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ON THE REFLEXIVITY THEOREM OF BROWN-CHEVREAU

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ABSTRACT. In this paper we give a complete proof of the Brown-Chevreau reflexivity theorem announced in [6].

KEYWORDS: *Reflexive operators, H^∞-functional calculus, absolutely continuous contraction.*

AMS SUBJECT CLASSIFICATION: Primary 47A15; Secondary 47A60.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded, linear operators on $\mathcal{H}$. In 1988, Scott Brown and Bernard Chevreau proved the following beautiful theorem about reflexivity of operators in $\mathcal{B}(\mathcal{H})$ (cf. [6]). (For a review of pertinent notation and terminology, see below.)

THEOREM 1. *Every absolutely continuous contraction in $\mathcal{B}(\mathcal{H})$ having an isometric H^∞-functional calculus is reflexive.*

This theorem has been used several times in subsequent work to obtain reflexivity theorems for operators (see [12], [16]), and an outline of the proof was given in [6] and [9].

Unfortunately, a complete proof of this theorem has never been published, to the author's knowledge. Thus the purpose of this note is to give such a complete proof of Theorem 1. The main contributions of the present author are a new Lemma 3 and a new Lemma 5.

2. NOTATION AND TERMINOLOGY

We begin by reviewing some notation and terminology for the convenience of the reader. If T is in $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ is a (closed) subspace of $\mathcal{H}$, then $T_{\mathcal{M}}$ denotes the compression of T to $\mathcal{M}$, i.e., $T_{\mathcal{M}} = P_{\mathcal{M}}T|\mathcal{M}$, where $P_{\mathcal{M}}$ denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. Also the spectrum of T, the point spectrum of T, the essential spectrum of T, and the resolvent set of T will be denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_e(T)$, and $\rho(T)$, respectively.

It is well known (cf. [13]) that $\mathcal{B}(\mathcal{H})$ is the dual space of Banach space $\mathcal{C}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ equipped with the trace-norm $\|\cdot\|_1$, and the duality is implemented by the bilinear form $\langle T, L\rangle = \text{trace}(TL)$, $T \in \mathcal{B}(\mathcal{H})$, $L \in \mathcal{C}_1(\mathcal{H})$. If T is an operator in $\mathcal{B}(\mathcal{H})$, $\mathcal{A}_T$ will denote the dual algebra generated by T, (i.e., the smallest weak*-closed algebra containing T and the identity operator on $\mathcal{H}$), $\mathcal{Q}_T$ $(= \mathcal{C}_1/{}^{\perp}\mathcal{A}_T)$ the natural predual of $\mathcal{A}_T$, and $\mathcal{W}_T$ the smallest weak operator topology (WOT) closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing T and the identity operator on $\mathcal{H}$. The elements of $\mathcal{Q}_T$ will be denoted by $[L]_T$ when $L \in \mathcal{C}_1(\mathcal{H})$. As usual $\text{Lat}(T)$ denotes the lattice of invariant subspaces of T, $\text{Alg Lat}(T) = \{X \in \mathcal{B}(\mathcal{H}) : \text{Lat}(T) \subset \text{Lat}(X)\}$, and T is reflexive if $\mathcal{W}_T = \text{Alg Lat}(T)$.

Let $\mathbf{N}$ denote the set of positive integers, $\mathbf{D}$ the open unit disc in $\mathbf{C}$, and $\mathbf{T} = \partial\mathbf{D}$. If E is a measurable subset of $\mathbf{T}$ (with respect to normalized Lebesgue measure m on $\mathbf{T}$), a set $\Lambda \subset \mathbf{D}$ is said to be dominating for E if almost every point of E is a nontangential limit of a sequence of points from Λ, and the set of all nontangential limits of Λ on $\mathbf{T}$ will be denoted by $NTL(\Lambda)$. For any $1 \leqslant p \leqslant \infty$, the spaces $\mathbf{L}^p (:= \mathbf{L}^p(\mathbf{T}))$ and $\mathbf{H}^p (:= \mathbf{H}^p(\mathbf{T}))$ are the usual Lebesgue and Hardy function spaces on $\mathbf{T}$, relative to the measure m, and if h is a function in $\mathbf{H}^\infty$, the analytic extension of h to $\mathbf{D}$ will be denoted again by h.

If T is an absolutely continuous contraction in $\mathcal{B}(\mathcal{H})$, the $\mathbf{H}^\infty$-functional calculus (cf. [4], [15]) Φ_T of T is a weak*-continuous, norm decreasing, algebra homomorphism of $\mathbf{H}^\infty$ onto a weak*-dense subalgebra of $\mathcal{A}_T$, and there exists a bounded, linear, one-to-one map $\varphi_T : \mathcal{Q}_T \to \mathbf{L}^1/\mathbf{H}^1_0$ such that $\varphi_T^* = \Phi_T$. If h is a function in $\mathbf{H}^\infty$, $\Phi_T(h)$ will be denoted by $h(T)$, and if f is in $\mathbf{L}^1$, the image of f under the natural projection from $\mathbf{L}^1$ onto $\mathbf{L}^1/\mathbf{H}^1_0$ will be denoted by $[f]_{\mathbf{L}^1/\mathbf{H}^1_0}$. If U in $\mathcal{B}(\mathcal{K})$ is the (absolutely continuous) minimal unitary dilation of T and E is the spectral measure of U, for any vectors x and y in $\mathcal{H}$, the Radom-Nykodym derivative of the complex measure $\langle E(\cdot)x, y\rangle$ on $\mathbf{T}$ with respect to m is denoted by $x \stackrel{T}{\cdot} y$ and it is well known (cf [4]) that $\varphi_T([x \otimes y])_T) = [x \stackrel{T}{\cdot} y]_{\mathbf{L}^1/\mathbf{H}^1_0}$, where $x \otimes y$ denotes the usual rank one operator in $\mathcal{B}(\mathcal{H})$. A simple computation yields that the n^{th} Fourier coefficient of the function $x \stackrel{T}{\cdot} y$ is $\langle T^{-n}x, y\rangle$ if n is negative and

$\langle T^{*n}x, y\rangle$ if n is nonnegative, so, in particular the sequence $\{T\}_{n=1}^{\infty}$ converges to 0 in the wot.

Recall that the minimal isometric dilation U_+ of T is the restriction of U to the invariant subspace $\mathcal{K}_+ = \bigvee_{n\geqslant 0} U^n\mathcal{H}$, and $\mathcal{K}_+$ can be decomposed as $\mathcal{K}_+ =_* \oplus\mathcal{R}$, corresponding to the Wold decomposition of U_+ as $U_+ = S_* \oplus R$, where S_* is a unilateral shift operator and R is unitary. Similarly, if $\mathcal{K}_* = \bigvee_{n\geqslant 0} U^{*n}\mathcal{H}$, then $B := U_{\mathcal{K}_*}$ is the minimal coisometric extension of T, and the Wold decomposition of B^* yields $\mathcal{K}_* = \oplus\mathcal{R}_*$, $B^* = S \oplus R_*^*$, where S is a unilateral shift operator and R_* is unitary. The orthogonal projections from $\mathcal{K}_+$ onto $_*$ and $\mathcal{R}$ will be denoted by Q_* and A, and similarly, the orthogonal projections from $\mathcal{K}_*$ onto and $\mathcal{R}_*$ by Q and A_*.

The unitary operators R and R_* being absolutely continuous, there exist measurable subsets Σ and Σ_* of $\mathbf{T}$, such that m_Σ, m_{Σ_*} (defined by $m_\Sigma(E) = m(E\cap\Sigma)$ for any measurable subset E of $\mathbf{T}$, and similarly for m_{Σ_*}) are scalar spectral measures for R and R_*, respectively.

If $\mathcal{M}$ is a semi-invariant subspace for T, the minimal unitary dilation of $T_{\mathcal{M}}$ (acting on $\mathcal{K}^{\mathcal{M}}$) will be denoted by $U^{\mathcal{M}}$. The meaning of the notation $\mathcal{K}_+^{\mathcal{M}} :=:= (\mathcal{K}^{\mathcal{M}})_+$, $\mathcal{K}_*^{\mathcal{M}} := (\mathcal{K}^{\mathcal{M}})_*$, etc. is now clear.

