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# WEAK SEQUENTIAL CONVERGENCE IN THE DUAL OF AN ALGEBRA OF COMPACT OPERATORS

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ABSTRACT. It is shown that under certain mild conditions the dual of a commutative algebra of compact operators has the Schur property. This result is then applied to the algebra $H^\infty/\varphi H^\infty$ where $H^\infty$ is the algebra of bounded analytic functions on the disc, and $\varphi$ is an inner function.

KEYWORDS: *Schur property, compact operator, Hilbert space operators.*
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## 0. INTRODUCTION

A Banach space $X$ is said to have the Schur property if every sequence that converges weakly (i.e. in the $\sigma(X, X^*)$ topology) also converges in norm. The space $\ell^1$ has this property, and this will be shown as an easy corollary of the main result. In general, if $\mathcal{F}$ is a commutative algebra of compact operators satisfying a certain very mild condition, then it will be shown in Section 1 that $\mathcal{F}^*$ has the Schur property. Subsequent to the writing of this paper, Jonathan Arazy has shown this author a quicker proof of the theorem of Section one using known results from [3], [4], and some of the basic lemmas presented below. While the results of [3] and [4] are more general, they do require greater machinery to develop as well as some augmentation to handle this specific case, and so the original "from scratch" version is presented here.

In Section 2 a not so obvious application of the theorem will be given. Let $m$ denote the Lebesgue (arc length) measure on the unit circle. Let $H_0^1$ denote the closure in $L^1(m)$ of the polynomials with zero constant coefficient. Let $H^\infty$ denote the space of bounded holomorphic functions on the open unit disc. Whenever convenient this space will be viewed as a subspace of $L^\infty(m)$ via radial limits. The space $H^\infty$ is the dual of $L^1(m)/H_0^1$ (see [2]). Let $\varphi$ be any inner function (i.e. $\varphi \in H^\infty$ and $|\varphi| = 1$ a.e. $[m]$), and consider $H_0^1/\varphi H_0^1$. This space is the predual of $H^\infty/\varphi H^\infty$. It is already known that the predual of $H^\infty$ is weakly sequentially complete (see [1], [2], [5], [7], [9]), so the space $H_0^1/\varphi H_0^1$ is already known to be weakly sequentially complete. The result of Section 1 will be applied to $H_0^1/\varphi H_0^1$ to show an even stronger result for this space – it has the Schur property.

## 1. THE MAIN THEOREM

Let $\mathcal{F}$ be a normed closed commutative algebra of compact operators on a Hilbert space $\mathcal{H}$. It will be assumed from now on that $\mathcal{F}$ satisfies the following:

*(R) Restriction on $\mathcal{F}$*: The set $\mathcal{F}(\mathcal{H}) \equiv \{Fx : x \in \mathcal{H} \text{ and } F \in \mathcal{F}\}$ is dense in $\mathcal{H}$, and $\mathcal{F}^*(\mathcal{H}) \equiv \{F^*x : x \in \mathcal{H} \text{ and } F \in \mathcal{F}\}$ is dense in $\mathcal{H}$, where $*$ denotes the adjoint of the operator or operators involved.

The main theorem is easily stated.

THEOREM 1.1. *Let $\mathcal{C}$ be the dual of $\mathcal{F}$. If a sequence converges weakly in $\mathcal{C}$, then it converges in norm.*

The proof will be given at the end of this section. First some vague motivating remarks will be made and some lemmas will be proved. One particular proof of the Schur property for $\ell^1$ depends on the fact that if two elements $\xi, \eta \in \ell^\infty$ have disjoint supports, then $\|\xi + \eta\|_\infty = \max(\|\xi\|_\infty, \|\eta\|_\infty)$. For operators on a Hilbert space a similar statement holds. If $P_V, P_W$ are self adjoint projections onto subspaces $V, W$ (respectively) of $\mathcal{H}$, then for bounded operators $A, B$ on $\mathcal{H}$

$$(1) \qquad \|P_V A P_W + P_{V^\perp} B P_{W^\perp}\| = \max(\|P_V A P_W\|, \|P_{V^\perp} B P_{W^\perp}\|).$$

