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SUBDIAGONAL ALGEBRAS FOR SUBFACTORS

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1. INTRODUCTION

In this paper we study a relation between subdiagonal algebras and subfactors. After the systematic investigation in triangular algebras by Kadison and Singer [8], Arveson [1] introduced the notion of subdiagonal algebras to unify several aspects of non-selfadjoint operator algebras. Since then there has been many investigation on them. On the other hand the work of Jones [7] on index for subfactors opened a rich theory of subfactors which has been developped by several people. The aim of the paper is a first attempt to understand the existence (or non-existence) of analytic structure as subdiagonal algebras in the theory of subfactors.

Let N be a type II_1 factor, G a countable discrete group and $\alpha : G \to \operatorname{Aut} N$ an outer action. Then the crossed product $M = N \rtimes G$ is also a type II_1 factor. Theorem 2 shows that there exists a bijective correspondence between the set of all maximal subdiagonal algebras for $N \subset M$ and the set of all positive cones of total orderes on G. If we regard a subfactor as a quantization of a group, then Theorem 2 means that a subdiagonal algebra is a quantization of (a positive cone of) a total order on a group. A totally ordered group G must be torsion free, in particular the order of G must be infinite, unless $G = \{1\}$. Therefore it is reasonable to conjecture that if N is a subfactor of M with $[M : N] < \infty$, then there exist no subdiagonal algebras with respect to the canonical conditional expectation $E : M \to N$ determined by the trace unless $M = N$. We shall confirm the conjecture in the case of subfactors N of a hyperfinite II_1 factor M with $[M : N] \leqslant 4$. Here the assumption of factorness is essential. In fact, if M is the $n \times n$ full matrix algebra and N is the diagonals (so N is abelian), then the set of all upper triangular matrices is a maximal subdiagonal algebra. In general, when N contains a Cartan subalgebra of M, the complete study of subdiagonal algebras was done by Muhly, Saito and Solel [12].

2. CROSSED PRODUCT CASE

Let M be a finite von Neumann algebra with a faithful normal normalized trace τ. We recall the definition of (σ-weakly closed) subdiagonal algebras by Arveson [1]. Let $\mathcal{A} \ni 1$ be a σ-weakly closed subalgebra of M and E a faithful normal conditional expectation from M onto $N = \mathcal{A} \cap \mathcal{A}^*$ such that $\tau(E(x)) = \tau(x)$ for $x \in M$. Then $\mathcal{A}$ is called a maximal subdiagonal subalgebra of M with respect to E if the following conditions are satisfied:

(1) $\mathcal{A} + \mathcal{A}^*$ is a σ-weakly dense in M,

(2) $E(xy) = E(x)E(y)$ for $x, y \in \mathcal{A}$,

(3) $\mathcal{A}$ is maximal among those subalgebras of M satisfying (1) and (2).

We note that Exel showed that any σ-weakly closed finite subdiagonal algebra is automatically maximal in his excellent paper [4]. Let $\eta : M \to L^2(M, \tau)$ be the canonical injection. Let $H^2 = [\eta(\mathcal{A})]_2$ be the $\|\cdot\|_2$-closure of $\eta(\mathcal{A})$ in $L^2(M)$. Put $\mathcal{I} = \{a \in \mathcal{A} | E(a) = 0\}$. Let $e_N : L^2(M) \to L^2(N)$ be a Jones projection. We also consider other projections $e_{\mathcal{A}} : L^2(M) \to H^2 = [\eta(\mathcal{A})]_2$ and $e_{\mathcal{I}} : L^2(M) \to [\eta(\mathcal{I})]_2$. We sometimes identify $\eta(M)$ with M.

LEMMA 1. *Let M be a type II_1-factor and N a subfactor of M. Let $\mathcal{A}$ be a maximal subdiagonal algebra with respect to the conditional expectation $E : M \to N$ determined by the trace. Then $e_N, e_{\mathcal{A}}, e_{\mathcal{I}}, J_M e_{\mathcal{A}} J_M$ and $J_M e_{\mathcal{I}} J_M$ are all in $N' \cap$ $\cap \langle M, e_N \rangle$ and we have that*

