$.

The ε in (2) can be omitted if $\frac{1}{2}\mathrm{cpr}(B\otimes M_2)$ is not an integer.

The proof requires the following lemma, which will also be needed for the proof of the next proposition.

LEMMA 3.4. *Let A be a unital C^*-algebra, let $p \in A$ be a projection, and let $u \in U(A)$. Then for each $\varepsilon > 0$ there is $v \in U(A)$ such that v commutes with p and*

(1) $\mathrm{cel}(uv^*) \leqslant \mathrm{cpl}(A) + \varepsilon$.

(2) *uv^* is within ε of a product of at most $\mathrm{cpr}(A)$ $*$-symmetries.*

Proof. By Theorem 1.9 of [10], there is $w \in U(A)$ with $\mathrm{cel}(w) \leqslant \mathrm{cpl}(A)+\varepsilon$ such that $wpw^* = upu^*$. Set $v = w^*u$, so that $uv^* = w$. Then v commutes with p and $\mathrm{cel}(uv^*) \leqslant \mathrm{cpl}(A)+\varepsilon$. This gives (1), and (2) now follows from the proof of Theorem 2.4 (1) of [10]. ∎

Proof of Proposition 3.3. (1) Let $u_0 \in U_0(B\otimes M_2)$ and let $\varepsilon > 0$. Use Lemma 3.4 (1) to find $u_1 \in U(B\otimes M_2)$ which commutes with

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and satisfies $\mathrm{cel}(u_0u_1^*) \leqslant \mathrm{cpl}(B\otimes M_2)+\varepsilon$. We can write

$$u_1 = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$$

with $v_1, v_2 \in U(B)$. The hypothesis $B \to M_2(B)$ injective on U/U_0 implies that $v_1v_2 \in U_0(B)$. Using the midpoint of a suitable path, choose $w_1, w_2 \in U_0(B)$ such that $w_1w_2 = v_1v_2$ and $\mathrm{cel}(w_1), \mathrm{cel}(w_2) \leqslant \frac{1}{2}\mathrm{cel}(B) + \varepsilon$. Define

$$u_2 = \begin{pmatrix} v_1^* w_1 v_1 & 0 \\ 0 & w_2 \end{pmatrix}.$$

The relation $w_1w_2 = v_1v_2$ gives $v_2w_2^* = v_1^*w_1 = (w_1^*v_1)^*$. Therefore

$$u_1u_2^* = \begin{pmatrix} w_1^*v_1 & 0 \\ 0 & v_2w_2^* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & (w_1^*v_1)^* \\ w_1^*v_1 & 0 \end{pmatrix} = s_1s_2,$$

where s_1 and s_2 are $*$-symmetries. We can write $u_1u_2^* = s_1s_2 = (\mathrm{i}s_1)(-\mathrm{i}s_2)$, where $\mathrm{i}s_1$ and $-\mathrm{i}s_2$ have spectrum in $\{\pm\mathrm{i}\}$ and therefore have exponential length at most $\frac{\pi}{2}$. Thus

$$\mathrm{cel}(u_1u_2^*) \leqslant \mathrm{cel}(\mathrm{i}s_1) + \mathrm{cel}(-\mathrm{i}s_2) \leqslant \pi.$$

Also clearly

$$\mathrm{cel}(u_2) \leqslant \max(\mathrm{cel}(v_1^*w_1v_1), \mathrm{cel}(w_2)) = \max(\mathrm{cel}(w_1), \mathrm{cel}(w_2)) \leqslant \frac{1}{2}\mathrm{cel}(B) + \varepsilon.$$

Putting our estimates together gives

$$\mathrm{cel}(u_0) \leqslant \mathrm{cpl}(B \otimes M_n) + \frac{1}{2}\mathrm{cel}(B) + \pi + 2\varepsilon.$$

Take the infimum over $\varepsilon > 0$ and then the supremum over $u_0 \in U_0(B \otimes M_2)$ to get the result.

(2) The basic idea of the proof of (2) is the same as the proof of (1), so we only describe the differences. We choose the same u_1 as before, noting that $u_0u_1^*$ is within ε of a product y_0 of at most $l = \mathrm{cpr}(B \otimes M_2)$ $*$-symmetries by Lemma 3.4 (2). If there are fewer than l of them, we add for convenience enough trivial ones (factors of 1) to bring the length of the product to exactly l. We approximate v_1v_2 to within ε by a product of at most $\mathrm{cer}(B)$ exponentials, and divide this product as nearly in half as possible to obtain w_1 and w_2. Then u_2 is within ε of a product z of at most $m = \left\langle \frac{1}{2}\mathrm{cer}(B) \right\rangle$ exponentials. Also, $u_1u_2^*$ is a product of two $*$-symmetries as before. Therefore

$$u_0 = (u_0u_1^*)(u_1u_2^*)u_2 = x(y_0u_1u_2^*)z = xyz,$$

where $\|x - 1\| < 2\varepsilon$ and $y = y_0u_1u_2^* = s_1 \cdot \dots \cdot s_{l+2}$ is a product of $l + 2$ $*$-symmetries.

