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DUAL PROPERTIES AND JOINT SPECTRA FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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ABSTRACT. If L is a solvable Lie algebra of operators acting on a Banach space E, we study the action of the opposite algebra of L, L', on E^*. Moreover, we extend Slodkowski joint spectra $\sigma_{\delta,k}$, $\sigma_{\pi,k}$ and study its usual spectral properties.

KEYWORDS: *Lie algebra, ideals of Lie algebra, n-tuple of commuting operators, joint spectrum, Slodkowski joint spectrum.*

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1. INTRODUCTION

In [1] we define a joint spectrum for a finite dimensional complex solvable Lie algebra of operators L acting on a Banach space E and we denote it by $\mathrm{Sp}(L, E)$. We also prove $\mathrm{Sp}(L, E)$ is a compact non void subset of $L^{2^\perp} = \{f \in L \mid f(L^2) = 0\}$. Besides, if I is an ideal of L, the projection property holds. Furthermore, if L is a commutative algebra, our spectrum reduces to Taylor joint spectrum ([5]).

Let a be an n-tuple of commuting operators acting on E, $a = (a_1, \ldots, a_n)$. Let a^* be the adjoint n-tuple of a, i.e., $a^* = (a_1^*, \ldots, a_n^*)$ where a_i^* is the adjoint operator of a_i. Then a^* is an n-tuple of commuting operators acting on E^*, the dual space of E. If $\sigma(a)$ (resp. $\sigma(a^*)$) denotes the Taylor joint spectrum of a (resp. a^*), it is well known that $\sigma(a) = \sigma(a^*)$. If we consider a solvable non commutative Lie algebra of operators L contained in $\mathcal{L}(E)$, the space of bounded linear maps on E, its dual, $L^* = \{x^* \mid x \in L\}$ defines a solvable Lie subalgebra of $\mathcal{L}(E^*)$ with the opposite bracket of L. One may ask if the joint spectra of L and L^* in the sense of [1] coincide. In the solvable non commutative case, in general, the answer is no.

We study this problem and prove $\mathrm{Sp}(L,E)$ and $\mathrm{Sp}(L^*,E^*)$ are related: one is obtained from the other by a translation, i.e., $\mathrm{Sp}(L,E)=\mathrm{Sp}(L^*,E^*)+c$, where c is a constant. Moreover, we characterize this constant in terms of the algebra and prove $c=0$ in the nilpotent case.

In the second part of our work, we study $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$ the Slodkowski spectra of [4]. We extend then to the case of solvable Lie algebras of operators and verify the usual spectral properties: they are compact, non void sets and the projection property for ideals still holds.

The paper is organized as follows. In Section 2 we review several definitions and results of [1]. In Section 3 we study the relation among $\mathrm{Sp}(L,E)$ and $\mathrm{Sp}(L^*,E^*)$, the dual property. In Section 4 we extend Slodkowski spectrum and prove its spectral properties.

2. PRELIMINARIES

We shall briefly recall several definitions and results related to the spectrum of a solvable Lie algebra of operators, ([1]).

From now on, L denotes a complex finite dimensional solvable Lie algebra. E denotes a Banach space on which L acts as right continuous operators, i.e., L is a Lie subalgebra of $\mathcal{L}(E)$.

Let f be a character of L and suppose $n=\dim L$. Let us consider the following complex, $(E\otimes\wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L and

$$d_p(f): E\otimes\wedge^p L\to E\otimes\wedge^{p-1}L$$

$$\begin{aligned}(1)\qquad d_p(f)e\langle x_1\wedge\cdots\wedge x_p\rangle &= \sum_{k=1}^{k=p}(-1)^{k+1}e(x_k-f(x_k))\langle x_1\wedge\cdots\wedge\widehat{x}_k\cdots\wedge x_p\rangle\\ &+\sum_{1\leqslant k<l\leqslant p}(-1)^{k+l}e\langle[x_k,x_l]\wedge x_1\cdots\wedge\widehat{x}_k\cdots\wedge\widehat{x}_l\cdots\wedge x_p\rangle\end{aligned}$$

where $\widehat{}$ means deletion. If $p\leqslant 0$ or $p\geqslant n+1$, we also define $d_p(f)\equiv 0$.