$$e_N + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = 1.$$

Proof. Since $N\mathcal{A} \subset \mathcal{A}$, $e_{\mathcal{A}}$ is in N'. Similarly $\mathcal{A}N \subset \mathcal{A}$ implies that $e_{\mathcal{A}} \in$ $\in (J_M N J_M)' = \langle M, e_N \rangle$. We have $e_{\mathcal{A}} = e_N + e_{\mathcal{I}}$ because $H^2 = L^2(N) \oplus [\eta(\mathcal{I})]_2$. Thus $e_N, e_{\mathcal{A}}$ and $e_{\mathcal{I}}$ are in $N' \cap \langle M, e_N \rangle$. Since $\eta(\mathcal{A}^*) = J_M \eta(\mathcal{A})$, we have that $e_{\mathcal{A}^*} = J_M e_{\mathcal{A}} J_M$, and similarly we have $e_{\mathcal{I}^*} = J_M e_{\mathcal{I}} J_M$. Since $\langle M, e_N \rangle = J_M N' J_M$, these operators also in $N' \cap \langle M, e_N \rangle$. Furthermore $L^2(M) = [\eta(\mathcal{A}^*)]_2 \oplus L^2(N) \oplus [\eta(\mathcal{A})]_2$ shows that $J_M e_{\mathcal{I}} J_M + e_N + e_{\mathcal{I}} = 1$. ∎

We recall a construction of subdiagonal algebras in crossed products due to Arveson [1]. Let G be a countable discrete group. Consider a subsemigroup S of G satisfying $S \cap S^{-1} = \{1\}$ and $S \cup S^{-1} = G$. Define the relation $x \leqq y$ in G by $x^{-1}y \in S$. Then $\leqq$ is a total order on G and is invariant under left multiplication. Clearly $S = \{x \in G | x \geqq 1\}$. Any (left-invariant) total order can be realized as above by such a subsemigroup S and we say that S is the positive cone of a total order on G.

Let N be a type II_1-factor and $\alpha : G \to \operatorname{Aut} N$ an outer action. Consider a crossed product $M = N \rtimes G$. Let $a = \sum_g a_g \lambda_g$ be a "Fourier" expansion of $a \in M$.

Then there exists a conditional expectation $E : M \rightarrow N$ such that $E\left(\sum_g a_g\lambda_g\right) = a_1$ and $\tau \circ E = \tau$. We also note that $N' \cap M = \mathbb{C}$. This E is the unique conditional expectation of M onto N, (see [2] and [3, 1.5.5]). Let S be a positive cone of a total order on G. Let $\mathcal{A}$ be the σ-weak closure of the set of all finite sums $\sum_g a_g\lambda_g$, where $a_g \in N$ and $a_g = 0$ except for finitely many $g \in S$. Then by a result of Exel [4], $\mathcal{A}$ is a maximal subdiagonal algebra of M with respect to $E : M \rightarrow N$. By Arveson [1, Corollary 2.2.4], we have that

$$\mathcal{A} = \{x \in M | E(x\lambda_g) = 0 \text{ for all } g \in S \setminus \{1\}\} =$$

$$= \left\{x = \sum_g a_g\lambda_g \in M | a_g = 0 \text{ for all } g \lneqq 1\right\} = H^2 \cap M.$$

The following theorem shows that any maximal subdiagonal algebra of M has this form:

THEOREM 2. *Let N be a type II_1 factor, G a countable discrete group and $\alpha : G \rightarrow \operatorname{Aut} M$ an outer action. Consider a II_1-factor $M = N \rtimes_\alpha G$ and the unique conditional expectation $E : M \rightarrow N$. Then there exists a bijective corespondence between the set of all maximal subdiagonal algebras $\mathcal{A}$ of M with respect to E and the set of all positive cones S of (left-invariant) total orders on G.*

Proof. Let S be a positive cone of a total order on G. Then the corresponding maximal subdiagonal algebra $\mathcal{A}$ is given by the σ-weak closure of the set of all finite sums $\sum_g a_g\lambda_g$, where $a_g \in N$ and $a_g = 0$ except for finitely many $g \in S$ as discussed above.

Conversely let $\mathcal{A}$ be a maximal subdiagonal algebra of M with respect to E. Then the projection $e_{\mathcal{A}} : L^2(M) \rightarrow H^2$ is in $N' \cap \langle M, e_N\rangle$ by Lemma 1. If we identify $\langle M, e_N\rangle$ with $N \otimes B(\ell^2(G))$, then N is described by diagonal operators

$$\left\{\bigoplus_g \alpha_g^{-1}(n) \in N \otimes B(\ell^2(g)) | n \in N\right\}.$$

Since α is an outer action, we have that

$$N' \cap \langle M, e_n\rangle \cong \ell^\infty(G).$$

Therefore we can identify the projection $e_{\mathcal{A}}$ with a characteristic function $\chi_S \in \ell^\infty(G)$ for some subset $S \subset G$. This implies that

$$H^2 = \bigoplus_{g\in S} [N\lambda_g]_2.$$

By [16, Theorem 1], we have that $\mathcal{A} = H^2 \cap M$. Therefore

$$S = \{g \in G | \lambda_g \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a subdiagonal algebra, we have that $\mathcal{A}\mathcal{A} \subset \mathcal{A}$, $\mathcal{A} + \mathcal{A}^*$ is a σ-weakly dense in M and $\mathcal{A} \cap \mathcal{A}^* = N$. These properties implies that $SS \subset S$, $S \cup S^{-1} = G$ and $S \cap S^{-1} = \{1\}$. Thus S is a positive cone of a total order on G.