If l is even, then

$$\mathrm{cel}(y) \leqslant \mathrm{cel}(\mathrm{i}s_1) + \mathrm{cel}(-\mathrm{i}s_2) + \cdots + \mathrm{cel}(\mathrm{i}s_{l+1}) + \mathrm{cel}(-\mathrm{i}s_{l+2}) \leqslant \frac{(l+2)\pi}{2}.$$

Therefore y is within ε of a product of $\frac{l+2}{2}$ exponentials by Theorem 2.8 (iv) of [14]. So u_0 is within 3ε of a product of $\frac{l}{2}+1+m$ exponentials, proving (2) in this case.

If l is odd, we will assume $2\arcsin\left(\frac{3\varepsilon}{2}\right) < \frac{\pi}{2}$. As above, there is c, a product of $\frac{l+1}{2}$ exponentials, such that $\|c - s_2 s_3 \cdot \cdots \cdot s_{l+2}\| < \varepsilon$. Then $\|u_0 - s_1 cz\| < 3\varepsilon$, so $\|s_1 - u_0 z^* c^*\| < 3\varepsilon$. We will show that $u_0 z^* c^*$ is an exponential, thus writing u_0 as a product of $\left\langle \frac{l}{2} \right\rangle + 1 + m$ exponentials. This will prove (2) without the ε, as required in the final part of the theorem for this case, and complete the proof.

We can write $is_1 = \exp(ia)$ with $\|a\| \leqslant \frac{\pi}{2}$. Therefore $iu_0 z^* c^* = \exp(ih)$ for some selfadjoint h, by Corollary 2.4 of [14]. It follows that $u_0 z^* c^* = \exp(i(h - \left(\frac{\pi}{2}\right) \cdot \cdot 1))$, as desired. ■

PROPOSITION 3.5. *Let A be a unital C^*-algebra, and let $p, q \in A$ be projections with $p+q = 1$ and q Murray-von Neumann equivalent to a subprojection of p. Assume $pAp \to A$ is injective on U/U_0. Then:*

(1) $\mathrm{cel}(A) \leqslant \mathrm{cpl}(A) + \mathrm{cel}(pAp) + \pi$.

(2) $\mathrm{cer}(A) \leqslant \left\langle \frac{1}{2}\mathrm{cpr}(A) \right\rangle + \mathrm{cer}(pAp) + 1 + \varepsilon$.

If $\mathrm{cpr}(A)$ is odd, the ε in (2) can be omitted.

Proof. (1) Let $r \leqslant p$ be a projection which is Murray-von Neumann equivalent to q. Write elements of A as 3×3 matrices relative to the decomposition $1 = (p-r)+r+q$, and use the equivalence of r and q to identify the subalgebra $(r+q)A(r+q)$ defined by the lower right 2×2 block with $qAq \otimes M_2$.

Let $u_0 \in U_0(A)$ and let $\varepsilon > 0$. Use Lemma 3.4 (1) to find $u_1 \in U(A)$ which commutes with p and satisfies $\mathrm{cel}(u_0 u_1^*) \leqslant \mathrm{cpl}(A) + \varepsilon$. With respect to our matrix decomposition, we have

$$u_1 = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & v \end{pmatrix}.$$

Set

$$u_2 = \begin{pmatrix} c_{11} & c_{22} & 0 \\ vc_{21} & vc_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & v^* \\ 0 & v & 0 \end{pmatrix} u_1.$$

Then $u_1 u_2^*$ is a product of two $*$-symmetries. Furthermore, $pu_2p \in U_0(pAp)$ because $pAp \to A$ is injective on U/U_0, so that $\mathrm{cel}(pu_2p) \leqslant \mathrm{cel}(pAp)$. We therefore get, just as in the proof of the previous proposition,

$$\mathrm{cel}(u_0) \leqslant \mathrm{cel}(u_0 u_1^*) + \mathrm{cel}(u_1 u_2^*) + \mathrm{cel}(u_2) \leqslant$$

$$\leqslant \mathrm{cpl}(A) + \varepsilon + \pi + \mathrm{cel}(pu_2p) \leqslant \mathrm{cpl}(A) + \mathrm{cel}(pAp) + \pi + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary this proves (1).

