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A CRITERION FOR ESSENTIAL SELF-ADJOINTNESS

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1. ASSUMPTIONS AND MAIN RESULTS

Let $\mathfrak{H}$ be a separable Hilbert space. $\{A\}_{n=0}^{\infty}$ denotes a sequence of bounded and self-adjoint operators on $\mathfrak{H}$. We consider a linear operator T in $\mathfrak{H}$, densely defined and symmetric on $\mathfrak{D}(T) = \mathfrak{D}_0 \subset \mathfrak{H}$.

NOTATION. If a linear operator S is defined and closeable on $\mathfrak{D}_0$ we denote its adjoint by S^* and its closure S^{**} by S^c. The restriction to $\mathfrak{D}_0$ of an operator B defined on some larger domain is occasionally denoted by $B|\mathfrak{D}_0$.

For the operators A_n and T we formulate the following conditions:

$\mathbf{A}_1$: For all n there exist k, m with

$$A_k A_n = A_n, \quad A_m A_k = A_k. \tag{1.1}$$

$\mathbf{A}_2$: For all n, $[T, A_n] \equiv A_n - A_n T$ is defined on $\mathfrak{D}_0$ and bounded.

$\mathbf{A}_3$: For all n, the operator $A_n T A_n|\mathfrak{D}_0$ is essentially self-adjoint.

$\mathbf{A}_4$: For all $\psi \in \mathfrak{D}(T^*)$, there is a sequence $\{A_{n_k}\}_{k=0}^{\infty}$ converging weakly to 1 such that

$$\lim_{k\to\infty} (T^* A_{n_k}\psi, \psi - A_{n_k}\psi) = 0 \tag{1.2}$$

One of the main results which can be proved under the conditions above (or under similar conditions discussed below) is the following:

THEOREM 1.1. *Assume* $\mathbf{A}_1$–$\mathbf{A}_4$. *Then* T *is essentially self-adjoint on* $\mathfrak{D}_0$.

As an illustration consider a Hilbert space which is an infinite orthogonal sum of closed subspaces. Define

$$\mathfrak{H} = \bigoplus_{k=0}^{\infty} \mathfrak{H}^{(k)}, \quad \mathfrak{H}_n = \bigoplus_{k=0}^{n} \mathfrak{H}^{(k)}, \tag{1.3}$$

and let A_n be the projection onto $\mathfrak{H}_n$. Hence A_n converges (strongly) to 1 and Assumption $\mathbf{A}_1$ is trivially satisfied. Let T be symmetric on some dense subset $\mathfrak{D}_0$. Then Theorem 1.1 gives conditions which assure that the essential self-adjointness of the restrictions $A_n T A_n$ of T is equivalent to the essential self-adjointness of T (the equivalence follows with the help of Theorem 3.1 and Remark 3.3 below).

Another example is provided by the Dirac operator the Hilbert space $L^2(\mathbf{R}^3)^4$. The theorem contains as a special case a famous result of Chernoff [1, 2] and Jörgens [3] stating that the essential self-adjointess of the Dirac operator on $C_0^\infty(\mathbf{R}^3)^4$ depends only on the local properties of the potential. In this case the A_n are multiplication operators with suitable C_0^∞-functions. See Section 4 for a discussion and a simplified proof of this result.

We conclude this section with some remarks on the assumptions.

REMARK 1.1. Assumption $\mathbf{A}_1$ is satisfied, e.g., if the A_n form an increasing sequence: $A_m A_n = A_n$ for all $m > n$. One might think of a sequence of projections, but we do not require $A_n^2 = A_n$. From $A_n^* = A_n$ for all n we easily conclude that the operators A_n, A_m, and A_k in $\mathbf{A}_1$ commute.

