J. OPERATOR THEORY
33(1995), 117–158

CHARACTERIZATION OF JORDAN ELEMENTS IN Ψ^*-ALGEBRAS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. We show that, given a Ψ^*-algebra $\mathcal{A} \subseteq L(H)$, H a Hilbert space, and an operator $J \in \mathcal{A}$ which is a Jordan operator of $L(H)$, then J also admits a Jordan decomposition within $\mathcal{A}$. The constructive proof of this fact indicates that the structure of the projections of a Ψ^*-algebra is very rich. We use this construction of obtain local similarity cross sections for Jordan elements $J \in \mathcal{A}$ within the Ψ^*-algebra $\mathcal{A}$.

KEYWORDS: *Similarity orbits, Jordan operator, Ψ^*-algebras, pseudo invertibility, local cross sections.*

AMS SUBJECT CLASSIFICATION: Primary 46H35, 46K99, 47D25, 47C10, 58B99; Secondary 46A13, 54A10, 55R10, 57S25.

INTRODUCTION

In Hilbert space theory it was a longer development to give an answer to the following question: For which $T \in L(H)$, the algebra of bounded linear operators on a Hilbert space H, does there exist a continuous local cross section to the conjugation operation

$$\pi^T : L(H)^{-1} \longrightarrow S(T), \quad \pi^T(g) := gTg^{-1},$$

where $L(H)^{-1}$ denotes the group of invertible operators and $S(T) := \{gTg^{-1} : g \in L(H)^{-1}\}$ the similarity orbit. After long and profound steps of investigation, the answer was finally given by D.A. Herrero and L. A. Fialkow and it is

THEOREM 0.1. (see [10]). *π^T admits continuous local cross sections iff T is a nice Jordan operator (see Definition 1.3 below).*

This implies that such a T has a particular simple structure, namely T is algebraic and the ranges $R(q(T))$ are closed for every polynomial q deviding the minimal polynomial of T (see [19], 7.13). The requirement that $R(a)$ is closed for $a \in L(H)$ is equivalent to the pseudo-invertibility of a and a is called a regular element of $L(H)$ in this case (see Definition 1.5 below). In [23] and [24] Theorem 0.1 was sharpened in the following direction (see also the remarks in [1]):

THEOREM 0.2. *If $T \in L(H)$ admits continuous local similarity cross sections then $S(T)$ is a locally-$L(H)$-rational manifold and π^T has a local cross section, which is a rational morphism from the manifold $S(T)$ into $L(H)^{-1}$ (see the Remark 4.14.(i) below).*

Now, $L(H)$ has of course a lot of interesting subalgebras $\mathcal{A}$. We think of C^*-algebras and algebras of pseudo-differential operators of order 0 on a compact manifold as well as algebras of rational matrix functions within the C^*-algebra of continuous matrix functions on the circle. These algebras look very different (certainly from a topological point of view), but they all have the important algebraic property of spectral invariance within $L(H)$ (i.e. $e \in \mathcal{A}$ and $\mathcal{A} \bigcap L(H)^{-1}$) and they are $*$-invariant.

These, properties, together with rather weak assumption on the topology of $\mathcal{A}$, often imply that there is a good perturbation theory for $\mathcal{A}$ although there are no implicit function theorems known in many cases. This was one of the main ideas of the article of B. Gramsch [13] who studied extensively the regular elements of $\mathcal{A}$ and gave the following definition of a Ψ^*-algebra $\mathcal{A} \subseteq L(H)$:

(i) $\mathcal{A}$ is a topological algebra, continuously embedded in $L(H)$,
(ii) $\mathcal{A}$ is a Fréchet space,
(iii) $\mathcal{A} = \mathcal{A}^*$,
(iv) $e \in \mathcal{A}$ and $\mathcal{A} \bigcap L(H)^{-1} = \mathcal{A}^{-1}$,

having applications to algebras of pseudo-differential operators in mind, which in many cases fulfill these properties (see [17], [29], [9], [13] for further references on this subject).