(2) The proof is obtained from the proof of (1) by the same sorts of modifications used to obtain the proof of (2) from the proof of (1) for the previous proposition. We omit the details. ■

THEOREM 3.6. *Let A be a unital C^*-algebra, and let $p_1, p_2, q_1, q_2 \in A$ be projections with $p_1 + p_2 + q_1 + q_2 = 1$. Assume that p_1 is Murray-von Neumann equivalent to p_2, that each q_i is Murray-von Neumann equivalent to a subprojection of p_i, and that $p_1Ap_1 \to A$ is injective on U/U_0. Then*

(1) $\mathrm{cel}(A) \leqslant \mathrm{cpl}(A) + \max_{i=1,2} \mathrm{cpl}((p_i + q_i)A(p_i + q_i)) + \frac{1}{2}\mathrm{cel}(p_1Ap_1) + 2\pi.$

(2) $\mathrm{cer}(A) \leqslant \left\langle \frac{1}{2}\mathrm{cpr}(A) + \frac{1}{2}\max_{i=1,2} \mathrm{cpr}((p_i + q_i)A(p_i + q_i)) \right\rangle + \left\langle \frac{1}{2}\mathrm{cer}(p_1Ap_1) \right\rangle + {}$
$+ 2 + \varepsilon.$

In (2), the ε can be omitted if the term inside the first $\langle \cdots \rangle$ is not an integer. If $q_2 = 0$, then the projective length or rank of $(p_2 + q_2)A(p_2 + q_2)$ can be omitted from the maximums.

Proof. (1) This is obtained by putting together the pieces of the two preceding propositions in the following way. Given $u_0 \in U_0(A)$, we first choose u_1 which commutes with $p_1 + q_1$ so that $\mathrm{cel}(u_0u_1^*) \leqslant \mathrm{cpl}(A) + \varepsilon$, as in the first step of the proof of Proposition 3.3. The argument in the first steps of the proof of Proposition 3.5, carried out in parallel on $(p_1 + q_1)u_1(p_1 + q_1)$ and $(p_2 + q_2)u_1(p_2 + q_2)$, produces u_2 of the form

$$u_2 = \mathrm{diag}(v_1, 1, v_2, 1)$$

with respect to the decomposition $1 = p_1 + q_1 + p_2 + q_2$, satisfying

$$\mathrm{cel}(u_1u_2^*) \leqslant \max_{i=1,2} \mathrm{cpl}((p_i + q_i)A(p_i + q_i)).$$

Now $v_1v_2 \in U_0(p_1Ap_1)$ because $p_1Ap_1 \to A$ is injective on U/U_0. Therefore we can apply the rest of the proof of Proposition 3.3, in the ij entries of this matrix for $i, j = 1, 3$, to get $\mathrm{cel}(u_2) \leqslant \frac{1}{2}\mathrm{cel}(p_1Ap_1) + \pi$. This proves (1).

(2) We do all the steps in the same order as in (1), using parts (2) of Propositions 3.3 and 3.5. We combine all of the $*$-symmetries at the end so as to get

$$\left\langle \frac{1}{2}\mathrm{cpr}(A) + \frac{1}{2}\max \mathrm{cpr}((p_i + q_i)A(p_i + q_i)) \right\rangle$$

rather than

$$\left\langle \frac{1}{2}\mathrm{cpr}(A) \right\rangle + \left\langle \frac{1}{2}\max \mathrm{cpr}((p_i + q_i)A(p_i + q_i)) \right\rangle.$$

The remark on the case $q_2 = 0$ is clear. ∎

We note that a slightly weaker theorem can be obtained as a direct corollary of Propositions 3.3 and 3.5. It requires more inclusions to be injective on U/U_0 and gives a larger bound in (2) if both $\mathrm{cpr}(A)$ and $\max \mathrm{cpr}((p_i + q_i)A(p_i + q_i))$ are odd.

As discussed before Proposition 2.10 of [10], we would really like to eliminate the cpr and cpl terms, and get $\mathrm{cer}(B \otimes M_2) \leqslant \mathrm{cer}(B)$ or even $\mathrm{cer}(B \otimes M_2) \leqslant \frac{1}{2}\mathrm{cer}(B) + b$ for some constant b. Such a formula, however, cannot hold in general, as is clear from the proof of Theorem 3.9 below.