In what follows, the concept of essential set will be needed, so we recall the definition here (cf. [8]). If T is an absolutely continuous contraction, a subset E of $\mathbf{T}$ is essential for T if either it has Lebesgue measure 0, or for any function h in $\mathbf{H}^\infty$, $\|h(T)\| \geqslant \|h\|_E$ where $\|h\|_E = \|h|E\|_\infty$. It is known (cf. [8]) that for T as above there exists a maximal (up to a set of measure 0) essential set E_T for T such that if E is any essential set for T then $m(E\setminus E_T) = 0$. Let us also recall (cf. [8], Lemma 3.5), that

$$NTL(\sigma(T)\cap\mathbf{D}) \subset E_T. \tag{1}$$

As usual the classs $\mathbb{A} = \mathbb{A}(\mathcal{H})$ (cf. [4]) will denote the set of all absolutely continuous contractions T in $\mathcal{B}(\mathcal{H})$ for which Φ_T is an isometry. In this case Φ_T is a weak $*$-homeomorphism of $\mathbf{H}^\infty$ onto $\mathcal{A}_T$ (cf. [7]), and φ_T is an isometry of $\mathcal{Q}_T$ onto $\mathbf{L}^1/\mathbf{H}_0^1$. For any f in $\mathbf{L}^1$, $\varphi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}_0^1})$ is denoted by $[f]_T$. If $\lambda\in\mathbf{D}$ and $\mathbf{P}_\lambda$ is the associated Poisson kernel on $\mathbf{T}$ (i.e., $\mathbf{P}_\lambda(t) := \frac{1-|\lambda|^2}{|1-\lambda e^{it}|^2}$), we write

$$[C_\lambda]_T = \varphi_T^{-1}([\mathbf{P}\lambda]_{L^1/\mathbf{H}_0^1}),$$

and it is easy to check that for any function h in $\mathbf{H}^\infty$,

$$\langle h(T), [C_\lambda]_T\rangle = h(\lambda).$$

For T in $\mathbf{A}$, it is known (cf. [1], [8]) that for any $[L]_T$ in Q_T and for any $\varepsilon > 0$, there exist vectors x and I in $\mathcal{H}$ such that

$$[L]_T = [x \otimes y]_T, \quad \|x\|\,\|y\| \leqslant (1+\varepsilon)\|[L]_T\|.$$

Furthermore, the class $\mathbf{A}_{1,\aleph_0} = \mathbf{A}_{1,\aleph_0}(\mathcal{H})$ is the set of all contractions T in $\mathbf{A}$ such that for any sequence $\{[L_n]_T\}_{n=1}^{\infty}$ in Q_T there exist x and $\{y_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $[x \otimes y_n]_T = [L_n]_T$, $n \in \mathbf{N}$, and by duality one may define the class $\mathbf{A}_{\aleph_0,1} = \mathbf{A}_{\aleph_0,1}(\mathcal{H})$ to be the set of all contractions in $\mathcal{B}(\mathcal{H})$ such that $T^* \in \mathbf{A}_{1,\aleph_0}$.

Let us also recall that if T is a contraction in $\mathbf{A}$, $\mathcal{M}$ is a semi-invariant subspace for T, and $\theta \in [0,1)$, then $\mathcal{E}_\theta^l(\mathcal{A}_T, \mathcal{M})$ ($\mathcal{E}_\theta^l(\mathcal{A}_T)$ if $\mathcal{M} = \mathcal{H}$) denotes the set of all $[L]_T$ in Q_T for which there exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{M}$ such that

(i) $\overline{\lim\limits_{n\to\infty}}\|[L]_T - [x_n \otimes y_n]_T\| \leqslant \theta$, and

(ii) $\{x_n\}_{n=1}^{\infty}$ converges weakly to 0, and for any $w \in \mathcal{M}$, $\|[w \otimes y_n]_T\| \to 0$.

Similarly one defines $\mathcal{E}_\theta^r(\mathcal{A}_T, \mathcal{M})$ ($\mathcal{E}_\theta^r(\mathcal{A}_T)$ if $\mathcal{M} = \mathcal{H}$) by changing (ii) in the definition above to

(ii)′ $\{y_n\}_{n=1}^{\infty}$ converges weakly to 0, and for any $w \in \mathcal{M}$, $\|[x_n \otimes w]_T\| \to 0$.

Note that if $\lambda \in \mathbf{D}$ and $[C_\lambda]_T \in \mathcal{E}_0^l(\mathcal{A}_T)$, it is known that in the definitions above one may take $x_n = y_n$ and $\|x_n\| = 1$ for any $n \in \mathbf{N}$ (cf. [4]).

If $\Lambda(\theta, \mathcal{M}, T)$ denotes the set of all λ in $\mathbf{D}$ for which there exist x and y in the unit ball of $\mathcal{M}$ such that $\|[C_\lambda]_T - [x \otimes y]_T\| \leqslant \theta$, let us recall that (cf. [8], Lemma 3.5)

$$NTL(\Lambda(\theta, \mathcal{M}, T)) \subset E_{T_\mathcal{M}}. \tag{2}$$

3. PROOF OF THEOREM 1

The proof will follow the steps mentioned in [6]. From now on we fix a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$, a contraction T in $\mathbf{A}(\mathcal{H})$ and an operator $\mathbf{A}$ in $\operatorname{Alg\,Lat}(T)$. Since $\mathcal{A}_T = \{h(T) : h \in \mathbf{H}^\infty\}$ and $\mathcal{A}_T \subset \mathcal{W}_T \subset \operatorname{Alg\,Lat}(T)$ (actually for $T \in \mathbf{A}$, $\mathcal{A}_T = \mathcal{W}_T$; cf. [4], Proposition 2.09), in order to show that T is reflexive it is sufficient to produce a function g in $\mathbf{H}^\infty$ such that $g(T) = A$, or, equivalently,

$$\langle g(T)x, y\rangle = \langle Ax, y\rangle, \quad x, y \in \mathcal{H}. \tag{3}$$

If $T \in \mathbb{A}_{1,\aleph_0} \cup \mathbb{A}_{\aleph_0,1}$, then T is reflexive (cf. [10], Theorem 6.2), so in what follows we may and do assume that

$$T \notin \mathbb{A}_{1,\aleph_0} \cup \mathbb{A}_{\aleph_0,1}. \tag{4}$$

The next three lemmas will produce a function g in $\mathbf{H}^\infty$ such that (3) holds for all vectors x and y for which $[x \otimes y]_T$ is in $\mathcal{E}_\Omega$ (see the definition below).

Let us recall the following lemma from [8], which is the main ingredient in the definition of g.

LEMMA 2. *Let* $\lambda \in \mathbf{D}$.

(i) *If* $\lambda \in \sigma_e(T) \cup \rho(T)$, *then* $[C_\lambda]_T \in \mathcal{E}_0^l(\mathcal{A}_T) \cap \mathcal{E}_0^r(\mathcal{A}_T)$.

(ii) *If* $\mathcal{M}_*$ *is an invariant subspace for* T^* *and* λ *is a limit point of* $\sigma_p(T_{\mathcal{M}_*})$, *then* $[C_\lambda]_T \in \mathcal{E}_0^l(\mathcal{A}_T)$.

Define now $\Omega_1 = \{\lambda \in \mathbf{D} : [C_\lambda] \notin \mathcal{E}_0^l(\mathcal{A}_T)\}$ and $\Omega = \Omega_1 \setminus \sigma_p(T)$. Since $\mathcal{E}_0^l(\mathcal{A}_T)$ is closed and the function $\lambda \to [C_\lambda]_T$ is continuous on $\mathbf{D}$, it follows that Ω_1 is open. If Ω_1 is empty, T is in $\mathbb{A}_{\aleph_0,1}$ (cf. [10], Theorem 6.2), which contradicts (4). Thus Ω_1 is not empty. If $\lambda \in \Omega_1 \cap \sigma_p(T)$, by Lemma 2 (ii), λ is isolated in $\sigma_p(T)$, so Ω is open and dense in Ω_1. Let us also note that by Lemma 2 any λ in Ω is in $(\rho_e(T) \cap \sigma(T)) \setminus \sigma_p(T)$, hence $\ker(\overline{\lambda} - T^*)$ is not zero. Therefore for any λ in Ω there exists a complex number $\tilde{g}(\lambda)$ such that

$$A^*_{\ker(\overline{\lambda}-T^*)} = \overline{\tilde{g}(\lambda)} I_{\ker(\overline{\lambda}-T^*)}.$$

The first step in showing that $\tilde{g}$ is the restriction of an $\mathbf{H}^\infty$-function to Ω is the following lemma (cf. [6]).