Here $\perp$ denotes the orthogonal complement of a space. In general there won't be any projections in $\mathcal{F}$, but certain pairs of elements in $\mathcal{F}$ will be aproximated by elements of the form $P_V A P_W$ and $P_{V^\perp} B P_{W^\perp}$. This will allow the formation of an infinite sum of certain elements in $\mathcal{F}$ in order to find a new element $A \in \mathcal{F}^{**}$. Given a sequence in $\mathcal{C}$ which tends weakly, but not in norm, to zero, this element $A$ can be arranged so that its evaluation on the sequence in $\mathcal{C}$ will not tend to zero (a contradiction). The reader familiar with the proof for $\ell^1$ may recognize this approach.

The space of all bounded operators on the Hilbert space $\mathcal{H}$ will be denoted $\mathcal{B}(\mathcal{H})$. The space of all compact operators on $\mathcal{H}$ will be denoted by $\mathcal{K}$. The space of all trace class operators on $\mathcal{H}$ will be denoted $\mathcal{C}_1$. In natural fashion $\mathcal{C}_1 \approx \mathcal{K}^*$ the dual of $\mathcal{K}$, and $\mathcal{B}(\mathcal{H}) \approx \mathcal{C}_1^*$ (see [8]). The topology $\mathcal{B}(\mathcal{H})$ inherits as the dual of $\mathcal{C}_1$ will be referred to as the weak* or ultraweak topology on $\mathcal{B}(\mathcal{H})$.

In what follows, the closed unit ball of any Banach space $X$ will be denoted by $B_X$. For any $x \in X$ and $x^* \in X^*$, the symbol $[x^*](x)$ will be used to denote the evaluation of the linear functional $x^*$ on $x$. The notation $[x](x^*)$ will represent $x$ (viewed as a naturally embedded element in $X^{**}$) evaluated at $x^*$. The square brackets will usually enclose an element of an algebra, and it will be helpful to realize this at times.

For $x \in \mathcal{H}$, define $B_{\mathcal{N}}x \equiv \{Nx : N \in \mathcal{N} \text{ and } \|N\| \leqslant 1\}$.

LEMMA 1.2. *Let $\mathcal{N}$ be an ultraweakly closed subspace of $\mathcal{B}(\mathcal{H})$. Then $\{x \in \mathcal{H} : B_{\mathcal{N}}x$ is norm compact$\}$ is norm closed.*

*Proof.* For each $x \in \mathcal{H}$, define $\varphi_x : \mathcal{N} \to \mathcal{H}$ by $\varphi_x(A) = Ax$ for all $A \in \mathcal{N}$. Note that if $B_{\mathcal{N}}x$ is compact then $\varphi_x$ is a compact operator from $\mathcal{N}$ into $\mathcal{H}$. Assume $x_i \to x$ in norm in $\mathcal{H}$, and $B_{\mathcal{N}}x_i$ is compact for $i = 1, 2, \ldots$. Then $\varphi_{x_i} \to \varphi_x$ in norm, and each $\varphi_{x_i}$ is a compact operator; so, $\varphi_x$ is a compact operator. Note that $\varphi_y : (\mathcal{N}, \text{ultraweak}) \to (\mathcal{H}, \text{weak})$ is continuous (for all $y \in \mathcal{H}$), so $\varphi_y(B_{\mathcal{N}})$ is weakly compact and therefore norm closed for all $y \in \mathcal{H}$. Thus, $\varphi_x(B_{\mathcal{N}}) = B_{\mathcal{N}}x$ is norm closed and relatively compact, and therefore norm compact. ∎

The following proposition will not be used in the rest of the paper but is of independent interest. The proof follows directly from Lemma 1.2.