We now apply our results to large matrix algebras. We will eventually take $B = C(X)$, and we usually have $\mathrm{cel}(C(X) \otimes M_n) = \infty$. Therefore from now on we restrict to the exponential rank case. The following theorem can be viewed as a weak analog for exponential rank of results proved in [1] (see Theorems 2 and 20) for lengths of products of commutators and triangular matrices. It is weaker because of the appearance of the term $b = \sup_{k \geqslant n} \mathrm{cpr}(B \otimes M_k)$. However, computations done here and in [10] suggest that this number is often small.

THEOREM 3.7. *Let B be a unital C^*-algebra, let $n \geqslant \mathrm{tsr}(B)$, and let b be an integer such that $\mathrm{cpr}(B \otimes M_k) \leqslant b + \varepsilon$ for all $k \geqslant n$. Then for $r \geqslant 0$ and $2^r n \leqslant k \leqslant 2^{r+1} n$, we have*

$$\mathrm{cer}(B \otimes M_k) \leqslant \left\langle \frac{3b}{2} \right\rangle + [2^{-r}\mathrm{cer}(B \otimes M_n)] + 4 + \varepsilon.$$

Proof. We first prove by induction on r that

$$(*) \qquad \mathrm{cer}(B \otimes M_{2^r n}) \leqslant \left[\left(3 + 2\left\langle \frac{b}{2} \right\rangle\right)(1 - 2^{-r}) + 2^{-r}\mathrm{cer}(B \otimes M_n)\right] + \varepsilon.$$

For $r = 0$ this is trivial. So suppose $(*)$ holds for some $r \geqslant 0$. Since $n \geqslant \mathrm{tsr}(B)$, Proposition 3.2 (1) implies that $B \otimes M_{2^r n} \to B \otimes M_{2^{r+1} n}$ is injective on U/U_0. Therefore Proposition 3.3 (2) applies to $B \otimes M_{2^r n}$, yielding

$$(**) \qquad \mathrm{cer}(B \otimes M_{2^{r+1} n}) \leqslant \left\langle \frac{b}{2} \right\rangle + \left\langle \frac{1}{2}\mathrm{cer}(B \otimes M_{2^r n}) \right\rangle + 1 + \varepsilon.$$

Using $(*)$ for r, we get

$$\left\langle \frac{1}{2}\mathrm{cer}(B \otimes M_{2^r n}) \right\rangle \leqslant \frac{1}{2} + \frac{1}{2}\left[\left(3 + \left\langle \frac{b}{2} \right\rangle\right)(1 - 2^{-r}) + 2^{-r}\mathrm{cer}(B \otimes M_n)\right] \leqslant$$

$$\leqslant \frac{1}{2} + \left(\frac{3}{2} + \frac{1}{2}\left\langle \frac{b}{2} \right\rangle\right)(1 - 2^{-r}) + 2^{-r-1}\mathrm{cer}(B \otimes M_n).$$

Substituting this in (**) yields an expression which simplifies to (*) for $r+1$, except without the brackets $[\cdots]$. However, since $\mathrm{cer}(B \otimes M_{2^{r+1}n}) \in \{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots\}$, we can round down to the next integer, that is, insert the brackets. This completes the inductive proof of (*).

Now let $r \geqslant 1$ and $2^r n \leqslant k \leqslant 2^{r+1}n$. Let $p_1, p_2, q_1, q_2 \in M_k \subset B \otimes M_k$ be orthogonal projections with $\mathrm{rank}(p_1) = \mathrm{rank}(p_2) = 2^{r-1}n$ and $\mathrm{rank}(q_1), \mathrm{rank}(q_2) \leqslant \leqslant 2^{r-1}n$, such that $p_1 + p_2 + q_1 + q_2 = 1$. Then $p_i(B \otimes M_k)p_i \cong B \otimes M_{2^{r-1}n}$. Since $r \geqslant 1$ and $n \geqslant \mathrm{tsr}(B)$, the inclusion $p_1(B \otimes M_k)p_1 \to B \otimes M_k$ is injective on U/U_0 by Proposition 3.2 (1). Therefore Theorem 3.6 (2) and (*) yield

$$\mathrm{cer}(B \otimes M_k) \leqslant b + \left\langle \frac{1}{2} \left[\left(3 + 2 \left\langle \frac{b}{2} \right\rangle \right) (1 - 2^{-r+1}) + 2^{-r+1} \mathrm{cer}(B \otimes M_n) \right] \right\rangle + 2 + \varepsilon.$$

