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DECOMPOSITION OF COMPLETELY POSITIVE MAPS

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ABSTRACT. We give a solution to the decomposition problem for completely positive maps by generalizing Choquet's theorem in the context of CP-convexity theory, i.e., if A is a separable C*-algebra and H is a separable Hilbert space, then every CP-state $\psi \in Q_H(A)$ can be represented by a CP-*measure* λ_ψ supported by the CP-extreme elements $D_H(A)$ of $Q_H(A)$.

KEYWORDS: *Completely positive maps, CP-convexity, Choquet's theorem.*

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INTRODUCTION

The notion of completely positive maps for C*-algebras was introduced by W. F. Stinespring ([26]) with its representation theorem, and developed further in its structure theory by W.B. Arveson ([3], [4]) with applications to operator theory, and since then it found various applications in operator theory, operator algebras and mathematical physics (cf. e.g., [2], [6], [9], [10], [11], [22], [23], [24], and also see more extensive references given in these monographs). However, despite its proven importance, a fundamental problem, what is called the decomposition problem, has remained unsolved, i.e., "How can completely positive maps be decomposed into *pure* elements?". The purpose of this paper is to provide a consistent decomposition theory for completely positive maps by generalizing Choquet's theorem in the context of "CP-convexity", which is a natural operator convexity for complete positive maps introduced and developed in [12]-[16].

To illustrate the situation of our problem, we shall assume that A is a separable C*-algebra, H is a separable Hilbert space, and let $\mathrm{CP}(A, B(H))$ be the set of all completely positive maps from A to $B(H)$ (the C*-algebra of all bounded linear operators on H), and let us denote by $Q_H(A)$ the unit ball of the cone $\mathrm{CP}(A, B(H))$. Then $Q_H(A)$ is a compact convex set with the BW-topology (cf. Section 1), hence, as a naive approach, one may try to apply Choquet's theorem to represent $\psi \in Q_H(A)$ by a boundary measure supported by the set $\partial_e Q_H(A)$ of all extreme points of $Q_H(A)$ (see e.g. [1] for Choquet's theorem). However, as we can easily see, every representation $\pi \in \mathrm{Rep}(A : H)$ of A on H is an extreme point of $Q_H(A)$ in scalar convexity (see Appendix 1), while π may not be pure (or irreducible) and so it may be decomposed into a direct sum of subrepresentations, i.e., $\pi = \bigoplus_\alpha \pi_\alpha = \sum_\alpha p_\alpha \pi_\alpha p_\alpha$ where p_α is the projection of H_π onto H_{π_α}. Thus even the direct sum decomposition of representations is beyond the grasp of the scalar convexity theory, so we realize that the scalar convexity is not suitable for describing the decomposition of completely positive maps.

As an alternative method, we may apply the decomposition theory of representations to this problem. Let $\psi \in \mathrm{CP}(A, B(H))$ be represented as $\psi = V^* \pi V$ by the Stinespring representation theorem where π is a representation of A and $V \in B(H, H_\pi)$ (cf. Section 1). From the construction of this canonical representation, we can assume that H_π is separable (since A and H are separable), so that it can be decomposed into irreducible representations, e.g.,

$$\pi = \int_Z^\oplus \pi(\zeta)\, \mathrm{d}\mu(\zeta) \qquad \text{and} \qquad H_\pi = \int_Z^\oplus H_\pi(\zeta)\, \mathrm{d}\mu(\zeta).$$

Then, it would be quite natural to claim that ψ could be decomposed into pure CP-maps of the form

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta)\, \mathrm{d}\mu(\zeta),$$

where $V(\zeta)$ could be defined by

$$V(\zeta)\, h := (Vh)(\zeta) \quad \text{for } h \in H.$$

Unfortunately, however, one can realize that $V(\zeta)$ defined above is not bounded in general, therefore the integrand $V(\zeta)^* \pi(\zeta) V(\zeta)$ may not be a pure CP-map in $\mathrm{CP}(A, B(H))$. In Section 3, we shall discuss a special class of CP-maps which allows the above disintegration, where we shall have to assume some strong continuity of $V \in B(H, H_\pi)$.

The difficulties illustrated above seem to suggest that they are caused by the lack of fundamental concept to describe decompositions of CP-maps. In our previous works ([12], [14]), we introduced the notion of "CP-convexity" for completely positive maps, where we defined that $\psi \in \mathrm{CP}(A, B(H))$ is a CP-*convex combination* of a bounded family of CP-maps $(\psi_\alpha)_{\alpha\in\Lambda} \subset \mathrm{CP}(A, B(H))$, if

$$\psi = \sum_{\alpha\in\Lambda} S_\alpha^* \psi_\alpha S_\alpha \quad \text{with } S_\alpha \in B(H) \text{ such that } \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leqslant I_H,$$

where the sum converges in the BW-topology. This convexity obviously includes scalar convexity and the direct sum of representations as particular cases, and, as we shall show in this paper, it indeed can describe all possible decompositions of completely positive maps in the sense of a measure generalization of CP-convex combination, which we shall call "CP-measure" and develop a theory of integration with respect to this measure in Section 3. Thus the decomposition problem for completely positive maps can now be solved by generalizing Choquet's theorem in the context of CP-convexity theory as we shall show in Section 4.

We should note that the CP-convex combination is a natural concept in the C*-algebraic approach in quantum physics, where an interaction between physical systems can be described by a completely positive map ψ, which is called "operation", and the coefficient "$S_\alpha^*(\,\cdot\,)S_\alpha$" in the above decomposition is called "effect" which represents the weight of the operation represented by the CP-map ψ_α (cf. [22], [23]). Therefore the decomposition of completely positive maps has the natural physical meaning of decomposing the interactions into minimal ones.

We shall briefly summarize the content of this paper. In Section 1, we review some basic definitions and results on CP-maps and CP-convexity. In Section 2, we first discuss particular decompositions, i.e., decompositions by CP-convex combination (Proposition 2.1) and disintegration with respect to a scalar measure (Theorem 2.4), which we shall use in Section 4. In Section 3, we develop a measure and integration theory inherent to CP-convexity, where the definition of our integral is more simple and straightforward than the usual method in the vector measure theory in Banach spaces, and it would be more suitable for operator algebras. Based on this measure and integration theory, we shall prove our main result, the CP-Choquet theorem, in Section 4.

We finally note that the CP-Choquet theorem is essentially used in [18] (though not explicitly stated, but converted into the terms of irreducible decomposition) to study the general Stone-Weierstrass problem for separable C*-algebras which originally motivated our present work. We expect that the CP-measure and integration and the CP-Choquet theorem will find further useful applications in operator theory, operator algebras and mathematical physics.

1. PRELIMINARIES ON CP-MAPS AND CP-CONVEXITY

The definition and basic properties of completely positive maps are given in references e.g. [3], [24], [26], [27]. For our purpose in the present paper, it suffices to recall the representation theorem due to Stinespring [26] (which could be considered as our definition of CP-maps), and some basic properties.

Let A be a C*-algebra, and H be a Hilbert space, then every completely positive map $\psi \in \mathrm{CP}(A, B(H))$ from A to $B(H)$ can be represented as

$$\psi = V^* \pi(a) V \qquad \text{for all } a \in A,$$

where π is a representation of A on a Hilbert space K and $V \in B(H, K)$ is a bounded linear operator from H to K ([26]). We can assume the minimality condition $K = [\pi(A)VH]$ without loss of generality, with which the Stinespring representation is unique up to unitary intertwining operators ([3]). For topologies in $\mathrm{CP}(A, B(H))$, we consider the norm topology, where we note that $\|\psi\| = \|V\|^2$, and also the BW-topology which is defined as the pointwise weak convergence topology; thus a bounded net $(\psi_\alpha) \subset \mathrm{CP}(A, B(H))$ converges to $\psi \in \mathrm{CP}(A, B(H))$ in the BW-topology if and only if $\psi_\alpha(a)$ converges to $\psi(a)$ in the weak topology in $B(H)$ for each $a \in A$. If A is a W*-algebra, we denote by $\mathrm{CP}(A, B(H))_n$, putting the suffix n, the set of all normal (i.e., $\sigma(A, A_*)$ - $\sigma(B(H), B(H)_*)$ continuous) CP-maps from A to $B(H)$. Note that $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$ is normal if and only if π is normal. In particular, when $A = B(H)$, we write $\mathrm{CP}(B(H))_n :=$ $\mathrm{CP}(B(H), B(H))_n$.

The cone $\mathrm{CP}(A, B(H))$ has a natural ordering induced by the cone itself, i.e., $\varphi \leqslant \psi$ if and only if $\psi - \varphi \in \mathrm{CP}(A, B(H))$. A CP-map $\psi \in \mathrm{CP}(A, B(H))$ is defined to be *pure* if $0 \leqslant \varphi \leqslant \psi$ for $\varphi \in \mathrm{CP}(A, B(H))$ implies $\varphi = c\psi$ with $0 \leqslant c \leqslant 1$. Then, $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$ is pure if and only if π is irreducible (cf. [3]). We denote by $P_H(A)$ the set of all pure elements of $\mathrm{CP}(A, B(H))$. We call $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$ to be *approximately unital* if $V^*V = I_H$, and we define $S_H(A) := \{\psi = V^*\pi V \in \mathrm{CP}(A, B(H))\,;\, V^*V = I_H\}$. We denote by $PS_H(A)$ the set of all pure approximately unital CP-maps, i.e., $PS_H(A) := P_H(A) \bigcap S_H(A) =$ $\{\psi = V^*\pi V \in P_H(A)\,;\, V^*V = I_H\}$. For the scalar case, we denote by $P(A)$ the pure states of A, i.e., $P(A) := PS_{\mathbf{C}}(A)$.