LEMMA 3. *Let* λ *be in* Ω, *and let* x *and* y *be vectors in* $\mathcal{H}$ *such that* $[C_\lambda]_T = [x \otimes y]_T$. *Then* $\langle Ax, y\rangle = \tilde{g}(\lambda)$.

Proof. By a standard argument one may suppose that $x = y$, and we set $\mathcal{M} = \bigvee_{n \geqslant 0} T^n x$. Since $(\lambda - T)$ is a one-to-one Fredholm operator and $\mathcal{M}$ is invariant for T, $(\lambda - T_{\mathcal{M}})$ is a one-to-one Fredholm operator, and since $[C_\lambda]_T = [x \otimes x]_T$, it follows that $T^*_{\mathcal{M}} x = \overline{\lambda} x$ and $\dim(\ker(\overline{\lambda} - T^*_{\mathcal{M}})) = 1$, so $i(\lambda - T_{\mathcal{M}}) = -1$. Since A is in $\operatorname{Alg}\operatorname{Lat}(T)$, it follows that $A(\lambda - T)\mathcal{M} \subset (\lambda - T)\mathcal{M}$, hence $A^*_{\mathcal{M}} x$ is in $\ker(\overline{\lambda} - T^*_{\mathcal{M}})$, and thus $A^*_{\mathcal{M}} x = \alpha x$ for some complex number α. Now

$$\langle x, Ax\rangle = \langle x, A_{\mathcal{M}} x\rangle = \langle A^*_{\mathcal{M}} x, x\rangle = \langle \alpha x, x\rangle = \alpha \|x\|^2 = \alpha,$$

so

$$A^*_{\mathcal{M}} x = \langle x, Ax\rangle x. \tag{5}$$

(i) If x is not orthogonal to $\ker(\overline{\lambda}-T^*)$, there exists u in $\ker(\overline{\lambda}-T^*)$ such that $\langle x,u\rangle \neq 0$, and using the fact that $\mathcal{M}$ is invariant for T, we obtain that $T^*_{\mathcal{M}}P_{\mathcal{M}}u = \overline{\lambda}P_{\mathcal{M}}u$. Hence $P_{\mathcal{M}}u$ is in $\ker(\overline{\lambda}-T^*_{\mathcal{M}})$, so $P_{\mathcal{M}}u = \alpha' x$ for some complex number α', and the same computations as above yield $\alpha' = \langle u,x\rangle$, so

$$P_{\mathcal{M}}u = \langle u,x\rangle x.$$

Therefore

$$\begin{aligned}\langle Ax,x\rangle &= \left\langle Ax, \frac{1}{\langle u,x\rangle}P_{\mathcal{M}}u\right\rangle = \frac{1}{\langle x,u\rangle}\langle Ax, P_{\mathcal{M}}u\rangle \\ &= \frac{1}{\langle x,u\rangle}\langle Ax,u\rangle = \frac{1}{\langle x,u\rangle}\langle x,A^*u\rangle = \frac{1}{\langle x,u\rangle}\langle x,\overline{\tilde{g}(\lambda)}u\rangle = \tilde{g}(\lambda).\end{aligned}$$

(ii) If x is orthogonal to $\ker(\overline{\lambda}-T^*)$, we assert that for any sequence $\{\lambda_n\}_{n=1}^{\infty}$ in Ω such that $\lambda_n \to \lambda$ and $\lambda_n \neq \lambda$ for every n, then x is not orthogonal to $\bigvee_{n\geqslant 1}\ker(\overline{\lambda}_n - T^*)$. Indeed, assume that x is orthogonal to $\bigvee_{n\geqslant 1}\ker(\overline{\lambda}_n - T^*)$. Since $\mathcal{M} = \bigvee_{n\geqslant 0} T^n x$, it follows that $\mathcal{M}$ is orthogonal to $\bigvee_{n\geqslant 0}\ker(\overline{\lambda}_n - T^*)$, so $\ker(\overline{\lambda}_n - T^*) = \ker(\overline{\lambda}_n - T^*_{\mathcal{M}^\perp})$ for every n, and since $\lambda_n \to \lambda$, without loss of generality we may assume that $i(\lambda - T) = i(\lambda_n - T)$ and $i(\lambda - T_{\mathcal{M}}) = i(\lambda_n - T_{\mathcal{M}}) = -1$ for every n. Hence for every $n \in \mathbf{N}$,

$$\begin{aligned} i(\lambda_n - T) &= -\dim(\ker(\overline{\lambda}_n - T^*)) = i(\lambda_n - T_{\mathcal{M}}) + i(\lambda_n - T_{\mathcal{M}^\perp}) \\ &= -1 + \dim(\ker(\lambda_n - T_{\mathcal{M}^\perp})) - \dim(\ker(\overline{\lambda}_n - T^*_{\mathcal{M}^\perp})) \\ &= -1 + \dim(\ker(\lambda_n - T_{\mathcal{M}^\perp})) - \dim(\ker(\overline{\lambda}_n - T^*)),\end{aligned}$$

from which it follows that for any $n \in \mathbf{N}$, $\dim(\ker(\lambda_n - T_{\mathcal{M}^\perp})) = 1$. Hence $\lambda \in\in (\sigma_p(T_{\mathcal{M}^\perp}))'$, and by Lemma 2 (ii) we get $[C_\lambda]_T \in \mathcal{E}^l_0(A_T)$, which is a contradiction.

Thus we may find a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in Ω converging to λ for each $n \in \mathbf{N}$ a vector x_n in $\ker(\overline{\lambda}_n - T^*)$ such that $\|x_n\| = 1$ and $\langle x, x_n\rangle \neq 0$.

We show now that $\tilde{g}(\lambda_n) \to \tilde{g}(\lambda)$. We have

$$\begin{aligned}\|(\overline{\lambda}-T^*)(I - P_{\ker(\overline{\lambda}-T^*)})x_n\| &= \|(\overline{\lambda}-T^*)x_n\| \\ &= \|(\overline{\lambda}-\overline{\lambda}_n)x_n\| = |\lambda - \lambda_n|.\end{aligned}$$

from which it follows that $(\overline{\lambda}-T^*)(I-P_{\ker(\overline{\lambda}-T^*)})x_n \to 0$, so $(I-P_{\ker(\overline{\lambda}-T^*)})x_n \to 0$ since $(\overline{\lambda}-T^*)$ is bounded from below on $(\ker(\overline{\lambda}-T^*))^\perp$. Moreover,

$$\begin{aligned}\tilde{g}(\lambda_n) &= \langle Ax_n, x_n\rangle = \langle Ax_n, P_{\ker(\overline{\lambda}-T^*)}x_n\rangle + \langle Ax_n, (I - P_{\ker(\overline{\lambda}-T^*)})x_n\rangle \\ &= \langle x_n, A^*P_{\ker(\overline{\lambda}-T^*)}x_n\rangle + \langle Ax_n, (I - P_{\ker(\overline{\lambda}-T^*)})x_n\rangle \\ &= \tilde{g}(\lambda)\langle x_n, P_{\ker(\overline{\lambda}-T^*)}x_n\rangle + \langle Ax_n, (I - P_{\ker(\overline{\lambda}-T^*)})x_n\rangle \\ &= \tilde{g}(\lambda)\|P_{\ker(\overline{\lambda}-T^*)}x_n\|^2 + \langle Ax_n, (I - P_{\ker(\overline{\lambda}-T^*)})x_n\rangle,\end{aligned}$$

so

$$\tilde{g}(\lambda_n) \to \tilde{g}(\lambda). \tag{6}$$

To conclude the proof we set, for each positive integer n,

$$\begin{gathered} u_n = \frac{P_{\mathcal{M}} x_n}{\|P_{\mathcal{M}} x_n\|}, \quad z_n = P_{\ker(\bar{\lambda} - T^*_{\mathcal{M}})} u_n = \langle u_n, x \rangle x, \\ y_n = (I - P_{\ker(\bar{\lambda} - T^*_{\mathcal{M}})}) u_n = u_n - \langle u_n, x \rangle x. \end{gathered} \tag{7}$$

Taking into account that $\mathcal{M}$ is invariant for T, $T^* x_n = \bar{\lambda}_n x_n$, and $A^* x_n = \overline{\tilde{g}(\lambda_n)} x_n$, we obtain that

$$T^*_{\mathcal{M}} u_n = \bar{\lambda}_n u_n, \quad A^*_{\mathcal{M}} u_n = \overline{\tilde{g}(\lambda_n)} u_n. \tag{8}$$

