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AN EQUIVALENT DESCRIPTION OF NON-QUASIANALYTICITY THROUGH SPECTRAL THEORY OF C_0-GROUPS

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ABSTRACT. Consider a weight ω on $\mathbf{R}$ with the following property.

(ne) For any C_0-group $\mathcal{T} := (T(t))_{t\in\mathbf{R}}$ on a Banach space E satisfying $\|T(t)\| \leqslant \omega(t)$ for all $t \in \mathbf{R}$, there holds $\sigma(A) \neq \emptyset$ for the generator A of $\mathcal{T}$.

It is well-known that a non-quasianalytic weight ω (i.e., $\int_{-\infty}^{+\infty} \frac{\log\omega(t)}{1+t^2}\,dt < +\infty$) shares (ne). Assuming that ω is not a non-quasianalytic weight, we construct a C_0-group $\mathcal{T} := (T(t))_{t\in\mathbf{R}}$ of translations on some weighted Hardy space such that $\|T(t)\| \leqslant \omega(t)$ for all $t \in \mathbf{R}$, but $\sigma(A) = \emptyset$ for the generator A of $\mathcal{T}$. This shown that (ne) is equivalent to the non-quasianalyticity of the weight.

KEYWORDS: *C_0-group, spectrum, non-quasianalitic weight, H^p spaces.*

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1. INTRODUCTION

By a *weight* we mean a measurable function ω on $\mathcal{R}$ such that

$$1 \leqslant \omega(s+t) \leqslant \omega(s) + \omega(t) \quad \text{for all} \quad s, t \in \mathbf{R}.$$

Following A. Beurling a weight ω is said to be *non-quasianalytic* if $\int_{-\infty}^{+\infty} \frac{\log\omega(t)}{1+t^2}\,dt < +\infty$. There are many equivalent characterizations of non-quasianalyticity in Harmonic Analysis (see the Introduction in [10]).

Recently, some new applications of non-quasianalyticity to spectral theory of C_0-groups have been obtained (see [7] and [9] for example). Let $\mathcal{T} := (T(t))_{t\in\mathbf{R}}$

be a C_0-group with generator A on a Banach space E which is dominated by some non-quasianalytic weight ω, i.e.,

$$(*) \qquad \|T(t)\| \leqslant \omega(t) \quad \text{for all} \quad t \in \mathbf{R}.$$

It is shown in [9] that $(*)$ implies the following Weak Spectral Mapping Theorem, i.e.,

$$(WSMT) \qquad \sigma(T(t)) = \overline{\exp(t \cdot \sigma(A))} \quad \text{for all} \quad t \in \mathbf{R}.$$

Concerning this result we have asked whether the condition $(*)$ is optimal, i.e., we have the following question (see [9], Problem 4.1).

PROBLEM. *Let ω be a weight on $\mathbf{R}$ such that*

$$\omega(t)^{\frac{1}{t}} \to 1 \quad \text{as} \quad t \to \pm\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\log \omega(t)}{1+t^2}\, dt = +\infty.$$

Is there a C_0-group $T := (T(t))_{t\in\mathbf{R}}$ on some Banach space E satisfying $\|T(t)\| \leqslant \omega(t)$ for all $t \in \mathbf{R}$ for which the WSMT fails?

In this paper we will solve this problem and thus give an equivalent description of non-quasianalyticity through spectral theory of C_0-groups. The crucial difficulty in solving the problem is the lack of sufficiently many examples of C_0-groups whose generator has empty spectrum. The Hille-Phillips example ([6], Section 23.16) seems to be unique. But there the growth bound is greater than zero, while we will construct an example having growth bound zero.

2. THE CONSTRUCTION

Our construction is heavily based on some new properties of the following variant of the classical Poisson integral. A particular case is considered in ([10], Section 3).