Let $H_*(E\otimes\wedge L, d(f))$ denotes the homology of the complex $(E\otimes\wedge L, d(f))$.

DEFINITION 1. Let L and E be as above, the set $\{f \in L^*, f(L^2) = 0 \mid H_*(E \otimes \wedge L, d(f)) \neq 0\}$ is the spectrum of L acting on E, and it is denoted by $\mathrm{Sp}\,(L, E)$.

THEOREM. *If L is a commutative Lie algebra,* $\mathrm{Sp}\,(L, E)$ *reduces to Taylor joint spectrum.*

THEOREM. $\mathrm{Sp}\,(L, E)$ *is a compact non void subset of* L^*.

THEOREM. (Projection property). *Let I be an ideal of L and π the projection map from L^* onto I^*, then*

$$\mathrm{Sp}\,(I, E) = \pi(\mathrm{Sp}\,(L, E)).$$

As in [1], we consider an $n-1$ dimensional ideal of L, L_{n-1}, and we decompose $E \otimes \wedge^p L$ in the following way

$$E \otimes \wedge^p L = (E \otimes \wedge^p L_{n-1}) \oplus (E \otimes \wedge^{p-1} L_{n-1}) \wedge \langle x_n \rangle$$

where $x_n \in L$ and is such that $L_{n-1} \oplus \langle x_n \rangle = L$.

If $\tilde{f}$ denotes the restriction of f to L_{n-1}, we may consider the complex $(E \otimes \otimes \wedge^p L_{n-1}, d(\tilde{f}))$.

As L_{n-1} is an ideal of codimension 1 of L, we may decompose the operator $d_p(f)$ as follows

$$\begin{gathered} d_p(f) : E \otimes \wedge^p L_{n-1} \to E \otimes \wedge^{p-1} L_{n-1} \\ d_p(f) = \tilde{d}_p(\tilde{f}) \\ d_p(f) : E \otimes \wedge^{p-1} L_{n-1} \wedge \langle x_n \rangle \to E \otimes \wedge^{p-1} L_{n-1} \oplus E \otimes \wedge^{p-2} L_{n-1} \langle x_n \rangle \end{gathered} \tag{2}$$

$$d_p(f)(a \wedge \langle x_n \rangle) = (-1)^p L_p(a) + (\tilde{d}_{p-1}(\tilde{f})(a)) \wedge \langle x_n \rangle \tag{3}$$

where $a \in E \otimes \wedge^{p-1} L_{n-1}$, and L_p is the bounded linear endomorphism defined by:

$$\begin{aligned} L_p e \langle x_1 \wedge \cdots \wedge x_{p-1} \rangle = & -e(x_n - f(x_n)) \langle x_1 \wedge \cdots \wedge x_{p-1} \rangle + \\ & + \sum_{1 \leqslant k \leqslant p-1} (-1)^{k+1} e(\langle [x_k, x_n] \rangle \wedge x_1 \cdots \wedge \hat{x}_k \cdots \wedge x_{p-1} \rangle \end{aligned} \tag{4}$$

where $\hat{}$ means deletion and x_i $(1 \leqslant i \leqslant p-1)$ belongs to L_{n-1}.

Now we consider the following morphism defined in [3] and [2].

Let $\theta(x_n)$ be the derivation of $\wedge L$ extension of the map $\mathrm{ad}\,(x_n)$

$$\operatorname{ad}(x_n)(y) = [x_n, y], \; y \in L$$

$$\theta(x_n)\langle x_1 \wedge \cdots \wedge x_p\rangle = \sum_{i-1}^{p} \langle x_1 \wedge \cdots \wedge \operatorname{ad}(x_n)(x_i) \wedge \cdots \wedge x_p\rangle \tag{5}$$

$\theta(x_n)$ satisfies:

$$\theta(x_n)(ab) = (\theta(x_n)a)b + a(\theta(x_n)b) \tag{6}$$

$$\theta(x_n)w = w\theta(x_n) \tag{7}$$

where w is the map

$$w(\langle x_1 \wedge \cdots \wedge x_p\rangle) = (-1)^p \langle x_1 \wedge \cdots \wedge x_p\rangle. \tag{8}$$

Let u belong to $\wedge L$ and $\varepsilon(u)$ be the $\wedge L$ endomorphism: $\varepsilon(u)v = u \wedge v$. As $(\wedge L)^*$ may be identified with $\wedge L^*$, let $\iota(u)$ be the dual map of $\varepsilon(u)$ ($\iota(u) : \wedge L^* \to \wedge L^*$).