Now coming back to Jordan operators, we want to analyse the following question:

Given $T \in \mathcal{A} \subseteq L(H)$ with the property that

$$L(H)^{-1} \ni g \stackrel{\pi^T}{\longmapsto} gTg^{-1} \in S_{L(H)^{-1}}(T) = \{hTh^{-1} : h \in L(H)^{-1}\}$$

admits continuous local cross sections, do there exist such sections also to the restricted operation

$$\mathcal{A}^{-1} \ni g \overset{\pi^T}{\longmapsto} gTg^{-1} \in S_{\mathcal{A}^{-1}}(T) = \{hTh^{-1} : h \in \mathcal{A}^{-1}\} \quad ?$$

In order to construct local similarity cross sections in a wide class of subalgebras $\mathcal{A} \subseteq L(H)$ we do the following:

(a) we construct a Jordan decomposition within $\mathcal{A}$ for operators in $\mathcal{A}$, which are Jordan operators in $L(H)$;

(b) we construct local similarity cross sections which are $*$-rational functions and thus keep values within $\mathcal{A}$.

It turns out that this program works for Ψ^*-algebras and moreover in larger classes of spectrally invariant $*$-subalgebras, which need not be complete in their topology. So we give in 1.1 a weaker notion of Ψ^*-algebras $\mathcal{A}$, which involves the regular elements of $\mathcal{A}$ and $L(H)$. Our Theorem is

THEOREM 0.3. (see 4.12, 4.14). *Let $\mathcal{A} \subseteq L(H)$ be a Ψ^*-algebra defined as in 1.1 below. If $T \in \mathcal{A}$ and $\pi^T : L(H)^{-1} \to S_{L(H)^{-1}}(T)$ has continuous local cross sections then also $\pi^T : \mathcal{A}^{-1} \to S_{\mathcal{A}^{-1}}(T)$ admits continous local cross sections. Furthermore, there exist local similarity cross sections which are rational morphisms from the locally rational manifolds $S_{\mathcal{A}^{-1}}(T)$ into $\mathcal{A}^{-1}$.*

The constructions within the proof of 0.3 give the following more explicit statements (see Lemma 4.8, Theorem 0.3, Remark 4.7):

THEOREM 0.4. *Let the assumptions of Theorem 0.3 be fulfilled. Then*

(i) *There exists a neighborhood W_1 of T in $\mathcal{A}$ and a $*$-rational map $\omega_1 : W_1 \to \mathcal{A}^{-1}$ such that*

$$\varphi_1 := \omega_1|_{W_1 \cap S_{\mathcal{A}^{-1}}(T)} : W_1 \bigcap S_{\mathcal{A}^{-1}}(T) \to \mathcal{A}^{-1}$$

is a local cross section of π^T.

(ii) *If the Ψ^*-algebra $\mathcal{A}$ is sequentially complete and locally convex then there exist local similarity cross sections using the holomorphic functional calculus in one variable and rational operations. In this case there exists a neighborhood W_2 of T in $\mathcal{A}$ and a holomorphic map $\omega_2 : W_2 \to \mathcal{A}^{-1}$ (holomorphic in the sense of infinite dimensional holomorphy) such that*

$$\varphi_2 := \omega_2|_{W_2 \cap S_{\mathcal{A}^{-1}}(T)} : W_2 \bigcap S_{\mathcal{A}^{-1}}(T) \to \mathcal{A}^{-1}$$

is a local cross section of π^T.

We have $\varphi_1 = \varphi_2$ on $W_1 \bigcap W_2 \bigcap S_{\mathcal{A}^{-1}}(T)$ in our construction.

For the nilpotent case (or $\sigma(T) = \{\lambda_0\}$) we obtain the following (see Theorem 4.13):

THEOREM 0.5. *Let $T \in \mathcal{A}$ be a nice Jordan operator with $\sigma(T) = \{\lambda_0\}$. Then the conjugation operation $\pi^T : \mathcal{A}^{-1} \to S_{\mathcal{A}^{-1}}(T)$ admits local cross sections, which are restrictions of rational functions to the similarity orbit.*

To prove these results we need to find a Jordan decomposition within $\mathcal{A}$ for elements of $\mathcal{A}$, which are at the same time Jordan operators in $L(H)$. This is done in Sections 2 and 3 and uses extensively calculations on regular elements of $\mathcal{A}$ and $L(H)$. Our considerations lead to the following:

THEOREM 0.6. (see Theorems 2.1, 3.2, 3.10). *$T \in \mathcal{A}$ has a Jordan decomposition within $\mathcal{A}$ iff it has one in $L(H)$. The set of Jordan operators of $\mathcal{A}$ is the union of similarity orbits, each orbit having a locally-$\mathcal{A}$-rational manifold structure. If $\sigma(T) \subset \mathbf{R}$ (for example T nilpotent) then T^* is similar to T by a group element of $\mathcal{A}$.*

The last section deals with a functional analytic description of the homogeneous topology on the similarity orbit of a Jordan operator of $\mathcal{A}$. This topology usually differs from the underlying topology of $\mathcal{A}$. For example, if $\mathcal{A} = L(H)$, then these topologies coincide iff the Jordan operator T is nice. To characterize the homogeneous topology for general Jordan operators we introduce a gap topology on the similarity orbit (see Definition 4.1). We show the equivalence of these topologies (see Theorem 4.2). We further show that the gap conditions, measured in the weaker topology of $L(H)$ instead of $\mathcal{A}$, also give an equivalent topology to the homogeneous topology (see Theorem 4.11). This is a very good example of how spectral invariance and the analysis of regular operators lead to perturbation results. At the end we have all the tools and structure theorems to finally prove Theorem 0.3.

1. PRELIMINARIES

NOTATION 1.1. In this paper we consider continuously embedded topological subalgebras $\mathcal{A}$ (multiplication is assumed to be jointly continuous in both variables) of the algebra $L(H)$ of bounded linear operators on the Hilbert space H with the following properties:

(i) $\mathcal{A} = \mathcal{A}^*$,

(ii) $e \in \mathcal{A}$ and $\mathcal{A} \bigcap L(H)^{-1}$ (spectral invariance),

(iii) $\mathcal{A}$ is with continuous inversion,

(iv) the regular elements $\mathcal{R}(\mathcal{A})$ and $\mathcal{R}(L(H))$ of $\mathcal{A}$ and $L(H)$ (see Definition 1.5 below) satisfy $\mathcal{A} \bigcap \mathcal{R}(L(H)) = \mathcal{R}(\mathcal{A})$.

These assumptions should always hold throughout this article. If (i) - (iv) is fulfilled for $\mathcal{A} \subseteq L(H)$, we call $\mathcal{A}$ a *Ψ^*-algebra.*

Note that (ii) implies that the group $\mathcal{A}^{-1}$ is open, even in the norm topology of $L(H)$. So if $\mathcal{A}$ is Fréchet then it is known that inversion is always continuous within $\mathcal{A}$ (see [32]). If (i), (ii), (iii) are assumed together with a holomorphic functional calculus in one variable on $\mathcal{A}$ ($\mathcal{A}$ for example locally convex and sequentially complete), then (iv) can always be obtained (see Remark 1.8 below).

We don't assume $\mathcal{A}$ to be Fréchet (as in the definition of Ψ^*-algebras in [13] in connection with pseudo-differential operators), so that our results can also be applied for example to the algebra $\mathcal{A} = \mathcal{H}(S) \otimes_{\varepsilon} \mathcal{B} \subseteq C(S, \mathcal{B})$, where $S \subset \mathbf{R}^N$ is a compact identification set and $\mathcal{B}$ a C^*-algebra (see [26], Anhang C). We even don't make use of completeness, so that also algebras of rational matrix functions can be considered (see Examples 2.20, 4.15).

For the following notations we only assume (i) - (iii): in $\mathcal{A}$ we denote by $\mathcal{P}(\mathcal{A})$ the set of all projections and by $\mathcal{P}_{\perp}(\mathcal{A})$ the orthogonal projections. In $\mathcal{P}(\mathcal{A})$ we have the equivalence relation

$$p \sim q, \quad p, q \in \mathcal{P}(\mathcal{A}) \ :\Longleftrightarrow pq = q \ \text{ and } \ qp = p \iff R(p) = R(q).$$

$\Gamma(\mathcal{A})$ is the set of the equivalence classes, X_p, $p \in \mathcal{P}(\mathcal{A})$. It is a homogeneous space under the similarity operation on representants and $\Gamma(\mathcal{A})$ will be considered in this homogeneous topology (see [13], Section 2). It is known that for $\mathcal{A} = L(H)$ the homogeneous topology on $\Gamma(L(H))$ is equivalent to the gap topology on closed subspaces, i.e. $d(X, Y) = \|P_X - P_Y\|$ where P_X and P_Y are the corresponding orthogonal projections on the closed subspaces $X, Y \subseteq H$ (see for example [21], [13], 4.13, [26], 1.4.3).