PROPOSITION 1.3. *If $\mathcal{A}$ is an ultraweakly closed commutative subalgebra of $\mathcal{B}(\mathcal{H})$, then $\{x \in \mathcal{H} : B_{\mathcal{A}}x$ is norm compact$\}$ is a closed invariant subspace of $\mathcal{A}$.*

Lemma 1.2 will now be applied to $\mathcal{F}$. Let $\mathcal{A}$ be the ultraweak closure of $\mathcal{F}$ in $\mathcal{B}(\mathcal{H})$. Note for later reference that $\mathcal{A} \approx \mathcal{F}^{**}$.

LEMMA 1.4. *For each $x \in \mathcal{H}$, the set $B_{\mathcal{F}}x$ is relatively norm compact in $\mathcal{H}$.*

*Proof.* Note that $B_{\mathcal{A}}x \supset B_{\mathcal{F}}x$ for each $x \in \mathcal{H}$. It will be shown that $B_{\mathcal{A}}x$ is norm compact for each $x \in \mathcal{H}$. By 1.2, it is enough to show that $B_{\mathcal{A}}x$ is norm compact for all $x$ in a dense subset of $\mathcal{H}$. Let $\mathcal{S} = \{x \in \mathcal{H} : Ky = x$ for some $K \in \mathcal{F}$ and $y \in \mathcal{H}\}$. By the restriction (R) placed on $\mathcal{F}$ at the beginning of this section, $\mathcal{S}$ is norm dense in $\mathcal{H}$. Let $x = Ky \in \mathcal{S}$. Then $B_{\mathcal{A}}x = B_{\mathcal{A}}Ky = KB_{\mathcal{A}}y$ which is norm compact since $B_{\mathcal{A}}y$ is weakly compact as shown in the proof of Lemma 1.2, and $K$ is a compact operator.

REMARK 1. Lemma 1.4 holds for $\mathcal{F}^*$ by the second part of the restriction (R) place on $\mathcal{F}$.

REMARK 2. The property of $B_{\mathcal{A}}x$ being compact is not unique to algebras containing compact operators. Let $\nu$ be a positive Borel measure whose support is the closed unit disc in the complex plane, with $\nu$(unit circle) $= 0$. The operator of multiplication by $z$ on $L^2(\nu)$ (i.e. $f(z) \to zf(z)$) generates the algebra $H^\infty$. Given any $f \in L^2(\nu)$, $B_{H^\infty}f$ is norm compact in $L^2(\nu)$.

Again, we will let $P_V$ and $P_W$ denote the projections of $\mathcal{H}$ onto closed subspaces $V$ and $W$ respectively.

LEMMA 1.5. *Given a finite dimensional subspace $V$ of $\mathcal{H}$, and given $\varepsilon > 0$, there exists a norm closed subspace $\mathcal{G}$ of $\mathcal{F}$ of finite codimension such that $\|GP_V\| < \varepsilon$ for all $G \in \mathcal{G}$ with $\|G\| \leqslant 1$.*

*Proof.* Select an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ for $V$. As before, for each $x_i$, $i = 1, 2, \ldots, n$ define $\varphi_i : \mathcal{F} \to \mathcal{H}$ by $\varphi_i(F) = Fx_i$ for all $F \in \mathcal{F}$. Then each $\varphi_i$ is a compact map by Lemma 1.4. Hence there is a norm closed subspace $\mathcal{H}_i$ of $\mathcal{H}$ of finite codimension such that $\sup\{\|x\| : x \in \mathcal{H}_i \bigcap \varphi_i(B_{\mathcal{F}})\} \leqslant \varepsilon/n$. Note that $\varphi_i^{-1}(\mathcal{H}_i)$ is norm closed and of finite codimension in $\mathcal{F}$ for $i = 1, 2, \ldots, n$. So $\mathcal{G} \equiv \bigcap_{i=1}^{n} (\varphi_i^{-1}(\mathcal{H}_i))$ is norm closed and of finite codimension in $\mathcal{F}$. Let $G \in \mathcal{G}$ with $\|G\| = 1$. Then $\|Gx_i\| < \varepsilon/n$ for $i = 1, 2, \ldots, n$. So for any $\sum_{i=1}^{n} c_i x_i \in V$ of norm one it holds that

$$\left\| G \sum_{i=1}^{n} c_i x_i \right\| \leqslant \sum_{i=1}^{n} |c_i| \frac{\varepsilon}{n} < \varepsilon \quad \text{since } \sum_{i=1}^{n} |c_i|^2 = 1.$$