Besides, we consider $\theta^*(x_n)$ the dual map of $-\theta(x_n)$.

As $\varepsilon(u \wedge v) = \varepsilon(u)\varepsilon(v), \iota(u \wedge v) = \iota(u)\iota(v)$.

As in [3] and [7] we define an isomorphism ρ

$$\begin{aligned} &\rho : \wedge L^* \to \wedge L \\ &\rho(a) = \iota(a) \cdot w \end{aligned} \tag{9}$$

where $a \in \wedge L^*$, $w = \langle x_1 \wedge \cdots \wedge x_{n-1} \wedge x_n\rangle$ and $\{x_1 \ldots x_n\}$ is a basis of L_{n-1}; ρ applies $\wedge^p L^*$ isomorphically onto $\wedge^{n-p} L$.

3. THE DUAL PROPERTY

Let L and E be as in Section 2. Let E^* be the space of continuous functionals on E. Let L' be the solvable Lie algebra defined as follows: as vector space, $L = L'$, and the bracket of L' is the opposite of the one of L, that is: $[x,y]' = -[x,y] = [y,x]$. L' is a complex finite dimensional solvable Lie algebra and $L^{2^\perp} = L'^{2^\perp}$.

As L acts as right continuous operators on E, the space $L^* = \{x^*, \; x \in L\}$ has the Lie structure of the algebra L' and acts as right continuous operators on E^*.

Observe that in the definition of ε, ι and ρ, we only consider the structure of L as vector space. As L and L' coincide as vector spaces, then $\wedge L = \wedge L'$ and we may consider

$$\rho : \wedge L^* \to \wedge L'.$$

If L_{n-1} is an ideal of codimension 1 of $L, L'_{n-1} = L_{n-1}$ is an ideal of codimension 1 of L'. Moreover, if $x_n \in L$ is such that $L_{n-1} \oplus \langle x_n \rangle = L$, then $L'_{n-1} \oplus \langle x_n \rangle = L'$.

Let $\theta'(x_n)$ be the derivation of $\wedge L'$ extension of the map ad (x_n), ad $(x_n)(y) = [x_n, y_n]'$. By ([2], V, 3), there exists a basis of $L, \{x_i\}$, $1 \leqslant i \leqslant n$, such that

(10) $$[x_j, x_i] = \sum_{h=1}^{i} c_{ij}^h x_h \quad (i < j)$$

and $L_{n-1} = \{x_i\}$, $1 \leqslant i \leqslant n-1$. If $w = \langle x_1 \wedge \cdots \wedge x_n \rangle$, then

$$\theta(x_n)w = \theta(x_n)\langle x_1 \wedge \cdots \wedge x_n \rangle = \sum_{i=1}^{n} \langle x_1 \wedge \cdots \wedge x_{i-1} \wedge [x_n, x_i] \wedge \cdots \wedge x_n \rangle$$
$$= \left(\sum_{i-1}^{n-1} c_{in}^i \right) \langle x_1 \wedge \cdots \wedge x_n \rangle = (\operatorname{trace} \operatorname{ad}(x_n))w.$$

As in [3] and [7], if $a \in \wedge L^*$, we have

(11) $$\rho\theta^*(x_n) = \theta(x_n)\rho(a) - \iota(a)\theta(x_n)w.$$

Then

(12) $$\rho\theta^*(x_n) = \theta\rho - (\operatorname{trace} \operatorname{ad}(x_n))\rho.$$

As L' has the opposite bracket of L,

(13) $$\rho\theta^*(x_n) = -(\theta(x_n) + \operatorname{trace} \operatorname{ad}(x_n))\rho.$$

Let us consider the maps $1_{E^*} \otimes \theta(x_n), 1_{E^*} \otimes \rho, 1_{E^*} \otimes \theta(x_n), 1_{E^*} \otimes \theta'(x_n)$ and let us still denote then by $\theta^*(x_n), \rho, \theta(x_n), \theta'(x_n)$ respectively. We observe that formulas (11), (12), (13) remain true.