LEMMA 1. *Let ρ be a measurable function on $\mathbf{R}$ such that $|\rho(t)| \leqslant K(1+|t|^\alpha)$ for some constants $K > 0, \alpha > 2$ and all $t \in \mathbf{R}$. For $z = x + iy, y > 0$, define*

$$(1) \qquad u(x+iy) := \frac{y}{\pi} \int_{-\infty}^{+\infty} \left(\frac{1}{(t-x)^2+y^2} - \frac{1}{t^2+1} \right) \rho(t)\, dt.$$

Then u is harmonic in the upper half-plane $\operatorname{Im} z > 0$ and

$$(2) \qquad \lim_{y\to 0} u(x+iy) = \rho(x) \quad \textit{almost everywhere.}$$

If, in additional, ρ is continuous, then

$$\lim_{z\to x_0} u(z) = \rho(x_0) \quad \textit{for all} \quad x_0 \in \mathbf{R}.$$

Proof. For fixed $R > 1$, let $\rho_R(t) := \rho(t)$ for $|t| \leqslant R; \rho_R(t) := \rho(-R)$ for $t \leqslant -R$ and $\rho_R(t) := \rho(R)$ for $t \geqslant R$. Define u_R by (1) using ρ_R instead of ρ. By ([5], Lemma 19.2.1) u_R is harmonic in $y > 0$ and $\lim_{y\to 0} u_R(x+\mathrm{i}y) = \rho_R(x)$ almost everywhere. Moreover, if ρ is continuous, then ρ_R is a bounded, continuous function and hence $\lim_{z\to x_0} u_R(z) = \rho_R(x_0)$ for all $x_0 \in \mathbf{R}$. For $x^2+y^2 \leqslant r^2$ we estimate (see [10], Section 3)

$$\begin{aligned}|u(z)-u_R(z)| &\leqslant Ky \int\limits_{|t|\geqslant R} \frac{1+r^2+2r|t|}{(t^2+1)[(t-x)^2+y^2]}|t|^\alpha\,\mathrm{d}t \\ &\leqslant Ky(1+r)^2 \cdot \int\limits_{|t|\geqslant R} \frac{|t|^{\alpha-1}}{(t-x)^2+y^2}\,\mathrm{d}t \to 0\end{aligned}$$

uniformly for $x^2+y^2 \leqslant r^2$. Lemma 1 follows from these facts. ∎

Now let $\rho(\cdot)$ be a measurable, non-negative, subadditive function on $\mathbf{R}$ such that for some constant $M > 0$

$$(3) \qquad \rho(t) \leqslant M(|t|+1) \quad \text{for all} \quad t \in \mathbf{R}.$$

Let u be the harmonic function defined by (1) and let v be any harmonic conjugate of u. Define

$$(4) \qquad G(z) := \exp(-u(z) - \mathrm{i}v(z)), \quad \operatorname{Im} z > 0.$$

Then G is analytic on $\operatorname{Im} z > 0$. Using Fatou's theorem and property (2) we obtain

$$(5) \qquad \lim_{y\to 0} |G(z)| = \exp(-\rho(x)) \quad \text{almost everywhere.}$$

For $1 \leqslant p < \infty$, let H^p be the classical Hardy space over the half-plane $\operatorname{Im} z > 0$ (we refer to [3], Chapter 11 for more information). We define H^p_ρ to be the weighted Hardy space consisting of all analytic functions f such that $f \cdot G \in H^p$ and take

$$(6) \qquad \|f\|_{p,\rho} := \left\{\sup_{y>0} \int_{-\infty}^{\infty} |f(x+\mathrm{i}y)G(x+\mathrm{i}y)|^p\,\mathrm{d}x\right\}^{\frac{1}{p}} < \infty.$$

We first claim that $f_0(z) := (z+\mathrm{i})^{-2}G(z)^{-1} \in H^p_\rho$. Let $f \in H^p_\rho$. It follows from ([3], p.p.189-190) that the boundary function $f(x) := \lim_{y\to 0} f(x+\mathrm{i}y)$ exists almost everywhere and thus by (5),

$$\|f\|_{p,\rho} := \left\{\int_{-\infty}^{\infty} |f(x)|^p e^{p\rho(x)}\,\mathrm{d}x\right\}^{\frac{1}{p}}. \tag{7}$$

For each $f \in H^p_\rho$ we define the translations of f by

$$T(\zeta)f(z) := f(z+\zeta) \quad \text{for} \quad \operatorname{Im} z > 0 \quad \text{and} \quad \operatorname{Im}\zeta \geqslant 0. \tag{8}$$

It is easily seen that all functions $T(\zeta)f(\operatorname{Im}\zeta \geqslant 0)$ are analytic on the half-plane $\operatorname{Im} z > 0$. In order to show that these functions are in H^p_ρ (i.e., each $T(\zeta)$ is well-defined on H^p_ρ) we need some more properties of u. In what follows M denotes the constant from (3).