REMARK 1.2. The commutator $[T, A_n]$ is assumed to be defined on its natural domain. Since A_n is defined on all of $\mathfrak{H}$, we have $\mathfrak{D}([T, A_n]) = \mathfrak{D}(TA_n) \cap \mathfrak{D}(T)$. Assumption $\mathbf{A}_2$ requires that $\mathfrak{D}([T, A_n]) = \mathfrak{D}_0 = \mathfrak{D}(T)$ which means

$$A_n \mathfrak{D}_0 \subset \mathfrak{D}_0. \tag{1.4}$$

Boundedness of $[T, A_n]$ on a dense domain implies the existence of a unique bounded extension $[T, A_n]^c$ to all of $\mathfrak{H}$.

REMARK 1.3. Theorem 1.1 can be formulated as a perturbation theoretic result. Let $T = H_0 + V$, where H_0 is essentially self-adjoint and V is symmetric on $\mathfrak{D}_0$. If we replace $\mathbf{A}_3$ by the assumption that $H_0 + A_n V A_n$ be essentially self-adjoint on $\mathfrak{D}_0$, then we may conclude the essential self-adjointness of T. See Section 5 for details.

REMARK 1.4. The expression $T^* A_{n_k} \psi$ occuring in $\mathbf{A}_4$ is well defined for all $\psi \in \mathfrak{D}(T^*)$, provided $\mathbf{A}_2$ holds (see Lemma 2.1 below). If $\mathbf{A}_4$ holds, then

$$\|A_{n_k}\| \leqslant K, \tag{1.5}$$

where the constant K is independent of k (any weakly convergent sequence of operators is bounded). If the commutators $[T, A_{n_k}]^c$ are also bounded uniformly in k, then we can replace $\mathbf{A}_4$ by a more convenient assumption:

LEMMA 1.1. *Assume* $\mathbf{A}_2$. *Then* $\mathbf{A}_4$ *is implied by*

$\mathbf{A}_4'$: *For all* $\psi \in \mathfrak{H}$ *there is a subsequence* $\{A_{n_k}\}_{k=0}^{\infty}$ *converging strongly to* 1 *such that*

$$\|[T, A_{n_k}]^c \psi\| \leqslant C(\psi),$$

where $C(\psi) > 0$ *is independent of* k.

By the uniform boundednesss principle, Assumption $\mathbf{A}_4'$ implies $\|[T, A_{n_k}]^c\| \leqslant K'$, uniformly in k.

2. LEMMAS AND PROOFS

We start by proving some elementary technical lemmas. As always we assume that A_n is self-adjoint and bounded and that T is symmetric on $\mathfrak{D}_0$.

LEMMA 2.1. *Assume* $\mathbf{A}_2$. *Then* $A_n \mathfrak{D}(T^*) \subset \mathfrak{D}(T^*)$ *and*

$$[T, A_n]^c \psi = T^* A_n \psi - A_n T^* \psi \quad \text{for all } \psi \in \mathfrak{D}(T^*). \tag{2.1}$$

Proof. Let $\psi \in \mathfrak{D}(T^*)$. For all $\varphi \in \mathfrak{D}_0$

$$(A_n \psi, T\varphi) = (\psi, (TA_n - [T, A_n])\varphi) = ((A_n T^* - [T, A_n]^*)\psi, \varphi). \tag{2.2}$$

Hence, by the definition of the adjoint operator, $A_n \psi \in \mathfrak{D}(T^*)$ and

$$T^* A_n \psi = (A_n T^* - [T, A_n]^*)\psi. \tag{2.3}$$

Since $i[T, A_n]$ is obviously symmetric on $\mathfrak{D}_0$, its bounded extension is self-adjoint, i.e.,

$$[T, A_n]^* = -[T, A_n]^c. \tag{2.4}$$

Combining this with equation (2.3) completes the proof of the Lemma 2.1. ■

LEMMA 2.2. *Assume* $\mathbf{A}_1$–$\mathbf{A}_3$. *Then* $A_n \mathfrak{D}(T^*) \subset \mathfrak{D}(T^c)$ *and*

$$T^c A_n \psi = (A_m T A_m)^c A_n \psi + [T, A_m]^c A_n \psi \quad \text{for all } \psi \in \mathfrak{D}(T^*), \tag{2.5}$$

where m *is chosen according to Assumption* $\mathbf{A}_1$.