Let us decompose, as in Section 2, $E^* \otimes \wedge^p L^* (E^* \otimes \wedge^{n-p} L'$, respectively) as the sum:

$$E^* \otimes \wedge^p L^*_{n-1} \oplus E^* \otimes \wedge^{p-1} L^*_{n-1} \wedge \langle x_n \rangle$$
$$E^* \otimes \wedge^{n-p} L'_{n-1} \oplus E^* \otimes \wedge^{n-p-1} L'_{n-1} \wedge \langle x_n \rangle \text{ (respectively)}$$

where L^*_{n-1} is the subspace of L^* generated by $\{y_j\}$, $1 \leqslant j \leqslant n-1$ and $\{y_j\}$, $1 \leqslant j \leqslant n$ is the dual basis of $\{x_j\}$, $1 \leqslant j \leqslant n$.

A standard calculation shows the following facts:

(14) $$\rho(E^* \otimes \wedge^p L^*_{n-1}) = (E^* \otimes \wedge^{n-p-1} L'_{n-1}) \wedge \langle x_n \rangle$$

(15) $$\rho(E^* \otimes \wedge^{p-1} L_{n-1} \wedge \langle y_n \rangle) = (E^* \otimes \wedge^{n-p} L'_{n-1})$$

(16) $$\rho | E^* \otimes \wedge^p L^*_{n-1} = \rho_{n-1} \wedge \langle x_n \rangle$$

(17) $$\rho | E^* \otimes \wedge^{p-1} L_{n-1} \wedge \langle y_n \rangle = \rho_{n-1} (-1)^{n-p}$$

where ρ_{n-1} is the isomorphism associated to the algebra L_{n-1}.

PROPOSITION 1. *Let L, L', E, L_{n-1}, ρ, be as above and f belong to $L^{2^\perp}$ and g to $L'^{2^\perp}$ such that: $f(x_n) + \operatorname{trace}\theta(x_n) = g(x_n)$. Then, the following diagram commutes:*

$$\begin{array}{ccc}
E^* \otimes \wedge^p L^*_{n-1} & \xrightarrow{L^*_{p+1}} & E^* \otimes \wedge^p L^*_{n-1}(y_n) \\
\rho \downarrow & & \downarrow \rho \\
E^* \otimes \wedge^{n-p-1} L'_{n-1} \wedge \langle x_n \rangle & \xrightarrow{L'_{n-p}} & E^* \otimes \wedge^{n-p-1} L'_{n-1}
\end{array}$$

where L'_{n-p} is the operator involved in the definition of $d'_{n-p}(g)$ and L^*_{p+1} is the adjoint operator of L_{p+1}, map involved in the definition of $d_{p+1}(f)$, (see (3), (4)).

Proof. By (4), (5)

$$L_{p+1} = -(x_n - f(x_n)) + \theta(x_n).$$

Then,

$$L^*_{p+1} = -(x^*_n - f(x_n)) + \theta^*(x_n).$$

As the bracket of L' is the opposite of the one of L, by (4), (5)

$$L'_{n-p} = -(x^*_n - g(x_n)) + \theta(x_n).$$

Then

$$L'_{n-p}\rho = -(x^*_n - g(x_n))\rho + \theta'(x_n)\rho.$$

On the other hand, by (13)

$$\begin{aligned}
\rho L^*_{n+p} &= -(x^*_n - f(x_n))\rho - \rho\theta^*(x_n) \\
&= -(x^*_n - f(x_n))\rho + \theta'(x_n)\rho + \operatorname{trace}\operatorname{ad}(x_n))\rho \\
&= -(x^*_n - (f(x_n) + \operatorname{trace}\theta(x_n)))\rho + \theta'\rho \\
&= -(x^*_n - g(x_n))\rho + \theta'(x_n)\rho = L'_{n-p}\rho. \quad \blacksquare
\end{aligned}$$