We denote by $\mathrm{Rep}\,(A)$ [resp. $\mathrm{Rep}_c(A), \mathrm{Irr}\,(A)$] the set of all [resp. cyclic, irreducible] representations of A, and write $\mathrm{Rep}\,(A : H)$ [resp. $\mathrm{Rep}_c(A : H), \mathrm{Irr}\,(A : H)$] to specify the Hilbert space H on which the representations are confined. Note that $\mathrm{Rep}\,(A : H) = \{\pi \in \mathrm{Rep}\,(A);\, H_\pi \subset H\}$, where H_π denotes the essential subspace of π. For $\pi \in \mathrm{Rep}\,(A : H)$, p_π denotes the projection of H onto H_π. More generally, we define the *support* (or *essential subspace*) H_ψ of ψ to be the support of $\psi(A)$ in H, or more precisely, if $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$, then $H_\psi := [|V|H]$ (the essential support of V), and we denote by p_ψ the projection of H onto H_ψ. We also define the *cyclic dimension* $\alpha_c(A)$ [resp. *irreducible dimension* $\alpha_i(A)$] by

$$\alpha_c(A) \ \ [\text{resp. } \alpha_i(A)] := \sup\{\dim H_\pi \, ; \, \pi \in \mathrm{Rep}_c(A) \ \ [\text{resp. Irr}\,(A)]\}.$$

We next review some definitions and results on CP-convexity. A CP-map $\psi \in \mathrm{CP}(A, B(H))$ is called a *CP-state* if ψ is a contraction, and the *CP-state space* of A for H is defined by $Q_H(A) := \{\psi \in \mathrm{CP}(A, B(H)); \|\psi\| \leqslant 1\}$, which generalizes the quasi-state space $Q(A)$ in the scalar convexity theory. If $\psi \in Q_H(A)$ is a CP-convex combination of $(\psi_\alpha)_{\alpha\in\Lambda} \subset Q_H(A)$, which was defined in Introduction, we briefly write

$$\psi = \text{CP-}\sum_{\alpha\in\Lambda} S_\alpha^*\psi_\alpha S_\alpha,$$

assuming that the condition "$S_\alpha \in B(H)$ such that $\sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leqslant I_H$" is implicitly satisfied. Note that the CP-state space $Q_H(A)$ is a *CP-convex* set, by which we mean that it is closed under the operation of CP-convex combination. For a subset $D \in Q_H(A)$, the *CP-convex hull* of D, denoted by CP-*conv* D, is defined as the set of all CP-convex combinations of elements in D, or equivalently the smallest CP-convex set including D. For instance, if $\dim H \geqslant \alpha_c(A)$, then $Q_H(A) = \text{CP-}conv\,\mathrm{Rep}_c(A : H)$ ([14], Proposition 1.4.A).

The notion of extreme points can be generalized in this CP-convexity as follows: A nonzero CP-state $\psi \in Q_H(A)$ is defined to be a *CP-extreme state* if $\psi = \text{CP-}\sum_\alpha S_\alpha^*\psi_\alpha S_\alpha$ with $\psi_\alpha \in Q_H(A)$ implies that ψ_α is unitarily equivalent to ψ, i.e., $\psi_\alpha = U_\alpha\psi U_\alpha^*$, and $S_\alpha = c_\alpha U_\alpha$ $(c_\alpha \in \mathbb{C})$ where U_α is a partial isometry such that $U_\alpha^* U_\alpha = p_\psi$ and $U_\alpha U_\alpha^* = p_{\psi_\alpha}$. We denote the set of all CP-extreme states by $D_H(A)$. We have shown in [16] that $D_H(A)$ can be characterized as

$$\begin{aligned} D_H(A) = \{u^*\pi u \in P_H(A); \, u^*u = I_H \ \ \textit{with} \ \dim H < \dim H_\pi, \aleph_0\} \\ \bigcup \{u^*\pi u \in P_H(A)\, ; \, uu^* = I_{H_\pi} \ \ \textit{with} \ \dim H \geqslant \dim H_\pi\}, \end{aligned}$$

i.e., the CP-extreme boundary $D_H(A)$ consists of pure approximately unital CP-maps (which appear only when H is finite dimensional) and irreducible representations of A on H. Note that it can be expressed according to the dimension of H as $D_H(A) = \mathrm{Irr}\,(A : H)$ if $\dim H \geqslant \inf\{\alpha_i(A), \aleph_0\}$, and $D_H(A) = PS_H(A) \bigcup \mathrm{Irr}\,(A : H)$ if $\dim H < \inf\{\alpha_i(A), \aleph_0\}$, and especially $D_{\mathbb{C}}(A) = P(A)$ (cf. [16]).

As another different way to define CP-extreme states, we could assume that S_α ($\alpha \in \Lambda$) are all positive, and require that $\psi_\alpha = \psi$ and $S_\alpha = c_\alpha p_\psi$ ($c_\alpha \in \mathbb{C}$) for ψ to be CP-extreme, then we have slightly different CP-extreme states

$$E_H(A) := \{u^*\pi u \in P_H(A);\ u^*u = I_H \ \text{ or } \ uu^* = I_{H_\pi}\},$$

which consists of all pure approximately unital CP-maps and irreducible representations of A on H, i.e., $E_H(A) = PS_H(A) \bigcup \operatorname{Irr}(A : H)$ (cf. [16], [17]). Note that $D_H(A) \subset E_H(A)$ in general, and the above subtle difference yields the facts that $D_H(A)$ is convenient for algebraic theory (cf. [16]), while $E_H(A)$ is useful for convexity arguments (cf. Section 2). However, if A and H are separable, then $D_H(A)$ satisfies the both needs (cf. Remark 1 in Section 2), and shall be used in Section 4.

We shall briefly review duality theorems in CP-convexity. A function $\gamma : Q_H(A) \to B(H)$ is defined to be *CP-affine*, if

$$\psi = \text{CP-}\sum_\alpha S_\alpha^* \psi_\alpha S_\alpha \quad \text{with } \psi_\alpha \in Q_H(A) \quad \text{implies} \quad \gamma(\psi) = \sum_\alpha S_\alpha^* \gamma(\psi_\alpha) S_\alpha.$$

We denote by $AC(Q_H(A), B(H))$ the set of all bounded BW-w continuous CP-affine functions from $Q_H(A)$ to $B(H)$. Then the CP-duality theorem ([14], Theorem 2.2.A) states that, if $\dim H \geqslant \alpha_c(A)$, we have

$$A \cong AC(Q_H(A), B(H)) \qquad (*\text{-isomorphism}),$$

where the product is defined on $\operatorname{Rep}_c(A : H)$ (cf. [14] for details), which generalizes Kadison's function representation theorem with recovering the full C*-structure. (We note that if $\dim H < \alpha_c(A)$, then the above isomorphism is reduced to just an order isomorphism.) We can also generalize the Gelfand-Naimark theorem for non-commutative C*-algebras on the CP-extreme boundary, i.e., if H is an infinite dimensional Hilbert space with $\dim H \geqslant \alpha_c(A)$, then A is $*$-isomorphic to the set of all those $B(H)$-valued functions on $D_H(A) = \operatorname{Irr}(A : H)$ which are BW-w uniformly continuous and preserve the unitary equivalence relations. This CP-Gelfand-Naimark theorem and the CP-duality theorem are connected by the abstract Dirichlet problem, and we can also generalize the spectral theorem for non-normal operators, which generalizes the same situation of the commutative case (cf. [16]).

2. SPECIAL DECOMPOSITIONS

We shall first consider the decomposition of CP-maps by a CP-convex combination of pure CP-maps. We first note the following basic facts, where we shall call a representation π to be *subatomic* if π is unitarily equivalent to a subrepresentation of a direct sum of irreducible representations.

PROPOSITION 2.1. *Let A be a C^*-algebra, H be a Hilbert space, and let $\psi = V^*\pi V \in Q_H(A)$. Then the following conditions are equivalent:*

(i) *π is subatomic.*

(ii) $\psi = \sum_i \psi_i$ with $\psi_i \in P_H(A)$.

(iii) $\psi \in$ CP-*conv* $E_H(A)$.

(iv) $\psi \in$ CP-*conv* $D_H(A)$ *if* $\dim H < \aleph_0$ *or* $\dim H \geqslant \alpha_i(A)$.

