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BOOK REVIEW:

Pseudo-Differential Calculus and Mathematical Physics, **Michael Demuth, Elmar Schrohe, Bert-Wolfgang Schulze** (Editors), Akademie Verlag, Berlin 1994, 391 pag., ISBN 3-05-501625-4.

KEYWORDS: *Asymptotics, pseudo-differential boundary value problem, uniform transmission property, submultiplicativity, conical singularities.*

AMS SUBJECT CLASSIFICATION: 35S15

This present volume is the first of a series "Advances in Partial Differential Equations"; its origins lies in the work of the Max Planck Research Group for Partial Differential Equations and Complex Analysis at the Potsdam University. It contains 7 papers, dealing mainly with pseudo-differential calculus for boundary value problems on manifolds with conical singularities, and two topics from mathematical physics.

In the sequel, each paper will be reviewed individually.

1) "The Variable Discrete Asymptotics in Pseudo-Differential Boundary Value Problems I" by B.-W. Schulze, deals with the variable branching asymptotics of solutions to general elliptic pseudo-differential boundary value problems. The BVP's on a C^∞ manifolds with C^∞ boundary are regarded here as particular edge problems, preparing the way to the general case in a forthcoming paper.

The presence of singularities at the boundary requires a pseudo-differential calculus which does not specialize to the transmission property in BVP for the model cone $\mathbf{R}_+$, because the solutions near proper edges have usually non-trivial asymptotics. It is based on previous results of the author which showed the form of the asymptotics (depending of a point y). But as the distribution and variation of multiplicities of poles may be chaotic when varying y, the author formulate this C^∞-dependence of y in terms of analytic functional on the complex plane. After defining precisely the discrete asymptotics on $\mathbf{R}_+ \ni t$, for $t \to 0$ dependent on a parameter $y \in \Omega$ (Ω an open set in $\mathbf{R}^q$) in the Section 2 the author studies what he calls the boundary symbols, which are operator-valued symbols having the form of block-matrices, whose left upper corners consists of interior ΨDo + Green + Mellin operator families. Green is taken from Boutet de Monvel calculus of ΨDo with transmission property, the operators considered here being the analogue of the singular Green operators of Boutet de Monvel. Then the author studies the Mellin smoothing boundary symbolds (with variable discrete asymptotics). Finally, the next paragraph studies the boundary symbols, which by no means are obvious extensions of Boutet de Monvel's. The last part of the article contains a study of variable discrete asymptotics in weighted "wedge" Sobolev spaces. The paper is self-contained but the results are too technical and complicated to be reproduced here.

2) The 2-d paper "Boundary Value Problems in Boutet de Monvel's Algebra for Manifold with Conical Singularities I" by E. Schrohe and B.-W. Schulze, begins the development of a pseudo-differential calculus for BVP on manifolds with finitely many conical singularities. Outside the singular set, Boutet de Monvel's calcul is used, but near a singularity, by identifying the manifold with $X\times[0,\infty)/X\times\{0\}$ where X is a smooth compact manifold with boundary, the authors introduce operators of Mellin type on $\mathbf{R}_+$ with values in Boutet de Monvel's algebra. For this, the paper develop a parameter-dependent version of Boutet de Monvel calculus and a class of weighted Sobolev spaces with discrete asymptotics based on the Mellin transform.

Finally, the authors introduce Green operators, the residual operators with respect to the calculus (for $\dim X = 0$ and Taylor asymptotics near $t = 0$ they coincide with Boutet de Monvel's singular Green operators of type zero). The 4-th Section of the paper, after a study of t-independent Mellin symbols with values in Boutet de Monvel's algebra, analyses the algebra of operators of the form $A = \sum t^j A_j + G$ with smoothing A_j and G a Green operator. This algebra will turn out to be an ideal in the final operator algebra. The idea of the paper is to focus on ellipticity and construction of parametrices in terms of the full pseudo-differential algebra, with a as small as possible class of residual elements. The paper ends with an appendix which collects some technical results.

3) The next paper by E. Schrohe "A Characterization of the Uniform Transmission Property for Pseudo-Differential Operators", extends a previous work of Grubb-Hörmander, Math. Scand. 67(1990), 273–289. The author shows that the ΨDo with the uniform two-sided transmission property can be characterized by the behaviour of their iterated commutators with multipliers and vector fields tangential to the boundary on ordinary and wedge Sobolev space.