THEOREM 1. *Let L, L', E, and ρ be as above. Let $\{x_i\}$, $1 \leqslant i \leqslant n$ be the basis of L defined in (10). Let f belong to $L'^{2^\perp}$ and g to $L'2^\perp$ such that:*

$$g = f + (\operatorname{trace}\tilde{\theta}(x_1) \ldots \operatorname{trace}\tilde{\theta}(x_n))$$

where $\tilde{\theta}(x_i)$ is the restriction of $\theta(x_i)$ to L_i, the ideal generated by $\{x_i\}$, $1 \leqslant i \leqslant$ $\leqslant n$. Then, if we consider the complex adjoint of $(E \otimes \wedge L, d(f))$ and the complex $(E \otimes \wedge L', d(g))$, for each p:

$$d'_{n-p}(g)\rho w = \rho d^*_{p+1}(f)$$

that is, the following diagram commutes:

$$\begin{array}{ccc} E^* \otimes \wedge^p L & \xrightarrow{d^*_{p+1}} & E^* \otimes \wedge^{p+1} L \\ \downarrow{\scriptstyle \rho w} & & \downarrow{\scriptstyle \rho} \\ E^* \otimes \wedge^{n-p} L' & \xrightarrow{d'_{n-p}} & E^* \otimes \wedge^{n-p-1} L' \end{array}$$

where w is the map of (8).

Proof. By means of an induction argument the proof may be derived from Proposition 1. ∎

THEOREM 2. *Let L, L' and E be as above. If we consider the basis of L, $\{x_i\}$ $1 \leqslant i \leqslant n$ defined in (10), in terms of the dual basis in $L^* = (L')$ we have:*

$$\operatorname{Sp}(L, E) + (\operatorname{trace} \tilde{\theta}(x_1) \cdots \tilde{\theta}(X_n)) = \operatorname{Sp}(L', E^*)$$

where $\tilde{\theta}(x_i)$ is as in Proposition 1.

Proof. Is a consequence of Theorem 1 and ([4], 2.1). ∎

THEOREM 3. *If L is a nilpotent Lie algebra, then*

$$\operatorname{Sp}(L, E) = \operatorname{Sp}(L', E^*).$$

Proof. By ([2], V, 1) the basis $\{X_i\}$ $1 \leqslant i \leqslant n$ of (10) may be chosen such that

$$[x_j, x_i] = \sum_{h=1}^{i-1} c_{ji}^h x_h \quad (i < j)$$

Then $\tilde{\theta}(x_i) = 0$. ∎

REMARK. Let L be a solvable Lie algebra. Let $n = \dim(L)$ and $k = \dim(L^2)$. By ([2], V, 3), the basis of (10) may be chosen such that $\{x_j\}$, $1 \leqslant j \leqslant k$ generates L^2. As L^2 is a nilpotent ideal of L, $\operatorname{trace} \tilde{\theta}(x_i) = 0$, $1 \leqslant i \leqslant k$. Then, we have the following proposition.

PROPOSITION 2. *Let L, L' and E as usual. Let us consider a basis of L as in the previous remark. Then*

$$\operatorname{Sp}(L, E) + (0 \ldots 0, \operatorname{trace} \tilde{\theta}(x_{k+1}) \ldots \operatorname{trace} \tilde{\theta}(x_n)) = \operatorname{Sp}(L', E^*).$$

Then, as we have seen, if L is a niopotent Lie algebra, the dual property is essentially the one of the commutative case. However, if L is a solvable non nilpotent Lie algebra, this property fails. For example, if L is the algebra

$$L = \langle x_1 \rangle \oplus \langle x_2 \rangle [x_2, x_1] = x_1.$$

Then,

$$\operatorname{trace} \theta(x_2) = 1, \quad \operatorname{trace} \theta(x_1) = 0$$

and

$$\operatorname{Sp}(L, E) + (0, 1) = \operatorname{Sp}(L', E^*).$$

4. THE EXTENSION OF SLODKOWSKI JOINT SPECTRA

Let L and E be as in Section 2. We give an homological version of Slodkowski spectrum instead of the cohomological one of [4].

Let $\Sigma_p(L, E)$ be the set: $\{f \in L^{2^\perp} \mid H_p(E \otimes \wedge L, d(f)) \neq 0\} = \{f \in L^{2^\perp} \mid R(d_{p+1}(f)) \neq \ker(d_p(f)\}$.