Proof. (i) $\Rightarrow$ (ii) : Assume that π is subatomic, i.e., there exists a family of irreducible representations (π_i) and an isometry $U : H_\pi \to \bigoplus_i H_{\pi_i}$ such that $\pi = U^*(\bigoplus_i \pi_i) U$. Then, denoting by p_i the projection of $\bigoplus_i H_{\pi_i}$ onto H_{π_i}, we have

$$\begin{aligned}\psi = V^*\pi V = V^*U^*(\bigoplus_i \pi_i)\, UV &= V^*U^*(\bigoplus_i p_i)(\bigoplus_i \pi_i)(\bigoplus_i p_i)\, UV \\ &= \sum_i (p_i UV)^*\pi_i\,(p_i UV) = \sum_i V_i^*\pi_i V_i,\end{aligned}$$

where we set $V_i := p_i UV$. Then $\psi = \sum_i \psi_i$ with $\psi_i := V_i^*\pi_i V_i \in P_H(A)$.

(ii) $\Rightarrow$ (iii) : Assume that $\psi = \sum_i \psi_i$ with $\psi_i = V_i^*\pi_i V_i \in P_H(A)$. Let $V_i = v_i|V_i|$ be the polar decomposition of V_i, and define a unitary extension $\tilde{v}_i :$ $H \to H_{\pi_i}$ of v_i such that $\tilde{v}_i^*\tilde{v}_i = I_H$ (if $\dim H \leqslant \dim H_\pi$), or $\tilde{v}_i\tilde{v}_i^* = I_{H_{\pi_i}}$ (if $\dim H \geqslant \dim H_{\pi_i}$). Then, we have

$$\psi = \sum_i |V_i|\tilde{v}_i^*\pi_i\tilde{v}_i|V_i| \quad \text{where } \tilde{v}_i^*\pi_i\tilde{v}_i \in E_H(A) \text{ and } \sum_i |V_i|^2 \leqslant I_H,$$

where the last inequality holds since $\sum_i |V_i|^2 = \sum_i V_i^*V_i = V^*V \leqslant I_H$. This shows that $\psi \in$ CP-*conv* $E_H(A)$.

(iii) $\Rightarrow$ (i) : Assume that $\psi \in$ CP-*conv* $E_H(A)$, i.e.,

$$\psi = \sum_i S_i^*\psi_i S_i \quad \text{with } \psi_i \in E_H(A) \text{ and } S_i \in B(H) \text{ such that } \sum_i S_i^* S_i \leqslant I_H.$$

Let $\psi_i = V_i^*\pi_i V_i$ be the Stinespring representation of ψ_i, where $\pi_i \in \mathrm{Irr}\,(A)$ since $\psi_i \in E_H(A)$. Then

$$\psi = V^*\pi V = \sum_i S_i^* V_i^*\pi_i V_i S_i = (\bigoplus_i V_i S_i)^*(\bigoplus_i \pi_i)(\bigoplus_i V_i S_i) = W^*\rho W$$

where $W := \bigoplus_i V_i S_i$ and $\rho := \bigoplus_i \pi_i|_{[(\bigoplus_i \pi_i)(A)WH]}$ is introduced to represent the minimal representation. Since the minimal representation is unique up to unitary equivalence, there exists a unitary operator $U : H_\pi \to H_\rho$ such that

$$\pi = U^* \rho U \qquad \text{and} \qquad V = U^* W.$$

Hence, $\pi = U^* (\bigoplus_i \pi_i) U$, so π is subatomic.

(iii) $\Rightarrow$ (iv) : If $\dim H < \aleph_0$, then $D_H(A) = E_H(A)$, so it is trivial. Assume that $\dim H \geqslant \alpha_i(A)$, then $D_H(A) = \mathrm{Irr}(A : H)$ (cf. Section 1). Let $\varphi \in E_H(A)$, then $\varphi = u^* \pi u$ with $\pi \in \mathrm{Irr}(A)$ and $u^* u = I_H$ or $uu^* = I_{H_\pi}$. Since $\dim H \geqslant \alpha_i(A)$, there exists a co-isometry $v : H \to H_\pi$ with $vv^* = I_{H_\pi}$, so that $\varphi = (v^* u)^* (v^* \pi v)(v^* u)$ where $v^* \pi v \in \mathrm{Irr}(A : H) = D_H(A)$ and $v^* u \in B(H)$ with $(v^* u)^*(v^* u) = u^*(vv^*)u = u^* I_{H_\pi} u = u^* u \leqslant I_H$. This means that $\varphi \in$ CP-*conv* $D_H(A)$, i.e., $E_H(A) \subset$ CP-*conv* $D_H(A)$. Hence $\psi \in$ CP-*conv* $E_H(A) \subset$ CP-*conv* $D_H(A)$.

(iv) $\Rightarrow$ (iii), under the condition $\dim H < \aleph_0$ or $\dim H \geqslant \alpha_i(A)$, is trivial since $D_H(A) \subset E_H(A)$. ∎

We recall that a C*-algebra A is defined to be *scattered* if the enveloping W*-algebra A^{**} is atomic, which can also be characterized by the above equivalent conditions with letting $H = \mathbf{C}$ and $\psi \in Q(A)$, replacing $\mathrm{CP}(A : B(H))$ with $(A^*)^+$, and especially by the fact that any representation of A is subatomic (cf. [21]). Then observe that CP-*conv* $E_H(A)$ generalizes σ-*conv* $(P(A) \bigcup \{0\}) := \left\{ \sum_{i=1}^{\infty} \lambda_i \omega_i \,;\, \omega_i \in P(A),\ \lambda_i > 0 \text{ with } \sum_{i=1}^{\infty} \lambda_i \leqslant 1 \right\}$, which is the atomic part of the quasi-state space $Q(A)$. Hence, we immediately have the following.

COROLLARY 2.2. *Let A be a C*-algebra and H be a Hilbert space. Then A is scattered if and only if one of the following equivalent conditions is satisfied.*

(i) *Any $\psi \in \mathrm{CP}(A, B(H))$ can be written as $\psi = \sum \psi_i$ with $\psi_i \in P_H(A)$.*

(ii) *$Q_H(A) =$ CP-conv $E_H(A)$.*

(iii) *$Q_H(A) =$ CP-conv $D_H(A)$ if $\dim H < \aleph_0$ or $\dim H \geqslant \alpha_i(A)$.*

REMARK 1. Assume that A and H are separable, then $\alpha_i(A) \leqslant \aleph_0$, so that $\dim H < \aleph_0$ or $\dim H = \aleph_0 \geqslant \alpha_i(A)$. Hence, we can drop the condition "if $\dim H < \aleph_0$ or $\dim H \geqslant \alpha_i(A)$" from Proposition 2.1 (iv) and Corollary 2.2 (iii) in this case.

REMARK 2. If H is separable, then the decomposition by CP-convex combination is a countable sum. In fact, let $\psi = CP\text{-}\sum_{i \in I} S_i^* \psi_i S_i$ with $\psi_i \in Q_H(A)$ for

$i \in I$. Note here that $B(H)$ is σ-finite since H is separable, so that there exists a faithful normal state $\omega_0 \in B(H)_*$. If A has a unit e (or if A is not unital, using an approximate unit), we have $0 < \omega_0(\psi(e)) = \omega_0\Big(\sum_{i\in I} S_i^*\psi_i(e)S_i\Big) \leqslant \omega_0\Big(\sum_{i\in I} S_i^*S_i\Big) = \sum_{i\in I}\omega_0(S_i^*S_i) \leqslant \omega_0(I_H) = 1$, so that $\sum_{i\in I} S_i^*S_i$ is a countable sum on the faithful normal ω_0, hence I is countable.

REMARK 3. K. Kraus ([22]) showed the above decomposition into pure CP-maps for the particular case $\psi \in \mathrm{CP}(C(H), B(H)) = \mathrm{CP}(B(H))_n$, where $C(H)$ denotes the C*-algebra of all compact operators on H, which is a typical example of a scattered C*-algebra.

We shall next investigate a class of CP-maps which can be disintegrated into pure CP-maps with respect to a scalar measure. We have to prepare some definitions and notations of countably Hilbert nuclear spaces, for which we refer to [20].

Let $\Omega \subset H_\Omega \subset \Omega'$ be a rigged Hilbert space. Recall that the nuclear topology of the countably Hilbert nuclear space Ω is defined by a chain of countably many Hilbert spaces Ω_n with norms $\|\cdot\|_n$ $(n = 1, 2, 3, \ldots)$, such that

$$\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \cdots \leqslant \|\cdot\|_n \leqslant \|\cdot\|_{n+1} \leqslant \cdots$$

and

$$\Omega \subset \cdots \subset \Omega_{n+1} \subset \Omega_n \subset \cdots \subset \Omega_2 \subset \Omega_1 \subset H_\Omega.$$

Since the norm $\|\cdot\|$ of H_Ω is continuous with respect to the nuclear topology on Ω by definition, there exists $n \in \mathbf{N}$ such that the embedding $T : \Omega \to H_\Omega$ is a nuclear (or trace class) operator with respect to the norm $\|\cdot\|_n$. We shall denote by $r(\Omega)$ the minimum of this integer n, i.e.,

$$r(\Omega) := \min\{n \in \mathbf{N}\,;\ T : \Omega \hookrightarrow H_\Omega \text{ is a nuclear operator with respect to } \|\cdot\|_n\},$$

and call the *rank* of the rigged Hilbert space $\Omega \subset H_\Omega \subset \Omega'$. In the following, we shall use the abbreviations $\Omega_r := \Omega_{r(\Omega)}$ and $\|\cdot\|_r := \|\cdot\|_{r(\Omega)}$. It is known that if H is a separable Hilbert space, then there exists a countably Hilbert nuclear space Ω such that $\Omega \subset H \subset \Omega'$ becomes a rigged Hilbert space.

