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SPECTRA FOR COMPACT GROUP ACTIONS

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1. INTRODUCTION AND PRELIMINARIES

Let (G, A, α) be a C^*-dynamical system. For G abelian, the principal tool for investigating the structure of the crossed product algebra $G \times_\alpha A$ has been the Connes spectrum [17, Section 8.11]. As Rieffel remarks in the introduction of [21], "it is a very interesting open question as to how the Connes spectrum should best be defined in the non-abelian case". Furthermore, on page 40 of [16], Landstad remarks that in a "good" definition of the spectrum, the kind of result one would want to generalize is the theorem of Olesen and Pedersen characterizing the primeness of the crossed product algebra in terms of the Connes spectrum [17, Theorem 8.11.10]. In this paper, we present definitions of both the Connes spectrum and the strong Connes spectrum, in the case of a compact group action, which do generalize both [17, Theorem 8.11.10], and the main theorem of Kishimoto [11, Theorem 3.5] on the simplicity of crossed product algebras.

Let (G, A, α) be a C^*-dynamical system, with G compact. For $\pi \in \hat{G}$, the space of equivalence classes of irreducible unitary representations of G, we denote by H_π the finite-dimensional Hilbert space on which π acts, by $d(\pi)$ the dimension of H_π, and by χ_π the normalized character of π, that is, $\chi_\pi(g) = d(\pi)\mathrm{trace}(\pi_g^{-1})$. To each $\pi \in \hat{G}$ we can associate a projection $P_\alpha(\pi) : A \to A$, defined by

$$P_\alpha(\pi)(a) = \int_G \chi_\pi(g)\alpha_g(a)\mathrm{d}g, \quad a \in A.$$

The range $A_1(\pi)$ of $P_\alpha(\pi)$, i.e., $\{a \in A : P_\alpha(\pi)(a) = a\}$, is the spectral subspace of A associated with π.

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A notion of spectrum for the action α of G on A, involving the spectral subspaces $A_1(\pi)$, was defined and discussed by Evans and Sund [2] and Katayama [10]. Landstad [16] and Peligrad [18] observed that the spectral subspaces

$$A_2(\pi) \equiv \{V \in A \otimes B(H_\pi) : (\alpha_g \otimes \mathrm{id})(V) = V(1_A \otimes \pi_g),\ g \in G\}$$

were more useful for studying the properties and ideal structure of the crossed product algebra $G \times_\alpha A$. In this paper we shall consider two types of spectra whose definitions involve $A_2(\pi)$. Note that $A_2(\pi)^* A_2(\pi)$ is a two-sided ideal in $(A \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi}$, the fixed-point subalgebra of $A \otimes B(H_\pi)$ under the tensor product action of G. ∎

DEFINITION 1.1. (a) $\mathrm{Sp}(\alpha) \equiv \{\pi \in \hat{G} : \mathrm{cl}(A_2(\pi)^* A_2(\pi))$ is an essential ideal in $(A \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi}\}$.

(b) $\widetilde{\mathrm{Sp}}(\alpha) \equiv \{\pi \in \hat{G} : \mathrm{cl}(A_2(\pi)^* A_2(\pi)) = (A \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\,\pi}$.

Corresponding to the above two Arveson type spectra for the action α, we have two Connes type spectra. Let $\mathcal{H}^\alpha(A)$ denote the family of all non-zero α-invariant hereditary C^*-subalgebras B of A.

DEFINITION 1.2. (a) The Connes spectrum $\Gamma(\alpha) \equiv \bigcap\{\mathrm{Sp}(\alpha|B) : B \in \mathcal{H}^\alpha(A)\}$.

(b) The strong Connes spectrum $\tilde{\Gamma}(\alpha) \equiv \bigcap\{\widetilde{\mathrm{Sp}}(\alpha|B) : B \in \mathcal{H}^\alpha(A)\}$.

It is easy to see that if G is abelian, $A_2(\pi)$ can be identified with $A_1(\pi)$ and $\Gamma(\alpha)$ coincides with the usual Connes spectrum of α [17, Chapter 8], while, by the proof of [20, Theorem 7.2.7], $\tilde{\Gamma}(\alpha)$ coincides with the strong Connes spectrum as defined by Kishimoto [11]. This fact, as well as our two main theorems, that $G \times_\alpha A$ is prime $\Leftrightarrow$ A is G-prime and $\Gamma(\alpha) = \hat{G}$ (Theorems 2.2, 3.8), while $G \times_\alpha A$ is simple $\Leftrightarrow$ A is G-simple and $\tilde{\Gamma}(\alpha) = \hat{G}$ (Theorem 2.5, 3.4), justify our terminology (see [17, Theorem 8.11.10] and [11, Theorem 3.5]). Note that our definition of $\Gamma(\alpha)$ differs from that of [2] and [10].

In section two we present proofs, more or less of an algebraic nature, of our main theorems. The proofs use both parts of the hypothesis concerning the action α of G on A (i.e., that A is G-prime (resp., G-simple) and that the spectrum is full) to deduce that the fixed-point algebra A^α is prime (resp., simple) (Proposition 2.1, 2.4). From this and results of [18], the conclusion concerning the structure of $G \times_\alpha A$ follows. In section three we present different proofs of our main theorems. The nature of these proofs is more representation-theoretic, and separates out the roles played by the two parts of the hypothesis on the action α. Specifically, we prove that $\Gamma(\alpha) = \hat{G}$ if and only if every non-zero ideal J of $G \times_\alpha A$ is an essential subideal of an ideal of the form $G \times_\alpha I$, I a G-invariant ideal of A (Theorem 3.7), while $\tilde{\Gamma}(\alpha) = \hat{G}$ if and only if every ideal J of $G \times_\alpha A$ is of the form $G \times_\alpha I$, I a G-invariant ideal of A (Theorem 3.3). For the prime case especially, these proofs use methods of non-commutative duality theory and results of [5] concerning the structure of ideals of $G \times_\alpha A$ invariant under the dual coaction. Unfortunately, our proof of the prime case also requires that the dynamical system be separable.

We feel that there should be a relevant notion of spectrum for the action of a non-abelian, non-compact group also, and results which would relate the spectrum to the structure of the crossed product algebra. Perhaps a preliminary step in developing such a notion would be a unified proof, valid for both compact groups and abelian groups, relating the spectral properties of the action to properties of the crossed product algebra. Such a proof is not yet at hand, and the main reason we present both types of proofs of our main theorems is that it is not yet clear to us which of our methods (if any) might contribute to a unified proof, covering both the abelian and compact cases, as well as to an extended theory.

Finally, in Section 4, we discuss various circumstances in which $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$, and present in Section 5 several counterexamples to natural questions and conjectures. We close this section by establishing additional notation, and by making explicit two results implicitly contained in [18].

For each pair $\pi_1, \pi_2 \in \hat{G}$ we let S_{π_1,π_2} denote the closed subspace $\chi_{\pi_1} * (G \times_\alpha A) * * \chi_{\pi_2}$ of $G \times_\alpha A$, and for $\pi \in \hat{G}$, we denote the hereditary C^*-subalgebra $S_{\pi,\pi}$ of $G \times_\alpha A$ simply as S_π (note that χ_π is a projection in the multiplier algebra $M(G \times_\alpha A)$). A function $x \in L^1(G, A)$ is called central if $\alpha_r(x(r^{-1}sr)) = x(s)$ for all $r, s \in G$. The C^*-subalgebra of $G \times_\alpha A$ generated by the (images of the) central functions will be denoted I. There is a faithful conditional expectation P of $G \times_\alpha A$ onto I given, for $x \in L^1(G, A)$, by

$$(Px)(s) = \int_G \alpha_r(x(r^{-1}sr))dr, \quad s \in G.$$

Following [18], let $I_\pi = I \cap S_\pi$ and let $I(\pi) = C(G) * \chi_\pi$, where $C(G)$ is considered a subspace of $M(G \times_\alpha A)$. Then $I(\pi)$ is $*$-isomorphic to the algebra of all $d(\pi) \times d(\pi)$ scalar matrices, and there is a $*$-isomorphism of $I(\pi) \otimes I_\pi$ onto S_π (see [18, Proposition 2.7]). We denote by $\imath$ the trivial representation of G.

PROPOSITION 1.3. *Let $\pi \in \hat{G}$. Then $\overline{\pi} \in \mathrm{Sp}(\alpha)$ if and only if the closure $\mathrm{cl}(S_{\pi,\imath} * S_{\imath,\pi})$ of the ideal $S_{\pi,\imath} * S_{\imath,\pi}$ in S_π is essential in S_π.*

Proof. Let $\mathrm{cl}(S_{\pi,\imath} * S_{\imath,\pi})$ be essential in S_π, and assume $\overline{\pi} \notin \mathrm{Sp}(\alpha)$, i.e., that $\mathrm{cl}(A_2(\overline{\pi})^* A_2(\overline{\pi}))$ is not essential in $(A \otimes B(H_{\overline{\pi}}))^{\alpha \otimes \mathrm{ad}\,\overline{\pi}}$. It follows then from [18, Lemma 2.10] that there exists $c \geqslant 0$ in I_π with $P(c\,\mathrm{cl}(S_{\pi,\imath} * S_{\imath,\pi})c) = cP(\mathrm{cl}(S_{\pi,\imath} * *S_{\imath,\pi}))c = 0$. Thus $c\,\mathrm{cl}(S_{\pi,\imath} * S_{\imath,\pi})c = 0$ which, since $c \in S_\pi$, contradicts our hypothesis.