4) "Submultiplicativity of Boutet de Monvel's Algebra for Boundary Value Problems" by B. Gramsch and E. Schrohe, shows that the algebra of operators of order and type zero in Boutet de Monvel's calculus is a submultiplicative Frèchet subalgebra of $\mathcal{L}(L^2(X))$ where X either a compact manifold with boundary either a half-space $\mathbf{R}^n_+$.

The idea of proof relies on a classical result of Beals, and the result of the previous reviewed paper by Schrohe. Similarly the singular Green operators of order -1 and type zero have a characterization by means of the behaviour of their iterated commutators with tangential vector fields and multipliers on wedge Sobolev spaces.

The next paper

5) "A Singular Elliptic Estimate and Application", by M. Lesch obtains an a priori estimate called by the author the singular elliptic estimate for an elliptic operator defined on a compact manifold with an isolated conic singularity; this estimate, if it holds, ensures that the given elliptic operators is Fredholm the operators e^{-tp^*p} and e^{-tpp^*} are of trace class and gives the asymptotics for $\mathrm{Tr}\,(\varphi e^{-tp^*p})$ where φ is a cut-off function near the singularity. The last section gives different examples for which the singular elliptic estimate holds (in particular these include the cases of conical singularities and the Atiyah–Patodi–Singer boundary conditions.

6) The paper "Reduction and Eigenstates in Deformation Quantization" by B. Fedosov attacks another topic. More specifically, let (M, ω_M) be a symplectic manifold, $W_D(M)$ the algebra of flat sections in the Weyl algebra boundle $W(D)$ with respect to an abelian connection D; this is an intermediate step between the Poisson-Lie algebra $C^\infty(M)$ (describing a classical mechanical system) and an operator algebra $\mathcal{L}(H)$ on a Hilbert space H (describing a quantum system). This three-level approach – i.e. classical – quantum deformation (here $W_D(M)$) - -operator has some advantages. The second transition obstruction are of topological nature and, as the author has shown in a previous work, they appear as indices of elliptic elements in W_D. Now the author applies this approach to the problem of the Hamiltonian reduction. For this, the first studies a special construction of deformation quantization for fibering spaces, which is interesting for its own sake. The main result of the author is that if $P = P(x, y, h) = p(x)$ (which belongs to $W_P(M)$) is a Hamiltonian whose leading term is such that $M_0 = \{p = 0\}$ is connected and compact and the orbit space is smooth, then in the Weyl algebra boundle $W(D)$ with the above orbits manifold, there exist an abelian connection $D = D_B$ (such that the reduced algebra is isomorphic to $W_D(B)$ (and the Weyl curvature is also described). The last part of this work is essentially a discussion

which leads to the conclusion that using deformation quantum algebra and the index theorem in these algebras one can get most of the topological and spectral characteristics obtained by the operator approach.

7) The last paper of this volume, "On the Spectral Theory of the Schrödinger operator with Electromagnetic Potential" by A. Mohamed and G. Raikov is an almost 100 pages long and interesting survey on the topics anounced in the title, containing also many contributions due to the authors.

Let $H = (ih \nabla + \mu A)^2 + gV$ the Schrödinger operator with electromagnetic potential (here $V : \mathbf{R}^m \to \mathbf{R}$ is the electric potential and $A : \mathbf{R}^m \to R^m$ is the electromagnetic potential. The authors outline especially the spectral properties of H due to the presence of this electromagnetic potential; in fact the classical-mechanics approximations concerning the spectrum of H are completely different in the case $\mu = 0$ and $\mu \neq 0$. The paper is enough self-contained to be a pleasant reading. The topics discussed are centered around two conjectures (concerned with the essential spectrum of H and the asymptotics of the total multiplicities of the eigenvalues of H), and can be in summarized as follows: 1) localization of the essential spectrum of H, and particularly conditions for the compactness of the resolvent of H; 2) Investigation of the discrete spectrum behaviour of H near the possible "accumulation points"; 3) Study of the spectrum of H in the semiclassical limit (on both the strong and weak electric field limits, and in the magnetic field limits).

In conclusion, this volume is an interesting and stimulating one.

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