Proof. Let $\psi \in \mathfrak{D}(T^*)$. By Lemma 2.1, $A_n\psi \in \mathfrak{D}(T^*)$. Choose m, k according to $\mathbf{A}_1$ and note that $A_m A_n = A_m A_k A_n = A_k A_n = A_n$. Hence for $\varphi \in \mathfrak{D}_0$,

$$(A_n\psi, A_m T A_m \varphi) = (A_m T^* A_n \psi, \varphi). \tag{2.6}$$

Since by Assumption $\mathbf{A}_3$

$$(A_m T A_m|\mathfrak{D}_0)^* = (A_m T A_m|\mathfrak{D}_0)^c, \tag{2.7}$$

equation (2.6) implies

$$A_n\psi \in \mathfrak{D}((A_m T A_m|\mathfrak{D}_0)^c). \tag{2.8}$$

Hence there is a sequence $\{\chi_j\}_{j=0}^{\infty}$ with $\chi_j \in \mathfrak{D}_0$ for all j, $\lim \chi_j = A_n\psi$, such that $\{A_m T A_m \chi_j\}_{j=0}^{\infty}$ is convergent. But then the sequence$\{\xi_j\}_{j=0}^{\infty}$ with $\xi_j = A_k\chi_j$ has the same properties: $\xi_j \in \mathfrak{D}_0$ and (by continuity of A_n) $\lim \xi_j$ exists. Furthermore

$$A_m T A_m \xi_j = A_k A_m T A_m \chi_j + A_m[T, A_k] A_m \chi_j, \tag{2.9}$$

which converges, as $j \to \infty$. Therefore

$$T\xi_j = T A_m \xi_j = A_m T A_m \xi_j + [T, A_m]\xi_j \tag{2.10}$$

again converges in $\mathfrak{H}$, as $j \to \infty$. Hence $A_n\psi = \lim \xi_j \in \mathfrak{D}(T^c)$ and

$$T^c A_n \psi = \lim_{j\to\infty} T\xi_j, \tag{2.11}$$

which is just what we wanted to prove. ∎

Proof of Theorem 1.1. Let $\psi \in \mathfrak{D}(T^*)$. By Lemma 2.2, $A_n\psi \in \mathfrak{D}(T^c)$, and

$$(\psi, T^*\psi) = (T^c A_{n_k}\psi, A_{n_k}\psi) + (T^c A_{n_k}\psi, \psi - A_{n_k}\psi) + (\psi - A_{n_k}\psi, T^*\psi), \tag{2.12}$$

Using $\mathbf{A}_4$ we find

$$|(T^c A_{n_k}\psi, \psi - A_{n_k}\psi)| + |(\psi - A_{n_k}\psi, T^*\psi)| \underset{n\to\infty}{\longrightarrow} 0. \tag{2.13}$$

Note that

$$(T^c A_{n_k}\psi, A_{n_k}\psi) = (A_{n_k}\psi, T^c A_{n_k}\psi) \tag{2.14}$$

is real, because the closure of a symmetric operator is always symmetric. We conclude that

$$(\psi, T^*\psi) = \lim_{k\to\infty} (A_{n_k}\psi, T^c A_{n_k}\psi) \tag{2.15}$$

is real as a limit of real numbers. Hence (by the polarization identity) T^* is symmetric which proves that T is essentially self-adjoint. ■

Proof of Lemma 1.1. Assuming $\mathbf{A}_2$ we find with the help of Lemma 2.1 that for all $\psi \in \mathfrak{D}(T^*)$

$$\|T^* A_{n_k}\psi\| \leqslant \|A_{n_k}\| \, \|T^*\psi\| + \|[T, A_{n_k}]^c\psi\|. \tag{2.16}$$