For the converse, assume $\overline{\pi} \in \mathrm{Sp}(\alpha)$. By [15, Lemma 3] and [18, Lemma 2.10],

$P(\mathrm{cl}(S_{\pi,i} * S_{i,\pi}))$ is an essential ideal in I_π. The proof of Lemma 2.10 in [18] also shows that $P(\mathrm{cl}(S_{\pi,i} * S_{i,\pi})) = I_\pi \cap (\mathrm{cl}(S_{\pi,i} * S_{i,\pi}))$. The image of I_π in $I(\pi) \otimes I_\pi$ by the isomorphism of [18, Proposition 2.7] is $\chi_\pi \otimes I_\pi = \{\mathrm{diag}(x, \ldots, x) : x \in I_\pi\}$. Thus $\{\mathrm{diag}(y, \ldots, y) : y \in I_\pi \cap \mathrm{cl}(S_{\pi,i} * S_{i,\pi})\}$ is essential in $\{\mathrm{diag}(x, \ldots, x) : x \in I_\pi\}$. Clearly this implies that the image of $\mathrm{cl}(S_{\pi,i} * S_{i,\pi})$ in $I(\pi) \otimes I_\pi$ is essential in this algebra, hence $\mathrm{cl}(S_{\pi,i} * S_{i,\pi})$ is essential in S_π.

PROPOSITION 1.4. *Let $\pi \in \hat{G}$. Then $\overline{\pi} \in \widetilde{\mathrm{Sp}}(\alpha)$ if and only if $\mathrm{cl}(S_{\pi,i} * S_{i,\pi}) = S_\pi$.*

Proof. We omit the proof as it is completely similar to the proof of Proposition 1.3. ∎

2. SPECTRA AND IDEALS (PART 1)

For $\pi \in \hat{G}$, we represent elements of $A \otimes B(H_\pi)$ as $d(\pi) \times d(\pi)$ matrices over A, i.e., as $[a_{ij}]$ with $1 \leqslant i, j \leqslant d(\pi)$ and $a_{ij} \in A$. Also, we write $[\pi_{ij}(g)]$ for the matrix of $\pi(g)$ in a fixed orthonormal basis of H_π, and $P_{ij}(\pi)$ for the map of A into A given by $P_{ij}(\pi)(a) = d(\pi) \int_G \overline{\pi_{ij}(g)} \alpha_g(a) \mathrm{d}g$, $a \in A$.

PROPOSITION 2.1. *Let (G, A, α) be a C^*-dynamical system, with G compact. If A is G-prime and $\Gamma(\alpha) = \hat{G}$, then the fixed-point subalgebra $A^\alpha \subset A$ is prime.*

Proof. If A^α is not prime, then there exist two non-zero positive elements $a_0, a_1 \in \in A^\alpha$ such that $a_1 A^\alpha a_0 = (0)$. As A is G-prime, $a_1 A a_0 \neq (0)$, and as the span of the spectral subspaces $\{A_1(\pi) : \pi \in \hat{G}\}$ is dense in A, $a_1 A_1(\pi_0) a_0 \neq (0)$ for some $\pi_0 \in \hat{G}$. Let $B = \mathrm{cl}(a_0 A a_0)$, so that $B \in \mathcal{H}^\alpha(A)$. Using the fact that any $[a_{ij}] \in A_2(\pi_0)$ is of the form $a_{ij} = P_{ij}(\pi_0)(a)$ for some fixed $a \in A$ [18, Lemma 2.2], it is easy to see that $B_2(\pi_0) = (\mathrm{cl}(a_0 A a_0))_2(\pi_0) = \mathrm{cl}(a_0 A_2(\pi_0) a_0)$, where, for $x, y \in A$, by $x[a_{ij}]y$ we mean the matrix with ij^{th} element $x a_{ij} y$. As $\pi_0 \in \Gamma(\alpha)$, we have (recall Definitions 1.1(a) and 1.2(a)) that $\mathrm{cl}(B_2(\pi_0)^* B_2(\pi_0)) = \mathrm{cl}(a_0 A_2(\pi_0)^* a_0^2 A_2(\pi_0) a_0)$ is an essential ideal of $(B \otimes B(H_{\pi_0}))^{\alpha \otimes \mathrm{ad}\, \pi_0} = ((\mathrm{cl}(a_0 A a_0)) \otimes B(H_{\pi_0}))^{\alpha \otimes \mathrm{ad}\, \pi_0}$. However, as $a_1 A^\alpha a_0 = (0)$ and as $A_2(\pi_0) a_0^2 A_2(\pi_0)^* \subset A^\alpha \otimes B(H_{\pi_0})$, we have

$$(*) \qquad (a_0 A_2(\pi_0)^* a_1^2)(A_2(\pi_0) a_0^2 A_2(\pi_0)^*) a_0^2 A_2(\pi_0) a_0 = (0).$$

Regroup the terms in the left-hand side of $(*)$ as

$$(a_0 A_2(\pi_0)^* a_1^2 A_2(\pi_0) a_0)(a_0 A_2(\pi_0)^* a_0^2 A_2(\pi_0) a_0).$$

As, by above, $\mathrm{cl}(a_0 A_2(\pi_0)^* a_0^2 A_2(\pi_0) a_0) = \mathrm{cl}(B_2(\pi_0)^* B_2(\pi_0))$ is an essential ideal of $(\mathrm{cl}(a_0 A a_0) \otimes B(H_{\pi_0}))^{\alpha \otimes \mathrm{ad}\, \pi_0}$, and as $a_0 A_2(\pi_0)^* a_1^2 A_2(\pi_0) a_0 \subset (a_0 A a_0 \otimes B(H_{\pi_0}))^{\alpha \otimes \mathrm{ad}\, \pi_0} =$

$= a_0(A \otimes B(H_{\pi_0}))^{\alpha\otimes \mathrm{ad}\,\pi_0} a_0$, we have that $a_1 A_2(\pi_0) a_0 = (0)$. It follows clearly that $a_1 A_1(\pi_0) a_0 = (0)$, a contradiction. Hence A^α is prime.

THEOREM 2.2. *Let (G, A, α) be a C^*-dynamical system with G compact. The following are equivalent:*

(1) $G \times_\alpha A$ is prime;

(2) A is G-prime and $\Gamma(\alpha) = \hat{G}$.

Proof. (2) $\Rightarrow$ (1) For each $\pi \in \hat{G}$, the C^*-algebras $\mathrm{cl}(A_2(\pi)^* A_2(\pi))$ and $\mathrm{cl}(A_2(\pi) A_2(\pi)^*)$ are strongly Morita equivalent ($A_2(\pi)$ being an imprimitivity bimodule). By Proposition 2.1, $A^\alpha \otimes B(H_\pi)$ is prime, hence so is the ideal $\mathrm{cl}(A_2(\pi) A_2(\pi)^*)$, and the Morita equivalent algebra $\mathrm{cl}(A_2(\pi)^* A_2(\pi))$. By definition of $\Gamma(\alpha)$, $\mathrm{cl}(A_2(\pi)^* A_2(\pi))$ is essential in $(A \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$, and thus $(A \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is prime also. The result follows by [18, Corollary 3.12].

(1) $\Rightarrow$ (2) That A is G-prime follows trivially. For $B \in \mathcal{H}^\alpha(A)$, $G \times_\alpha B$ is a hereditary C^*-subalgebra of $G \times_\alpha A$, and thus $G \times_\alpha B$ is prime. By [18, Corollary 3.12] again, $(B \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ is prime, and $B_2(\pi) \neq (0)$ (since $B_1(\pi) \neq 0$). Thus the non-zero ideal $\mathrm{cl}(B_2(\pi)^* B_2(\pi))$ is essential in $(B \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$ $\forall \pi \in \hat{G}$, and we are done. ∎

To prove an analogous result about simple crossed products, we need a preliminary lemma. For $[a_{ij}] \in A \otimes B(H_\pi)$, define $\mathrm{tr}[a_{ij}] = \sum_{i=1}^{d(\pi)} a_{ii}$.

LEMMA 2.3. *Let (G, A, α) be a C^*-dynamical system with G compact, and let J be an ideal in A^α. Then $(\mathrm{cl}(AJA))^\alpha =$ closed linear span$\{\mathrm{tr}(A_2(\pi) J A_2(\pi)^*) : \pi \in$ $\in \hat{G}\}$.*

Proof. As $A_2(\pi) A_2(\pi)^* \subseteq A^\alpha \otimes B(H_\pi)$, it is immediate that $\mathrm{tr}(A_2(\pi) J A_2(\pi)^*) \subseteq$ $\subseteq (AJA)^\alpha$. For the opposite inclusion, let $\pi^1, \pi^2 \in \hat{G}$ and let $a \in A_1(\pi^1)$, $b \in$ $\in A_1(\pi^2)$, $w \in J$. Using the notation introduced at the begining of this section, we have that $a = P_\alpha(\pi^1)(a) = \sum_i P_{ii}(\pi^1)(a)$, and thus by [18, Remark 2.1(iii)],

$$\alpha_s(a) = \sum_{i,j} \pi^1_{ji}(s) P_{ij}(\pi^1)(a), \quad s \in G,$$

and similarly for b. Thus

$$P(awb^*) = \int_G \alpha_s(awb^*) ds =$$

$$= \sum_{i,j,k,l} \int_G \pi^1_{ji}(s) P_{ij}(\pi^1)(a) w \, \overline{\pi^2_{kl}(s)} \, (P_{lk}(\pi^2)(b))^* ds =$$

$$= \delta_{\pi^1,\pi^2} \cdot \frac{1}{d(\pi^1)} \sum_{ij} P_{ij}(\pi^1)(a) w (P_{ij}(\pi^2)(b))^*,$$

by the orthogonality relations. For $\pi^1 = \pi^2 = \pi$, $\sum_{ij} P_{ij}(\pi)(a)(P_{ij}(\pi)(b))^* =$ $= \mathrm{tr}([x_{ij}]w[y_{ij}]^*)$ with $x_{ij} = P_{ij}(\pi)(a)$ and $y_{ij} = P_{ij}(\pi)(b)$, so $P(awb^*) \in$ $\in \mathrm{tr}(A_2(\pi)JA_2(\pi)^*)$, and equality holds. ■