The same computations as above yield $y_n \to 0$, or equivalently,

$$\|y_n\| = (1 - |\langle u_n, x \rangle|^2)^{1/2} \to 0.$$

By (8), (5), and (7), we obtain

$$\begin{aligned} \tilde{g}(\lambda_n) &= \langle A_{\mathcal{M}} u_n, u_n \rangle = \langle A_{\mathcal{M}} u_n, z_n \rangle + \langle A_{\mathcal{M}} u_n, y_n \rangle \\ &= \langle x, u_n \rangle \langle u_n, A^*_{\mathcal{M}} x \rangle + \langle A_{\mathcal{M}} u_n, y_n \rangle \\ &= |\langle u_n, x \rangle|^2 \langle A x, x \rangle + \langle A_{\mathcal{M}} u_n, y_n \rangle, \end{aligned}$$

so

$$\tilde{g}(\lambda_n) \to \langle A x, x \rangle. \tag{9}$$

By (6) and (9) we get $\tilde{g}(\lambda) = \langle Ax, x \rangle$ and the proof is complete. ∎

Define now $\mathcal{E}_\Omega$ to be the linear span in Q_T of the set $\{[C_\lambda]_T : \lambda \in \Omega\}$. Taking into account that $Q_T^* = \mathcal{A}_T$, if h is an H^∞-function such that, for every $[L]_T$ in $\mathcal{E}_\Omega$, $\langle h(T), [L]_T \rangle = 0$, it follows that $h(\lambda) = 0$ for any λ in Ω, so h is identically zero. Hence $\mathcal{E}_\Omega$ is dense in Q_T. Furthermore, define for any $\lambda \in \Omega$,

$$\varphi_A([C_\lambda]_T) = \tilde{g}(\lambda),$$

and note that φ_A can be extended by linearity to all of $\mathcal{E}_\Omega$ since $\{[C_\lambda]_T\}_{\lambda \in \mathbb{D}}$ is linearly independent in Q_T.

The next lemma comes from [6] and shows that $\tilde{g}$ is the restriction of an H^∞-function to Ω.

LEMMA 4. *If x and y are vectors in $\mathcal{H}$ such that $[x \otimes y]_T$ is in $\mathcal{E}_\Omega$, then*

$$\varphi_A([x \otimes y]_T) = \langle Ax, y \rangle.$$

In particular φ_A is bounded on $\mathcal{E}_\Omega$.

By Lemma 4, φ_A can be uniquely extended to a bounded linear functional on Q_T, and using again the fact that $Q_T^* = \mathcal{A}_T$, one gets that there exists an $\mathbf{H}^\infty$-function g such that for any vectors x and y in $\mathcal{H}$ for which $[x \otimes y]_T$ is in $\mathcal{E}_\Omega$, $\langle g(T), [x \otimes y]_T \rangle == \varphi_A([x \otimes y]_T)$, or equivalently, (3) is true for any vectors x and y such that $[x \otimes y]_T \in\in \mathcal{E}_\Omega$.

To complete the proof of Theorem 1 we need to show that (3) is true for any vectors x and y in $\mathcal{H}$, and this it will be done in the next few lemmas.

Define

$$(10) \qquad \mathcal{H}_1 = \bigvee_{\lambda \in \Omega} \ker(\overline{\lambda} - T^*), \quad \mathcal{H}_0 = \mathcal{H}_1^\perp.$$

For any vector x in $\mathcal{H}$ we will write the decomposition of x relative to $\mathcal{H}_0$ and $\mathcal{H}_1$ as $x = x^0 + x^1$.

REMARK. By Lemma 4, it follows that (3) is true for any vectors $x \in \mathcal{H}$ and $y \in \mathcal{H}_1$, so the proof of Theorem 1 will be complete once we show that (3) is also true for any vectors $x \in \mathcal{H}$, $y \in \mathcal{H}_0$.

Note that by the definition of $\mathcal{H}_1$, $T^*_{\mathcal{H}_1} \in C_0$, and hence if $\mathbf{m}(E_{T^*_{\mathcal{H}_1}}) = 1$, then $T^*_{\mathcal{H}_1} \in \mathbf{A}$, so $T^*_{\mathcal{H}_1} \in \mathbf{A}_{\aleph_0,1}$ (cf. [3], Theorem 2, and [10], Theorem 6.2). Then obviously $T^* \in \mathbf{A}_{\aleph_0,1}$, which contradicts (4). Hence $\mathbf{m}(E_{T^*_{\mathcal{H}_1}}) < 1$ and the following lemma shows that $E_{T^*_{\mathcal{H}_1}}$ has positive Lebesgue measure.

LEMMA 5. *For any $\theta \in (0,1)$ the set $\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\}$ dominates $\mathbf{T} \setminus E_{T^*_{\mathcal{H}_1}}$.*

Proof. We need to show that $\mathbf{T} \setminus E_{T^*_{\mathcal{H}_1}} \subset NTL\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\}$, or equivalently, that $T \setminus NTL(\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ is essential for $T^*_{\mathcal{H}_1}$. Since $T \setminus NTL\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\} \subset NTL(\mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$, it suffices to show that $NTL(\mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ is essential for $T^*_{\mathcal{H}_1}$. We assert that

$$(11) \qquad \mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}_\theta^r(\mathcal{A}_{T^*}, \mathcal{H}_0)\} \subset \sigma(T^*_{\mathcal{H}_1} \cup \Lambda((1-\theta^2)^{1/2}, \mathcal{H}_1, T^*)$$

which completes the proof since

$$\begin{aligned} NTL(\mathbb{D}\setminus\{\lambda\in\mathbb{D}:[C_\lambda]_{T^*}\in\mathcal{E}^r_\theta(\mathcal{A}_{T^*},\mathcal{H}_0)\})\subset\\ \subset NTL(\sigma(T^*_{\mathcal{H}_1})\cup\Lambda((1-\theta^2)^{1/2},\mathcal{H}_1,T^*))\\ = NTL(\sigma(T^*_{\mathcal{H}_1}))\cup NTL(\Lambda((1-\theta^2)^{1/2},\mathcal{H}_1,T^*))\subset E_{T^*_{\mathcal{H}_1}}, \end{aligned}$$

where the last inclusion follows from (1) and (2).

To prove (11) let us first note that if $\lambda\in\Omega_1$, then $\overline{\lambda}\in\sigma(T^*_{\mathcal{H}_1})$ since Ω is dense in Ω_1, and if γ is in Ω, $\overline{\gamma}\in\sigma_p(T^*_{\mathcal{H}_1})\subset\sigma(T^*_{\mathcal{H}_1})$. If $\lambda\in\mathbb{D}\setminus(\sigma(T^*_{\mathcal{H}_1})\cup\Lambda((1-\theta^2)^{1/2},\mathcal{H}_1,T^*))$ then $\overline{\lambda}\notin\Omega_1$, so $[C_{\overline{\lambda}}]_T\in\mathcal{E}^l_0(\mathcal{A}_T)$, and hence $[C_\lambda]_{T^*}\in\mathcal{E}^r_0(\mathcal{A}_{T^*})$. Therefore there exists a sequence $\{x_n\}_{n=1}^\infty$ of unit vectors in $\mathcal{H}$ such that

$$\|[C_\lambda]_{T^*}-[x_n\otimes x_n]_{T^*}\|\to 0,\quad \|[x_n\otimes w]_{T^*}\|\to 0\quad\text{for any } w\in\mathcal{H}.$$

Without loss of generality we may suppose that $\alpha=\lim\limits_{n\to\infty}\|x_n^0\|$ exists, and since $\lambda\notin\Lambda((1-\theta^2)^{1/2},\mathcal{H}_1,T^*)$, it follows that for any positive integer n,

$$(1-\theta^2)^{1/2}\leqslant\|[C_\lambda]_{T^*}-[x_n^1\otimes x_n^1]_{T^*}\|=\|[C_\lambda]_{T^*}-[x_n^1\otimes x_n]_{T^*}\|\leqslant$$

$$\leqslant\|[C_\lambda]_{T^*}-[x_n\otimes x_n]_{T^*}\|+\|x_n^0\otimes x_n]_{T^*}\|\leqslant\|[C_\lambda]_{T^*}-[x_n\otimes x_n]_{T^*}\|+\|x_n^0\|.$$

From the above relation it follows that $(1-\theta^2)^{1/2}\leqslant\alpha$, and since for any $n\in\mathbb{N}$,

$$\begin{aligned} \|[C_\lambda]_{T^*}-[x_n^0\otimes x_n^0]_{T^*}\|&=\|[C_\lambda]_{T^*}-\|x_n\otimes x_n^0]_{T^*}\|\\ &\leqslant\|[C_\lambda]_{T^*}-[x_n\otimes x_n]_{T^*}\|+\|x_n\otimes x_n^1]_{T^*}\|\\ &\leqslant\|[C_\lambda]_{T^*}-[x_n\otimes x_n]_{T^*}\|+\|x_n^1\| \end{aligned}$$

we obtain

$$\overline{\lim_{n\to\infty}}\|[C_\lambda]_{T^*}-[x_n^0\otimes x_n^0]_{T^*}\|\leqslant\overline{\lim_{n\to\infty}}\|x_n^1\|=(1-\alpha^2)^{1/2}\leqslant\theta.$$

Moreover, for any w in $\mathcal{H}_0$,

$$\|[x_n^0\otimes w]_{T^*}\|=\|[x_n\otimes w]_{T^*}\|\to 0,$$

so $[C_\lambda]_{T^*}\in\mathcal{E}^r_\theta(\mathcal{A}_{T^*},\mathcal{H}_0)$, and the proof is complete. ∎

As an application of the previous lemma we get the following.