LEMMA 2. *For $z = x + \mathrm{i}y$ with $x \in \mathbf{R}$ and $y > 0$,*

$$|u(x+\mathrm{i}y)| \leqslant 5M + M(|x|+y) + 2My\log(1+x^2+y^2). \tag{9}$$

Proof. We first assume $x \geqslant 0$ and estimate $|u(x+\mathrm{i}y) - u(\mathrm{i}y)|$. It follows from (3) that

$$\begin{aligned} M^{-1}|u(x+\mathrm{i}y) - u(\mathrm{i}y)| &\leqslant \frac{y}{\pi}\int_{-\infty}^{\infty}\left|\frac{1}{(t-x)^2+y^2} - \frac{1}{t^2+y^2}\right|\cdot(|t|+1)\,\mathrm{d}t \\ &\leqslant 2 + \frac{y}{\pi}[\log(x^2+y^2) - 2\log y] + \frac{x}{\pi}(\pi + 2\arctan\frac{x}{2y} - 2\arctan\frac{x}{y}) \\ &\leqslant 2 + x + \frac{y}{\pi}\log(x^2+y^2) - \frac{2}{\pi}y\log y \leqslant 3 + x + \frac{y}{\pi}\log(x^2+y^2), \end{aligned}$$

where we use the inequality $y\log y \geqslant -\mathrm{e}^{-1}$ for all $y > 0$. Thus

$$|u(x+\mathrm{i}y)| \leqslant 3M + Mx + My\log(x^2+y^2) + |u(\mathrm{i}y)|.$$

A similar estimate holds for $x < 0$. But

$$\begin{aligned} |u(\mathrm{i}y)| &\leqslant \frac{My}{\pi}\int_{-\infty}^{\infty}\left|\frac{1}{t^2+y^2} - \frac{1}{t^2+1}\right|\cdot(|t|+1)\,\mathrm{d}t \\ &= \frac{M}{\pi}\left|-2y\log y + \frac{\pi}{2} - \frac{\pi}{2}y\right| \leqslant M + My + My|\log y| \\ &\leqslant 2M + My + My\log(1+x^2+y^2), \end{aligned}$$

where we use again the inequality $|y\log y| \leqslant \mathrm{e}^{-1}$ for $0 < y \leqslant 1$. Therefore

$$|u(x+\mathrm{i}y)| \leqslant 5M + M(|x|+y) + 2My\log(1+x^2+y^2)$$

as desired. ∎

LEMMA 3. *The following estimate holds*

(10) $\quad u(x+\mathrm{i}y)-u(\mathrm{i}y)\leqslant \rho(x)+M\cdot\inf\{|x|,y\}+M \quad$ *for all* $\quad x\in\mathbf{R}, y>0.$

Proof. Fix $y>0$ and consider $x>0$. Then the function $\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)$ is positive for $t>x/2$ and negative for $t<x/2$. Thus, from the definition of u we have