PROPOSITION 2.4. *Let (G, A, α) be a C^*-dynamical system with G compact. If A is G-simple and $\tilde{\Gamma}(\alpha) = \hat{G}$, then the fixed point algebra $A^\alpha \subseteq A$ is simple.*

Proof. Let J be a non-zero closed two-sided ideal in A^α. We first prove that for each $\pi \in \hat{G}$, $\mathrm{tr}(A_2(\pi)JA_2(\pi)^*) \subseteq J$. Let $B = \mathrm{cl}(JAJ)$, so that $B \in \mathcal{H}^\alpha(A)$. By hypothesis and the definition of $\tilde{\Gamma}(\alpha)$, $\mathrm{cl}(B_2(\pi)^*B_2(\pi)) = (B \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$. Using [18, Lemma 2.2], one can easily check that $B_2(\pi) = \mathrm{cl}(JA_2(\pi)J)$, so that $\mathrm{cl}(B_2(\pi)^*B_2(\pi)) = \mathrm{cl}(JA_2(\pi)^*JA_2(\pi)J)$, while obviously $(B \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi} =$ $= \mathrm{cl}(J(A \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}J)$. (Again, by $j[a_{ik}]$ we mean the matrix $[ja_{ik}]$). Thus

$$\mathrm{cl}(JA_2(\pi)^*JA_2(\pi)J) = \mathrm{cl}(J(A \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}J).$$

From this we deduce that

$$\mathrm{cl}(A_2(\pi)JA_2(\pi)^*JA_2(\pi)JA_2(\pi)^*) = \mathrm{cl}(A_2(\pi)J(A \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}JA_2(\pi)^*).$$

As $A_2(\pi)JA_2(\pi)^* \subseteq A^\alpha \otimes B(H_\pi)$, we have $A_2(\pi)JA_2(\pi)^*JA_2(\pi)JA_2(\pi)^* \subseteq J \otimes$ $\otimes B(H_\pi)$, and thus

$$J \otimes B(H_\pi) \supseteq A_2(\pi)J(A \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}JA_2(\pi)^* \supseteq A_2(\pi)JA_2(\pi)^*,$$

so that $\mathrm{tr}(A_2(\pi)JA_2(\pi)^*) \subseteq \mathrm{tr}(J \otimes B(H_\pi)) \subseteq J$. ■

By Lemma 2.3, $(\mathrm{cl}(AJA))^\alpha \subseteq J$. As A is G-simple, $\mathrm{cl}(AJA) = A$, so that $A^\alpha = (\mathrm{cl}(AJA))^\alpha \subseteq J$. Thus $J = A^\alpha$ and A^α is simple.

THEOREM 2.5. *Let (G, A, α) be a C^*-dynamical system with G compact. The following are equivalent:*

(1) *$G \times_\alpha A$ is simple;*

(2) *A is G-simple and $\tilde{\Gamma}(\alpha) = \hat{G}$.*

Proof. (2) ⇒ (1). By Proposition 2.4, A^α is simple. Hence the non-zero ideal $\mathrm{cl}(A_2(\pi)A_2(\pi)^*) \subseteq A^\alpha \otimes B(H_\pi)$ is simple, and so is the Morita equivalent algebra $\mathrm{cl}(A_2(\pi)^*A_2(\pi))$. By hypothesis $\mathrm{cl}(A_2(\pi)^*A_2(\pi)) = (A \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$, so the latter algebras are all simple, and the result follows from [18, Corollary 3.7].

(1) ⇒ (2). That A is G-simple follows trivially. For $\pi \in \hat{G}$ and $B \in \mathcal{H}^\alpha(A)$, $G \times_\alpha B$ is a hereditary C^*-subalgebra of $G \times_\alpha A$, and hence is also simple. By [18, Corollary 3.7], $B_2(\pi) \neq 0$ and $(B \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$ is simple. Hence the non-zero ideal $\mathrm{cl}(B_2(\pi)^*B_2(\pi)) = (B \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$ and $\pi \in \tilde{\Gamma}(\alpha)$. ■

3. SPECTRA AND IDEALS (PART 2)

Recall that representations L of $G \times_\alpha A$ are in one-to-one correspondence with covariant pairs of representations $\langle V, \tau \rangle$ of (G, A, α). For $f \in L^1(G, A)$, the correspondence is determined by $L(f) = \int_G \tau(f(s))V(s)ds$, and, by abuse of notation, we shall simply write $L = \langle V, \tau \rangle$. Denoting also by L the extension of L to the multiplier algebra $M(G \times_\alpha A)$, we have, for $f \in L^1(G)$, that $L(f) = V(f) = \int_G f(s)V(s)ds$. In particular, for $\pi \in \hat{G}$ (G compact), $L(\chi_\pi) = P_\pi$, the projection of the Hilbert space H_L on which L acts onto the subspace of H_L on which V acts as π. It follows that the irreducible representations of S_π correspond precisely to the irreducible representations $L = \langle V, \tau \rangle$ of $G \times_\alpha A$ for which $V \supseteq \pi$, while the irreducible representations of $\mathrm{cl}(S_{\pi,i} * S_{i,\pi})$ correspond precisely to the irreducible representations $L = \langle V, \tau \rangle$ of $G \times_\alpha A$ for which $V \supseteq \pi$ and $V \supseteq i$. These observations lead to the following:

LEMMA 3.1. *Let $\pi \in \hat{G}$. Then*

(a) $\overline{\pi} \in \mathrm{Sp}(\alpha)$ *if and only if* $\{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : V \supseteq \pi \text{ and } V \supseteq i\}$ *is dense in* $\{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : V \supseteq \pi\}$, *and*

(b) $\overline{\pi} \in \widetilde{\mathrm{Sp}}(\alpha)$ *if and only if* $\{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : V \supseteq \pi \text{ and } V \supseteq i\} = \{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : V \supseteq \pi\}$.

Proof. Propositions 1.3 and 1.4. ∎

By the above lemma, $\widetilde{\mathrm{Sp}}(\alpha) = \hat{G}$ if and only if for each $L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{}$, $V \supseteq i$. Clearly this is equivalent to requiring that the closure of the ideal $(G \times_\alpha A)\chi_i(G \times_\alpha A)$ equal $G \times_\alpha A$. An action with this latter property is said to be saturated (see Section 7 of [20] for this and other equivalent formulations), and clearly the condition $\tilde{\Gamma}(\alpha) = \hat{G}$ is equivalent to the action being hereditarily saturated ([20, Section 7]). The approach to relating the ideal structure of $G \times_\alpha A$ to the spectrum of the action which we present in this section of the paper had its origin in the following question, raised verbally by N. C. Phillips to the first-named author: is hereditary saturation equivalent to every ideal of $G \times_\alpha A$ being of the form $G \times_\alpha I$, I a G-invariant ideal of A?

We first investigate simplicity and the strong Connes spectrum, since this problem is a bit more straightforward. If V is a representation of G on a Hilbert space H, we denote by H_i the subspace of H on which V acts as the identity representation i, so that $H_i = \{x \in H : V(s)x = x \ \forall s \in G\}$.

PROPOSITION 3.2. *Let (G, A, α) be a C^*-dynamical system with G compact, and assume that $\tilde{\Gamma}(\alpha) = \hat{G}$ (i.e., that the action is hereditarily saturated). If $L = \langle V, \tau \rangle$ is a representation of $G \times_\alpha A$ on H, and a is an element of A with $\tau(a)|H_i = 0$, then $\tau(a) = 0$.*

Proof. Let $S = \{x \in A : \tau(x)|H_i = 0\}$. Then S is a closed left ideal of A, and, as $\tau(\alpha_g x)|H_i = V(g)\tau(x)V(g^{-1})|H_i = V(g)\tau(x)|H_i$, S is α-invariant. Thus $B \equiv \mathrm{cl}(S^*S)$ is an α-invariant hereditary subalgebra of A, and if $\tau(S) \neq 0$, then the restriction $\tilde{L}$ of L to $G \times_\alpha B$ is non-zero. Letting K denote the non-zero essential subspace of this restriction, we have a non-degenerate representation $\tilde{L}$ of $G \times_\alpha B$ on K. Clearly $\tilde{L} = \langle \tilde{V}, \tilde{\tau} \rangle$ where $\tilde{V}$ is the restriction of V to K, and $\tilde{\tau}$ is the double restriction of τ, from A to B, and from acting on H to acting on K. As by hypothesis $\widetilde{\mathrm{Sp}}(\alpha|B) = \hat{G}$, it follows from Lemma 3.1 and the subsequent remarks that $\tilde{V}$ acts as the identity on a non-zero subspace $K_i \subseteq K$. Clearly $K_i \subseteq H_i$. However, as $\tau(S)|H_i = 0$, it also follows that $H_i \subseteq K^\perp$, so $K_i \subseteq H_i^\perp$ also, a contradiction to the assumption that $\tau(S) \neq 0$.