DEFINITION 2. Let L and E be as in Section 2

$$\sigma_{\delta,k}(L, E) = \bigcup_{0 \leqslant p \leqslant n} \Sigma_p(L, E)$$

$$\sigma_{\pi,k} = \bigcup_{k \leqslant p \leqslant n} \Sigma_p(L, E) \bigcup \{f \in L^{2^\perp} \mid R(d_k(f) \text{ is closed}\}$$

where $0 \leqslant k \leqslant n$.

We shall see $\sigma_{\delta,k}(L, E)$ and $\sigma_{\pi,k}(L, E)$ are compact non void sets of L^* and they verify the projection property for ideals.

Observe: $\sigma_{\delta,n}(L, E) = \sigma_{\pi,0}(L, E) = \operatorname{Sp}(L, E)$.

Let us see $\sigma_{\delta,k}(L, E)$ has the usual property of a spectrum.

THEOREM 4. *Let L and E be as usual. Then $\sigma_{\delta,k}(L, E)$ is a compact set of L^*.*

Proof. As $\sigma_{\delta,k}(L, E)$ is contained in $\operatorname{Sp}(L, E)$, it is enough to prove that $\sigma_{\delta,k}(L, E)$ is closed in L^*. Let us consider the complex $(E \otimes \wedge^p L, d_k(f))$ $(0 \leqslant k \leqslant p + 1)$

$$E \otimes \wedge^{p+1} L \xrightarrow{d_{p+1}(f)} E \otimes \wedge^p L \cdots \xrightarrow{d_1(f)} E \otimes \wedge^0 L \to 0.$$

This complex is a parametrized chain complex of Banach spaces on $L^{2^\perp}$ in the sense of [5] and [2]. By ([5], 2.1), $\{f \in L^{2^\perp}/(E \otimes \wedge^k L, d_k(f))$ $(0 \leqslant k \leqslant p+1)$ is not exact$\}$ is a closed set of $L^{2\perp}$, and then in L^*. However, this set is exactly $\sigma_{\delta,k}$. ∎

Let L_{n-1} be an ideal of codimension 1 of L and x_n in L such that $L_{n-1} \oplus \langle x_n \rangle = L$. As in [5] and [1] we consider a short exact sequence of complex. Let f belong to L^* and denote $\tilde{f}$ its restriction to L_{n-1}.

$$(13) \qquad 0 \to (E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{i} (E \otimes \wedge L, d(f)) \xrightarrow{p} (E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \to 0$$

where p is the following map: As in Section 2 we decompose $E \otimes \wedge^k L$

$$E \otimes \wedge^k L = E \otimes \wedge^k L_{n-1} \oplus E \otimes \wedge^{k-1} L_{n-1} \langle x_n \rangle$$
$$p(E \otimes \wedge^k L_{n-1}) = 0$$
$$p(e\langle x_1 \wedge \cdots \wedge x_{k-1} \wedge x_n \rangle) = (-1)^{k-1} e\langle x_1 \wedge \cdots \wedge x_{k-1} \rangle$$

where $x_i \in L_{n-1}$, $1 \leqslant i \leqslant k-1$.

REMARK (19).

(i) $d(f) | E \otimes \wedge L_{n-1} = \tilde{d}(\tilde{f})$,

(ii) As in [1] $H_p(E \otimes \wedge L, d(f)) = \operatorname{tor}_p^{(UL)}(E, C(f))$,

(iii) As in [5] and [1] we have a long exact sequence of $U(L)$ modules, where $U(L)$ is the universal algebra of L

$$\to H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{i_*} H_p(E \otimes \wedge L, d(f)) \xrightarrow{p_*}$$
$$\to H_{p-1}(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \xrightarrow{\delta_{*p}} H_{p-1}(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \to$$

where δ_{*p} is the connecting operator.