DEFINITION 2.3. Let A be a C*-algebra and H be a Hilbert space, and let $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$, and assume that there exists a countably Hilbert nuclear space Ω such that $\Omega \subset H_\pi \subset \Omega'$ is a rigged Hilbert space. Then, ψ is defined to be *pre-nuclear* if $V \in B(H, \Omega_r)$, i.e., V is a bounded linear operator from the Hilbert space H to the Hilbert space Ω_r with the $\|\cdot\|_r$-norm. We also define ψ to be *nuclear* if $\psi \in B(H, \Omega_n)$ for all $n \in \mathbf{N}$, i.e., $V : H \to \Omega = \bigcap_{n=1}^{\infty} \Omega_n$ is continuous with respect to the nuclear topology.

REMARK 4. As an example of nuclear CP-maps, we cite the class of *finite rank* CP-maps, i.e., $\psi = V^*\pi V$ where $\dim VH < \infty$. One can take a countably Hilbert nuclear space Ω such that $VH \subset \Omega$ and $\Omega \subset H_\pi \subset \Omega'$ is a rigged Hilbert space.

REMARK 5. If $\psi = V^*\pi V$ is pre-nuclear, then V is a trace class (nuclear) operator from H to H_π, i.e., $V \in T(H, H_\pi)$, since V can be decomposed as $V : H \xrightarrow{V} \Omega_r \overset{T}{\hookrightarrow} H_\pi$ where T denotes the nuclear embedding.

Now, we can prove the following decomposition theorem for pre-nuclear CP-maps.

THEOREM 2.4. *Let A be a separable C^*-algebra and H be a Hilbert space. Let $\psi = V^*\pi V \in \mathrm{CP}(A, B(H))$ where π is a representation of A on a separable Hilbert space H_π, and assume that ψ is a pre-nuclear CP-map with respect to a rigged Hilbert space structure $\Omega \subset H_\pi \subset \Omega'$. Then, for any maximal abelian subalgebra $\mathcal{M}_\pi$ of $\pi(A)'$, there exist a standard measure space (Z, μ) and measurable families of irreducible representations $(\pi(\zeta))_{\zeta \in Z} \subset \mathrm{Irr}(A : H)$ and trace class (nuclear) operators $(V(\zeta))_{\zeta \in Z} \subset T(H, H_\pi(\zeta))$ such that*

$$\psi = \int_Z V(\zeta)^* \pi(\zeta)\, V(\zeta)\, \mathrm{d}\mu(\zeta) \qquad (\text{BW-}\textit{integral}),$$

i.e.,

$$(\psi(a), \rho) = \int_Z (V(\zeta)^* \pi(\zeta)(a)\, V(\zeta), \rho)\, \mathrm{d}\mu(\zeta) \quad \textit{for all} \quad a \in A \quad \textit{and} \quad \rho \in B(H)_* .$$

Proof. From the standard disintegration theorem (e.g., [8], Theorem 8.5.2), there exists a standard measure space (Z, μ) corresponding to $\mathcal{M}_\pi$, and a measurable family $(\pi(\zeta))_{\zeta \in Z}$ of irreducible representations of A on $(H_\pi(\zeta))_{\zeta \in Z}$ such that

$$\pi = \int_Z^{\oplus} \pi(\zeta)\, \mathrm{d}\mu(\zeta) \quad \text{and} \quad H_\pi = \int_Z^{\oplus} H_\pi(\zeta)\, \mathrm{d}\mu(\zeta).$$

Since H_π is equipped as a rigged Hilbert space $\Omega \subset H_\pi \subset \Omega'$, and H_π is represented as the direct integral of Hilbert spaces $(H_\pi(\zeta))_{\zeta \in Z}$, according to Gelfand-Vilenkin's theorem ([20], Section 4.4, Theorem 1$'$), there exists a family of nuclear operators $\{P(\zeta) : \Omega_r \to H_\pi(\zeta)\}_{\zeta \in Z}$ such that, for every $f \in \Omega_r$,

$$P(\zeta)(f) = f(\zeta) \qquad \text{for} \quad \mu\text{-a.e. } \zeta \in Z.$$

(The original theorem by Gelfand-Vilenkin is stated for Ω instead of Ω_r, however it can be easily seen that their proof is true for Ω_r, since it only depends on the existence of a nuclear map $T : \Omega_r \to H_\pi$ as they remark after the proof.)

By the assumption that ψ is pre-nuclear, $V \in B(H, \Omega_r)$, so that we can define

$$V(\zeta) := P(\zeta)\, V : H \to H_\pi(\zeta)\,,$$

where note that $V(\zeta)$ is a trace class operator from H to $H_\pi(\zeta)$, i.e., $V(\zeta) \in T(H, H_\pi(\zeta)) \subset B(H, H_\pi(\zeta))$, which ensures $V(\zeta)^* \pi(\zeta)\, V(\zeta) \in P_H(A)$. Then, for any $a \in A$ and $f, g \in H$, we have

$$\begin{aligned}
(\psi(a)f, g) &= (\pi(a)Vf, Vg) \\
&= \int_Z (\pi(\zeta)(a)(Vf)(\zeta), (Vg)(\zeta))\, \mathrm{d}\mu(\zeta) \\
&= \int_Z (\pi(\zeta)(a)V(\zeta)f, V(\zeta)g)\, \mathrm{d}\mu(\zeta) \\
&= \int_Z (V(\zeta)^* \pi(\zeta)(a)V(\zeta)f, g)\, \mathrm{d}\mu(\zeta).
\end{aligned}$$

Since any $\rho \in B(H)_*$ is of the form $\rho(x) = \sum_{i=1}^{\infty}(xf_i, g_i)$ for $x \in B(H)$ with $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|g_i\|^2 < \infty$ (e.g., [25], Corollary 1.15.4), it follows that

$$(\psi(a), \rho) = \int_Z (V(\zeta)^* \pi(\zeta)(a)\, V(\zeta), \rho)\, \mathrm{d}\mu(\zeta).$$

This completes the proof. ∎

3. CP-MEASURE AND INTEGRATION

In this section, we develop a measure and integration theory inherent to the notion of CP-convexity. To illustrate our motivation, we shall recall that a scalar σ-convex combination can be interpreted as the barycenter of an atomic probability measure. In analogy, a CP-convex combination

$$\psi = \sum_{\alpha \in \Lambda} S_\alpha^* \psi_\alpha S_\alpha \quad \text{where} \quad \psi_\alpha \in Q_H(A) \quad \text{and} \quad S_\alpha \in B(H) \text{ with } \sum_{\alpha\in\Lambda} S_\alpha^* S_\alpha \leqslant I_H$$

could be interpreted as a distribution of the weight of "effects" $S_\alpha^*(\cdot)S_\alpha \in Q_H(B(H))_n$ in the sense of the theory of operation. Therefore, in the scope of the CP-duality $A \cong AC(Q_H(A), B(H))$, we could claim that ψ is the barycenter of a $Q_H(B(H))_n$-valued measure, if the integral of the elements of $AC(Q_H(A), B(H))$ with respect to this measure is meaningful. We thus realize that we need to develop a CP-map valued measure and integration theory. In the following, we actually formulate the theory for more general setting with $P(A, B)$-valued measure where $P(A, B)$ denotes the set of all positive linear maps between order-unit spaces A and B, and define the integral of A-valued strongly, and weakly, measurable functions with respect to this $P(A, B)$-valued measure.

Let $(X, \mathcal{B})$ be a measurable space, A be a (real) order-unit space, where we shall denote the order-unit of A by e, and let B be a (real) dual order-unit space, i.e., there exists a base-norm space B_* such that $B = (B_*)^*$. (See [1], Chapter 1, Section 1 for the definitions of order-unit spaces and base-norm spaces. More generally, it would suffice to assume that B is an ordered real Banach space, and there exists an ordered real Banach space in separating order and norm duality with B, and that B is pointwise monotone σ-complete with respect to this duality.) We denote by $S(X, A)$ the set of all (countable valued) simple functions from X to A, i.e., $f \in S(X, A)$ is a function $f : X \to A$ of the form

$$f = \sum_{i=1}^{\infty} \chi_{E_i}\, a_i \quad \text{with} \quad (E_i)_{i=1}^{\infty} \subset \mathcal{B} \text{ (disjoint)} \quad \text{and} \quad (a_i)_{i=1}^{\infty} \subset A,$$

where χ_E denotes the characteristic function of $E \in \mathcal{B}$. We denote by $BS(X, A)$ the set of all bounded simple functions from X to A. It is straightforward to see that $BS(X, A)$ is a normed (real) linear space with the sup-norm; $\|f\| := \sup\{\|f(x)\|; x \in X\}$.

Let λ be a $P(A, B)$-valued BW-countably additive measure, i.e., $\lambda : \mathcal{B} \to P(A, B)$ is a map such that, if $E = \sum_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{B}$ (disjoint), then $\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i)$ converging in the BW-topology, i.e., for any $a \in A$ and $\rho \in B_*$, $(\lambda(E)a, \rho) = \sum_{i=1}^{\infty} (\lambda(E_i)a, \rho)$. We first claim the following lemma.