Using equation (1.5) and $\mathbf{A}_4'$ we find

$$\|T^* A_{n_k}\psi\| \leqslant K\|T^*\psi\| + C(\psi) \equiv C_1(\psi). \tag{2.17}$$

Hence we obtain

$$|(T^c A_{n_k}\psi, \psi - A_{n_k}\psi| \leqslant C_1(\psi)\|\psi - A_{n_k}\psi\|, \tag{2.18}$$

which tends to zero because the sequence A_{n_k} is assumed to converge strongly to 1. This proves Lemma 1.1. ■

3. SOME REMARKS AND FURTHER RESULTS

REMARK 3.1. A slight modification of the proof of Theorem 1.1 allows to replace the condition $\mathbf{A}_4$ resp $\mathbf{A}_4'$ with, e.g.,

$\mathbf{B}_4$: For some subsequence $\{A_{n_k}\}_{k=0}^{\infty}$ and for all $\psi \in \mathfrak{H}$,

$$\sum_{k=1}^{\infty} A_{n_k} = 1,$$

where the series converges in the strong sense, and

$$\left\| \sum_{k=1}^{j} [T, A_{n_k}]^c\psi \right\| \leqslant C(\psi),$$

where $C(\psi) > 0$ is independent of j.

Repeat the calculation equations (2.12)–(2.18) with A_n replaced by $\tilde{A}_j = \sum_{k=1}^{j} A_{n_k}$ to arrive at the conclusion of Theorem 1.1.

REMARK 3.2. An immediate consequence of the essential self-adjointness of T on $\mathfrak{D}_0$ and the boundedness of the commutator is the following.

COROLLARY 3.1. *Assume $\mathbf{A}_2$ and let T be essentially self-adjoint on $\mathfrak{D}_0$. Denote*

$$\psi(t) \equiv \exp(-itT^c)\psi, \quad \textit{for all } \psi \in \mathfrak{H}. \tag{3.1}$$

Then for all $\psi, \varphi \in \mathfrak{H}$ the function $t \to (\psi(t), A_n\varphi(t))$ is continuously differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi(t), A_n\varphi(t)) = (\psi(t), \mathrm{i}[T, A_n]^c\varphi(t)). \tag{3.2}$$

Proof. For convenience we give the proof of this simple fact: For $\psi, \varphi \in \mathfrak{H}$ choose $\{\psi_j\}, \{\varphi_j\}$ in $\mathfrak{D}_0$ with $\psi = \lim\psi_j$, $\varphi = \lim\varphi_j$. By the strong continuity of the unitary group, $f_j(t) \equiv (\psi_j(t), A_n\varphi_j(t))$ is continuous, and by $\mathbf{A}_2$ even continuously differentiable with $f_j'(t) \equiv (\psi_j(t), \mathrm{i}[T, A_n]\varphi_j(t))$. Define $g(t) \equiv (\psi(t), \mathrm{i}[T, A_n]^c\varphi(t))$. Then

$$|f_j'(t) - g(t)| \leqslant \|[T, A_n]^c\|(\|\varphi_j\|\,\|\psi_j - \psi\| + \|\psi\|\,\|\varphi_j - \varphi\|) < \varepsilon, \tag{3.3}$$

where ε can be chosen independently of t and is arbitrarily small for j large. Hence we can exchange the differentiation and the limit to conclude that $f(t) = \lim f_j(t)$ is differentiable with $f'(t) = \lim f_j'(t)$. ■

The following theorem is a converse of Theorem 1.1. We give conditions for the essential self-adjointness of A_nTA_n as a consequence of the self-adjointness of T^c. First we state a lemma similar to Lemma 2.1.