$$
\begin{aligned}
u(x+\mathrm{i}y)-u(\mathrm{i}y) &= \frac{y}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&= \frac{y}{\pi}\int_{\frac{x}{2}}^{\infty}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&+\frac{y}{\pi}\int_{-\infty}^{\frac{x}{2}}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&\leqslant \frac{y}{\pi}\int_{-\frac{x}{2}}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)\rho(t+x)\,\mathrm{d}t\\
&+\frac{y}{\pi}\int_{-\frac{x}{2}}^{\frac{x}{2}}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&\leqslant \frac{y}{\pi}\left\{\int_{-\frac{x}{2}}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)(\rho(t)+\rho(x))\,\mathrm{d}t\right.\\
&\left.+\int_{-\frac{x}{2}}^{\frac{x}{2}}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{t^2+y^2}\right)\rho(t)\,\mathrm{d}t\right\}\\
&\leqslant \rho(x)+\frac{y}{\pi}\int_{\frac{x}{2}}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&+\frac{y}{\pi}\int_{-\frac{x}{2}}^{\frac{x}{2}}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)\rho(t)\,\mathrm{d}t\\
&\leqslant \rho(x)+\frac{y}{\pi}\int_{\frac{x}{2}}^{\infty}\left(\frac{1}{t^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)M(t+1)\,\mathrm{d}t\\
&+\frac{y}{\pi}\int_{0}^{\frac{x}{2}}\left(\frac{1}{(t-x)^2+y^2}-\frac{1}{(t+x)^2+y^2}\right)M(t+1)\,\mathrm{d}t,\\
&\leqslant \rho(x)+\frac{Mx}{\pi}\left(\frac{\pi}{2}-\arctan\frac{x}{2y}\right)+M\ (\leqslant\rho(x)+Mx+M)\\
&= \rho(x)+\frac{M}{\pi}\int_{\frac{1}{2}}^{\infty}\frac{x^2y}{t^2x^2+y^2}\,\mathrm{d}t+M\leqslant\rho(x)+My+M,
\end{aligned}
$$

where for the second inequality we use the subadditivity of ρ and for the fourth the condition (3). Analogously one obtains

$$u(x+\mathrm{i}y)-u(\mathrm{i}y)\leqslant\rho(x)+M\cdot\inf\{|x|,y\}+M\quad\text{for}\quad x<0.$$

This yields (10). ∎

LEMMA 4. *For all* $x_0 \in \mathbf{R}, \xi \in \mathbf{R}$ *and* $\eta \geqslant 0$ *there holds*

$$\limsup_{z \to x_0} (u(z+\xi) - u(z)) \leqslant 8M + \rho(\xi)$$

and

$$\limsup_{z \to x_0} (u(z+i\eta) - u(z)) \leqslant 8M(1+\eta) + u(i\eta).$$

Proof. By [1], p.306 we can choose a continuous function ρ_1 such that

$$|\rho(t) - \rho_1(t)| \leqslant 2M \quad \text{for all} \quad t \in \mathbf{R}. \tag{11}$$

For ρ_1 let u_1 be the harmonic function on $\operatorname{Im} z > 0$ given by (1). Since ρ_1 is continuous, by Lemma 1 we have

$$\lim_{z \to x_0} u_1(z) = \rho_1(x_0) \quad \text{for all} \quad x_0 \in \mathbf{R}.$$

Moreover, by (11) we obtain that $|u(z) - u_1(z)| \leqslant 2M(1+y)$. Combining these estimates with Lemma 3 one can easily derive Lemma 4. ∎

Now the main lemma goes as follows.

LEMMA 5. *For* $\operatorname{Im}\zeta \geqslant 0$ *there holds*

$$L_\zeta := \sup\{u(z+\zeta) - u(z) : \operatorname{Im} z > 0\} < \infty. \tag{12}$$

Moreover $L_{i\eta} \leqslant u(i\eta) + 8M\eta + 8M$ *for all* $\eta \geqslant 0$.

Proof. We only need to show $L_\xi < +\infty (\xi \in \mathbf{R})$ and $L_{i\eta} < +\infty$ $(\eta \geqslant 0)$. Let $\eta \geqslant 0$ and consider

$$H_\eta(z) := G(z+i\eta)^{-1} G(z) \quad \text{for all} \quad \operatorname{Im} z > 0.$$

Then $H_\eta(\cdot)$ is an analytic function and by Lemma 2 we obtain

$$\limsup_{|z| \to +\infty} \frac{\log^+ |H_\eta(z)|}{|z|^{\frac{3}{2}}} = 0.$$

For $x_0 \in \mathbf{R}$ it follows from Lemma 4 that

$$\limsup_{z \to x_0} |H_\eta(z)| \leqslant \exp(u(i\eta) + 8M\eta + 8M). \tag{13}$$