LEMMA 3.1. *Let $(E_i)_{i=1}^{\infty} \subset \mathcal{B}$ be disjoint and $(a_i)_{i=1}^{\infty} \subset A$ be a bounded sequence. Then the series $\sum_{i=1}^{\infty} \lambda(E_i)\, a_i$ converges in $\sigma(B, B_*)$-topology.*

Proof. We first assume $(a_i) \subset A^+ := \{a \in A;\ a \geqslant 0\}$ and set $b_n = \sum_{i=1}^{n} \lambda(E_i)\, a_i$

for $n \in \mathbf{N}$, then b_n is an increasing sequence in B. Observe that

$$\begin{aligned} b_n &\leqslant \sum_{i=1}^{n} \lambda(E_i)\|a_i\|e \leqslant \Big(\sup_{1\leqslant i\leqslant n}\|a_i\|\Big)\sum_{i=1}^{n}\lambda(E_i)e \\ &= \Big(\sup_{1\leqslant i\leqslant n}\|a_i\|\Big)\,\lambda\Big(\sum_{i=1}^{n}E_i\Big)e \leqslant \Big(\sup_{i\in\mathbf{N}}\|a_i\|\Big)\,\lambda(X)e\ . \end{aligned}$$

Since (b_n) is a bounded and increasing sequence in B, and B is monotone σ-complete, (b_n) converges to $\text{l.u.b.}_n(b_n) = \sum_{i=1}^{\infty} \lambda(E_i)\,a_i \in B$ in $\sigma(B, B_*)$-topology.

In case that $(a_i) \subset A$ is more general, we decompose $a_i = a_i^{(1)} - a_i^{(2)}$ with $a_i^{(1)}, a_i^{(2)} \in A^+$, and we define

$$\sum_{i=1}^{\infty}\lambda(E_i)a_i := \sum_{i=1}^{\infty}\lambda(E_i)a_i^{(1)} - \sum_{i=1}^{\infty}\lambda(E_i)a_i^{(2)}.$$

This definition does not depend on the way of decomposition $a_i = a_i^{(1)} - a_i^{(2)}$; indeed, if $a_i = a_i^{(1)} - a_i^{(2)} = \tilde{a}_i^{(1)} - \tilde{a}_i^{(2)}$ where $a_i^{(j)}, \tilde{a}_i^{(j)} \in A^+$ $(j = 1, 2)$, then $a_i^{(1)} + \tilde{a}_i^{(2)} = \tilde{a}_i^{(1)} + a_i^{(2)}$, hence

$$\sum_{i=1}^{\infty}\lambda(E_i)(a_i^{(1)} + \tilde{a}_i^{(2)}) = \sum_{i=1}^{\infty}\lambda(E_i)(\tilde{a}_i^{(1)} + a_i^{(2)}),$$

i.e.,

$$\sum_{i=1}^{\infty}\lambda(E_i)a_i^{(1)} + \sum_{i=1}^{\infty}\lambda(E_i)\tilde{a}_i^{(2)} = \sum_{i=1}^{\infty}\lambda(E_i)\tilde{a}_i^{(1)} + \sum_{i=1}^{\infty}\lambda(E_i)a_i^{(2)},$$

so we have

$$\sum_{i=1}^{\infty}\lambda(E_i)a_i^{(1)} - \sum_{i=1}^{\infty}\lambda(E_i)a_i^{(2)} = \sum_{i=1}^{\infty}\lambda(E_i)\tilde{a}_i^{(1)} - \sum_{i=1}^{\infty}\lambda(E_i)\tilde{a}_i^{(2)}\ . \quad \blacksquare$$

REMARK 6. We note that the above lemma holds when λ is finitely additive at the order unit e.

We then define the integral of bounded simple functions as follows.

DEFINITION 3.2. Let $\lambda : \mathcal{B} \to P(A, B)$ be a BW-countably additive measure and let $f \in BS(X, A)$ which is expressed as

$$f = \sum_{i=1}^{\infty}\chi_{E_i}a_i \quad \text{with} \quad (E_i) \subset \mathcal{B} \quad \text{(disjoint)} \quad \text{and} \quad (a_i) \subset A \quad \text{(bounded)}.$$

Then, we define the *integral* of f with respect to λ by

$$\int_X f \, d\lambda := \sum_{i=1}^{\infty} \lambda(E_i) a_i \quad \in B.$$

REMARK 7. The above integral is well defined, i.e., it does not depend on any particular way of the expression of $f \in BS(X,A)$, which can be proved in a straightforward manner (see Appendix 2). We note that the BW-countably additivity of λ is necessary for this proof of well-definedness.

Let τ denote a topology in A, and $\mathcal{B}_\tau$ denote the Borel sets induced from the τ-topology in A. A function $f : X \to A$ is defined to be *τ- measurable* if its range $f(X)$ is τ-separable in A, and $f^{-1}(E) \in \mathcal{B}$ for all $E \in \mathcal{B}_\tau$. We denote by $M_\tau(X,A)$ [resp. $BM_\tau(X,A)$] the set of all [resp. bounded] τ-measurable functions from X to A. The following is a direct generalization of the well-known fact for norm-measurable functions (see e.g., [7], II. Corollary 3).

LEMMA 3.3. *Assume that any bounded part of A is metrizable in the topology τ. Then $BS(X,A)$ is dense in $BM_\tau(X,A)$ with the τ-uniformly convergence topology.*

Proof. It is enough to prove our assertion for the unit ball of $BM_\tau(X,A)$. We denote by $d(\cdot,\cdot)$ the metric in the unit ball of A in the topology τ, and we define $S_r(a) := \{b \in A; d(a,b) < r\}$ for $r > 0$. Let $f \in BM_\tau(X,A)$ with $\|f\| \leqslant 1$, and let $(a_k)_{k=1}^{\infty}$ be a τ-dense subset of $f(X)$. For each $n \in \mathbb{N}$, define $Y_k^{(n)} := f^{-1}\left(S_{\frac{1}{n}}(a_k) \bigcap f(X)\right) \in \mathcal{B}$ and set $X_k^{(n)} := Y_k^{(n)} \setminus \bigcup_{l<k} Y_l^{(n)} \in \mathcal{B}$. Then, for each fixed $n \in \mathbb{N}$, $X_k^{(n)}$ are mutually disjoint and $X = \bigcup_{k=1}^{\infty} X_k^{(n)}$. We define $f_n := \sum_{k=1}^{\infty} \chi_{X_k^{(n)}} a_k \in BS(X,A)$. Then f_n converges uniformly to f, because, for every $n \in \mathbb{N}$ and $x \in X$, there exists $k \in \mathbb{N}$ such that $x \in X_k^{(n)}$ and we have $d(f_n(x), f(x)) = d(a_k, f(x)) < \frac{1}{n}$. ∎

We first define the integral of bounded norm-measurable functions, for which we write $\tau = s$. The proof of the following lemma is immediate from the definition.

LEMMA 3.4. *The integral $\lambda(f) := \int f \, d\lambda$ for $f \in BS(X,A)$ defined in Definition 3.2 is a bounded linear map from $BS(X,A)$ to B with $\|\lambda\| = \|\lambda(X)e\|$.*

DEFINITION 3.5. Let A be an order-unit space and B be a dual order-unit space, and let $\lambda : \mathcal{B} \to P(A,B)$ be a BW-countably additive measure. Then, by Lemmas 3.3 and 3.4, the integration map $\lambda : BS(X,A) \to B$ has the unique bounded linear extension to $BM_s(X, A)$. We call this map the *strong integration* of elements in $BM_s(X, A)$ with respect to λ.

We next assume that A is a dual order-unit space, i.e., there exists a base-norm space A_* such that $A = (A_*)^*$, and then we shall define the integral of bounded weak* (i.e., $\sigma(A, A_*)$)-measurable functions, for which we shall write $\tau = w$. Since it seems hardly possible to formulate the general case for this weak integral, we shall consider a particular case as defined below, which shall be enough for our purpose in Section 4.

Let $\lambda : \mathcal{B} \to P(A, B)_n$ be a BW-countably additive measure where $P(A, B)_n$ denotes the set of all normal (i.e., $\sigma(A, A_*)$ - $\sigma(B, B_*)$ continuous) positive linear maps from A to B. For most of applications, for example W^*-algebras and JBW-algebras, this normality is equivalent to monotone continuity, so it would suffice to restrict ourselves to these cases.

DEFINITION 3.6. Let $\lambda : \mathcal{B} \to P(A, B)$ be a BW-countably additive measure. For each $a \in A$, $\rho \in B_*$, and $E \in \mathcal{B}$, define $\lambda_{a,\rho}(E) := (\lambda(E)a, \rho)$. If there exists a scalar measure ν such that $\lambda_{a,\rho} \ll \nu$ uniformly for $a \in A$ and $\rho \in B_*$, then we say that λ is *BW-absolutely continuous with respect to* ν.