LEMMA 3.1. *Assume $\mathbf{A}_2$. Then $A_n\mathfrak{D}(T^c) \subset \mathfrak{D}(T^c)$ and*

$$[T, A_n]^c\psi = T^cA_n\psi - A_nT^c\psi \quad \text{for all } \psi \in \mathfrak{D}(T^c). \tag{3.4}$$

Proof. By the definition of closure, for any $\psi \in \mathfrak{D}(T^c)$ there is a sequence $\{\psi_k\}_{k=0}^{\infty}$ with $\psi_k \in \mathfrak{D}_0$, $\lim\psi_k = \psi$, and $\lim T\psi_k = T^c\psi$. Using the continuity of A_n and $[T, A_n]$ we find

$$TA_n\psi_k = A_nT\psi_k + [T, A_n]\psi_k \underset{k\to\infty}{\longrightarrow} A_nT^c\psi + [T, A_n]^c\psi \tag{3.5}$$

Hence the sequnces $\{A_n\psi_k\}$ and $\{TA_n\psi_k\}$ both converge. This implies

$$\lim_{k\to\infty} A_n\psi_k = A_n\psi \in \mathfrak{D}(T^c), \quad T^cA_n\psi = \lim_{k\to\infty} TA_n\psi_k. \tag{3.6}$$

Now equation (3.4) follows immediately. ■

THEOREM 3.1. *Assume $\mathbf{A}_2$. Let T be essentially self-adjoint on $\mathfrak{D}_0$. In addition we assume for all ψ that $A_n^2\psi \in \mathfrak{D}(T^c)$ implies $A_n\psi \in \mathfrak{D}(T^c)$. Then $A_nTA_n|\mathfrak{D}_0$ is essentially self-adjoint.*

Proof. Denote $B = A_n T A_n | \mathfrak{D}_0$. Let $\psi \in \mathfrak{D}(B^*)$, $\varphi \in \mathfrak{D}_0$. Then

$$(B^*\psi, \varphi) = (\psi, A_n T A_n \varphi) = (A_n^2 \psi, T\varphi) + ([T, A_n]^* A_n \psi, \varphi) \tag{3.7}$$

shows that $A_n^2 \psi \in \mathfrak{D}(T^*) = \mathfrak{D}(T^c)$ and

$$T^c A_n^2 \psi = B^* \psi - [T, A_n]^* A_n \psi. \tag{3.8}$$

The additional assumption implies $A_n \psi \in \mathfrak{D}(T^c)$ and with Lemma 3.1 we obtain

$$T^c A_n^2 \psi = A_n T^c A_n \psi + [T, A_n]^c A_n \psi. \tag{3.9}$$

Using equation (2.4) we find that

$$B^* \psi = A_n T^c A_n \psi \quad \text{for all } \psi \in \mathfrak{D}(B^*). \tag{3.10}$$

Hence B^* is symmetric which is equivalent to the essential self-adjointness of B. ■

REMARK 3.3. The additional assumption in Theorem 3.1 is trivially satisfied if the operators A_n are projections.

4. AN EXAMPLE: DIRAC OPERATORS

In the Hilbert space $\mathfrak{H} = L^2(\mathbf{R}^3)^4$ of $\mathbf{C}^4$-valued square integrable functions on $\mathbf{R}^3$ we define the free Dirac operator as the closure of the operator

$$H_0 = -\mathrm{i}\alpha \cdot \nabla + \beta \quad \text{on } \mathfrak{D}_0 = C_0^\infty(\mathbf{R}^3 \setminus \{0\})^4. \tag{4.1}$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Hermitian 4×4 Dirac matrices (see [4] or details). It is well known that H_0 is essentially self-adjoint on $\mathfrak{D}_0$, its closure H_0^c is self-adjoint on the first Sobolev space

$$\mathfrak{D}(H_0^c) = H^1(\mathbf{R}^3)^4 \equiv \{\psi \in L^2(\mathbf{R}^3)^4 | \alpha \cdot \nabla \psi \in L^2(\mathbf{R}^3)^4\}. \tag{4.2}$$

Here ∇ denotes the distributional derivative.