THEOREM 3.3. *Let (G, A, α) be a C^*-dynamical system with G compact. Then $\tilde{\Gamma}(\alpha) = \hat{G}$ if and only if every closed ideal of $G \times_\alpha A$ is of the form $G \times_\alpha I$, I a G-invariant ideal of A.*

Proof. $\Rightarrow$) Let J be an ideal of $G \times_\alpha A$. By [7, Propositions 11, 12 and 13], it suffices to show that $J = \mathrm{Ind}(\mathrm{Res}\, J)$, as $\mathrm{Res}\, J$ is a G-invariant ideal of A and $\mathrm{Ind}(\mathrm{Res}\, J) = G \times_\alpha (\mathrm{Res}\, J)$. Rephrased in terms of representations (see [7, Proposition 9]), let $L = \langle V, \tau \rangle$ be a representation of $G \times_\alpha A$ on H, with kernel $L = J$, so that $\mathrm{Ind}(\mathrm{Res}\, J) = \mathrm{kernel}(\mathrm{Ind}(\mathrm{Res}\, L))$. Here $\mathrm{Ind}(\mathrm{Res}\, L)$ is the representation of $G \times_\alpha A$ on $L^2(G, H)$ corresponding to the covariant pair $\langle \lambda \otimes 1, \tilde{\tau} \rangle$, where λ is the left-regular representation of G on $L^2(G)$ and, for $a \in A$, $s \in G$ and $\zeta \in L^2(G, H)$, $(\tilde{\tau}(a)\zeta)(s) = = \tau(\alpha_{s^{-1}} a)\zeta(s)$.

By the discussion prior to Proposition 3.2, our hypothesis implies that $\mathrm{cl}((G \times_\alpha A)\chi_i(G \times_\alpha A)) = G \times_\alpha A$, so that the hereditary subalgebra S_i, which is precisely the algebra of constant functions from G into A^α [22], is a full hereditary subalgebra of $G \times_\alpha A$. By [17, Chapter 4.1.], to check that kernel $L = \mathrm{kernel}(\mathrm{Ind}(\mathrm{Res}\, L))$, we need only show that $L|S_i$ and $(\mathrm{IndRes}\, L)|S_i$, when restricted to their essential subspaces, have the same kernel. These essential subspace are H_i (the subspace of H on which V acts as the identitity) and $(L^2(G, H))_i =$ the space of constant functions from G to H. Accordingly, let $x \in S_i$ be the function $x(t) = a$, $\forall t \in G$, and $a \in A^\alpha$, and suppose $L(x)|H_i = 0$. As $L(x) = \int_G \tau(x(t))V(t)\mathrm{d}t$, we have $L(x)|H_i = \tau(a)|H_i$. By Proposition 3.2, $L(x)|H_i = 0$ if and only if $\tau(a) = 0$, which happens if and only if $\tilde{\tau}(a) = 0$ (as kernel τ is α-invariant). In turn, by the same argument as above, applied to $\mathrm{Ind}(\mathrm{Res}\, L)$, $\tilde{\tau}(a) = 0$ if and only if the restriction to $L^2(G, H)_i$ of $(\mathrm{Ind}(\mathrm{Res}\, L))(x) = 0$, and we are done.

$\Leftarrow$) If every ideal of $G \times_\alpha A$ is of the form $G \times_\alpha I$, I a G-invariant ideal of A, then as in the proof of Theorem 3.2 of [6], the assumption that the ideal $\mathrm{cl}((G \times_\alpha A)\chi_i(G \times_\alpha A))$

is of this form implies $\mathrm{cl}((G \times_\alpha A)\chi_i(G \times_\alpha A)) = G \times_\alpha A$, so the action is saturated and, by Proposition 1.4, $\widetilde{\mathrm{Sp}}(\alpha) = \hat{G}$. All that remains is to show that the assumption on ideals of $G \times_\alpha A$ implies a similar result for ideals of $G \times_\alpha B$, B an α-invariant hereditary subalgebra of A. By [13], $G \times_\alpha B$ is a hereditary subalgebra of $G \times_\alpha A$, and thus every ideal is of the form $(G \times_\alpha B) \cap (G \times_\alpha I)$, I a G-invariant ideal of A. We shall be done once we show that $(G \times_\alpha B) \cap (G \times_\alpha I) = G \times_\alpha (B \cap I)$. For x in $G \times_\alpha (B \cap I)$, write $x = \lim_n x_n$, x_n a continuous function $G \to B \cap I$. As $x_n \in (G \times_\alpha B) \cap (G \times_\alpha I)$, x also lies in $(G \times_\alpha B) \cap (G \times_\alpha I)$, and $(G \times_\alpha B) \cap (G \times_\alpha I) \supseteq G \times_\alpha (B \cap I)$. For containment in the other direction, let x lie in the C^*-algebra $(G \times_\alpha B) \cap (G \times_\alpha I)$, and let $x = uvw$, $u, v, w \in (G \times_\alpha B) \cap (G \times_\alpha I)$. Write $u = \lim_n u_n$, $v = \lim_n v_n$ and $w = \lim_n w_n$, with u_n, w_n continuous functions from G to B, and v_n continuous from G to I. Then $u_n v_n w_n$ is a continuous function on G with values in BIB. As I is an ideal in A, $BIB \subseteq I$, while $BIB \subseteq BAB \subseteq B$ as B is hereditary. Thus $BIB \subseteq B \cap I$, $u_n v_n w_n \in G \times_\alpha (B \cap I)$, and $x = uvw \in G \times_\alpha (B \cap I)$ also. ∎

THEOREM 3.4. *Let (G, A, α) be a C^*-dynamical system with G compact. The following are equivalent:*

(1) *$G \times_\alpha A$ is simple;*

(2) *A is G-simple and $\tilde{\Gamma}(\alpha) = \hat{G}$.*

Proof. Follows trivially from Theorem 3.3. ∎

We now turn our attention to the question of prime crossed products. Note that by Lemma 3.1(a) and the subsequent discussion, it is easy to see that $\mathrm{Sp}(\alpha) = \hat{G} \Leftrightarrow \Leftrightarrow \{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : V \supseteq i\}$ is dense in $(G \times_\alpha A)\hat{} \Leftrightarrow (G \times_\alpha A)\chi_i(G \times_\alpha A)$ is an essential ideal in $G \times_\alpha A$.

PROPOSITION 3.5. *Let (G, A, α) be a C^*-dynamical system, with G compact, and assume that $\Gamma(\alpha) = \hat{G}$. If $\mathcal{O}$ is a non-empty open subset of $(G \times_\alpha A)\hat{}$ and a is an element of A such that for all $L = \langle V, \tau \rangle \in \mathcal{O}$, $\tau(a)|(H_L)i = 0$, then for all $L = \langle V, \tau \rangle \in \mathcal{O}$, $\tau(a) = 0$.*

Proof. The proof is similar to that of Proposition 3.2. For convenience, if $L = = \langle V, \tau \rangle$ we write $\tau = \mathrm{Res}\, L$ and we denote the set of all representations τ of A which are part of a covariant pair $\langle V, \tau \rangle = L \in \mathcal{O}$ as $\mathrm{Res}\, \mathcal{O}$. Let $S = \{x \in A : : \tau(x)|(H_L)i = 0$ for all $\tau = \mathrm{Res}\, L \in \mathrm{Res}\, \mathcal{O}\}$. Then S is an α-invariant closed left ideal of A, and $B = \mathrm{cl}(S^*S)$ is an α-invariant hereditary subagebra of A. Suppose, for some $x_1 \in S$ and $\tau_1 = \mathrm{Res}\, L_1 \in \mathrm{Res}\, \mathcal{O}$, that $\tau_1(x_1) \neq 0$. Then $B \neq (0)$, and also, letting $W = \{L = \langle V, \tau \rangle \in (G \times_\alpha A)\hat{} : \tau(x_1) \neq 0\}$, we have that $\mathcal{O} \cap W$ is a non-empty open subset of $(G \times_\alpha A)\hat{}$. As $G \times_\alpha B$ is a non-zero hereditary subalgebra of $G \times_\alpha A$ [13], it follows from the above and [17, Propositions 4.1.9, 4.1.10 and

4.1.12] that $\{R \in (G \times_\alpha B)\hat{} : R = L|G \times_\alpha B$ acting on $L(G \times_\alpha B)H_L$, for some $L \in \mathcal{O} \cap W\}$ is a non-empty open subset of $(G \times_\alpha B)\hat{}$. If $L = \langle V, \tau\rangle$ then clearly $L(G \times_\alpha B)H_L = \tau(B)H_L \subseteq H_L$, and $R = \langle V', \tau'\rangle$ where V' equals V restricted to $\tau(B)H_L$, and τ' equals $\tau|B$ acting on $\tau(B)H_L$. Thus $H_R \subseteq H_L$ and $(H_R)_i \subseteq (H_L)_i$. On the other hand, for $L = \langle V, \tau\rangle \in \mathcal{O} \cap W$, $\tau(B)|(H_L)_i = 0$ so that $(H_L)_i \subseteq (H_R)^\perp$, and $(H_R)_i \subseteq H_R \subseteq ((H_L)_i)^\perp$. Thus $(H_R)_i = (0)$, contradicting the assumption that $\mathrm{Sp}(\alpha|B) = \hat{G}$ and thus that $(G \times_\alpha B)\chi_i(G \times_\alpha B)$ is essential in $G \times_\alpha B$. Thus, indeed, for all $L = \langle V, \tau\rangle \in \mathcal{O}$ and all x in S (and in particular for $x = a$), $\tau(x) = 0$. ■