LEMMA 3.7. *Let $\lambda : \mathcal{B} \to P(A, B)_n$ be a BW-countably additive measure which is BW-absolutely continuous with respect to a scalar measure ν. Let $(f_n) \subset BS(X, A)$ be a Cauchy sequence in the uniform w^*-convergence topology. Then $(b_n) = (\int_X f_n \, d\lambda)$ is a Cauchy sequence in $\sigma(B, B_*)$-topology.*

Proof. Since $\lambda_{a,\rho} \ll \nu$ for any $a \in A$ and $\rho \in B_*$, by Lebesgue-Radon-Nikodym's theorem, there exists a unique $F(a, \rho) \in L^1(X, \nu)$ such that, for any $E \in \mathcal{B}$,

$$\lambda_{a,\rho}(E) = (\lambda(E)a, \rho) = \int_E F(a, \rho)(x) \, d\nu(x).$$

We note that, since $\lambda(E)$ is normal (i.e., $\sigma(A, A_*)$-$\sigma(B, B_*)$ continuous), for any fixed $\rho \in B_*$ and $E \in \mathcal{B}$, the map $a \in A \mapsto \int_E F(a, \rho)(x) \, d\nu$ is $\sigma(A, A_*)$-continuous. Moreover, from the above definition, for each fixed $\rho \in (B_*)^+$, the map $a \in A \mapsto F(a, \rho) \in L^1(X, \nu)$ is positive linear and monotone continuous, so that $F(\cdot, \rho)(x) \in A_*$ for ν-a.e. $x \in X$.

Let $f_n = \sum_{i=1}^{\infty} \chi_{E_i} a_i$ with disjoint $(E_i) \subset B$ and bounded $(a_i) \subset A$, and observe that, for each $\rho \in B_*$,

$$\begin{aligned}(b_m - b_n, \rho) &= \Big(\int_X (f_m - f_n)\,\mathrm{d}\lambda, \rho\Big) \\ &= \sum_{i,j}\Big(\lambda\Big(E_i^{(m)} \bigcap E_j^{(n)}\Big)\Big(a_i^{(m)} - a_j^{(n)}\Big), \rho\Big) \\ &= \sum_{i,j} \int_{E_i \bigcap E_j} F\Big(a_i^{(m)} - a_j^{(n)}, \rho\Big)(x)\,\mathrm{d}\nu(x) \\ &= \int_X F\Big(f_m(x) - f_n(x), \rho\Big)(x)\,\mathrm{d}\nu(x).\end{aligned}$$

Since $(f_m(x) - f_n(x)) \to 0$ uniformly as $m, n \to \infty$ in $\sigma(A, A_*)$-topology, from the above remark, $F(f_m(x) - f_n(x), \rho)(x) \to 0$ for ν-a.e. $x \in X$, so we conclude that $\int_X F(f_m(x) - f_n(x), \rho)(x)\,\mathrm{d}\nu(x) \to 0$ as $m, n \to \infty$. Hence $(b_m - b_n) \to 0$ as $m, n \to \infty$ in $\sigma(B, B_*)$-topology, i.e., (b_n) is a Cauchy sequence in $\sigma(B, B_*)$-topology. ∎

Since B is weak* sequentially complete, the Cauchy sequence (b_n) of the above lemma converges to a unique element in B. We then have the following definition of the weak integral.

DEFINITION 3.8. Let A and B be dual order unit spaces, and assume that any bounded part of A is w^*- metrizable, and A_* consists of all monotone continuous linear functionals on A. Let $\lambda : B \to P(A, B)_n$ be a normal BW-countably additive measure which is BW-absolutely continuous with respect to a scalar measure. Then by Lemmas 3.3 and 3.7, the integration map $\lambda : BS(X, A) \to B$ has the unique extension to $BM_w(X, A)$. We call this map the *weak integration* of elements in $BM_w(X, A)$ with respect to λ.

REMARK 8. Note that the weak integral is an extension of the strong integral, and that, from the proof of Lemma 3.7, the weak integral of $f \in BM_w(X, A)$ with respect to λ is given by

$$\Big(\int_X f\,\mathrm{d}\lambda, \rho\Big) = \int_X F(f(x), \rho)(x)\,\mathrm{d}\nu(x) \quad \text{for } \rho \in B_*.$$

REMARK 9. It is straightforward to extend the integrals defined in this section for a complex space valued measurable functions. Let $A^{\mathbf{C}}$ [resp. $B^{\mathbf{C}}$] be the complex vector space generated by a real order unit space A [resp. B], i.e., $A^{\mathbf{C}} = A + \mathrm{i}A$ [resp. $B^{\mathbf{C}} = B + \mathrm{i}B$], and let $f \in BM_\tau(X, A^{\mathbf{C}})(= BM_\tau(X, A)^{\mathbf{C}})$ be written as

$$f = f_1 + \mathrm{i} f_2 \quad \text{with } f_1, f_2 \in BM_\tau(X, A),$$

and assume that the integrals of f_1, f_2 with respect to a $P(A, B)$ valued measure λ exist. Then, we can define

$$\int_X f \,\mathrm{d}\lambda := \int_X f_1 \,\mathrm{d}\lambda + \mathrm{i} \int_X f_2 \,\mathrm{d}\lambda.$$

In Section 4, we shall apply the above integral to the situation where $A^{\mathbf{C}} = B^{\mathbf{C}} = B(H)$ and λ is a $Q_H(B(H))_n$-valued measure where H is a separable Hilbert space, so that every bounded part of $B(H)$ is w^*-separable and metrizable (cf. [27], Proposition 2.7). In this particular case, we shall call it *CP-measure and integration*.

NOTE. In the present paper, we have only given the definitions of the integral of bounded measurable functions, which is enough for our purpose in Section 4, however, we can also discuss the integral for unbounded measurable functions (cf. [12]). The strong integral for unbounded measurable functions has some advantages compared with the usual method in vector measure theory in Banach spaces which uses semi-variations as a generalization of Bochner integral (cf. [5], [12], [19]).

4. CP-CHOQUET THEOREM

We shall prove the generalized Choquet theorem for the CP-state space $Q_H(A)$ using the CP-measure and integration defined in Section 3, where we set $X = Q_H(A)$ and $\mathcal{B} = \mathcal{B}_{BW}$ (the Borel sets induced from the BW-topology on $Q_H(A)$). If H is separable, then by the duality $A \cong AC(Q_H(A), B(H))$, every element $a \in A$ defines a w^*- measurable function $\hat{a} : \varphi \in Q_H(A) \mapsto \varphi(a) \in B(H)$, so we can define the weak integral $\int_{Q_H(A)} \hat{a} \,\mathrm{d}\lambda$ with respect to a CP-measure λ on $Q_H(A)$. We first prepare the following lemma.

LEMMA 4.1. *Let A be a separable C^*-algebra and H be a separable Hilbert space. Then the CP-extreme boundary $D_H(A)$ is a BW-measurable set, i.e., $D_H(A) \in \mathcal{B}_{BW}$.*

Proof. We first note that $\alpha_i(A) \leqslant \aleph_0$ since A is separable. If $\dim H \geqslant \alpha_i(A)$, then $D_H(A) = \mathrm{Irr}\,(A : H)$ (cf. Section 1), which is BW-measurable since $\mathrm{Irr}\,(A : H)$ is a G_δ-subset of $Q_H(A)$ (cf. [8], 3.7.4).

If $\dim H < \alpha_i(A)$, then $D_H(A) = PS_H(A) \bigcup \mathrm{Irr}\,(A : H)$ (cf. Section 1). We already know that $\mathrm{Irr}\,(A : H) \in \mathcal{B}_{BW}$, so we have to show $PS_H(A) \in \mathcal{B}_{BW}$.

Let $h \in H$ with $\|h\| = 1$, and define the following map

$$\Theta_h : \psi \in Q_H(A) \mapsto \Theta_h(\psi) := (\psi(\cdot)h, h) \in Q(A),$$

which is obviously BW-w* continuous. Recall that $P(A)$ is G_δ subset of $Q(A)$ ([25], Corollary 3.4.2), so $\Theta_h^{-1}(P(A)) \in \mathcal{B}_{BW}$, and note that

$$\begin{aligned}\Theta_h^{-1}(P(A)) &= \{V^*\pi V \in Q_H(A);\, (\pi(\cdot)Vh, Vh) \in P(A)\}\\ &= \{V^*\pi V \in P_H(A);\, \|Vh\| = 1\}.\end{aligned}$$

Let $(h_i)_{i=1}^{\infty}$ be a dense sequence in $S_1 := \{h \in H; \|h\| = 1\}$, then we have

$$\begin{aligned}PS_H(A) &= \{V^*\pi V \in P_H(A);\, (V^*Vh, h) = (h, h) \quad \text{for all } h \in H\}\\ &= \{V^*\pi V \in P_H(A);\, (V^*Vh, h) = \|Vh\|^2 = 1 \quad \text{for all } h \in S_1\}\\ &= \{V^*\pi V \in P_H(A);\, \|Vh_i\| = 1 \text{ for all } i \in \mathbf{N}\}\\ &= \bigcap_{i=1}^{\infty} \Theta_{h_i}^{-1}(P(A)) \in \mathcal{B}_{BW}. \quad \blacksquare\end{aligned}$$

Now we shall prove our main theorem.