Hence $\|GP_V\| < \varepsilon$. ∎

The lemma can be applied to $\mathcal{F}^*$ by the remark following Lemma 1.4. Given a subspace $W$ of $\mathcal{H}$ of finite codimension this will yield a subspace $\tilde{\mathcal{G}}$ of $\mathcal{F}$ of finite codimension such that $\|G^* P_W^*\| = \|P_W G\| < \varepsilon$ for all $G \in \tilde{\mathcal{G}}$ of norm one. Taking the intersection of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ yields (after renaming this intersection $\mathcal{G}$):

LEMMA 1.6. *Let $V$ and $W$ be finite dimensional subspaces of $\mathcal{H}$. Given $\varepsilon > 0$, there exists a norm closed subspace $\mathcal{G}$ of $\mathcal{F}$ of finite codimension such that for all $G \in \mathcal{G}$ of norm one, $\|GP_V\| = \varepsilon$ and $\|P_W G\| = \varepsilon$.*

One final lemma is needed. In this lemma, the number 1/3 can be replaced by any number $< 1/2$.

LEMMA 1.7. *Let $X$ be a Banach space. Let $S$ be any finite dimensional subspace of $X$. Suppose $\{b_i\}_{i=1}^{\infty}$ is a sequence in $X$ and $b_i \to 0$ weakly in $X$ and $\|b_i\| = 1$ for all $i = 1, 2, \ldots$. Then there exists $N$ such that for all $i > N$,* $\operatorname{dist}(b_i, S) > 1/3$.

*Proof.* Suppose the result is not true. By dropping to a subsequence, it can be assumed that there exists a sequence $\{\tilde{b}_i\}_{i=1}^{\infty} \subset S$ such that $\|b_i - \tilde{b}_i\| \leqslant 1/3$ for all $i = 1, 2, \ldots$. Now let $\tilde{b}$ be an adherent point of $\{\tilde{b}_i\}_{i=1}^{\infty}$. Note that $\|\tilde{b}\| \geqslant 2/3$. Now the weakly closed set $\tilde{b} + 1/2B_X$ contains an infinite number of the $b_i$, and $b_i \to 0$ weakly, so $0 \in \tilde{b}+1/2B_X$. This implies $\|\tilde{b}\| \leqslant 1/2$. This is a contradiction. ∎

*Proof of Theorem 1.1.* Suppose $\{b_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{C}$ with $\|b_i\| \not\to 0$ and $b_i \to 0$ in the weak (i.e. $\sigma(\mathcal{C}, \mathcal{C}^*)$) topology. A contradiction will be found. It will be assumed that $\|b_i\| = 1$ for all $i = 1, 2, \ldots$, by (first) restriction to a subsequence of $\{b_i\}_{i=1}^{\infty}$ bounded below and then (second) appropriate scalar multiplication. Without loss of generality, it only has to be shown that this (newly arranged) sequence does not converge weakly to zero.

Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers. Inductively choose $c_1, c_2, \ldots \in \{b_i\}_{i=1}^{\infty}$ and $K_1, K_2, \ldots \in \mathcal{F}$ as follows. Let $c_1 = b_1$ and choose $K_1 \in B_{\mathcal{F}}$ with $[K_1](c_1) > 1/3$. Next assume $c_1, c_2, \ldots, c_n$ and $K_1, K_2, \ldots, K_n$ have been chosen. Now $c_{n+1}$ and $K_{n+1}$ will be chosen. Let $V$ and $W$ be finite dimensional subspaces of $\mathcal{H}$ such that for $i = 1, 2, \ldots, n$

$$\|K_i P_{V^\perp}\| < \varepsilon_{n+1} \quad \text{and} \quad \|P_{W^\perp} K_i\| < \varepsilon_{n+1}. \tag{2}$$