Before proceeding, we briefly recall some results from [7] and [5] concerning ideals of A and of $G \times_\alpha A$. By [7, Proposition 11] and [5, Lemma 3.10], for each ideal J of $G \times_\alpha A$, there exists a unique smallest α-invariant ideal $\mathrm{Sub}\, J$ of A such that $G \times_\alpha \mathrm{Sub}\, J \supseteq J$. The ideal $G \times_\alpha \mathrm{Sub}\, J$ is invariant under the dual coaction $\hat{\alpha}$ of G on $G \times_\alpha A$ (see [5]), and furthermore, every ideal of $G \times_\alpha A$ invariant under $\hat{\alpha}$ is of the form $G \times_\alpha I$, I an α-invariant ideal of A [5, Theorem 3.4]. Thus, $J^{\hat{\alpha}}$, the smallest $\hat{\alpha}$-invariant ideal of $G \times_\alpha A$ containing J, equals $G \times_\alpha \mathrm{Sub}\, J$ [5, Proposition 3.11]. The map Res, introduced in the proof of Proposition 3.5, can be unambiguously defined on ideals also, so that if $L = \langle V, \tau\rangle$ is a representation of $G \times_\alpha A$, $\mathrm{Res}\,(\mathrm{kernel}\, L) = \mathrm{kernel}\, \tau$ [7, Proposition 9]. Before giving our characterization of the meaning of the condition $\Gamma(\alpha) = \hat{G}$ (Theorem 3.7), we need the following preparatory lemma:

LEMMA 3.6. *Let (G, A, α) be a separable C^*-dynamical system, and let J be an ideal of $G \times_\alpha A$. Let $a \in \mathrm{Sub}\, J$ and suppose that $\tau(a) = 0$ for all $\tau = \mathrm{Res}\, L$, $L \in \hat{J}$. Then $a = 0$.*

Proof. Of course, we identify $\hat{J}$ with $\{L = \langle V, \tau\rangle \in (G \times_\alpha A)\hat{} : L(J) \neq (0)\}$. Writing $\mathrm{PR}(G \times_\alpha A)$ for the space of primitive ideals of $G \times_\alpha A$, we first show that if $P \in \mathrm{PR}(G \times_\alpha A)$ and $P \not\supseteq G \times_\alpha \mathrm{Sub}\, J$, then there exists $Q \in \mathrm{PR}(G \times_\alpha A)$ such that $\mathrm{Res}\, Q = \mathrm{Res}\, P$ and $Q \not\supseteq J$. For if not, then $\bigcap\{Q \in \mathrm{PR}(G \times_\alpha A) : \mathrm{Res}\, Q = \mathrm{Res}\, P\} \supseteq J$. But by the proof of Proposition 4.2 of [5], $G \times_\alpha \mathrm{Res}\, P = \bigcap\{Q \in \mathrm{PR}(G \times_\alpha A) : \mathrm{Res}\, Q = \mathrm{Res}\, P\}$, and we would have $G \times_\alpha \mathrm{Res}\, P \supseteq J$. By the discussion preceding this lemma, it would follow that $\mathrm{Res}\, P \supseteq \mathrm{Sub}\, J$ and thus that

$$\begin{aligned} P &\supseteq G \times_\alpha \mathrm{Res}\, P \text{ ([5, Proposition 3.11])} \\ &\supseteq G \times_\alpha \mathrm{Sub}\, J, \text{ a contradiction.} \end{aligned}$$

Now if $a \neq 0$, there exists $L = \langle V, \tau\rangle \in (G \times_\alpha \mathrm{Sub}\, J)\hat{}$ with $(\mathrm{Res}\, L)(a) \neq 0$. Letting P be the kernel of L, considered as an element of $(G \times_\alpha A)\hat{}$, we have $P \not\supseteq G \times_\alpha \mathrm{Sub}\, J$. Choose Q as above, so that $Q \in \mathrm{PR}(G \times_\alpha A)$ with $\mathrm{Res}\, P = \mathrm{Res}\, Q$ and $Q \not\supseteq J$. Let $R = \langle W, \sigma\rangle \in (G \times_\alpha A)\hat{}$ with kernel $R = Q$. Then we can view R as an element of $\hat{J}$, so that by hypothesis, $(\mathrm{Res}\, R)(a) = 0$. As kernel $(\mathrm{Res}\, R) = \mathrm{Res}\,(\mathrm{kernel}\, R) = \mathrm{Res}\, Q =$

$= \operatorname{Res} P = \operatorname{kernel}(\operatorname{Res} L)$, $(\operatorname{Res} L)(a) = 0$ also, a contradiction. Thus $a = 0$. ∎

THEOREM 3.7. *Let (G, A, α) be a separable C^*-dynamical system, with G compact. Then $\Gamma(\alpha) = \hat{G}$ if and only if every ideal J of $G \times_\alpha A$ is essential in $J^{\hat{\alpha}} = G \times_\alpha \operatorname{Sub} J$.*

Proof. $\Rightarrow$) Let J be an ideal of $G \times_\alpha A$. As in the discussion prior to Proposition 3.5, the assumption that $\operatorname{Sp}(\alpha) = \hat{G}$ implies that $(G \times_\alpha A)\chi_i(G \times_\alpha A)$ is an essential ideal in $(G \times_\alpha A)$. Thus, to show J is essential in $G \times_\alpha \operatorname{Sub} J$, it suffices to show $J \cap ((G \times_\alpha A)\chi_i(G \times_\alpha A))$ is essential in $(G \times_\alpha \operatorname{Sub} J) \cap ((G \times_\alpha A)\chi_i(G \times_\alpha A))$. However, by the Morita equivalence of $(G \times_\alpha A)\chi_i(G \times_\alpha A)$ with $\chi_i(G \times_\alpha A)\chi_i = S_i$, it suffices to show that $J \cap S_i$ is essential in $(G \times_\alpha \operatorname{Sub} J) \cap S_i$. Recalling that S_i equals the set of all constant functions from G to A^α, we see it suffices to show that if $a \in \operatorname{Sub} J \cap A^\alpha$ and $(J \cap S_i)\tilde{a} = (0)$, then $\tilde{a} = 0$, where $\tilde{a} \in G \times_\alpha A$ is defined by $\tilde{a}(t) = a$, $t \in G$. Accordingly, assume $(J \cap S_i)\tilde{a} = (0)$, and let $L\langle V, \tau\rangle \in \hat{J} \subset (G \times_\alpha A)\hat{}$. Then $L(J \cap S_i)L(\tilde{a})|(H_L)_i = L(J \cap S_i)\tau(a)|(H_L)_i = 0$. As $a \in A^\alpha$, $\tau(a)(H_L)_i \subset (H_L)_i$. If $(H_L)_i \neq 0$, then $L|J \cap S_i$, acting on $L(J \cap S_i)H_L = (H_L)_i$, is irreducible, and if $\tau(a)|(H_L)_i \neq 0$, then $L(J \cap S_i)\tau(a)|(H_L)_i \neq 0$. Thus, if $L \in \hat{J}$ and $(H_L)_i \neq 0$, then $\tau(a)|(H_L)_i = 0$, while if $L \in \hat{J}$ and $(H_L)_i = 0$, clearly $\tau(a)|(H_L)_i = 0$. As a consequence, for all $L = \langle V, \tau\rangle \in \hat{J}$, $\tau(a)|(H_L)_i = 0$. Thus, by Proposition 3.5, $\forall L = \langle V, \tau\rangle \in \hat{J}$, $\tau(a) = 0$, and by Lemma 3.6, $a = 0$. Thus certainly $\tilde{a} = 0$, and we are done.

$\Leftarrow$) We assume every ideal J in $G \times_\alpha A$ is essential in $G \times_\alpha \operatorname{Sub} J$, and wish to show that $\Gamma(\alpha) = \hat{G}$. First we show that $\operatorname{Sp}(\alpha) = \hat{G}$, i.e., that $(G \times_\alpha A)\chi_i(G \times_\alpha A)$ is essential in $G \times_\alpha A$. By hypothesis, $(G \times_\alpha A)\chi_i(G \times_\alpha A)$ is essential in an ideal of $G \times_\alpha A$ of the form $G \times_\alpha I$, I a G-invariant ideal of A. However, by the proof of Theorem 3.2 of [6], the only such ideal of the form $G \times_\alpha I$ which also contains $\chi_i(G \times_\alpha A)\chi_i = S_i$ is $G \times_\alpha A$ itself. What remains to be shown is that for every non-zero α-invariant hereditary subalgebra $B \subset A$, every ideal K of $G \times_\alpha B$ is essential in an ideal of the form $G \times_\alpha I$, I an α-invariant ideal of B, so that by the argument above, $\operatorname{Sp}(\alpha|B) = \hat{G}$ also. As $G \times_\alpha B$ is a hereditary subalgebra of $G \times_\alpha A$ [13], $K = (G \times_\alpha B) \cap J$ for an ideal J of $G \times_\alpha A$, and J is essential in $G \times_\alpha L$, for L an α-invariant ideal of A. We shall show $K = (G \times_\alpha B) \cap J$ is essential in $(G \times_\alpha B) \cap (G \times_\alpha L)$, which, as in the proof of the second part of Theorem 3.3, equals $G \times_\alpha (B \cap L)$, and we shall be done. Let $x \in (G \times_\alpha B) \cap (G \times_\alpha L)$ and suppose $((G \times_\alpha B) \cap J)x = 0$. Let $\{e_\alpha\}$ be an approximate identity in $G \times_\alpha B$ and let $j \in J$. Then $e_\alpha^* j^* j e_\alpha \leqslant \|j\|^2 e_\alpha^* e_\alpha \in G \times_\alpha B$ so $e_\alpha^* j^* j e_\alpha \in (G \times_\alpha B) \cap J$ and thus $0 = x^* e_\alpha^* j^* j e_\alpha x = (j e_\alpha x)^*(j e_\alpha x)$, so $j e_\alpha x = 0$ and as $x \in G \times_\alpha B$, $\lim_\alpha j e_\alpha x = jx = 0$. As J is essential in $G \times_\alpha L$ and $x \in G \times_\alpha L$, $x = 0$.