(iv) As in [1] we observe if we regard the $U(L)$ module $H_P(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f}))$ as $U(L_{n-1})$ module, we obtain: $\operatorname{tor}_p^{U(L_{n-1})}(E, C(\tilde{f}))$. Then, as $U(L_{n-1})$ is a subalgebra with unit of $U(L)$

$$\tilde{f} \in \sum_p (L_{n-1}, E) \text{ if and only if } H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \neq 0 \text{ as } U(L) \text{ module.}$$

PROPOSITION 3. *Let L, L_{n-1}, E, f and $\tilde{f}$ be as above. Then, if f belongs to $\Sigma_p(L, E)$, $\tilde{f}$ belongs to $\Sigma_{p-1}(L_{n-1}, E) \cup \Sigma_p(L_{n-1}, E)$.*

Proof. By Remark (19).(iv), $\tilde{f} \notin \Sigma_p(L_{n-1}, E) \cup \Sigma_{p-1}, E$ if and only if $H_i(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) = 0$ as $U(L)$ module $(i = p, p-1)$. By Remark (19).(iii), $H_p(E \otimes \wedge L, d(f)) = 0$, i.e., $f \notin \Sigma(L, E)$. ∎

PROPOSITION 4. *Let L, L_{n-1}, E, be as usual. Let $\Pi : L^* \to L^*_{n-1}$ be the projection map. Then*

$$\Pi(\sigma_{\delta,k}(L, E)) \subseteq \sigma_{\delta,k}(L_{n-1}, E)).$$

Proof. Is a consequence of Proposition 3. ∎

PROPOSITION 5. *Let L, L_{n-1}, E, be as usual, and Π as in Proposition 4. Then:* $\sigma_{\delta,k}(L_{n-1},E)) = \Pi(\sigma_{\delta,k}(L,E))$.

Proof. By Proposition 4 it is enough to see

$$\sigma_{\delta,kj}(L_{n-1},E) \subseteq \Pi(\sigma_{\delta,k}(L,E)).$$

By refinning an argument of [1] and [5], we shall see that if $\tilde{f}$ belongs to $\Sigma_p(L_{n-1},E)$ there is an extension of $\tilde{f}$ to L^*, f, such that $f \in \Sigma_p(L,E)$.

First of all, as $\tilde{f} \in \Sigma_p(L_{n-1},E) \subseteq \mathrm{Sp}(L,E)$, by [1], Theorem 3, if g is an extension of $\tilde{f}$ to $L^*, g(L^2) = 0$, i.e., $g \in L^{2^\perp}$.

Let us suppose our claim is false, equivalently, $H_p(E \times \wedge L, d(f)) = 0\ \forall\ f \in L^{2^\perp}$, $\Pi(f) = \tilde{f}$.

Let us consider the connecting map associated to the long exact sequence of Remark (14).(iii), δ_*

$$\delta_{*p} : H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) \to H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})).$$

As $H_p(E \otimes \wedge L, d(f)) = 0\ \forall f \in L^{2^\perp}$, $\Pi(f) = \tilde{f}$, $\delta_{*p}(f)$ is a surjective map.

Let $k \in E \otimes \wedge^p L_{n-1}$ such that $\tilde{d}(\tilde{f})(k) = 0$. Then, it is well known that if $m \in E \otimes \wedge^{p+1} L$ is such that $p(m) = k : \delta_{*p}([k]) = [d_{p+1}(f)(m)]$, where p is the map defined in (18). Let us consider $m = k \wedge x_n \in E \otimes \wedge^p L_{n-1} \wedge \langle x_n \rangle$. Then: $p((-1)^p m) = k$, and

$$\delta_{*p}(f)([k]) = (-1)^p [d_{p+1}(f)(m)].$$

As

$$\begin{aligned} d_{p+1}(f)(m) = d_{p+1}(f)(k \wedge \langle x_n \rangle) &= (\tilde{d}(\tilde{f})(k)) \wedge \langle x_n \rangle + (-1)^{p+1} L_{p+1}(k) \\ &= (-1)^{p+1} L_{p+1}(k) \in E \otimes \wedge^p L_{n-1} \\ \delta_{*p}(f)[k] &= -[L_{p+1}(k)]. \end{aligned}$$