By Lemma 1.5 choose norm closed subspace $\mathcal{G}$ of finite codimension in $\mathcal{F}$ such that for all $G \in B_{\mathcal{G}}$ (i.e. the unit ball of $\mathcal{G}$)

$$\|GP_V\| < \varepsilon_{n+1} \quad \text{and} \quad \|P_W G\| < \varepsilon_{n+1}. \tag{3}$$

Now let $\mathcal{S}' = \mathcal{G}^\perp \equiv \{b \in \mathcal{C} : [G](b) = 0 \text{ for all } G \in \mathcal{G}\}$. Let $\mathcal{S}$ be the finite dimensional space in $\mathcal{C}$ spanned by $(\mathcal{S}', c_1, c_2, \ldots, c_n)$. For some $j > n$, $|[K_i](b_j)| < 1/2^{n+1}$ for all $i = 1, 2, \ldots, n$ and $\operatorname{dist}(b_j, \mathcal{S}) > 1/3$ (by Lemma 1.7). Set $c_{n+1} = b_j$. For later reference, note that

$$[K_i](c_{n+1}) < \frac{1}{2^{n+1}} \quad \text{for } i = 1, 2, \ldots, n. \tag{4}$$

Let $\mathcal{S}_\perp \equiv \{B \in \mathcal{F} : [B](b) = 0 \text{ for all } b \in \mathcal{S}\}$. Then $\mathcal{C}/\mathcal{S}$ is isometrically isomorphic to $(\mathcal{S}_\perp)^*$, and the coset $c_{n+1} + \mathcal{S}$ in $\mathcal{C}/\mathcal{S}$ has norm $> 1/3$. So there exists $K_{n+1}$ of norm one in $\mathcal{S}_\perp$ such that $|[K_{n+1}](c_{n+1})| > 1/3$. Multiplying $K_{n+1}$ by an appropriate scalar of modulus one, it may be assumed that $[K_{n+1}](c_{n+1}) > 1/3$. So

$$\|K_{n+1}\| \leqslant 1 \quad \text{and} \tag{5}$$

$$[K_{n+1}](c_{n+1}) > 1/3 \ \text{ and } \ [K_{n+1}](c_j) = 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{6}$$

Now $K_{n+1}$ annihilates $\mathcal{S}$, and so it annihilates $\mathcal{S}'$. But $(\mathcal{S}')_\perp = \mathcal{G}$, since $\mathcal{G}$ is norm closed. So $K_{n+1} \in \mathcal{G}$, and (by (3))

$$\|K_{n+1}P_V\| < \varepsilon_{n+1} \quad \text{and} \quad \|P_W K_{n+1}\| < \varepsilon_{n+1}. \tag{7}$$

This finishes the description of the introduction process. Next, some of the properties of $K_1, K_2, \ldots, K_n$ will be examined (see (2)). The spaces $V, W \subset \mathcal{H}$ will be just those given above in selecting $K_{n+1}$. Applying (2) and (7) yields the following two inequalities:

$$\begin{aligned} &\left\|P_W \sum_{i=1}^n K_i P_V - \sum_{i=1}^n K_i\right\| \\ &\qquad = \left\|P_{W^\perp} \sum_{i=1}^n K_i P_V + P_W \sum_{i=1}^n K_i P_{V^\perp} + P_{W^\perp} \sum_{i=1}^n K_i P_{V^\perp}\right\| \\ &\qquad \leqslant 3 \sum_{i=1}^n \varepsilon_{n+1} = 3n\varepsilon_{n+1}, \end{aligned} \tag{8}$$

$$\begin{aligned} &\|P_{W^\perp} K_{n+1} P_{V^\perp} - K_{n+1}\| \\ &\qquad = \|P_W K_{n+1} P_{V^\perp} + P_{W^\perp} K_{n+1} P_V + P_W K_{n+1} P_V\| < 3\varepsilon_{n+1}. \end{aligned} \tag{9}$$