Let $y > 0$. By the definition of u one can easily see that $u(iy + i\eta) - u(iy) \leqslant 0$ whenever $y + \eta \geqslant 1$. So

$$u(iy + i\eta) - u(iy) \leqslant 2 \sup\{|u(it)| : 0 < t \leqslant 1\}.$$

The later is finite by Lemma 2. Therefore

$$\sup_{y>0} |H_\eta(\mathrm{i}y)| = \exp[\sup_{y>0}(u(\mathrm{i}y+\mathrm{i}\eta) - u(\mathrm{i}y))] < +\infty.$$

By applying the famous Phragmen-Lindelöf theorem (see [2], Corollary 4.2) to the functions of H_η restricted to sectors $\{z : 0 < \arg z < \pi/2\}$ and $\{z : \pi/2 < \arg z < \pi\}$ respectively, we conclude first from the above estimates that H_η is bounded. Using the above Phragmen-Lindelöf theorem for the bounded function H_η again, by (13) we find that $L_{\mathrm{i}\eta} \leqslant u(\mathrm{i}\eta) + 8M\eta + 8M$. The proof of $L_\xi < +\infty$ goes similarly by using the estimate in Lemma 3. ∎

We show that each $T(\zeta)$ is well-defined. In fact, for $f \in H^p_\rho$ we have

$$|T(\zeta)f(z)G(z)| = |f(z+\zeta)G(z+\zeta)| \cdot \exp(u(z+\zeta) - u(z)).$$

By Lemma 5 we see that $T(\zeta)f \in H^p_\rho$ and thus $\{T(\zeta) : \operatorname{Im}\zeta \geqslant 0\}$ is a semigroup of bounded operators on H^p_ρ with norms

$$\|T(\zeta)\| \leqslant \exp(L_\zeta) \quad \text{for all} \quad \zeta \geqslant 0.$$

In particular, by Lemma 5 again,

$$\|T(\mathrm{i}\eta)\| \leqslant \exp(u(\mathrm{i}\eta) + 8M\eta + 8M) \quad \text{for all} \quad \eta \geqslant 0. \tag{14}$$

Moreover, it follows from (7) that

$$\|T(\xi)\| \leqslant \mathrm{e}^{\rho(\xi)} \quad \text{for all} \quad \xi \in \mathbf{R}.$$

To show the strong continuity we need a lemma whose proof is similar to the one of Lemma 1 in ([3], p.p. 21-22). Here we omit the details.

LEMMA 6. *Let ρ be given as above and $1 \leqslant p < \infty$. Suppose φ, φ_n $(n = 1, 2, \ldots)$ to be measurable functions on $\mathbf{R}$ such that $\varphi_n(x) \to \varphi(x)$ a.e. and*

$$\int_{-\infty}^{\infty} |\varphi_n(x)|^p \mathrm{e}^{-\rho(x)}\,\mathrm{d}x \to \int_{-\infty}^{\infty} |\varphi(x)|^p \mathrm{e}^{-\rho(x)}\,\mathrm{d}x.$$

Then

$$\int_{-\infty}^{\infty} |\varphi_n(x) - \varphi(x)|^p \mathrm{e}^{-\rho(x)}\,\mathrm{d}x \to 0.$$

Now, by using (7) and Lemma 6 one can easily verify that $(T(\xi))_{\xi\in\mathbf{R}}$ is a C_0-group on H^p_ρ with generator $A = \frac{\partial}{\partial\xi}$. Moreover, $(T(\mathrm{i}\eta))_{\eta\geqslant 0}$ is a C_0-semigroup on H^p_ρ with generator $B = \frac{\partial}{\partial\eta} = \mathrm{i}A$. This implies further that $\{T(\zeta) : \mathrm{Im}\,\zeta > 0\}$ is a holomorphic semigroup on H^p_ρ.