From the construction, the correspondence between positive cones S and subdiagonals $\mathcal{A}$ is bijective by noticing the fact that $\mathcal{A} = H^2 \cap M$. ■

REMARK. In Theorem 2, the factorness of N is essential. In fact let $N = \mathbf{C}^n$ and $G = \mathbf{Z}/n\mathbf{Z}$. The action α is given by cyclic permutations of coordinates of $\mathbf{C}^n$. Then the crossed product $M = N \rtimes G$ is isomorphic to the $n \times n$ full matrix algebra. There exist no total orders on G but the set $\mathcal{A}$ of all upper triangular matrices is a maximal subdiagonal algebra of M with respect to $E : M \to N$.

COROLLARY 3. *Let N be a type II_1-factor and $\alpha : G \to \operatorname{Aut} N$ an outer action of a finite group G. Consider a crossed product $M = N \rtimes_\alpha G$ and the conditional expectation $E : M \to N$. Then there exist no subdiagonal algebras of M with respect to E unless $G = \{1\}$.*

Prof. Since the order of G is finite, there exist no positive cones of total orders on G unless $G = \{1\}$. Therefore by Theorem 2 there exist no subdiagonal algebras of M with respect to E. ■

If the order of G is infinite, there occur several cases on the number of maximal subdiagonal algebras of $M = N \rtimes G$.

EXAMPLES. We use the notation as in Theorem 2.

(1) If $G = \mathbf{Z}$, then there exist exactly two maximal subdiagonal algebras of $M = N \rtimes G$ with respect to E corresponding to $\mathbf{Z}_+$ and $-\mathbf{Z}_+$ by Theorem 2. The subdiagonal algebra $\mathcal{A} = N \rtimes \mathbf{Z}_+$ is called an analytic crossed product and has been studied deeply, for example see [9], [10] and [11].

(2) If $G = \mathbf{Z}^2$, then there exist infinitely many maximal subdiagonal algebras $\mathcal{A}$ of $M = N \rtimes \mathbf{Z}^2$ with respect to E. In fact consider maximal subdiagonal algebras $\mathcal{A} = \mathcal{A}_\theta$ for positive irrational numbers θ corresponding to positive cones

$$S_\theta = \{(m, n) \in \mathbf{Z}^2 | \theta m + n \geqslant 0\}.$$

(3) Let $G = \prod_{n=1}^{\infty} G_n$, where $G_n \cong \mathbf{Z}/2\mathbf{Z}$. Then there exists no subdiagonal algebras of $M = N \rtimes G$ with respect to E, because a totally ordered group must be torsion free.

(4) Let $G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then there exists no subdiagonal algebras of $M = N \rtimes G$ with respect to E for the same reason as in (3).

3. FINITE INDEX CASE

A totally ordered group must be torsion free, in particular the order of G must be infinite unless $G = \{1\}$. If we regard a subfactor as a quantization of a group, then Theorem 2 means that a subdiagonal algebra is a quantization of a positive cone of a total order on a group. Therefore it is reasonable to conjecture that if N is a subfactor of M with $[M : N] < \infty$, then there exist no subdiagonal algebras with respect to the canonical conditional expectation $E : M \to N$ determined by the trace, unless $M = N$. With the help of the classification of subfactors of a hyperfinite II_1-factor (see [5], [6], [13], [14] and [15]), we shall confirm the conjecture in the case of subfactors N of a hyperfinite II_1 factor M with $[M : N] \leqq 4$.

THEOREM 4. *Let M be a hyperfinite II_1-factor and N a subfactor of M. If $[M : N] \leqq 4$, then there exist no maximal subdiagonal algebras of M with respect to the conditional expectation $E : M \to N$ determined by the trace, unless $M = N$.*

Proof. We assume that $M \neq N$. Hence we may suppose that

$$\dim N' \cap \langle M, e_N \rangle \geqq 3$$

by Lemma 1. Therefore by a result of the classification of subfactors mentioned above, the possible principal graph of the subfactor $N \subset M$ is one of $D_4, A_n^{(1)}, A_{\infty,\infty}, D_n^{(1)}$ and D_∞.