Next use (8) and (9) to get the first inequality below, and (1) and (5) to get the second,

$$\begin{aligned} \left\|\sum_{i=1}^n K_i + K_{n+1}\right\| &= \left\|\sum_{i=1}^n K_i - P_W \sum_{i=1}^n K_i P_V + P_W \sum_{i=1}^n K_i P_V \right. \\ &\qquad \left. + K_{n+1} - P_{W^\perp} K_{n+1} P_{V^\perp} + P_{W^\perp} K_{n+1} P_{V^\perp}\right\| \\ &\leqslant 3n\varepsilon_{n+1} + 3\varepsilon_{n+1} + \left\|P_W \sum_{i=1}^n K_i P_V + P_{W^\perp} K_{n+1} P_{V^\perp}\right\| \\ &\leqslant 3(n+1)\varepsilon_{n+1} + \max\left(\left\|\sum_{i=1}^n K_i\right\|, 1\right). \end{aligned} \tag{10}$$

Now set $A_n = \sum_{i=1}^{n} K_i$. By (10) and (5), the sequence $\{A_n\}_{n=1}^{\infty}$ is bounded provided that $\sum_{n=1}^{\infty} n\varepsilon_{n+1} < \infty$. Assuming the $\varepsilon_n$ have been so chosen, then $\{A_n\}_{n=1}^{\infty}$ has a weak* adherent point, $A$, in $\mathcal{F}^{**}$. Fix $j = 1, 2, \ldots$, and consider $c_j$. By (6)

$$[A](c_j) = \sum_{i=1}^{\infty}[K_i](c_j) = \sum_{i=1}^{j}[K_i](c_j).$$

But

$$\left|\sum_{i=1}^{j}[K_i](c_j)\right| \geqslant \frac{1}{3} - \frac{(j-1)}{2^j}$$

by (6) and (4), so $|[A](c_j)| > 1/4$ provided $j$ is sufficiently large. Hence $[A](c_j) \not\to 0$, and so $[A](b_j) \not\to 0$. The proof is complete. ∎

REMARK 3. The theorem also holds for any norm closed subspace $\mathcal{L}$ of $\mathcal{K}$ with the property that $B_{\mathcal{L}}x$ and $B_{\mathcal{L}^*}x$ are relatively norm compact for all $x \in \mathcal{H}$.

REMARK 4. Let $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space. The space of all compact operators that are diagonal with respect to the basis $\{\varphi_i\}_{i=1}^{\infty}$ is isomorphic to $c_0$. This algebra of compact operators is commutative and satisfies restriction (R). Hence the result for $\ell^1$ follows.

## 2. AN APPLICATION TO $H^{\infty}/\varphi H^{\infty}$.

Throughout this section let $\varphi$ denote an arbitrary fixed inner function. It will be shown that the space $H^{\infty}/\varphi H^{\infty}$ is isometrically isomorphic to the second conjugate space of a commutative algebra of compact operators (on a Hilbert space) satisfying (R). This result first appears in [8] and is repeated for completeness. First the notion of a dual algebra isomorphism will be defined. Suppose $\mathcal{A}_1$ is a Banach algebra which is the dual (as a Banach space) of a Banach space $X_1$, and suppose $\mathcal{A}_2$ is likewise a Banach algebra dual to Banach space $X_2$. Then a mapping from $\mathcal{A}_1$ into $\mathcal{A}_2$ is a dual algebra isomorphism if it is an algebra isomorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$, an isometry, and a homeomorphism when $\mathcal{A}_1$ and $\mathcal{A}_2$ are endowed with their respective weak* topologies.