COROLLARY 6. *The set $E_{T^*_{\mathcal{H}_1}}$ has positive Lebesgue measure.*

Proof. We will show that if $\mathbf{m}(E_{T^*_{\mathcal{H}_1}}) = 0$, then $T_{\mathcal{H}_0}$ is in $\mathbf{A}$. Once we have done this, the proof of the corollary is completed as follows. For any vectors x^0, y^0 in $\mathcal{H}_0$,

$$\|[x^0 \otimes y^0]_{T^*}\| = \|[x^0 \otimes y^0]_{T^*_{\mathcal{H}_0}}\|,$$

so for every $\theta \in (0,1)$,

$$\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*_{\mathcal{H}_0}} \in \mathcal{E}^r_\theta(\mathcal{A}_{T^*_{\mathcal{H}_0}})\} = \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}^r_\theta(\mathcal{A}_{T^*}, \mathcal{H}_0)\}.$$

By (12), for any $\theta \in (0,1)$, $NTL(\mathbf{D}\setminus\{\lambda \in \mathbf{D} : [C_\lambda]_T \in \mathcal{E}^r_\theta(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ has Lebesgue measure 0, so $\{\lambda \in \mathbf{D} : [C_\lambda]_T \in \mathcal{E}^r_\theta(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ dominates $\mathbf{D}$, which implies that $T^*_{\mathcal{H}_0}$ is in $\mathbf{A}_{1,\aleph_0}$ (cf. [10]). Then T^* is in $\mathbf{A}_{1,\aleph_0}$ and this contradicts (4).

Now we show that if $\mathbf{m}(E_{T^*_{\mathcal{H}_1}}) = 0$, then $T_{\mathcal{H}_0}$ is in $\mathbf{A}$. For this, let h be an $\mathbf{H}^\infty$-function, E_h be a set of Lebesgue measure 0 such that for every $e^{it} \notin E_h$ the limit $\lim\limits_{r\to 1^-} h(re^{it}) = h(e^{it})$ exists, and let $\{\theta_n\}_{n=1}^\infty$ be a sequence of positive numbers converging to 0. Again by (12), for any $n \in \mathbf{N}$, $NTL(\mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}^r_{\theta_n}(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ has Lebesgue measure 0, so $\Gamma := E_h \cup \bigcup\limits_{n=1}^{\infty} NTL(\mathbf{D}\setminus\{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}^r_{\theta_n}(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$ has Lebesgue measure 0. If e^{it} is not in Γ and $\{r_k\}_{k=1}^\infty$ is a sequence of positive numbers increasing to 1, then for each $n \in \mathbf{N}$, $r_k e^{it} \notin \mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}^r_{\theta_n}(\mathcal{A}_{T^*}, \mathcal{H}_0)\}$ for sufficiently large k (otherwise e^{it} is in $NTL(\mathbf{D} \setminus \{\lambda \in \mathbf{D} : [C_\lambda]_{T^*} \in \mathcal{E}^r_{\theta_n}(\mathcal{A}_{T^*}, \mathcal{H}_0)\})$, so $[C_{r_k e^{it}}]_{T^*} \in \mathcal{E}^r_{\theta_n}(\mathcal{A}_{T^*}, \mathcal{H}_0)$ if k is sufficiently large. For such k let $\{x^0_{p,k}\}_{p=1}^\infty$, $\{y^0_{p,k}\}_{p=1}^\infty$ be sequences in the unit ball of $\mathcal{H}_0$ such that $\overline{\lim\limits_{p\to\infty}} \|[C_{r_k e^{it}}]_{T^*} - [x^0_{p,k} \otimes y^0_{p,k}]_{T^*}\| \leqslant \theta_n$. Then

$$\begin{aligned} |h(r_k e^{it})| &\leqslant |h(r_k e^{it}) - (h(T^*_{\mathcal{H}_0})x^0_{p,k}, y^0_{p,k})| + |(h(T^*_{\mathcal{H}_0})x^0_{p,k}, y^0_{p,k})| \\ &\leqslant |h(r_k e^{it}) - (h(T^*)x^0_{p,k}, y^0_{p,k})| + \|h(T^*_{\mathcal{H}_0})\| \\ &\leqslant \|h\|_\infty \|[C_{r_k e^{it}}]_{T^*} - [x^0_{p,k} \otimes x^0_{p,k}]_{T^*}\| + \|h(T^*_{\mathcal{H}_0})\|, \end{aligned}$$

so

$$|h(r_k e^{it}) \leqslant \|h\|_\infty \overline{\lim_{p\to\infty}} \|[C_{r_k e^{it}}]_{T^*} - [x^0_{p,k} \otimes x^0_{p,k}]_{T^*}\| + \|h(T^*_{\mathcal{H}_0})\| \leqslant \theta_n \|h\|_\infty + \|h(T^*_{\mathcal{H}_0}\|.$$

Since $h(r_k e^{it}) \to h(e^{it})$, it follows that $|h(e^{it}) \leqslant \theta_n \|h\|_\infty + \|h(T^*_{\mathcal{H}_0})\|$, and taking into account the fact that $\theta_n \to 0$, we get $|h(e^{it})| \leqslant \|h(T^*_{\mathcal{H}_0})\|$ which proves the assertion. ∎

The following theorem is an easy adaptation of Theorem 10 in [2].

THEOREM 8. *Let T be an absolutely continuous contraction in $\mathcal{B}(\mathcal{H})$. If Γ is an essential set for T with possitive Lebesgue measure, then for any function f in the unit ball of $\mathbf{L}^1(\Gamma)$, there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ of vectors in the unit ball of $\mathcal{H}$ converging weakly to 0 such that $\|f - x_n \cdot y_n\|_1 \to 0$.*

Proof. It is known (cf. [3]) that there exists a Hilbert space $\mathcal{D}$ such that if U is the unitary operator of multiplication by the independent variable on $\mathbf{L}^2(\mathcal{D})$, then T is unitarily equivalent to the compression of U to some semi-invariant subspace $\tilde{\mathcal{H}}$ such that $\mathbf{L}^\infty(\mathcal{D}) \cap \tilde{\mathcal{H}}$ is dense in $\tilde{\mathcal{H}}$. So we may assume that $T = U_{\tilde{\mathcal{H}}}$, and let us note that Assumption 1 in [2] holds for any subset σ of Γ with positive Lebesgue measure. Indeed, given such a subset σ, a positive number ε, and a finite sequence $\{h_n\}_{n=1}^N$ of functions in $\tilde{\mathcal{H}}$, we want to find an essentially bounded function z in $\tilde{\mathcal{H}}$ such that

(i) $$\langle z, h_n \rangle = 0, \quad n = 1, 2, \dots, N,$$

(ii) $$\|\chi_{\mathbf{T}\setminus\sigma} z\| < \varepsilon \|\chi_\sigma z\|.$$

The proof of this is exactly the same as the proof of Corollary 10 in [3]; namely, let P be the orthogonal projection from $\tilde{H}$ onto $(\operatorname{span}\{h_n\}_{n=1}^N)^\perp$, let $\delta > 0$ such that $\delta/(1-2\delta) < \varepsilon^2$, and let u be any $\mathbf{H}^\infty$-function such that $|u(e^{it})| = 1$ for almost every $e^{it} \in \sigma$ and $|u(e^{it})| = \delta^{1/2}$ for almost every $e^{it} \in \mathbf{T} \setminus \sigma$. Then $\|u\|_\sigma = \|u\|_\infty$, so $\|u(T)\| = \|u(T)\|_e$ (cf. [7], Proposition 3.2). Since $1 - P$ is a finite dimensional projection, $\|u(T)\| = \|u(T)\|_e = \|u(T)P\|$, and $\mathbf{L}^\infty(\mathcal{D}) \cap P\tilde{\mathcal{H}}$ is dense in $P\tilde{\mathcal{H}}$, so one can find a function z in $\mathbf{L}^\infty(\mathcal{D}) \cap P\tilde{\mathcal{H}}$ with $\|z\| = 1$ and $\|u(T)z\| > (1-\delta)^{1/2}$. Obviously z satisfies (i), and since $\|uz\| = \|u(U)z\| \geqslant \|u(T)z\| > (1-\delta)^{1/2}$, by the choice of δ it follows that z also satisfies (ii).