Moreover, by equations (4) and (5): $L_{p+1}(k) = -k(x_n - f(x_n)) + \theta(x_n)(k)$. Then

$$\begin{aligned} 0 = \tilde{d}_p(\tilde{f})(d_{p+1}(f)(m)) &= \tilde{d}_p(\tilde{f})(-k(x_n - f(x_n)) + \tilde{d}_p(\tilde{f})\theta(x_n)(k) \\ &= (-\tilde{d}_p(\tilde{f})(k))(x_n - f(x_n)) + \tilde{d}_p(\tilde{f})\theta(x_n)(k) = \tilde{d}_p(\tilde{f})\theta(x_n)(k). \end{aligned}$$

Which implies:

$$\delta_{*p}(f) = [k](x_n - f(x_n)) - [\theta(x_n)k].$$

Let us consider the complex of Banach spaces and maps

$$E \otimes \wedge^{p+1} L \xrightarrow{d_{p+1}(f)} E \otimes \wedge^p L \xrightarrow{d_p(f)} E \otimes \wedge^{p-1} L. \tag{20}$$

Then, this is an analytically parametrized complex of Banach spaces on C exact $\forall\ f \in L^{2^\perp} \Pi(f) = \tilde{f}$ and exact at ∞ ([5], [2], [3]).

As δ_{*p} differs by a constant term of the connecting map of ([5], 1.3), the argument of ([5], 3.1) still applies to the complex (20). Then $H_p(E \otimes \wedge L_{n-1}, \tilde{d}(\tilde{f})) = 0$ as $U(L)$ module. By Remark (19).(iv) we finish our proof. ∎

THEOREM 5. *Let L and E be as usual. Let I bean ideal of L. Then*

$$\sigma_{\delta k}(I,E) = \Pi\sigma_{\delta k}(L,E)$$

where Π denotes the projection map.

Proof. By ([2], V, 3, Proposition 5) and an inductive argument we conclude the theorem. ∎

THEOREM 6. *Let L and E be as usual. Then $\sigma_{\delta k}(L,E)$ is a non void set of L^*.*

Proof. It is a consequence of ([2], V, 3), Theorem 5 and the one dimensional case. ∎

THEOREM 7. *Let L and E be as usual, and $L', \{x_i\}$, $1 \leqslant i \leqslant n$ and $\tilde{\theta}(x_i)$, $1 \leqslant i \leqslant n$ as in Theorem (2). Then, in terms of the dual basis of $\{x_i\}$, $1 \leqslant i \leqslant n$*

(i) $\sigma_{\delta k}(L,E) + (\text{trace}\,\tilde{\theta}(x_i)\ldots\text{trace}\,\tilde{\theta}(x_n)) = \sigma_{\pi k}(L',E^*)$;

(ii) $\sigma_{\pi k}(L,E) = \sigma_{\delta n-k}(L',E) - (\text{trace}\,\tilde{\theta}(x_1)\ldots\text{trace}\,\tilde{\theta}(x_n))$.

Proof. It is a consequence of ([4], 2.1) and Theorem (1). ∎

THEOREM 8. *Let L and E be as usual. Then $\sigma_{\pi k}(L,E)$ is a compact subset of L^* and if I is an ideal of L and Π the projection map from L^* onto I^**

$$\sigma_{\pi k}(L,E) = \Pi\sigma_{\pi k}(L,E).$$

Proof. It is a consequence of Theorem (4), (6), (7) and Proposition (3). ∎

REMARK. In [1] we see the projection property for subspace which are not ideals fails for $\mathrm{Sp}\,(L,E)$. As $\sigma_{\pi 0}(L,E) = \sigma_{\delta n}(L,E) = \mathrm{Sp}\,(L,E)$, the same result remain true, in general, for $\sigma_{\delta k}(L,E)$ and $\sigma_{\pi k}(L,E)$.

THEOREM 9. *Let $L, E, L', \{x_i\}$, $1 \leqslant i \leqslant n$, and $\tilde{\theta}(x_i)$, $1 \leqslant i \leqslant n$ as in Theorem (3). Then*

(i) $\sigma_{\delta k}(L,E) = \sigma_{\pi k}(L',E^*)$,

(ii) $\sigma_{\pi k}(L,E) = \sigma_{\delta n-k}(L',E^*)$.

Proof. It is a consequence of Theorem (7) and the fact: $\tilde{\theta}(x_i) = 0$ ([2],IV,1). ∎

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