∎

THEOREM 3.8. *Let (G, A, α) be a separable C^*-dynamical system, with G compact. The following are equivalent:*

(1) $G \times_\alpha A$ is prime;

(2) A is G-prime and $\Gamma(\alpha) = \hat{G}$.

Proof. (1)⇒(2). If the crossed product algebra $G \times_\alpha A$ is prime, then certainly A is G-prime, and as every ideal of $G \times_\alpha A$ is in fact essential in $G \times_\alpha A$, it follows from Theorem 3.7 that $\Gamma(\alpha) = \hat{G}$.

(2)⇒(1). Let J be an ideal of $G \times_\alpha A$. By Theorem 3.7 J is essential in $G \times_\alpha \mathrm{Sub}\, J$, and we need only show that if A is G-prime, then every ideal of the form $G \times_\alpha I$, I a non-zero G-invariant ideal of A, is essential in $G \times_\alpha A$. However, as I is essential in A, this follows from [14]. ∎

4. EQUALITY OF SPECTRA

Even when G is commutative, $\tilde{\Gamma}(\alpha)$ is rather difficult to compute. In this section we present several situations in which $\tilde{\Gamma}(\alpha) = \Gamma(\alpha)$. Of course, we always have $\tilde{\Gamma}(\alpha) \subseteq \Gamma(\alpha)$.

Before presenting the first situation in which equality of spectra holds, we need the following:

LEMMA 4.1. *If a C^*-algebra B is a finite direct sum of simple ideals, then each hereditary C^*-subalgebra of B is also the direct sum of a finite family of simple ideals.*

Proof. Let $B = \bigoplus_{j=1}^{m} I_j$, each I_j being simple, and let D be a hereditary C^*-subalgebra of B. For $d \in D_+$, one has $d = \sum_{j=1}^{m} d_j$ with $d_j \in (I_j)_+$ for $1 \leqslant j \leqslant m$. Thus $0 \leqslant d_j \leqslant d$ for each j, hence $\{d_j : 1 \leqslant j \leqslant m\} \subseteq D$ as D is hereditary. It follows that $D = \bigoplus_{j=1}^{m} (D \cap I_j)$, each non-zero summand $D \cap I_j$ being an ideal of D and also a simple C^*-algebra, as it is a hereditary subalgebra of the simple C^*-algebra I_j.

PROPOSITION 4.2. *Let (G, A, α) be a C^*-dynamical system, with G compact and with A G-simple. Suppose, for some $I \in \mathrm{PR}(A)$, that the isotropy subgroup $G_I = \{g \in G : \alpha_g(I) = I\}$ is finite. Then $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$.*

Proof. As A is G-simple, it follows from [7, Lemma 22] that the orbit of I in $\mathrm{PR}(A)$ is all of $\mathrm{PR}(A)$, hence the isotropy group of every primitive ideal of A is finite. We shall first prove that $\widetilde{\mathrm{Sp}}(\alpha) = \mathrm{Sp}(\alpha)$, and then, for $B \in \mathcal{H}^\alpha(A)$ and $J \in \mathrm{PR}(B)$, that G_J is finite. As B is G-simple, the first part of the proof will imply

$\widetilde{\mathrm{Sp}}(\alpha|B) = \mathrm{Sp}(\alpha|B)$, and hence $\tilde{\Gamma}(\alpha) = \Gamma(\alpha)$.

Let $\pi \in \hat{G}$, and denote by β the action $\alpha \otimes \mathrm{ad}\,\pi$ of G on $A \otimes B(H_\pi)$. Obviously the latter is G-simple under the action of β, $G_I = G_{I\otimes B(H_\pi)}$, and by [7, Lemma 22] not only is the β-orbit of $I \otimes B(H_\pi)$ equal to all of $\mathrm{PR}(A \otimes B(H_\pi))$, but also the map of G/G_I onto $\mathrm{PR}(A \otimes B(H_\pi))$, given by $gG_I \to \beta_g(I \otimes B(H_\pi))$, $g \in G$, is a homeomorphism. In particular, $\mathrm{PR}(A \otimes B(H_\pi))$ is Hausdorff, $I \otimes B(H_\pi)$ is a maximal ideal, and hence $(A \otimes B(H_\pi))/(I \otimes B(H_\pi))$ is simple. We can apply to the C^*-dynamical system $(A \otimes B(H_\pi), G, \beta)$ Theorem 2.13 (i) of [8] to obtain an isomorphism between $G \times_\beta (A \otimes B(H_\pi))$ and $(G_I \times_{\overline{\beta}} (A \otimes B(H_\pi))/(I \otimes B(H_\pi))) \otimes K(L^2(G/G_I))$. As G_I is finite and $(A \otimes B(H_\pi))/(I \otimes B(H_\pi))$ is simple, it follows from [21, Theorem 3.1] and the above isomorphism that $G \times_\beta (A \otimes B(H_\pi))$ is the direct sum of finitely many simple ideals. Now by [22] the fixed-point subalgebra $(A \otimes B(H_\pi))^\beta$ is $*$-isomorphic to the hereditary subalgebra $\chi_i(G \times_\beta (A \otimes B(H_\pi)))\chi_i$ of $G \times_\beta (A \otimes B(H_\pi))$ and is thus, by Lemma 4.1, also a finite direct sum of simple ideals. Obviously, then, any essential ideal of $(A \otimes B(H_\pi))^\beta$ coincides with $(A \otimes B(H_\pi))^\beta$, and $\widetilde{\mathrm{Sp}}(\alpha) = \mathrm{Sp}(\alpha)$.

Now let $B \in \mathcal{H}^\alpha(A)$. First observe that B is G-simple as every ideal of B is of the form $I \cap B$, I an ideal of A. If $I \cap B$ is α-invariant, then letting $I_1 = \bigcap_{g\in G} \alpha_g(I)$, we have $I \cap B = I_1 \cap B$. As A is G-simple, $I_1 = (0)$ or A, so $I_1 \cap B = (0)$ or B, and B is indeed G-simple.

Now if $I \in \mathrm{PR}(A)$ and $I \supseteq B$, then $\bigcap_{g\in G} \alpha_g(I) \supseteq B$. As $B \neq (0)$, the G-simplicity of A implies $\bigcap_{g\in G} \alpha_g(I) = A$, hence $I = A$. It follows that $\mathrm{hull}(B) = \emptyset$ and, by [17, Proposition 4.1.10], that $I \to I \cap B$ is a homeomorphism of $\mathrm{PR}(A)$ onto $\mathrm{PR}(B)$. As $\alpha_g(I \cap B) = \alpha_g(I) \cap B$, we have $I \cap B = \alpha_g(I \cap B)$ if and only if $I = \alpha_g(I)$. Thus, for $J \in \mathrm{PR}(B)$, G_J is indeed finite, so that $\widetilde{\mathrm{Sp}}(\alpha|B) = \mathrm{Sp}(\alpha|B)$ by the first part of the proof, and $\tilde{\Gamma}(\alpha) = \Gamma(\alpha)$. ∎

PROPOSITION 4.3. *Let (G, A, α) be a separable C^*-dynamical system with G compact. If A is G-simple and also a type* I *C^*-algebra, then $\tilde{\Gamma}(\alpha) = \Gamma(\alpha)$.*

Proof. Let $B \in \mathcal{H}^\alpha(A)$. As in the proof of Proposition 4.2, B is also G-simple. Now let $\pi \in \hat{G}$ and, as before, let $\beta = \alpha \otimes \mathrm{ad}\,\pi$, the action of G on $A \otimes B(H_\pi)$. As B is type I and G-simple, it follows from [4, Lemma 3.2] that $\mathrm{PR}(B)$, and hence also $\mathrm{PR}(B \otimes B(H_\pi))$, are Hausdorff and that, for $P \in \mathrm{PR}(B \otimes B(H_\pi))$, that the G-orbit of P equals all of $\mathrm{PR}(B \otimes B(H_\pi))$. Again, by [7, Lemma 22], we get that $gG_P \to \beta_g(P)$ is a homeomorphism of G/G_P onto $\mathrm{PR}(B \otimes B(H_\pi))$, and by [8, Theorem 2.13 (ii)] we have a $*$-isomorphism between $G \times_\beta (B \otimes B(H_\pi))$ and a quotient of $C^*(H) \otimes K(\mathcal{H})$, where H is a compact group and $\mathcal{H}$ is a Hilbert space. In particular, the primitive ideal space of $G \times_\beta (B \otimes B(H_\pi))$, like that of $C^*(H)$, is discrete. The primitive ideal space of $(B \otimes B(H_\pi))^\beta$is likewise discrete [22], hence an essential ideal of $(B \otimes B(H_\pi))^\beta$ must equall all of $(B \otimes B(H_\pi))^\beta$, and $\widetilde{\mathrm{Sp}}(\alpha|B) = \mathrm{Sp}(\alpha|B)$.