Now Proposition 2–9 in [2] can be carried out, with the minor modification that the sets σ considered in [2] must be subsets of Γ. ∎

In order to complete the proof of Theorem 1, let us recall the folowing two lemmas (cf. [8], Lemma 2.3 and Lemma 2.4).

LEMMA 9. *Let $T \in \mathbf{A}$ be a contraction, $\mathcal{H}_0$ be a semi-invariant subspace for T such that $T_{\mathcal{H}_0}$ is not a unitary operator, and $A_*^{\mathcal{H}_0}\mathcal{H}_0 = \mathcal{R}_*^{\mathcal{H}_0}$. Then given $y^0, x^0 \in \mathcal{H}_0$, $[K]_T \in \beta\,\overline{\mathrm{aco}}\,\mathcal{E}_\theta^r(\mathcal{A}_T, \mathcal{H}_0)$, $\theta \in (0,1)$, and $0 < \beta < \delta$, there exist $\{x_n^0\}_{n=1}^\infty$, $\{u_n^0\}_{n=1}^\infty$, $\{\tilde{u}_n^0\}_{n=1}^\infty$, $\{y_n^0\}_{n=1}^\infty$, sequences of vectors in $\mathcal{H}_0$ such that*

(i) $$\|[y^0 \otimes x^0]_T + [K]_T - [y_n^0 \otimes x_n^0]_T\| < \theta\delta,$$

(ii) $$\|y_n^0 \otimes y^0\| < 3\delta^{1/2}, \ \|x_n^0\| < \|x^0\| + \delta^{1/2},$$

(iii) $$\{\tilde{u}_n^0\}_{n=1}^\infty \text{ converges weakly to } 0,\ y_n^0 - y^0 = u_n^0 + \tilde{u}_n^0,$$ $$\|Q^{\mathcal{H}_0}\tilde{u}_n^0\| \to 0,\ \|[u_n^0 \otimes w]_T\| \to 0,\ w \in \mathcal{H}.$$

The dual of the above lemma is the following.

LEMMA 10. *Let $T \in \mathbf{A}$ be a contraction, $\mathcal{H}_1$ be a semi-invariant subspace for T such that $T_{\mathcal{H}_1}$ is not unitary operator, and $\overline{A^{\mathcal{H}_1}\mathcal{H}_1} = \mathcal{R}^{\mathcal{H}_1}$. Then given $y^1, x^1 \in \mathcal{H}_1$, $[K]_T \in \beta\,\overline{\text{aco}}\,\mathcal{E}^1_\theta(\mathcal{A}_T, \mathcal{H}_1)$, $\theta \in (0,1)$, and $0 < \beta < \delta$, there exist $\{x^1_n\}_{n=1}^\infty$, $\{u^1_n\}_{n=1}^\infty$, $\{\tilde{u}^1_n\}_{n=1}^\infty$, $\{y^1_n\}_{n=1}^\infty$, sequences of vectors in $\mathcal{H}_1$ such that*

(i) $$\|[y^1 \otimes x^1]_T + [K]_T - [y^1_n \otimes x^1_n]_T\| < \theta\delta,$$

(ii) $$\|x^1_n \otimes x^1\| < 3\delta^{1/2}, \ \|y^1_n\| < \|x^1\| + \delta^{1/2},$$

(iii) $$\{\tilde{u}^1_n\}_{n=1}^\infty \text{ converges weakly to } 0, \ y^1_n - y^1 = u^1_n + \tilde{u}^1_n,$$
$$\|Q^{\mathcal{H}_0}_* \tilde{u}^1_n\| \to 0, \ \|[w \otimes u^1_n]_T\| \to 0, \ w \in \mathcal{H}_1.$$

The last result needed for the proof of Theorem 1 is the following lemma (cf [8], Lemma 6.1), which shows that the approximation scheme in [8] also works for the deomposition of $\mathcal{H}$ as in (10).

LEMMA 11. *Let y^0, x^0, y^1, x^1 be vectors in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, let $[L]_{T^*} \in \mathcal{Q}_{T^*}$, and let δ be a positive number such that*

$$\|[L]_{T^*} - [(y^0 + y^1) \otimes (x^0 + x^1)]_{T^*}\| < \delta.$$

Then there exist $\tilde{y}^0_1, \tilde{x}^0_1, \tilde{x}^1_1, \tilde{y}^1_1$ in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, such that

$$\|[L]_{T^*} - [(\tilde{y}^0_1 + \tilde{y}^1_1) \otimes (\tilde{x}^0_1 + \tilde{x}^1_1)]_{T^*}\| < \delta/2,$$
$$\|\tilde{y}^0_1 - y^0\| < 3\delta^{1/2}, \ \|\tilde{x}^0_1\| < \|x^0\| + \delta^{1/2},$$
$$\|\tilde{x}^1_1 - x^1\| < 3\delta^{1/2}, \ \|\tilde{y}^1_1\| < \|y^1\| + \delta^{1/2},$$

Proof. Let f in $\mathbf{L}^1(\mathbf{T})$ be such that $[f]_{T^*} = [L]_{T^*} - [(y^0 + y^1) \otimes (x^0 + x^1)]_{T^*}$, $\|f\|_1 < \delta$, and define $f_0 := \chi_{\mathbf{T} \setminus E^*_{T_{\mathcal{H}_1}}} f$, $f_1 := \chi_{E^*_{T_{\mathcal{H}_1}}} f$, $\delta_0 := \|f_0\|_1$, and $\varepsilon := \delta - (\delta_0 + \delta_1)/5$. As in [8], the first term to be transformed is

$$[L_0]_{T^*} = [y^0 \otimes x^0]_{T^*} + [f_0]_{T^*}.$$

For this, there are two cases.

(i) If $T^*_{\mathcal{H}_0}$ is unitary, then $\mathcal{H}_0, \mathcal{H}_1$ are reducing for T, and $T_{\mathcal{H}_0}$ is also unitary. Since T^* is in $\mathbf{A}$, then obviously

$$\mathbf{T} \setminus E_{T^*_{\mathcal{H}_1}} \subset \Sigma,$$

so $f_0 \in \mathbf{L}^1(\Sigma)$. Then by [8], Proposition 2.1, or by [11], Theorem 3.11, there exist vectors u^0 and r^0 in $\mathcal{H}_0$, such that

$$\|y^0 \overset{T_{\mathcal{H}_0}}{\cdot} x^0 + f_0 - (y^0 + u^0) \overset{T_{\mathcal{H}_0}}{\cdot} r^0\|_1 < \varepsilon,$$

and

$$\|u^0\| \leqslant 2\|f_0\|_1^{1/2}, \quad \|r^0\| \leqslant \|x^0\| + \|f_0\|_1^{1/2}.$$

Hence if we define $\tilde{y}_1^0 = y^0 + u^0$, $\tilde{x}_1^0 = r^0$, and we take into account that

$$[\tilde{y}_1^0 \otimes x^1]_{T^*} = [y^0 \otimes x^1]_{T^*} = 0,$$

we obtain

$$\|[y^0 \otimes (x^0 + x^1)]_{T^*} + [f_0]_{T^*} - [\tilde{y}_1^0 \otimes (\tilde{x}_1^0 + x^1)]_{T^*}\| < \delta/2 + \varepsilon, \tag{13}$$

and

$$\|\tilde{y}_1^0 - y^0\| < 3\delta^{1/2}, \quad \|\tilde{x}_1^0\| < \|x^0\| + \delta^{1/2}. \tag{14}$$