Until now we made no restriction on the integral $\int_{-\infty}^{\infty} \frac{\rho(t)}{1+t^2}\,\mathrm{d}t$. To reach our goal we assume that the weight $\omega(t) := \mathrm{e}^{\rho(t)} (t \in \mathbf{R})$ does not satisfy the ***non-quasianalytic*** growth condition, i.e.,

$$\int_{-\infty}^{\infty} \frac{\rho(t)}{1+t^2}\,\mathrm{d}t = +\infty. \tag{16}$$

Then

$$\frac{u(\mathrm{i}y)}{y} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t^2+y^2} - \frac{1}{t^2+1}\right)\rho(t)\,\mathrm{d}t \to -\infty$$

as $y \to +\infty$. Thus by (14) we obtain

$$\lim_{\eta\to+\infty} \frac{\log\|T(\mathrm{i}\eta)\|}{\eta} = -\infty.$$

This immediately implies that $\sigma(A) = \emptyset$. So we have the following main result.

THEOREM 1. *Let ρ satisfy* (3) *and* (16). *For each $1 \leqslant p < \infty$, the semigroup $T := \{T(\zeta) : \mathrm{Im}\,\zeta \geqslant 0\}$ on H^p_ρ defined as above possesses the following properties:*

(i) *T is holomorphic on $\mathrm{Im}\,z > 0$ and strongly continuous on $\mathrm{Im}\,z > 0$.*

(ii) *$\|T(\xi)\| \leqslant \mathrm{e}^{\rho(\xi)}$ for $\xi \in \mathbf{R}$ and $\|T(\mathrm{i}\eta)\| \leqslant \exp(u(\mathrm{i}\eta) + 8M\eta + 8M)$ for $\eta > 0$.*

(iii) *For the generator A of the C_0-group $(T(x))_{x\in\mathbf{R}}$ one has $\sigma(A) = \emptyset$.*

(iv) *If $\int_0^\infty \frac{\rho(t)}{1+t^2}\,\mathrm{d}t = +\infty$, then the following alternative holds: for each pair $(f,\mu) \in H^p_\rho \times (H^p_\rho)^*$*

either $$\limsup_{\xi\to+\infty} \frac{\log^+\|T(\xi)f\|}{\rho(\xi)} = 1$$

or $$\langle T(\zeta)f, \mu\rangle = 0 \quad \textit{for all} \quad \mathrm{Im}\,\zeta \geqslant 0.$$

It remains to check (iv). To this end, we need an auxiliary lemma which is implicitly used in [4].

PROPOSITION 1. *Suppose that f is analytic on $\mathrm{Im}\,z > 0$ and there are constants $M_1 > 0$ and $1 \leqslant \alpha < 2$ such that*

$$\limsup_{z\to x_0} |f(z)| \leqslant M_1 \quad \textit{for all} \quad x_0 \in \mathbf{R} \quad \textit{and}$$

$$\limsup_{|z|\to\infty} |f(z)|\exp(-|z|^\alpha) < +\infty.$$

Then $\lim_{y\to+\infty} \frac{\log|f(\mathrm{i}y)|}{y} = -\infty$ implies $f \equiv 0$.

Proof. For $n \in \mathbf{N}$ we consider

$$f_n(z) := \mathrm{e}^{-\mathrm{i}nz} f(z), \quad \operatorname{Im} z > 0.$$

Then

$$\limsup_{z \to x_0} |f_n(z)| = \limsup_{z \to x_0} |f(z)| \leqslant M_1$$

for all $x_0 \in \mathbf{R}$ and

$$\limsup_{|z| \to \infty} |f_n(z)| \exp(-|z|^\alpha) < +\infty.$$

Moreover,

$$\lim_{y \to +\infty} \frac{\log |f_n(\mathrm{i}y)|}{y} = -\infty.$$

Thus f_n is bounded on $\mathrm{i}\mathbf{R}_+$. Applying the Phragmen-Lindelöf theorem to the functions of f_n restricted to sectors $\{z : 0 < \arg z < \pi/2\}$ and $\{z : \pi/2 < \arg z < \pi\}$ respectively, we find that f_n is bounded on $\operatorname{Im} z > 0$. Applying the Phragmen-Lindelöf theorem to f_n again, we obtain $|f_n(z)| \leqslant M_1$ for all $\operatorname{Im} z > 0$ and all $n \in \mathbf{N}$. This implies $f \equiv 0$. ∎