An action α of G on a C^*-algebra A is called pointwise unitary if for each $\tau \in \hat{A}$ there is a strongly continuous representation U of G on H_τ such that $\langle U, \tau \rangle$ is a covariant pair for the system (G, A, α). The action α is called locally unitary if for each $\tau_0 \in \hat{A}$ there exists a strictly continuous homomorphism $\nu : G \to U(M(A))$ and a neighborhood N of τ_0 such that for each $\tau \in N$, $\langle \overline{\tau} \circ \nu, \tau \rangle$ is a covariant pair for (G, A, α). These definitions were introduced in [19]. Clearly, every locally unitary action is pointwise unitary, and it is proven in [19, Proposition 1.3] that for an action of a second countable compact group on a separable continuous trace algebra, the converse is also true.

A C^*-algebra A is called n-homogeneous ($n < \infty$) if each irreducible representation of A has dimension n. Fell [3] has shown that an n-homogeneous C^*-algebra A is $*$-isomorphic to the C^*-algebra of all continuous sections vanishing at infinity of a locally trivial bundle over $\hat{A}$ whose fibres all equal the $n \times n$ matrix M_n. Conversely, each such C^*-algebra of continuous sections is an n-homogeneous algebra.

LEMMA 4.4. *Let A be the C^*-algebra of all continuous sections vanishing at infinity of a locally trivial bundle over the locally paracompact space $X = \hat{A}$, with fibres M_n. Suppose α is a pointwise unitary action of G on A and $\{N_i\}$ is an open cover of relatively compact subsets of X such that, for each i, the ideal of A corresponding to N_i is naturally isomorphic to $C_0(N_i, M_n)$. Then*

(1) $\Gamma(\alpha) = \bigcap_i \Gamma(\alpha | C_0(N_i, M_n))$,

(2) $\tilde{\Gamma}(\alpha) = \bigcap_i \tilde{\Gamma}(\alpha | C_0(N_i, M_n))$.

Proof. (1): Let $\pi \in \bigcap_i \Gamma(\alpha | C_0(N_i, M_n))$ and consider $B \in \mathcal{H}^\alpha(A)$. If $\mathrm{cl}(B_2(\pi)^* B_2(\pi))$ is not essential in $(B \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\, \pi}$, then there exists a non-zero positive element c in $(B \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\, \pi}$ such that $B_2(\pi)^* B_2(\pi) c = 0$. For some i, $c | N_i \neq 0$. Let $f : X \to [0, 1]$ be a continuous function that vanishes off N_i and is identically 1 on some compact subset of N_i on which c is not identically zero. We can view f as belonging to the center of A, and consider $p = [p_{k\ell}] = [\delta_{k\ell} f]$, $1 \leqslant k, l \leqslant d(\pi)$, as an element of $A \otimes B(H_\pi)$. Thus $pc \neq 0$ and the entries of pc belong to $B' = B \cap C_0(N_i, M_n)$, which can readily be seen to be a non-zero α-invariant hereditary C^*-subalgebra of $C_0(N_i, M_n)$. Since $pc \in (B' \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\, \pi}$ and $B_2'(\pi)^* B_2'(\pi) pc = 0$, we get a contradiction to $\pi \in \mathrm{Sp}(\alpha | B')$. Thus $\mathrm{cl}(B_2(\pi)^* B_2(\pi))$ is essential in $(B \otimes B(H_\pi))^{\alpha \otimes \mathrm{ad}\, \pi}$, and $\bigcap_i \Gamma(\alpha | C_0(N_i, M_n)) \subseteq \Gamma(\alpha)$. The converse inclusion is obvious, and (1) is established.

(2): As above, the inclusion $\tilde{\Gamma}(\alpha) \subseteq \bigcap_i \tilde{\Gamma}(\alpha|C_0(N_i, M_n))$ is readily checked, and we need only show, for $\pi \in \bigcap_i \tilde{\Gamma}(\alpha|C_0(N_i, M_n))$, that $\pi \in \tilde{\Gamma}(\alpha)$. Accordingly, let $B \in \mathcal{H}^\alpha(A)$, let $\varepsilon > 0$ and let $b \in (B \otimes B(H_\pi))^{\alpha\otimes \mathrm{ad}\,\pi}$. Let $\{\varphi_j\}$ be a partition of unity on X such that for each j, $\mathrm{cl}\{x \in X : \varphi_j(x) > 0\}$ is a compact subset of some $N_{i(j)}$. Denote $N_j' = \{x \in X : \varphi_j(x) > 0\}$ and let $\mathcal{F}$ be a finite set of indices such that for $j \notin \mathcal{F}$, $N_j' \cap K = \emptyset$, K being the compact set $\{x : \|b(x)\| \geqslant \varepsilon\}$. For each $j \in \mathcal{F}$ choose a function f_j in $C_0(X)$ such that $0 \leqslant f_j \leqslant 1$, $f_j|N_j' \equiv 1$, $f_j|(X \setminus N_{i(j)}) \equiv 0$, and denote by p_j the element $\mathrm{diag}(f_j, \ldots, f_j) \in C_0(N_{i(j)}) \otimes B(H_\pi)$, and by B^j the algebra $B \cap C_0(N_{i,(j)}, M_n)$. Obviously B^j is an α-invariant hereditary C^*-subalgebra of $C_0(N_{i(j)}, M_n)$ and $p_j b \in (B^j \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$. Hence there exists $h_j \in B_2^j(\pi)^* B_2^j(\pi)$ such that $\|p_j b - h_j\| < \varepsilon$. Letting $q_j = \mathrm{diag}(\varphi_j, \ldots, \varphi_j) \in C_0(N_{i(j)}, M_n) \otimes B(H_\pi)$, we have $h \equiv \sum_{j\in\mathcal{F}} q_j h_j \in B_2(\pi)^* B_2(\pi)$.

We claim that $\|b - h\| < 3\varepsilon$. Indeed, suppose $x \in K$. Then $\sum_{j\in\mathcal{F}} \varphi_j(x) = 1$, and if $x \in N_j'$ for some $j \in \mathcal{F}$, then $\|b(x) - h_j(x)\| < \varepsilon$. Thus $\|b(x) - h(x)\| = = \left\|\sum_{j\in\mathcal{F}} q_j(x)(b(x) - h_j(x))\right\| < \varepsilon$. If now $x \in N_j' \setminus K$ then from $\|b(x) - h_j(x)\| < \varepsilon$ we get $\|h_j(x)\| < \varepsilon + \|b(x)\| < 2\varepsilon$. Thus, if $x \in \left(\bigcup_{j\in\mathcal{F}} N_j'\right) \setminus K$, we have $\|b(x) - h(x)\| =$

$$= \left\|b(x) - \sum_{j\in\mathcal{F}} q_j(x)h_j(x)\right\| \leqslant \|b(x)\| + \left\|\sum_{j\in\mathcal{F}} q_j(x)h_j(x)\right\| < \varepsilon + \sum_{j\in\mathcal{F}} \varphi_j(x)\|h_j(x)\| < 3\varepsilon.$$

Finally, if $x \notin \bigcup_{j\in\mathcal{F}} N_j'$ then $\|b(x)\| < \varepsilon$ and $h(x) = 0$ so $\|b(x) - h(x)\| < \varepsilon$. We have proven $\mathrm{cl}(B_2(\pi)^* B_2(\pi)) = (B \otimes B(H_\pi))^{\alpha\otimes\mathrm{ad}\,\pi}$, and we conclude that $\pi \in \tilde{\Gamma}(\alpha)$. ∎

PROPOSITION 4.5. *Let α be a locally unitary action of a compact group G on an n-homogeneous ($n < \infty$) C^*-algebra A whose spectrum $\hat{A}$ is compact. Then $\Gamma(\alpha) = \tilde{\Gamma}(\alpha)$.*

Proof. We shall identify A with the C^*-algebra of all continuous sections vanishing at infinity of a locally trivial bundle of $n \times n$ matrix algebras over $\hat{A}$. We are going to show that each $\tau_0 \in \hat{A}$ has an open, relatively compact neighborhood N over which the bundle is trivial, and such that $\Gamma(\alpha|C_0(N, M_n)) = \tilde{\Gamma}(\alpha|C_0(N, M_n))$. The conclusion of the proposition then follows immediately from Lemma 4.4.

Let $\tau_0 \in \hat{A}$ and suppose that N' is a relatively compact open neighborhood of τ_0 over which the bundle defining A is trivial and for which there exists a strictly continuous homomorphism $\mu : G \to U(M(A))$ such that

$$\alpha_g(a)(x) = \mu_g(x)a(x)\mu_g(x)^*, \quad a \in A,\ g \in G,\ x \in N'.$$

Each $x \mapsto \mu_g(x)$ is a continuous function from N' to $U(M_n)$. For $\pi \in \hat{G}$ let $P_\pi(x) = = \int_G \chi_\pi(g)\mu_g(x)\mathrm{d}g \in M_n$. Then $x \mapsto P_\pi(x)$ is a projection-valued continuous function on N', and the representation $g \mapsto \mu_g(x)$ restricted to the range of $P_\pi(x)$ is equivalent to a multiple of π, say $n_\pi(x)\pi$, with $n_\pi(x) < \infty$. The representation $g \mapsto \mu_g(x)$ is equivalent to the direct sum $\sum_{\pi\in\hat{G}} \oplus n_\pi(x)\pi$, where only finitely many terms are non-zero. The continuity of P_π implies that $x \mapsto n_\pi(x)$ is locally constant. If $\mu(x_0)$ is equivalent to $W = \sum_{\pi\in\mathcal{F}} n_\pi(x_0)\pi$, with $\mathcal{F}$ a finite subset of $\hat{G}$, then there is an open neighborhood N'' of x_0 such that the functions $x \mapsto n_\pi(x)$, for $\pi \in \mathcal{F}$, are constant in N''. Thus, since $n = \sum_{\pi\in\mathcal{F}} n_\pi(x_0)d(\pi)$, for $x \in N''\mu(x)$ is equivalent to W, and we have shown that there is a representation W of G on $\mathbf{C}^n$ and a function $x \mapsto v(x)$ from $N'' \to U(M_n)$ such that $\mu_g(x) = v(x)W_g v(x)^*$, $g \in G$, $x \in N''$.