We denote by g the bounded operator of multiplication by a function $g \in C_0^\infty(\mathbf{R}^3)$,

$$\psi \to g\psi, \quad (g\psi)(x) = g(x)\psi(x), \qquad \text{all } \psi \in \mathfrak{H}, \text{ all } x \in \mathbf{R}^3. \tag{4.3}$$

It is easy to see that

$$[H_0, g]^c = -\mathrm{i}\alpha \cdot (\nabla g) \tag{4.4}$$

is a bounded operator of multiplication with a Hermitian matrix-valued C_0^∞-function.

LEMMA 4.1. *The operator gH_0g is essentially self-adjoint on $\mathfrak{D}_0$.*

Proof. Since β is a bounded operator in $\mathfrak{H}$, it is sufficient to consider the operator

$$T_0 \equiv -\mathrm{i}\alpha \cdot \nabla \tag{4.5}$$

when investigating the self-adjointness-properties. Again, T_0 is essentially self-adjoint on $\mathfrak{D}(T_0) = \mathfrak{D}_0$, self-adjoint on $\mathfrak{D}(T_0^c) = \mathfrak{D}(H_0^c)$, and $[T_0, g]^c = [H_0, g]^c$. Our proof uses the self-adjointness of T_0^c on $H^1(\mathbf{R}^3)^4$ and the boundedness of $[T_0, g]^c$. Denote $B = gT_0g|\mathfrak{D}_0$. As in the calculation leading to equation (3.8) we conclude for all $\psi \in \mathfrak{D}(B^*)$ that $g^2\psi \in \mathfrak{D}(T_0^*) = \mathfrak{D}(T_0^c)$ and

$$B^*\psi = -\mathrm{i}\alpha \cdot \nabla g^2\psi + \mathrm{i}g\alpha \cdot (\nabla g)\psi. \tag{4.6}$$

By the Leibniz rule which holds for the product of a distribution with a smooth function we obtain

$$-\mathrm{i}\alpha \cdot \nabla g^2\psi = -2\mathrm{i}g\alpha \cdot (\nabla g)\psi - \mathrm{i}g^2\alpha \cdot \nabla\psi. \tag{4.7}$$

This shows that for all $\psi \in \mathfrak{D}(B^*)$ the distribution $\alpha \cdot \nabla\psi \in H^{-1}(\mathbf{R}^3)^4$ satisfies $g^2\alpha \cdot \nabla\psi \in L^2(\mathbf{R}^3)^4$. Therefore we can perform the following calculation for arbitrary $\psi, \varphi \in \mathfrak{D}(B^*)$

$$(\varphi, B^*\psi) = (\varphi, -\mathrm{i}\alpha \cdot \nabla g^2\psi + \mathrm{i}g\alpha \cdot (\nabla g)\psi) = \tag{4.8}$$

$$= (-\mathrm{i}g^2\alpha \cdot \nabla\varphi, \psi) + (-\mathrm{i}g\alpha \cdot (\nabla g)\varphi, \psi) = \tag{4.9}$$

$$= (-\mathrm{i}\alpha \cdot \nabla g^2\varphi + \mathrm{i}g\alpha \cdot (\nabla g)\varphi, \psi) = (B^*\varphi, \psi). \tag{4.10}$$

Hence B^* is symmetric, i.e., B is essentially self-adjoint. ∎

Now, let us consider a function $f \in C_0^\infty(\mathbf{R})$ with the properties $f(0) = 1$, and $f(r) = 0$ if $r \geqslant 1$. Define the sequence of multiplication operators A_n, $n = 1, 2, \dots$ by

$$(A_n\psi)(x) = \begin{cases} \psi(x) & \text{if } |x| \leqslant n, \\ f(|x| - n)\psi(x) & \text{if } |x| \geqslant n. \end{cases} \tag{4.11}$$

The sequence $\{A_n\}$ satisfies $\mathbf{A}_1$ and $\mathbf{A}_4'$, the commutators $[H_0, A_n]^c$ are bounded uniformly in n by $\sup|f'(r)|$. By Lemma 4.1 all the operators $A_nH_0A_n$ are essentially self-adjoint on $\mathfrak{D}_0$. Hence the following corollary is an immediate consequence of Theorem 1.1.