Let $H^2$ denote the closure of $H^{\infty}$ in $L^2(m)$ where $m$ denotes Lebesgue measure on the unit circle. The symbol $\mathcal{M}$ will be used to denote the orthogonal complement of $\varphi H^2$ in $H^2$. The projection operator of $H^2$ onto $\mathcal{M}$ will be denoted $P$. For any $h \in H^{\infty}$, let $M_h$ denote the operator of multiplication by $h$

on $H^2$. Henceforth view $PM_h$ as an operator on $\mathcal{M}$ and set $T = PM_z$. The smallest algebra containing $\{T^m\}_{m=0}^{\infty}$ that is ultraweakly closed in $\mathcal{B}(\mathcal{M})$ will be called $\mathcal{A}_T$. The Banach space $\mathcal{A}_T$ is the dual of a quotient space of the space of trace class operators on $\mathcal{M}$. There is a natural map $\Lambda' : H^\infty \to \mathcal{A}_T$ given by $h \to PM_h$. The kernel of $\Lambda'$ is $\varphi H^\infty$. So this induces a map $\Lambda : H^\infty/\varphi H^\infty \to \mathcal{A}_T$. Propositions 2.2 and 2.3 of [12] show the following to be true:

THEOREM 2.1. (Sarason). *The map $\Lambda : H^\infty/\varphi H^\infty \to \mathcal{A}_T$ is a dual algebra isomorphism.*

From now on, $H^\infty/\varphi H^\infty$ and $\mathcal{A}_T$ will be identified when convenient. An element $h + \varphi H^\infty$ in $H^\infty/\varphi H^\infty$ will be called compact if $PM_h$ is a compact operator on $\mathcal{M}$. There is an easy way to tell if $h + \varphi H^\infty$ is compact. The set of continuous complex valued functions on the unit circle will be denoted by $C$. Viewing the functions of $H^\infty$ as elements of $L^\infty(m)$, the following result (from [12], Theorem 2) applies.

THEOREM 2.2. (Sarason). *If $h \in H^\infty$, then $h\overline{\varphi} \in H^\infty + C$ iff $PM_h$ is compact.*

The following now links the study of $H^\infty/\varphi H^\infty$ with the results of Section 1. We present a theorem from [8]. The proof given here is slightly different from that of [8] (another proof can be found in [10]). The compact elements of $H^\infty/\varphi H^\infty$ form an ideal which will be denoted $\mathcal{F}$.

THEOREM 2.3. (Kriete, Moore and Page). *The algebra $H^\infty/\varphi H^\infty$ is isometrically isomorphic to the second conjugate space of $\mathcal{F}$.*

*Proof.* The proof is a direct adaptation of the techniques used in the proof of Theorem 2.2 (see [12], pages 190-191). From this proof, it follows that $L^\infty/H^\infty$ is the dual of $H_0^1$, where $H_0^1$ is the closure in $L^1(m)$ of the polynomials with zero constant term. Also by [12], $H_0^1$ is the dual of $C/A$ where $A \equiv H^\infty \bigcap C$.

First the second predual of $\overline{\varphi}H^\infty/H^\infty$ will be described. To do this view $\overline{\varphi}H^\infty/H^\infty$ as a subspace of $L^\infty/H^\infty$. The annihilator of $\overline{\varphi}H^\infty/H^\infty$ in $H_0^1$ is $\varphi H_0^1$. The annihilator of $\varphi H_0^1$ in $C/A$ is $(\overline{\varphi}H^\infty \bigcap C)/A$. Hence $((\overline{\varphi}H^\infty \bigcap C)/A)^{**} \approx \overline{\varphi}H^\infty/H^\infty$.

Let $\Gamma' : (\overline{\varphi}H^\infty \bigcap C)/A \to \overline{\varphi}H^\infty/H^\infty$ denote the canonical embedding. If $w' \in \overline{\varphi}H^\infty \bigcap C$, then $w' + A$ is mapped by $\Gamma'$ to $w' + H^\infty$ (as observed in [12]).