Proof of (iv). Let $\langle f, \mu \rangle \in H^p_\rho \times (H^p_\rho)^*$ assume that

$$\limsup_{x \to +\infty} \frac{\log^+ \|T(x)f\|}{\rho(x)} < 1.$$

Then there exist constants $L > 1$ and $0 < \beta < 1$ such that

$$\|T(x)f\| \leqslant L\mathrm{e}^{\beta\rho(x)} \quad \text{for all} \quad x \geqslant 0. \tag{17}$$

For the continuous function $\log^+ |\langle T(t)f, \mu\rangle|$ ($t \in \mathbf{R}$) let U be the corresponding harmonic function given by (1). Then

$$\lim_{z \to x_0} U(z) = \log^+ |\langle T(x_0)f, \mu\rangle| \quad \text{for all} \quad x_0 \in \mathbf{R}.$$

Let V be any harmonic conjugate of U and consider

$$F(z) := \exp(-U(z) - \mathrm{i}V(z))\langle T(z)f, \mu\rangle, \quad \operatorname{Im} z > 0.$$

Then F is analytic on $\operatorname{Im} z > 0$ and for $x_0 \in \mathbf{R}$,

$$\limsup_{z \to x_0} |F(z)| = \exp(-\log^+ |\langle T(x_0)f, \mu\rangle|) \cdot |\langle T(x_0)f, \mu\rangle| \leqslant 1.$$

Moreover, for $y > 1$, by (ii) in Theorem 1, we find from the definition of U that

$$|F(\mathrm{i}y)| \leqslant \exp(-U(\mathrm{i}y) + u(\mathrm{i}y) + 8My + 8M) \cdot \|f\| \cdot \|\mu\|.$$

Using (17), one can easily verify that the assumption $\int_0^\infty \frac{\rho(t)}{1+t^2}\,\mathrm{d}t = +\infty$ implies that $\lim_{y \to +\infty} \frac{1}{y} \log |F(\mathrm{i}y)| = \infty$. Note that $\limsup_{|z| \to \infty} |F(z)| \exp(-|z|) < \infty$. Proposition 1 is applicable to F and thus $F \equiv 0$, i.e., $\langle T(\zeta)f, \mu\rangle = 0$ for all $\operatorname{Im} \zeta \geqslant 0$. ∎

REMARK 1. For the semigroup $\{J^\zeta : \mathrm{Re}\,\zeta \geqslant 0\}$ of fractional calculus given in ([6], p.663-670) we know that $\|J^{i\eta}\| = e^{\frac{\pi}{2}|\eta|}$ for $\eta \in \mathbf{R}$ and $\|J^\xi\| \leqslant \frac{1}{\xi\Gamma(\xi)}$ for $\xi > 0$. With a similar proof as the one of (iv) in Theorem 1 we find that the C_0-group $(J^{i\eta})_{\eta\in\mathbf{R}}$ also shares a "0-1 law", i.e.,

$$\limsup_{\eta\to\infty} \frac{\log^+ \|J^{i\eta} f\|}{\frac{\pi}{2}|\eta|} = 1 \quad \text{for all} \quad 0 \neq f \in L^2(0,1).$$

To finish this paper we solve the Problem mentioned in the Introduction.

Consider a weight ω on $\mathbf{R}$ with the following property.

(ne) For any C_0-group $(T(t))_{t\in\mathbf{R}}$ on a Banach space E satisfying $\|T(t)\| \leqslant \omega(t)$, $t \in \mathbf{R}$, there holds $\sigma(A) \neq \emptyset$ for the generator A of T.

As seen in the Introduction a non-quasianalytic weight satisfies (ne). But from the Theorem we see that the non-quasianalyticity of the weight is also necessary for (ne). So the following corollary is true.

COROLLARY 1. *A weight ω satisfies (ne) if and only if it is non-quasianalytic.*

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