On the right hand side, we first use the inequality $\left\langle \frac{m}{2} \right\rangle \leqslant \frac{m}{2} + \frac{1}{2}$ for $m \in \mathbb{Z}$, then drop the factor $1 - 2^{-r+1}$ and the brackets $[\cdots]$, and finally round down to the next integer plus ε (since $\mathrm{cer}(B \otimes M_k) \in \{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots\}$). This gives

$$\mathrm{cer}(B \otimes M_k) \leqslant \left[b + \frac{1}{2} + \left(\frac{3}{2} + \left\langle \frac{b}{2} \right\rangle \right) + 2^{-r} \mathrm{cer}(B \otimes M_n) + 2 \right] + \varepsilon =$$

$$= \left\langle \frac{3b}{2} \right\rangle + [2^{-r} \mathrm{cer}(B \otimes M_n)] + 4 + \varepsilon,$$

as desired.

If $r = 0$, the desired estimate is weaker than the one that follows from Proposition 3.5 by taking p of rank n and $q = 1 - p$. ■

COROLLARY 3.8. *Let X be a compact metric space of dimension d. Then for $n \geqslant 5d + 3$ and $2^r n \leqslant k \leqslant 2^{r+1}n$, we have*

$$\mathrm{cer}(C(X) \otimes M_k) \leqslant 10 + [2^{-r} \mathrm{cer}(C(X) \otimes M_n)] + \varepsilon,$$

and for $n \geqslant 2d + 4$ and $2^r n \leqslant k \leqslant 2^{r+1}n$, we have

$$\mathrm{cer}(C(X) \otimes M_k) \leqslant 13 + [2^{-r} \mathrm{cer}(C(X) \otimes M_n)] + \varepsilon.$$

Proof. This follows from the previous theorem, using the fact that $\mathrm{tsr}(C(X)) = = \left[\frac{d}{2}\right] + 1 \leqslant 2d + 4,\ 5d + 3$ ([12], Proposition 1.7), and using the estimates for b from Theorems 2.1 and 2.3. ■

It was shown in Theorem 3.4 of [9] that if X is finite dimensional, then

$$\limsup_{n \to \infty} \mathrm{cer}(C(X) \otimes M_n)) \leqslant 4.$$

This corollary gives some limits on the values of n for which $\operatorname{cer}(C(X)\otimes M_n)$ is large. It also provides an alternate proof of Corollary 3.5 of [9], and suitable modifications give an alternate proof of Theorem 4.7 of [9]. (This theorem plays a crucial role in [3] and [11].)

We finish by proving that the projective length and rank can in fact be arbitrarily large or even infinite.

THEOREM 3.9. *Let B_m denote the closed unit ball in $\mathbb{R}^m$. Then the C^*-algebra $A = C(B_{6l+2})\otimes M_2$ satisfies*

$$(2l-2)\pi < \operatorname{cpl}(A) < \infty \quad\text{and}\quad 4l-3 \leqslant \operatorname{cpr}(A) < \infty.$$

Proof. Let $B = C(B_{6l+2})$. Since $U(B)$ is connected, Propositions 3.2 (2) and 3.3 (2) give

$$\operatorname{cer}(A) \leqslant \left\langle \frac{1}{2}\operatorname{cpr}(A)\right\rangle + \left\langle \frac{1}{2}\operatorname{cer}(B)\right\rangle + 1 + \varepsilon.$$

Now $\operatorname{cer}(A) \geqslant 2l+1$ by Theorem 2.3 of [9], and $\operatorname{cer}(B) = 1$ because B is commutative. It follows that $\operatorname{cpr}(A) \geqslant 4l-3$. The lower bound on $\operatorname{cpl}(A)$ now follows from Theorem 2.4 (1) of [10]. Finiteness of both quantities follows from Corollary 2.2. ∎

Some slight improvements to the lower bounds are possible. Since we do not believe they are close to being sharp, we will not worry about the details.

COROLLARY 3.10. (Compare [9], Corollary 2.6.) *Let X be the Hilbert cube $[0,1]^{\mathbb{N}}$, and let $A = C(X)\otimes M_2$. Then $\operatorname{cpl}(A) = \infty$ and $\operatorname{cpr}(A) = \infty$.*

Proof. This follows from the previous theorem and Proposition 2.12 of [10]. ∎

Unfortunately, we have no information on the following question:

QUESTION 3.11. Are there analogs of the previous theorem and its corollary for $C(X)\otimes M_n$ with $n \geqslant 2$?

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