Now $(\overline{\varphi}H^\infty \bigcap C)/A \approx (H^\infty \bigcap \varphi C)/\varphi A$ and $\overline{\varphi}H^\infty/H^\infty \approx H^\infty/\varphi H^\infty$. So $\Gamma'$ gives rise to a natural injection $\Gamma : (H^\infty \bigcap \varphi C)/\varphi A \to H^\infty/\varphi H^\infty$, and $w + \varphi A$ is mapped by $\Gamma$ to $w + \varphi H^\infty$ for $w \in H^\infty \bigcap \varphi C$. Also $\Gamma$ is the canonical embedding of $(H^\infty \bigcap \varphi C)/\varphi A$ into its second dual, $H^\infty/\varphi H^\infty$. This embedded subspace will be shown to coincide with $\mathcal{F}$. If $w \in H^\infty \bigcap \varphi C$, then $\Gamma(w+\varphi A)$ is compact by Therem 2.2. Conversely, if $h+\varphi H^\infty \in H^\infty/\varphi H^\infty$ is compact with $h \in H^\infty$, then it has to be shown that $\Gamma(g+\varphi A) = h+\varphi H^\infty$ for some $g \in H^\infty \bigcap \varphi C$. By Theorem 2.2, $h\overline{\varphi} \in H^\infty + C$; or equivalently $h \in \varphi H^\infty + \varphi C$. So $h - \varphi f \in H^\infty \bigcap \varphi C$ for some $f \in H^\infty$. Therefore $h + \varphi H^\infty = \Gamma(h - \varphi f + \varphi A)$. And $\Gamma$ maps $(H^\infty \bigcap \varphi C)/\varphi A$ onto the ideal of compact elements in $H^\infty/\varphi H^\infty$. This finishes the proof since $\Gamma$ is an isometry. ∎

The proof of Theorem 2.3 provides a predual of $\overline{\varphi}H^\infty/H^\infty$ that is isometrically isomorphic to $H_0^1/\varphi H_0^1$ (i.e. mod out the annihilator of $\overline{\varphi}H^\infty/H^\infty$ in $H_0^1$). Now $\overline{\varphi}H^\infty/H^\infty \approx H^\infty/\varphi H^\infty$ and further use of the identifications made in the proof shows that $H_0^1/\varphi H_0^1$ is isometrically isomorphic to the dual of $\mathcal{F}$.

THEOREM 2.4. *If a sequence in $H_0^1/\varphi H_0^1$ converges weakly, than it converges in norm.*

*Proof.* Since $H_0^1/\varphi H_0^1 \approx \mathcal{F}^*$, it is only necessary to verify that $\mathcal{F}$ satisfies the restriction (R). The identity operator $I$ is in $\mathcal{A}_T \approx H^\infty/\varphi H^\infty$. By Theorem 2.3 then, there is a sequence $\{K_i\}_{i=1}^\infty$ in $\mathcal{B}_\mathcal{F}$ such that $K_i \to I$ ultraweakly. This means that for any $x \in \mathcal{M}$, it follows that $K_i x \to x$ and $K_i^* x \to x$ in norm; so, restriction (R) is satisfied. ∎

REMARK 5. Let $\{\xi_i\}_{i=1}^\infty$ be a sequence of complex numbers (all of norm $< 1$) that is dominating in the unit disc (i.e. for all $f \in H^\infty$, $\sup\{|f(\xi_i)|\}_{i=1}^\infty = \|f\|_\infty$). Now $\{\xi_i\}_{i=1}^\infty$ is an element of the algebra $\ell^\infty$ (under pointwise multiplication), and it generates a subalgebra dual algebra isomorphic to $H^\infty$. The techniques of Section 1 can be expanded without much difficulty to show that if $\mathcal{F}$ is a commutative algebra of compact operators with the identity $I$ in $\mathcal{A} \approx \mathcal{F}^{**}$, then $\mathcal{A}$ has a subalgebra that is dual algebra isomorphic to $H^\infty$. This is not hard to show, but it does require a rewrite of the entire proof and will not be provided here.

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