If the principal graph is D_4, then $M = N \rtimes \mathbb{Z}/3\mathbb{Z}$. Thus there exist no maximal subdiagonal algebras of M in the case by Corollary 3.

If the principal graph is $D_4^{(1)}$, then $M = N \rtimes \mathbb{Z}/4\mathbb{Z}$ or $M = N \rtimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. Thus the theorem holds in this case by Corollary 3.

Consider the case when the principal graph is $D_n^{(1)}$ $(n \geqq 5)$ or D_∞. Then there exist two outer automorphisms α and β with period 2 on a hyperfinite II_1-factor R such that

$$N = R^\beta \subset M = R \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}.$$

Since $N' \cap \langle M, e_N \rangle \cong \mathbb{C}^3$, $\{1, e_N^M, e_R^M\}$ forms a linear basis of it. Therefore the minimal projections of $N' \cap \langle M, e_N \rangle$ are $e_N^M, e_R^M - e_N^M$ and $1 - e_R^M$. We can also write

$$R = N \rtimes \mathbb{Z}/2\mathbb{Z} = N + Nv$$

for some implementing unitary v with $v^2 = 1$. Then

$$e_R^M = e_N^M + v e_N^M v^*.$$

Therefore we have that

$$e_N^M + v e_N^M v^* + (1 - e_R^M) = 1.$$

Suppose on the contrary that there were a subdiagonal algebra $\mathcal{A}$. Then Lemma 1 shows that

$$e_N^M + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = 1.$$

We may assume and do assume that $e_{\mathcal{I}} = v e_N^M v^*$ without loss of generality. We shall identify M with $\eta(M)$. Then by [16, Theorem 1] we have

$$\mathcal{I} = [\mathcal{I}]_2 \cap M = e_{\mathcal{I}} L^2(M) \cap M = (v e_N^M v^*) L^2(M) \cap M = [vN]_2 \cap M.$$

Therefore v is in $\mathcal{I}$. Since v is a unitary with $v^2 = 1$, we have $v = v^* \in \mathcal{I} \cap \mathcal{I}^* = \{0\}$. This is a contradiction. Thus there exist no subdiagonal algebras in the case of $D_n^{(1)}$ or D_∞.

Finally consider the case that the principal graph is one of $A_n^{(1)}$ or $A_{\infty,\infty}$. Then we see that $N' \cap \langle M, e_N \rangle$ is isomorphic to $M_4(\mathbf{C})$, $M_2(\mathbf{C}) \oplus M_2(\mathbf{C})$ or $\mathbf{C} \oplus M_2(\mathbf{C}) \oplus \mathbf{C}$. In any case there exists a faithful representation $\pi : N' \cap \langle M, e_N \rangle \to B(\mathbf{C}^4)$ such that $\operatorname{rank} \pi(e_N^M) = 1$ and $\operatorname{rank} \pi(I) = 4$. Let θ be the anti-automorphism on $N' \cap \langle M, e_N \rangle$ defined by $\theta(x) = J_M x^* J_M$. By Lemma 1 we have $e_N + e_{\mathcal{I}} + J_M e_{\mathcal{I}} J_M = I$. Since $\theta(e_{\mathcal{I}}) = J_M e_{\mathcal{I}} J_M$, we have

$$\operatorname{rank} \pi(e_{\mathcal{I}}) = \frac{4 - \operatorname{rank} \pi(e_N)}{2} = \frac{3}{2}.$$

This is a contradiction. Thus the assertion is verified for all cases. ∎

REMARK. There is another way to show the non-existence of subdiagonal algebras in general cases including non-hyperfinite factors. Consider the anti-automorphism $\theta(x) = J_M x^* J_M$ for $x \in N' \cap \langle M, e_N \rangle$. Suppose that the anti-automorphism θ preserves the trace, (for example, say, N is an extremal subfactor of II_1-factor M with a finite depth). Then Lemma 1 implies that

$$\tau(e_{\mathcal{I}}) = \frac{1 - [M : N]^{-1}}{2}.$$

We can also calculate the possible values of traces of projections in $N' \cap \langle M, e_N \rangle$ using the Perron-Frobenius eigenvector for the principal graph. In certain cases $\tau(e_{\mathcal{I}})$ does not belong to the possible values of trace of projections. In this way we can also show the non-existence of subdiagonal algebras easily in certain cases.

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