For $T \in \mathcal{A}$ we denote by $S_{\mathcal{A}^{-1}}(T) := \{gTg^{-1} : g \in \mathcal{A}^{-1}\}$ and $S_{L(H)^{-1}}(T) := \{gTg^{-1} : g \in L(H)^{-1}\}$ the similarity orbits of T in $\mathcal{A}$ or $L(H)$ respectively.

REMARK 1.2. Let $\mathcal{A} \subseteq L(H)$ with (i) and (ii). For $p \in \mathcal{P}(\mathcal{A})$, the unique orthogonal projection $p_{\perp}$ on $R(p)$ is given by the $*$-rational formula

$$p_{\perp} = pp^*(e - (p - p^*)^2)^{-1} \qquad (\text{also } = p(e + p - p^*)^{-1})$$

(see [20], Theorem A, p.20 and [28], 2.15) and is contained in $\mathcal{A}$ by the property of spectral invariance.

DEFINITION 1.3. We put

$$q_k := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \in L(\mathbf{C}^k)$$

(for a suitable orthonormal basis in $\mathbf{C}^k$) and call $J \in L(H)$ a *Jordan operator* (see [19], 7.4) if J is similar to an operator of the form

$$J = \bigoplus_{j=1}^{n} \Big[\lambda_j \mathbf{1}_{H_j} + \bigoplus_{i=1}^{k_j} q_{n_i^{(j)}}^{(\alpha_{ij})} \Big],$$

where $n \in \mathbf{N}$, $\lambda_1, \dots, \lambda_n$ distinct complex numbers and $1 \leqslant \alpha_{ij} \leqslant \infty$, $k_j \in \mathbf{N}$, $n_i^{(j)} \in \mathbf{N}$.

J is called *nice* if $\alpha_{ij} = \infty$ for at most one $i \in \{1, \dots, k_j\}$ for each $j \in \{1, \dots, n\}$.

It is known that $J \in L(H)$ is a Jordan operator in $L(H)$ iff J is algebraic and $R(q(J))$ is closed for every polynomial q dividing the minimal polynomial p of J (see [19], 7.13). We call $J\,\mathrm{Alg}(L(H))$ the set of all Jordan operators and $JN(L(H))$ the set of Jordan nilpotents in H.

To define $J \in \mathcal{A}$ to be Jordan operator in the Algebra $\mathcal{A}$, we want the existence of a system of projections in $\mathcal{A}$, such that these projections reduce J in a way it can be reduced as a Jordan operator in $L(H)$. We give the following rather long definition, which expresses the idea of a Jordan decomposition of $J \in \mathcal{A}$ within $\mathcal{A}$ in a set of algebraic relations, which we require to hold in $\mathcal{A}$:

DEFINITION 1.4. We call $J \in \mathcal{A}$ a *Jordan operator* (or *Jordan element*) within $\mathcal{A}$, if the following holds:

(i) There exists a natural number n and projections $p^{(1)}, \dots, p^{(n)} \in \mathcal{A}$ such that:

$$\text{(a) } p^{(1)} + \dots + p^{(n)} = e, \quad \text{(b) } p^{(i)} p^{(j)} = \delta_{ij} p^{(i)} \quad (i, j \in \{1, \dots, n\}).$$

(ii) For every $j \in \{1, \dots, n\}$ there exists a $k_j \in \mathbf{N}$ and projections $p_1^{(j)}, \dots,$ $\dots, p_{k_j}^{(j)} \in \mathcal{A}$ such that:

$$\text{(a) } p_1^{(j)} + \dots + p_{k_j}^{(j)} = p^{(j)}, \quad \text{(b) } p_k^{(j)} p_l^{(j)} = \delta_{kl} p_k^{(j)} \quad (k, l \in \{1, \dots, k_j\}).$$