THEOREM 4.2. *Let A be a separable C^*-algebra and H be a separable Hilbert space. Then, for any CP-state $\psi \in Q_H(A)$, there exists a CP-measure λ_ψ which is supported by $D_H(A)$ such that*

$$\psi(a) = \int_{Q_H(A)} \hat{a}\, d\lambda_\psi \qquad \textit{for all } a \in A.$$

(In this setting, we shall say that ψ is the CP-barycenter of λ_ψ, or ψ is represented by the boundary CP-measure λ_ψ.)

Proof. Let $\psi = V^*\pi V$ be the Stinespring representation, where we can assume that H_π is separable since A and H are separable (cf. [3], [26]; remember that, in the canonical construction, H_π is isomorphic to the completion of a quotient space of $A \otimes H$, and since A and H are separable by our assumption, we conclude that H_π is separable). By the decomposition theory for representations, there exits a measurable space (Z,μ) and a family of irreducible representations $(\pi(\zeta))_{\zeta\in Z}$ on $(H_\pi(\zeta))_{\zeta\in Z}$ such that

$$\pi = \int_Z^{\oplus} \pi(\zeta)\,\mathrm{d}\mu(\zeta) \quad \text{and} \quad H_\pi = \int_Z^{\oplus} H_\pi(\zeta)\,\mathrm{d}\mu(\zeta).$$

Let us define a partial isometry

$$U : H_\pi \longrightarrow L^2(Z,\mu,H)$$

where $U := \int_Z^{\oplus} u(\zeta)\,\mathrm{d}\mu(\zeta)$ and $\{u(\zeta) : H_\pi(\zeta) \to H\}_{\zeta\in Z}$ is a measurable field of isometries or co-isometries which are defined as follows. If H is an infinite dimensional Hilbert space, then it is known that there exists a measurable field of isometries $\{u(\zeta) : H_\pi(\zeta) \to H\}_{\zeta\in Z}$, so that $U = \int_Z^{\oplus} u(\zeta)\,\mathrm{d}\mu(\zeta)$ defines an isometry from H to $L^2(Z,\mu,H)$ (cf. [27], p.273). If H is finite dimensional, then ψ is finite rank, so pre-nuclear, hence, by Theorem 2.4, it has a decomposition of the form

$$\psi = \int_Z V(\zeta)^* \pi(\zeta) V(\zeta)\,\mathrm{d}\mu(\zeta) \qquad \text{(BW-integral)},$$

where $V(\zeta) = P(\zeta)V \in T(H, H_\pi(\zeta)) \subset B(H, H_\pi(\zeta))$. Let $V(\zeta) = v(\zeta)|V(\zeta)|$ be the polar decomposition of $V(\zeta)$, then $(v(\zeta))_{\zeta\in Z}$ is a measurable field of partial isometries from H to $H_\pi(\zeta)$. Then, we can take a measurable field $\{u(\zeta) : H_\pi(\zeta) \to H\}_{\zeta\in Z}$ of isometries or co-isometries, where $u(\zeta)$ is defined by a unitary extension of $v(\zeta)^*$ such that $u(\zeta)^*u(\zeta) = I_{H_\pi(\zeta)}$ or $u(\zeta)u(\zeta)^* = I_H$, according to $\dim H \geqslant \dim H_\pi(\zeta)$ or $\dim H < \dim H_\pi(\zeta)$ respectively. The existence of such a measurable field $(u(\zeta))_{\zeta\in Z}$, which extends $(v(\zeta)^*)_{\zeta\in Z}$, can be proved by the technique given in [27], Lemma 8.12 (the Gram-Schmidt orthogonalization of fundamental sequences), which was used to show the existence of the measurable field of isometries $(u(\zeta))_{\zeta\in Z}$ for infinite dimensional H. Then $U := \int_Z^{\oplus} u(\zeta)\,\mathrm{d}\mu(\zeta)$ defines a partial isometry from H_π to $L^2(Z,\mu,H)$.

Now, let us define

$$\tilde{\pi}(\zeta) := u(\zeta)\pi(\zeta)u(\zeta)^* \quad \text{for } \zeta \in Z.$$

Then it is obvious that $\tilde{\pi}(\zeta) \in D_H(A)$ for each $\zeta \in Z$, and we note that the map $\tilde{\pi} : \zeta \in Z \mapsto \tilde{\pi}(\zeta) \in D_H(A) \subset Q_H(A)$ is BW-measurable, i.e., for any $a \in A$ and $f \in H$, the function $\zeta \in Z \mapsto (\tilde{\pi}(\zeta)(a)f, f) = (\pi(\zeta)(a)u(\zeta)^*f, u(\zeta)^*f)$ is μ-measurable, which follows from the fact that the fields $(\pi(\zeta))_{\zeta\in Z}$ and $(u(\zeta)^*f)_{\zeta\in Z}$ are μ-measurable. For our later use, we shall define

$$\tilde{\pi}^{-1}(E) := \{\zeta \in Z;\ \tilde{\pi}(\zeta) \in E\} \quad \text{for } E \in \mathcal{B}_{BW},$$

which is a Borel set in Z for each $E \in \mathcal{B}_{BW}$.

We next define a representation $\kappa : B(H) \to L^\infty(Z, \mu, B(H))$ on $L^2(Z, \mu, H)$ by

$$\kappa(b)\, h := \int_Z^\oplus b\, h(\zeta)\, \mathrm{d}\mu(\zeta) \quad \text{for } b \in B(H) \text{ and } h = \int_Z^\oplus h(\zeta)\, \mathrm{d}\mu(\zeta) \in L^2(Z, \mu, H).$$

Note that κ is normal, i.e., $\sigma(B(H), B(H)_*)$ - $\sigma(L^\infty(Z, \mu, B(H)), L^1(Z, \mu, B(H)_*))$ continuous (cf. [25], Proposition 1.22.13). We also define a projection P_E on $L^2(Z, \mu, H)$ by

$$P_E := \int_Z^\oplus P_E(\zeta)\, \mathrm{d}\mu(\zeta) \quad \text{where} \quad P_E(\zeta) := \begin{cases} I_H & \text{for } \zeta \in \tilde{\pi}^{-1}(E), \\ 0 & \text{for } \zeta \notin \tilde{\pi}^{-1}(E). \end{cases}$$

We then define a CP-measure $\lambda_\psi : E \in \mathcal{B} \mapsto \lambda_\psi(E) \in Q_H(B(H))_n$ by

$$\lambda_\psi(E) := (P_E\, U\, V)^*\ \kappa\ (P_E\, U\, V) = V^* U^* P_E\ \kappa\ P_E\, U\, V,$$

where it is immediate to see the BW-countable additivity of λ_ψ.

We shall show that λ_ψ is BW-absolutely continuous with respect to a scalar measure. Note first that, for any $b \in B(H)$ and $\omega_{\xi,\eta} := (\cdot\, \xi, \eta) \in B(H)_*$ with $\xi, \eta \in H$, we have

$$\begin{aligned} (\lambda_\psi(E)\, b, \omega_{\xi,\eta}) &= (U^* P_E \kappa(b) P_E U V \xi, V\eta) \\ &= \int_{\tilde{\pi}^{-1}(E)} (u(\zeta)^* b\, u(\zeta)(V\xi)(\zeta), (V\eta)(\zeta))\, \mathrm{d}\mu(\zeta), \end{aligned}$$

so that, noting $\|u(\zeta)\| = 1$, we have

$$\begin{aligned} |(\lambda_\psi(E)\, b, \omega_{\xi,\eta})| &\leqslant \int_Z |(u(\zeta)^* b\, u(\zeta)(V\xi)(\zeta), (V\eta)(\zeta))|\, \mathrm{d}\mu(\zeta) \\ &\leqslant \|b\| \int_Z \|(V\xi)(\zeta)\|\, \|(V\eta)(\zeta)\|\, \mathrm{d}\mu(\zeta) \\ &\leqslant \|b\| \left(\int_Z \|(V\xi)(\zeta)\|^2\, \mathrm{d}\mu(\zeta) \right)^{\frac{1}{2}} \left(\int_Z \|(V\eta)(\zeta)\|^2\, \mathrm{d}\mu(\zeta) \right)^{\frac{1}{2}} \\ &\leqslant \|b\|\, \|V\xi\|\, \|V\eta\| \leqslant \|b\|\, \|V\|^2 \|\xi\|\, \|\eta\|. \end{aligned}$$

Since any $\rho \in B(H)_*$ is of the form

$$\rho = \sum_{n=1}^\infty \omega_{\xi_n, \eta_n} \quad \text{where} \quad \sum_{n=1}^\infty \|\xi_n\|^2 < \infty \quad \text{and} \quad \sum_{n=1}^\infty \|\eta_n\|^2 < \infty,$$

we have

$$|(\lambda_\psi(E)\, b, \rho)| \leqslant \sum_{n=1}^{\infty} |((\lambda_\psi(E) b, \omega_{\xi,\eta})| \leqslant \sum_{n=1}^{\infty} \|b\|\, \|V\|^2 \|\xi_n\|\, \|\eta_n\|$$

$$\leqslant \|b\|\, \|V\|^2 \left(\sum_{n=1}^{\infty} \|\xi_n\|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \|\eta_n\|^2\right)^{\frac{1}{2}} < \infty ,$$

so that we can define

$$\tilde{F}_\psi(b,\rho)(\zeta) := \sum_{n=1}^{\infty} (u(\zeta)^*\, b\, u(\zeta)(V\xi_n)(\zeta), (V\eta_n)(\zeta)) \in L^1(Z,\mu),$$

and

$$(\lambda_\psi)_{b,\rho}(E) = (\lambda_\psi(E)\, b, \rho) = \int_{\tilde{\pi}^{-1}(E)} \tilde{F}_\psi(b,\rho)(\zeta)\, \mathrm{d}\mu(\zeta).$$

Let us define $\nu := \mu \circ \tilde{\pi}^{-1}$, then ν is a scalar measure on $Q_H(A)$ supported by $\tilde{\pi}(Z) \subset D_H(A)$, and the above equality ensures that $(\lambda_\psi)_{b,\rho} << \nu$ uniformly for $b \in B(H)$ and $\rho \in B(H)_*$, i.e., λ_ψ is BW-absolutely continuous with respect to ν.