COROLLARY 4.1. *Let A_n be defined as above and let V be a symmetric operator on $\mathfrak{D}_0$ such that $\mathbf{A}_4$ holds with $T = V$, and assume that $A_n(H_0 + V)A_n$ is essentially self-adjoint on $\mathfrak{D}_0$. Then $H_0 + V$ is essentially self-adjoint on $\mathfrak{D}_0$.*

The assumptions of the corollary are trivially satisfied, if V is multiplication by a locally bounded Hermitian matrix-valued function (no matter how fast it grows at infinity), because in this case $A_n V A_n$ is a bounded perturbation of the essentially self-adjoint operator $A_n H_0 A_n$. As noted by Chernoff [2] this result is in marked contrast to the situation for the second-order Schrödinger operator and is related to the existence of a limiting velocity for the propagation of wavepackets according to the Dirac equation.

Because of the Kato-Rellich theorem only the relative boundedness of $A_n V A_n$ with respect to $A_n H_0 A_n$ is needed and one could also consider potentials with local singularities. Other examples include nonlocal potentials. See [3,4] for details.

A variant of the preceding proof is obtained by using a partition of unity $\{f_n\}$ on $\mathbf{R}^3$ with $\sup_{x,n} |\nabla f_n(x)| \leqslant M < \infty$ to define operators A_n satisfying $\mathbf{B}_4$.

5. PERTURBATION THEORY

Let H_0 and V be symmetric operators defined on a dense subset $\mathfrak{D}_0$ of some Hilbert space $\mathfrak{H}$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of bounded self-adjoint operators satisfying $\mathbf{A}_1$. Assume that $\mathbf{A}_2$ holds with T replaced by H_0 and V, respectively. Instead of $\mathbf{A}_3$ let us now assume

$\mathbf{C}_3$: H_0 is essentially self-adjoint on $\mathfrak{D}_0$, and for all n the operator $H_0 + A_n V A_n$ is essentially self-adjoint on $\mathfrak{D}_0$ with

$$\mathfrak{D}((H_0 + A_n V A_n)^c) = \mathfrak{D}(H_0^c). \tag{5.1}$$

The operator $T \equiv H_0 + V$ is well defined and symmetric on $\mathfrak{D}_0$ and we assume that it satisfies $\mathbf{A}_4$.

THEOREM 5.1. *Under the above assumptions, T is essentially self-adjoint on $\mathfrak{D}_0$.*

With the help of the following lemma, the proof of Theorem 5.1 is an easy modification of the proof of Theorem 1.1.

LEMMA 5.1. *Assume $\mathbf{A}_1$, $\mathbf{C}_3$, and $\mathbf{A}_2$ with H_0 and V. Then $\psi \in \mathfrak{D}(T^*)$ implies that $A_n\psi \in \mathfrak{D}(T^c)$ and*

$$T^c A_n \psi = H_0^c A_n \psi + V^c A_n \psi, \quad \text{for all } \psi \in \mathfrak{D}(T^*). \tag{5.2}$$

Proof. Let $\psi \in \mathfrak{D}(T^*)$. Since T satisfies $\mathbf{A}_2$ we find with Lemma 2.1 that $A_n\psi \in \mathfrak{D}(T^*)$. Hence for $\varphi \in \mathfrak{D}_0$ we obtain using $A_m A_n = A_n$

$$(T^* A_n \psi, \varphi) = (A_n \psi, \{T_m - [V, A_m]\}\varphi), \quad T_m \equiv H_0 + A_m V A_m. \tag{5.3}$$