We claim now that, after passing to a smaller neighborhood of x_0 if necessary, the function v can be chosen to be continuous. Indeed, let U_1 be the unitary group of the commutant of $\{W_g : g \in G\}$. We want to show that the map $x \mapsto v(x)U_1$ from N'' to $U(M_n)/U_1$ is continuous. Let $\{x_i\}$ be a net converging to x in N''. The continuity of the above map will be proven once we show that for each subnet $\{x_{i_j}\}$ of $\{x_i\}$ such that $\{v(x_{i_j})\}$ converges, we have $\lim_j v(x_{i_j})U_1 = v(x)U_1$. Suppose $v(x_{i_j}) \to s \in U(M_n)$. Then $\mu_g(x) = sW_g s^*$, $g \in G$, hence $v(x)^{-1}s \in U_1$ which means that $sU_1 = v(x)U_1$. Now, by [1, p. 110], there is a neighborhood of $v(x_0)U_1$ in $U(M_n)/U_1$ on which the quotient map $U(M_n) \mapsto U(M_n)/U_1$ admits a continuous section. The claim made at the beginning of this paragraph is established by composing this continuous section with $x \mapsto v(x)U_1$.

From the above, it follows that we have a representation W of G on $\mathbf{C}^n$, a neighborhood $N \subseteq N'$, and a continuous map $v : N \to U(M_n)$ such that

$$\mu_g(x) = v(x)W_g v(x)^*, \quad g \in G,\ x \in N.$$

One sees immediately that $\Gamma(\alpha|C_0(N, M_n)) = \Gamma(\operatorname{ad}\mu|C_0(N, M_n)) = = \Gamma(\operatorname{ad}W|C_0(N, M_n))$, and that $\tilde{\Gamma}(\alpha|C_0(N, M_n)) = \tilde{\Gamma}(\operatorname{ad}W|C_0(N, M_n))$. More generally, if β is any action of G on a C^*-algebra and γ is an automorphism of that algebra, then the spectra of β and of $\gamma \circ \beta \circ \gamma^{-1}$ are equal. Now $\operatorname{ad}W$ on $C_0(N, M_n)$ can be identified with $\mathrm{id} \otimes \operatorname{ad}W$ on $C_0(N) \otimes M_n$, and clearly $\Gamma(\mathrm{id} \otimes \operatorname{ad}W) = \Gamma(\operatorname{ad}W)$ and $\tilde{\Gamma}(\mathrm{id} \otimes \operatorname{ad}W) = \tilde{\Gamma}(\operatorname{ad}W)$, where in the last two equalities $\operatorname{ad}W$ is viewed as acting on M_n. By Proposition 4.3, $\Gamma(\operatorname{ad}W) = \tilde{\Gamma}(\operatorname{ad}W)$, and we get $\Gamma(\alpha|C_0(N, M_n)) = = \tilde{\Gamma}(\alpha|C_0(N, M_n))$ as desired.

5. COUNTEREXAMPLES

We conclude the paper with two counter-examples to what might be considered natural conjectures. The first concerns equality of the spectra $\Gamma(\alpha)$ and $\tilde{\Gamma}(\alpha)$, which, in light of Propositions 4.3 and 4.5, one might be tempted to speculate would be equal for all actions on type I C^*-algebras. The second concerns equality of the spectrum of an action α of G on A and of the double-dual action $\hat{\hat{\alpha}} = \alpha \otimes \operatorname{ad}\lambda$ of G on $A \otimes K(L^2(G))$, λ being the left-regular representation of G. For G abelian, one has $\tilde{\Gamma}(\alpha) = \tilde{\Gamma}(\hat{\hat{\alpha}})$ [11, Lemma 3.1] and $\Gamma(\alpha) = \Gamma(\hat{\hat{\alpha}})$ [17, Proposition 8.11.6].

The first example shows that we can have $\Gamma(\alpha) \neq \tilde{\Gamma}(\alpha)$ even for an action of a commutative compact group on a commutative C^*-algebra. It also provides a counterexample to a claim made in [9].

EXAMPLE 5.1. Let α be the action of $G = \mathbf{Z}_2 = \{0, 1\}$ on $A = C([-1, 1])$ given by

$$\alpha_1(f)(t) = f(-t), \qquad f \in A,\ t \in [-1, 1].$$

For $\chi \in \hat{G} = \{\chi_0, \chi_1\}$ let M_χ be the operator of multiplication by χ on $L^2(G)$. Then $\chi \mapsto M_\chi$ is a unitary representation of $\hat{G}$. Denote by γ the action $\operatorname{ad}\lambda$ of G on $K(L^2(G)) = B(L^2(G))$. By [12, Proposition 5.3] (see also [21, Proposition 4.3]) $G \times_\alpha A$ is naturally isomorphic to the fixed-point algebra $(A \otimes B(L^2(G)))^{\alpha \otimes \gamma}$. The dual action of $\hat{G}$ on $G \times_\alpha A$ is mapped by this isomorphism to the restriction of $\operatorname{id} \otimes \operatorname{ad} M$ to $(A \otimes B(L^2(G)))^{\alpha \otimes \gamma}$. If we identify $A \otimes B(L^2(G))$ with $\{[f_{ij}] : f_{ij} \in C([-1, 1]),\ 1 \leqslant i, j \leqslant 2\}$ then

$$(A \otimes B(L^2(G)))^{\alpha \otimes \gamma} = \{[f_{ij}] : f_{11}(t) = f_{22}(-t), f_{12}(t) = f_{21}(-t),\ -1 \leqslant t \leqslant 1\}$$

and $(\operatorname{id} \otimes M_{\chi_1})[f_{ij}] = [g_{ij}]$, where $g_{ii} = f_{ii}$ and $g_{ij} = -f_{ij}$ if $i \neq j$. Consider the following one-dimensional irreducible representations of $(A \otimes B(L^2(G)))^{\alpha \otimes \gamma}$:

$$\pi_1([f_{ij}]) = f_{11}(0) + f_{12}(0),$$

$$\pi_2([f_{ij}]) = f_{11}(0) - f_{12}(0),$$

and their kernels $P_k = \pi_k^{-1}(0)$, $k = 1, 2$. Clearly $(\operatorname{id} \otimes M_{\chi_1})(P_1) = P_2 \not\subset P_1$. Thus, by [11, Lemma 3.4] $\tilde{\Gamma}(\alpha) = \{\chi_0\}$. On the other hand, every non-zero α-invariant hereditary C^*-subalgebra of $C([-1, 1])$, that is, every ideal of $C([-1, 1])$ that corresponds to a symmetric closed subset of $[-1, 1]$ different from $[-1, 1]$ itself, contains a non-zero odd function. It follows that $\Gamma(\alpha) = \{\chi_0, \chi_1\}$ and we have $\Gamma(\alpha) \neq \tilde{\Gamma}(\alpha)$ for this action.

The next example shows that, unlike the situation for G commutative, $\Gamma(\alpha)$ need not equal $\Gamma(\hat{\alpha})$, while $\tilde{\Gamma}(\alpha)$ need not equal $\tilde{\Gamma}(\hat{\alpha})$.

EXAMPLE 5.2. Let $G = S_3$ and let i, χ and π denote the three elements of $\hat{G}$, with i being the trivial representation, χ the non-trivial one-dimensional representation, and $d(\pi) = 2$. As in [18, Example 3.9], if $A = M_2$ and α is the action $\mathrm{ad}\,\pi$ of S_3 on M_2, then $G \times_\alpha A$ is not simple, so $\Gamma(\alpha)$ (which equals $\tilde{\Gamma}(\alpha)$ by Proposition 4.2) is a proper subset of $\hat{G}$. Always $i \in \Gamma(\alpha)$, and here $\chi \in \Gamma(\alpha)$. Indeed, $\mathrm{ad}\,\pi$ is equivalent with $\pi \otimes \pi \cong i \oplus \chi \oplus \pi$ so that A^α is one-dimensional and thus the one-dimensional representation $\chi \in \Gamma(\alpha)$. Thus $\Gamma(\alpha) = \tilde{\Gamma}(\alpha) = \{i, \chi\}$. However, for the action $\hat{\hat{\alpha}} \cong \alpha \otimes \mathrm{ad}\,\lambda$ on $G \times_{\hat{\alpha}} (G \times_\alpha A) \cong A \otimes K(\ell^2(G))$, the fact that $\lambda \cong i \oplus \chi \oplus 2\pi$ while $\pi \otimes \pi \cong i \oplus \chi \oplus \pi$ implies that $\pi \otimes \lambda$ contains i. Letting p be the projection in $B(H_\pi \otimes \ell^2(G))$ corresponding to i, one has $p(A \otimes K(\ell^2(G)))p$ is an $\hat{\hat{\alpha}}$-invariant hereditary subalgebra on which $\hat{\hat{\alpha}}$ acts trivially, hence $\chi \notin \Gamma(\hat{\hat{\alpha}}) \cup \tilde{\Gamma}(\hat{\hat{\alpha}})$, and $\Gamma(\hat{\hat{\alpha}}) = \tilde{\Gamma}(\hat{\hat{\alpha}}) = \{i\}$.

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