We can now apply Definition 3.8 to define the CP-integral $\int_{Q_H(A)} \hat{a}\, \mathrm{d}\lambda_\psi$ for any $a \in A$. We first note that, since $(\lambda_\psi)_{b,\rho} << \nu$, for any $b \in B(H)$ and $\rho = \omega_{\xi,\eta} \in B(H)_*$, there exists $F_\psi(b, \omega_{\xi,\eta}) \in L^1(Q_H(A), \nu)$ such that, for any $E \in \mathcal{B}_{BW}$,

$$(\lambda_\psi)_{b,\omega_{\xi,\eta}}(E) = (\lambda_\psi(E) b, \omega_{\xi,\eta}) = \int_E F_\psi(b, \omega_{\xi,\eta})(\varphi)\, \mathrm{d}\nu(\varphi),$$

from which we can deduce in particular that

$$\int_E F_\psi(b, \omega_{\xi,\eta})(\varphi)\, \mathrm{d}\nu(\varphi) = \int_{\tilde{\pi}^{-1}(E)} \tilde{F}_\psi(b, \omega_{\xi,\eta})(\zeta)\, \mathrm{d}\mu(\zeta) .$$

It now follows from the proof of Lemma 3.7 (cf. Remark 8 after Definition 3.8) that for any $a \in A$ and $\omega_{\xi,\eta} \in B(H)_*$, we have

$$\begin{aligned}
\Big(\int_{Q_H(A)} \hat{a}\, \mathrm{d}\lambda_\psi, \omega_{\xi,\eta}\Big) &= \int_{\tilde{\pi}(Z)} F_\psi(\varphi(a), \omega_{\xi,\eta})(\varphi)\, \mathrm{d}\nu(\varphi) \\
&= \int_Z \tilde{F}_\psi(\tilde{\pi}(\zeta)(a), \omega_{\xi,\eta})(\zeta)\, \mathrm{d}\mu(\zeta) \\
&= \int_Z \Big(u(\zeta)^* \tilde{\pi}(\zeta)(a) u(\zeta)(V\xi)(\zeta), (V\eta)(\zeta)\Big)\, \mathrm{d}\mu(\zeta) \\
&= \int_Z \Big(\pi(\zeta)(a) u(\zeta)^* u(\zeta)(V\xi)(\zeta), u(\zeta)^* u(\zeta)(V\eta)(\zeta)\Big)\, \mathrm{d}\mu(\zeta) .
\end{aligned}$$

We note here that, if H is infinite dimensional, then $u(\zeta)^* u(\zeta) = I_{H_\pi(\zeta)}$, and if H is finite dimensional, then $u(\zeta)^* u(\zeta) \geqslant P_{[V(\zeta)H]}$ from our definition of $u(\zeta)$, where $P_{[V(\zeta)H]}$ denotes the projection of $H_\pi(\zeta)$ onto $[V(\zeta)H] = [P(\zeta)VH] = [(VH)(\zeta)]$. Hence, we have

$$\Big(\int\limits_{Q_H(A)} \hat{a}\, \mathrm{d}\lambda_\psi, \omega_{\xi,\eta} \Big) = \int\limits_Z \big(\pi(\zeta)(a)(V\xi)(\zeta), (V\eta)(\zeta)\big)\, \mathrm{d}\mu(\zeta)$$
$$= \big(\pi(a)V\xi, V\eta\big) = \big(\psi(a), \omega_{\xi,\eta}\big)\, .$$

Since $\xi, \eta \in H$ are arbitrary, we have proved

$$\psi(a) = \int\limits_{Q_H(A)} \hat{a}\, \mathrm{d}\lambda_\psi,$$

where λ_ψ is supported by $D_H(A)$ from our definition. This completes the proof. ∎

REMARK 10. In the above theorem, if H is infinite dimensional, then $D_H(A) = \mathrm{Irr}\,(A : H)$, hence, in particular, if $\psi = \pi \in \mathrm{Rep}\,(A : H)$, then this CP-Choquet theorem provides an analytic expression of the algebraic decomposition $\pi = \int_Z^\oplus \pi(\zeta)\, \mathrm{d}\mu(\zeta)$. On the other hand, if $\dim H = 1$, then $D_H(A) = P(A)$, and the CP-measure λ_ψ reduces to the usual Choquet's boundary measure. Thus the CP-Choquet theorem interpolates the gap between the algebraic decomposition and Choquet's theorem.

REMARK 11. From Corollary 2.2 and Theorem 4.2, a separable C*-algebra A is scattered if and only if every $\psi \in Q_H(A)$ can be represented by an *atomic* boundary CP-measure.

REMARK 12. We note that $\lambda_\psi(\cdot) I_H$ defines a POV (positive operator valued) measure such that $0 \leqslant \lambda_\psi(E)\, I_H \leqslant V^*V$ for $E \in \mathcal{B}_{BW}$, where $\lambda_\psi(D_H(A))\, I_H = V^*V$ and $\lambda_\psi(\emptyset)\, I_H = 0$. If we put the weight $I - V^*V$ at the origin 0, then this provides a resolution of the identity I_H into positive operators (effects) on $D_H(A) \bigcup \{0\}$.

APPENDIX

1. $\mathrm{Rep}\,(A : H) \subset \partial_e(Q_H(A))$: Let $\pi \in \mathrm{Rep}\,(A : H)$ and assume that $\pi = \sum_i c_i \psi_i$ with $\psi_i \in Q_H(A)$ and $c_i > 0$, $\sum_i c_i = 1$. Since $c_i\psi_i \leqslant \pi$, by Arveson's theorem ([3], Theorem 1.4.2), $c_i\psi_i = T_i\pi$ with $T_i \in \pi(A)', 0 \leqslant T_i \leqslant p_\pi$, and, since $\psi_i \in Q_H(A)$, $\|\psi_i\| = \|c_i^{-1} T_i \pi\| = \|c_i^{-1} T_i\| \leqslant 1$, so $\|T_i\| \leqslant c_i$, i.e., $0 \leqslant T_i \leqslant c_i p_\pi$. If A has an identity e (or if A is not unital, using an approximate unit), $p_\pi = \pi(e) =$

$\sum_i c_i\psi_i(e) = \sum_i T_i\pi(e) = \sum_i T_i p_\pi \leqslant (\sum_i c_i p_\pi)p_\pi = p_\pi$. Hence we have $T_i = c_i p_\pi$, i.e., $\psi_i = \pi$.

2. *Definition 3.2 is well defined* : It suffices to prove this for $f \in BS(X, A)^+$ from the definition of the integral. Suppose that f has two different expressions

$$f = \sum_{i=1}^{\infty} \chi_{E_i} a_i = \sum_{j=1}^{\infty} \chi_{F_j} b_j \quad \text{where } X = \bigcup_{i=1}^{\infty} E_i = \bigcup_{j=1}^{\infty} F_j \text{ (disjoint)}$$
$$\text{and} \quad (a_i), (b_j) \subset A^+ \text{ (bounded)}.$$

We set $G_{ij} := E_i \bigcap F_j$ and $c_{ij} := a_i = b_j$ (on G_{ij}). Then, $E_i = \bigcup_{j=1}^{\infty} G_{ij}$ (disjoint) and $F_j = \bigcup_{i=1}^{\infty} G_{ij}$ (disjoint), so that, for any $\rho \in (B_*)^+$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \lambda(E_i)a_i, \rho\right) &= \sum_{i=1}^{\infty}\left(\lambda\left(\bigcup_{j=1}^{\infty} G_{ij}\right)a_i, \rho\right) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}(\lambda(G_{ij})c_{ij}), \rho\right) \\
&= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}(\lambda(G_{ij})c_{ij}, \rho)\right) = \sum_{j=1}^{\infty}\left(\lambda\left(\bigcup_{i=1}^{\infty} G_{ij}\right)b_j, \rho\right) \\
&= \left(\sum_{j=1}^{\infty} \lambda(F_j)b_j, \rho\right),
\end{aligned}$$

since λ is BW-countably additive and the sums $\sum_{i=1}^{\infty}$ and $\sum_{j=1}^{\infty}$ are exchangeable as all terms are positive. Hence, noting that $\rho \in B_*$ is arbitrary, we have shown

$$\sum_{i=1}^{\infty} \lambda(E_i)a_i = \sum_{j=1}^{\infty} \lambda(F_j)b_j.$$

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