This shows that $A_n\psi \in \mathfrak{D}(T_m^*) = \mathfrak{D}(T_m^c) = \mathfrak{D}(H_0^c)$, where we have used $\mathbf{C}_3$. By definition of closure, there is a sequence $\chi_j \in \mathfrak{D}_0$ with $\chi_j \to A_n\psi$ and $H_0^c\chi_j \to$ $\to H_0^c A_n\psi$, i.e., $\{\chi_j\}$ converges in the Hilbert space $\mathfrak{D}(H_0)$ equipped with the graph norm $\|\psi\|_0^2 = \|H_0^c\psi\|^2 + \|\psi\|^2$. Since T_m^c is closed on $(\mathfrak{D}(H_0), \|\cdot\|_0)$, it is bounded and hence the sequence $\{T_m^c\chi_j\}$ is again convergent. As in the proof of Lemma 2.2 we can replace χ_j by the sequence $\xi_j = A_k\chi_j$, which has the same properties, if A_k is chosen according to $\mathbf{A}_1$. In particular, the sequences $\{T_m\xi_j\}$ and $\{H_0\xi_j\}$ are convergent. But then

$$V\xi_j = VA_m\xi_j = A_mVA_m\xi_j + [V, A_m]\xi_j = (T_m - H_0)\xi_j + [V, A_m]^c\xi_j \tag{5.4}$$

is convergent, i.e., $\lim \xi_j = A_n\psi \in \mathfrak{D}(V^c)$. Finally,

$$T\xi_j = H_0\xi_j + V\xi_j \to H_0^c A_n\psi + V^c A_n\psi, \tag{5.5}$$

which implies $A_n\psi \in \mathfrak{D}(T^c)$ together with equation (5.2). ■

THEOREM 5.2. *Let H_0 be essentially self-adjoint on $\mathfrak{D}_0$ and X be self-adjoint on $\mathfrak{D}(X)$, such that $[H_0^c, X]$ is well defined on $\mathfrak{D}_0$ and bounded. Let V be a real-valued function on* $\mathbf{R}$, *which is locally bounded. Define $V(X) = \int V(\lambda)\mathrm{d}E_X(\lambda)$ and assume $\mathfrak{D} \subset \mathfrak{D}(V(X))$. Then $H_0 + V(X)$ is essentially self-adjoint on $\mathfrak{D}_0$.*

Proof. Let g be a real-valued function, which can be written as the Fourier-transform of a function $\tilde{g}$, such that $(1 + |\xi|)\tilde{g}(\xi)$ is integrable:

$$g(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda\xi}\tilde{g}(\xi)\mathrm{d}\xi. \tag{5.6}$$

Define $g(X)$ by the weak integral

$$g(X)\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}X\xi}\psi\tilde{g}(\xi)\mathrm{d}\xi, \quad \text{for all } \psi \in \mathfrak{H}. \tag{5.7}$$

Then

$$\|[H_0^c, g(X)]^c\| \leqslant \frac{1}{\sqrt{2\pi}}\|[H_0^c, X]^c\| \int_{-\infty}^{\infty} |\xi|\,|\tilde{g}(\xi)|\mathrm{d}\xi. \tag{5.8}$$

Let $f \in C_0^\infty(\mathbf{R})$ be real-valued, with $f(\lambda) = 1$, if $|\lambda| \leqslant 1/2$ and $f(\lambda) = 0$, if $|\lambda| \geqslant 1$. Define $A_n := f(X/n) = \int_{-n}^{n} f(\lambda/n)\mathrm{d}E_X(\lambda)$, where E_X is the spectral measure of X.

Now it is easy to see that $T = H_0$ and A_n satisfy the assumptions $\mathbf{A}_1$–$\mathbf{A}_4$, and even $\mathbf{A}_4'$. By our assumptions, $V(X) = \int V(\lambda) \mathrm{d}E_X(\lambda)$ is a densely defined self--adjoint operator which commutes with all A_n. Moreover, $A_n V(X) A_n$ is bounded and symmetric on $\mathfrak{D}_0$. Hence $H_0 + A_n V(X) A_n$ is essentially self-adjoint on $\mathfrak{D}_0$. Essential self-adjointness of $H_0 + V(X)$ now follows immediately from Theorem 5.1. ∎

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