(ii) If $T^*_{\mathcal{H}_0}$ is not unitary, let us first note that if $\tilde{B} = \tilde{S}^* \oplus \tilde{R}_*$ is the minimal coisometric extension of T^* (acting on $\tilde{}\oplus \tilde{\mathcal{R}}_*$), then the minimal coisometric extension $\tilde{B}^{\mathcal{H}_0}$ of $T^*_{\mathcal{H}_0}$ is the compression of $\tilde{B}$ to a $*$-invariant subspace ($\mathcal{H}_0$ is invariant for T), so for any h_0 in $\mathcal{H}_0$, $\|\tilde{A}_*^{\mathcal{H}_0} h_0\| \leqslant \|\tilde{A}_* h_0\|$, from which it follows that

$$\|\tilde{Q} h_0\| \leqslant \|\tilde{Q}^{\mathcal{H}_0} h_0\|. \tag{15}$$

Let us also remark that if $\overline{\tilde{A}_*^{\mathcal{H}_0}\mathcal{H}_0} \neq \tilde{\mathcal{R}}_*^{\mathcal{H}_0}$, then $T^*_{\mathcal{H}_0}$ is in $\mathbb{A}$ (cf. [15], Proposition 2.2.5), and by the proof of Corollary 6, $T^* \in \mathbb{A}_{1,\aleph_0}$, which again contradicts (4). Hence $\overline{\tilde{A}_*^{\mathcal{H}_0}\mathcal{H}_0} = \tilde{\mathcal{R}}_*^{\mathcal{H}_0}$, and from Lemma 6 and Lemma 1.2 in [5],

$$[f_0]_{T^*} \in \delta_0 \,\overline{\text{aco}}\,\{[C_\lambda]_{T^*} : [C_\lambda]_{T^*} \in \mathcal{E}^r_{1/4}(\mathcal{A}_{T^*}, \mathcal{H}_0)\}.$$

So we may apply Lemma 9 to T^* by taking $[K]_{T^*} = [f_0]_{T^*}$, $\delta = \delta_0$, and find the sequences $\{y_n^0\}_{n=1}^\infty$, $\{x_n^0\}_{n=1}^\infty$, $\{u_n^0\}_{n=1}^\infty$, $\{\tilde{u}_n^0\}_{n=1}^\infty$ of vectors in $\mathcal{H}_0$ such that

$$\|[y^0 \otimes x^0]_{T^*} + [f_0]_{T^*} - [y_n^0 \otimes x_n^0]_{T^*}\| < \delta_0/2, \tag{16}$$

$$\|y_n^0 - y^0\| < 3\delta_0^{1/2}, \quad \|x_n^0\| < \|x^0\| + \delta_0^{1/2}, \text{ and} \tag{17}$$

$$y_n^0 - y^0 = u_n^0 + \tilde{u}_n^0, \quad \|\tilde{Q}^{\mathcal{H}_0}\tilde{u}_n^0\| \to 0, \quad \|[u_n^0 \otimes w]_{T^*}\| \to 0, \ w \in \mathcal{H}. \tag{18}$$

Taking into account that $\|T^{*n}x^1\| \to 0$, it follows that x^1 is in $\tilde{}$, so

$$[\tilde{u}_n^0 \otimes x^1]_{T^*} = [\tilde{Q}\tilde{u}_n^0 \otimes x^1]_{T^*}. \tag{19}$$

On the other hand, by (15) and (18), $\|\tilde{Q}\tilde{u}_n^0\| \to 0$, so

$$\|[\tilde{Q}\tilde{u}_n^0 \otimes x^1]_{T^*}\| \to 0.$$

Then again by (18),

$$\|[(u_n^0 + \tilde{u}_n^0) \otimes x^1]_{T^*}\| = \|[(y_n^0 - y^0) \otimes x^1]_{T^*}\| \to 0,$$

so if N is large enough,

$$\|[(y_N^0 - y^0) \otimes x^1]_{T^*}\| < \varepsilon. \tag{20}$$

If we define $\tilde{y}_1^0 = y_N^0$, $\tilde{x}_1^0 = x_N^0$, then by (17),

$$\|\tilde{y}_1^0 - y^0\| < 3\delta^{1/2}, \quad \|\tilde{x}_1^0\| < \|x^0\| + \delta^{1/2},$$

and by (16) and (20),

$$\begin{aligned}
&\|[y^0 \otimes (x^0 + x^1)]_{T^*} + [f_0]_{T^*} - [\tilde{y}_1^0 \otimes (\tilde{x}_1^0 + b^1)]_{T^*}\| \\
&\quad \leqslant \|[y^0 \otimes x^0]_{T^*} + [f_0]_{T^*} - [\tilde{y}_1^0 \otimes \tilde{x}_1^0]_{T^*}\| + \|[(\tilde{y}_1^0 - y^0) \otimes x^1]_{T^*}\| < \delta_0/2 + \varepsilon.
\end{aligned}$$

Hence in both cases one may find $\tilde{y}_1^0, \tilde{x}_1^0$, vectos in $\mathcal{H}_0$ such that (13) and (14) hold.

Now we consider the term $[L_1]_{T^*} = [\tilde{y}_1^1 \otimes \tilde{x}_1^1]_{T^*} + [f_1]_{T^*}$. Obviously $T^*_{\mathcal{H}_1}$ is not unitary, and $\tilde{\mathcal{R}}^{\mathcal{H}_1} = 0$. We shall prove that $[f_1]_{T^*} \in \delta_1\mathcal{E}_0^l(\mathcal{A}_{T^*}, \mathcal{H}_1)$. Once we show this the proof is completed as follows. By Lemma 10 (applied to T^* by taking $[K]_{T^*} = [f_1]_{T^*}$ and $\delta = \delta_1$) we can find sequences $\{y_n^1\}_{n=1}^\infty$, $\{x_n^1\}_{n=1}^\infty$, $\{u_n^1\}_{n=1}^\infty$, $\{\tilde{u}_n^1\}_{n=1}^\infty$ of vectors in $\mathcal{H}_1$ such that

$$\|[y^1 \otimes x^1]_{T^*} + [f_1]_{T^*} - [y_n^1 \otimes x_n^1]_{T^*}\| < \delta_1/2, \tag{21}$$

$$\|x_n^1 - x^1\| < 3\delta_1^{1/2}, \quad \|y_n^1\| < \|y^1\| + \delta_1^{1/2}, \tag{22}$$

and

$$\begin{gathered}
\{\tilde{u}_n^1\}_{n=1}^\infty \text{ converges weakly to } 0, \quad x_n^1 - x^1 = u_n^1 + \tilde{u}_n^1, \\
\|\tilde{Q}^{\mathcal{H}_1}\tilde{u}_n^1\| \to 0, \quad \|[w \otimes u_n^1]_{T^*}\| \to 0, \ w \in \mathcal{H}.
\end{gathered} \tag{23}$$

Since for any positive integer $n, x_n^1, x^1 \in \mathcal{H}_1 \subset$,

$$\begin{aligned}
\|[\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{T^*}\| &= \|[\tilde{Q}\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{T^*}\| \\
&= \|[\tilde{Q}\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{\tilde{B}}\| = \|[\tilde{Q}\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{\tilde{S}^*}\|.
\end{aligned}$$

Hence,

$$\|[\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{T^*}\| = \|[\tilde{Q}\tilde{y}_1^0 \otimes (x_n^1 - x^1)]_{\tilde{S}^*}\| \to 0 \tag{24}$$

since $\{x_n^1 - x^1\}_{n=1}^\infty$ converges weakly to 0 and $\tilde{S}^* \in C_0$ (cf. [7]).

Thus if M is large enough, by (23) and (24) it follows that

$$\|[\tilde{y}_1^0 \otimes (x_M^1 - x^1)]_{T^*}\| < \varepsilon. \tag{25}$$

Defining $\tilde{x}_1^1 = x_M^1$, $\tilde{y}_1^1 = y_M^1$, by (22)

$$\|\tilde{x}_1^1 - x^1\| < 3\delta_1^{1/2}, \quad \|\tilde{y}_1^1\| < \|y^1\| + \delta_1^{1/2},$$

and by (21) and (25),

$$\begin{aligned} &\|[(\tilde{y}_1^0 + y^1) \otimes x^1]_{T^*}\| + [f_1]_{T^*} - [(\tilde{y}_1^0 + \tilde{y}_1^1) \otimes \tilde{x}_1^1]_{T^*}\| \\ &\leqslant \|[y^1 \otimes x^1]_{T^*} + [f_1]_{T^*} - [\tilde{y}_1^1 \otimes \tilde{x}_1^1]_{T^*}\| + \|[\tilde{y}_1^0 \otimes (\tilde{x}_1^1 - x^1)]_{T^*}\| \\ &< \delta_1/2 + \varepsilon. \end{aligned} \tag{26}$$

Putting together (13) and (26) we obtain

$$\|[L]_{T^*} - [(\tilde{y}_1^0 + \tilde{y}_1^1) \otimes (\tilde{x}_1^0 + \tilde{x}_1^1)]_{T^*}\| < \delta_0/2 + \delta_1/2 + 2\varepsilon < \delta.$$

Now we prove that $[f_1]_{T^*} \in \delta_1 \mathcal{E}_0^l(\mathcal{A}_{T^*}, \mathcal{H}_1)\}$. To show this, let us note that since $E_{T^*_{\mathcal{H}_1}}$ is essential for $T^*_{\mathcal{H}_1}$ and $\mathbf{m}(E_{T^*_{\mathcal{H}_1}})$ is not zero (cf. Corollary 6), by Theorem 8 there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ of vectors in the unit ball of $\mathcal{H}_1$, converging weakly to 0, such that

$$\|f_1 - \delta_1(x_n \overset{T}{\cdot} y_n)\|_1 \to 0.$$

Then

$$\|[f_1]_{T^*} - \delta_1[x_n \otimes y_n]_{T^*}\| \to 0$$

and since $T^*_{\mathcal{H}_1}$ is in C_0, one obtains as in (24) that for any vector w in $\mathcal{H}$,

$$\|[w \otimes y_n]_{T^*}\| = \|[\tilde{Q}w \otimes y_n]_{\tilde{B}}\| = \|[\tilde{Q}w \otimes y_n]_{\tilde{S}^*}\| \to 0,$$

since $\tilde{S}^* \in C_0$ and $\{y_n\}_{n=1}^\infty$ converges weakly to 0. Hence $[f_1]_{T^*} \in \delta_1 \mathcal{E}_0^l(\mathcal{A}_{T^*}, \mathcal{H}_1)$, and the proof of the lemma is complete. ∎

Proof of Theorem 1. By the remark preceeding Lemma 5, it is sufficient to show that (3) is true for any vectors $x \in \mathcal{H}$, $y^0 \in \mathcal{H}_0$. Take first vectors x^0 and y^0 in $\mathcal{H}_0$, let $[L]_T = [x^0 \otimes y^0]_T$ be in Q_T, and let $\{[x_n \otimes y_n]\}_{n=1}^{\infty}$ be a sequence in $\mathcal{E}_\Omega$ such that

$$\|[x_n \otimes y_n]_T - [x^0 \otimes y^0]_T\| = \|y_n \otimes x_n]_{T^*} - [y^0 \otimes x^0]_{T^*}\| \overset{n\to\infty}{\longrightarrow} 0.$$

An easy application of Lemma 11 yields the existence of bounded sequences of vectors $\{\tilde{x}_n^0\}_{n=1}^{\infty}$, $\{\tilde{y}_n^0\}_{n=1}^{\infty}$, $\{\tilde{x}_n^1\}_{n=1}^{\infty}$, $\{\tilde{y}_n^1\}_{n=1}^{\infty}$, in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, such that

$$[x_n \otimes y_n]_T = [(\tilde{x}_n^0 + \tilde{x}_n^1) \otimes (\tilde{y}_n^0 + \tilde{y}_n^1)]_T,$$
$$\|\tilde{x}_n^1\| \to 0, \quad \|\tilde{y}_n^0 - y^0\| \to 0.$$

By passing to subseqences if necessary, without loss of generality we may assume that there exist vectors $\tilde{x}^0$ in $\mathcal{H}_0$, $\tilde{y}^1$ in $\mathcal{H}_1$, such that $\{\tilde{x}_n^0\}_{n=1}^{\infty}$ converges weakly to $\tilde{x}^0$ and $\{\tilde{y}_n^1\}_{n=1}^{\infty}$ converges weakly to $\tilde{y}^1$. Then

$$\langle g(T)x_n, y_n\rangle = \langle g(T)(\tilde{x}_n^0 + \tilde{x}_n^1), (\tilde{y}_n^0 + \tilde{y}_n^1)\rangle = \langle g(T)\tilde{x}_n^0, \tilde{y}_n^0\rangle\langle g(T)\tilde{x}_n^1, (\tilde{y}_n^0 + \tilde{y}_n^1)\rangle,$$

and since $\|\tilde{x}_n^1\| \to 0$ and $\{\tilde{x}_n^0\}_{n=1}^{\infty}$ converges weakly to $\tilde{x}^0$, it follows that

$$\langle g(T)x_n, y_n\rangle \to \langle g(T)\tilde{x}^0, y^0\rangle.$$

Similarly one obtains

$$\langle Ax_n, y_n\rangle \to \langle A\tilde{x}^0, y^0\rangle,$$

and by Lemma 4 it follows that

$$\langle g(T)\tilde{x}^0, y^0\rangle = \langle A\tilde{x}^0, y^0\rangle. \tag{27}$$

On the other hand,

$$\|[(\tilde{x}_n^0 + \tilde{x}_n^1) \otimes (\tilde{y}_n^0 + \tilde{y}_n^1)]_T - [x^0 \otimes y^0]_T\| \to 0,$$

and as above we obtain

$$[\tilde{x}^0 \otimes y^0]_T = [x^0 \otimes y^0]_T.$$

Since $A \in \operatorname{Alg}\operatorname{Lat}(T)$, it follows that

$$\langle Ax^0, y^0\rangle = \langle A\tilde{x}^0, y^0\rangle. \tag{28}$$

Putting together (27) and (28), we obtain

$$\langle Ax^0, y^0\rangle = \langle g(T)x^0, y^0\rangle. \tag{29}$$

Let us take now any vectors $x^1 \in \mathcal{H}_1$, $y^0 \in \mathcal{H}_0$. As above we can find sequences $\{\tilde{x}_n^0\}_{n=1}^\infty$, $\{\tilde{y}_n^0\}_{n=1}^\infty$, $\{\tilde{x}_n^1\}_{n=1}^\infty$, $\{\tilde{y}_n^1\}_{n=1}^\infty$, of vectors in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, and vectors $\tilde{x}^0 \in \mathcal{H}_0$ and $\tilde{y}^1 \in \mathcal{H}_1$ such that $\{\tilde{x}_n^0\}_{n=1}^\infty$ converges weakly to $\tilde{x}^0$, $\{\tilde{y}_n^1\}_{n=1}^\infty$, converges weakly to $\tilde{x}^0$, and

$$[x_n \otimes y_n]_T = [(\tilde{x}_n^0 + \tilde{x}_n^1) \otimes (\tilde{y}_n^0 + \tilde{y}_n^1)]_T,$$

$$\|\tilde{x}_n^1 - x^1\| \to 0, \quad \|\tilde{y}_n^0 - y^0\| \to 0.$$

Then

$$\langle g(T)x_n, y_n\rangle = \langle g(T)(\tilde{x}_n^0 + \tilde{x}_n^1), (\tilde{y}_n^0 + \tilde{y}_n^1)\rangle = \langle g(T)\tilde{x}_n^0, \tilde{y}_n^0\rangle + \langle g(T)\tilde{x}_n^1, (\tilde{y}_n^0 + \tilde{y}_n^1)\rangle,$$

so

$$\langle g(T)x_n, y_n\rangle \to \langle g(T)\tilde{x}^0, y^0\rangle + \langle g(T)x^1, (y^0 + \tilde{y}^1)\rangle.$$

Similarly one obtains

$$\langle Ax_n, y_n\rangle \to \langle A\tilde{x}^0, y^0\rangle + \langle Ax^1, (y^0 + \tilde{y}^1)\rangle.$$

By Lemma 4, it follows that

$$\langle g(T)\tilde{x}^0, y^0\rangle + \langle g(T)x^1, (y^0 + \tilde{y}^1)\rangle = \langle A\tilde{x}^0, y^0\rangle + \langle Ax^1, (y^0 + \tilde{y}^1)\rangle,$$

and since $\tilde{y}^1$ is in $\mathcal{H}_1$, by (3) and (27) we obtain

$$\langle g(T)x^1, y^0\rangle = \langle Ax^1, y^0\rangle,$$

and the proof is complete. ∎

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