(iii) For every $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, k_j\}$ there exists an $n_i^{(j)} \in \mathbb{N}$ and projections $p_{i,1}^{(j)}, \dots, p_{i,n_i^{(j)}}^{(j)} \in \mathcal{A}$ such that:

(a) $1 \leqslant n_1^{(j)} < n_2^{(j)} < \dots < n_{k_j}^{(j)}$,

(b) $p_{i,1}^{(j)} + \dots + p_{i,n_i^{(j)}}^{(j)} = p_i^{(j)}$,

(c) $p_{i,k}^{(j)} p_{i,l}^{(j)} = \delta_{kl} p_i^{(j)} \quad (k, l \in \{1, \dots, n_i^{(j)}\})$.

(iv) For every $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, k_j\}$ there exist elements (matrix units in C^*-algebras) $I_{r,s}^{j,i} \in p_{i,r}^{(j)} \mathcal{A} p_{i,s}^{(j)} \subseteq \mathcal{A}$ $(r, s \in \{1, \dots, n_i^{(j)}\})$ such that:

$$\text{(a) } I_{r,r}^{j,i} = p_{i,r}^{(j)}, \quad \text{(b) } I_{r,s}^{j,i} \cdot I_{s,t}^{j,i} = I_{r,t}^{j,i} \quad (r, s, t \in \{1, \dots, n_i^{(j)}\}).$$

(v) There exist n distinct complex numbers $\lambda_1, \dots, \lambda_n$ such that the following relations hold:

(a) $p^{(i)} J p^{(j)} = 0$ for $i, j \in \{1, \dots, n\}$ such that $i \neq j$,

(b) $p_k^{(j)} J p_l^{(j)} = 0$ for $k \neq l$ $(j \in \{1, \dots, n\}, k, l \in \{1, \dots, k_j\}$,

(c) $p_{i,r}^{(j)} J p_{i,s}^{(j)} = 0$ for $r < s - 1$ and $r > s$ $(r, s \in \{1, \dots, n_i^{(j)}\})$,

(d) $p_{i,r}^{(j)} J p_{i,r}^{(j)} = \lambda_j p_{i,r}^{(j)}$ $\forall r \in \{1, \dots, n_i^{(j)}\}$,

(e) $p_{i,r}^{(j)} J p_{i,r+1}^{(j)} = I_{r,r+1}^{j,i}$ $\forall r \in \{1, \dots, n_i^{(j)} - 1\}$.

In this case J can be written as

$$J = \sum_{j=1}^{n} \sum_{i=1}^{k_j} \sum_{r=1}^{n_i^{(j)}} \lambda_j p_{i,r}^{(j)} + \sum_{j=1}^{n} \sum_{i=1}^{k_j} \sum_{r=1}^{n_i^{(j)}-1} I_{r,r+1}^{j,i} = \sum_{j=1}^{n} \left[\lambda_j p^{(j)} + \sum_{i=1}^{k_j} \sum_{r=1}^{n_i^{(j)}-1} I_{r,r+1}^{j,i} \right].$$

We denote by $J \operatorname{Alg}(\mathcal{A})$ the set of all Jordan operators and by $JN(\mathcal{A})$ the set of Jordan nilpotents in $\mathcal{A}$.

One aim of this paper is to show that all these algebraic relations, which I for example essentially needed in the construction of $*$-rational local similarity cross section, in fact can be realized within $\mathcal{A}$ only if $J \in \mathcal{A} \bigcap J \operatorname{Alg}(L(H))$ (see Theorem 3.2 below, $\mathcal{A} \subseteq L(H)$ a Ψ^*-algebra). Symbolically

$$\mathcal{A} \bigcap J \operatorname{Alg}(L(H)) = J \operatorname{Alg}(\mathcal{A}).$$

DEFINITION 1.5. Let $\mathcal{A} \subseteq L(H)$ with (i) - (iii) of Notation 1.1. We call $a \in \mathcal{A}$ a *regular element* (or *relatively invertible* or *pseudo-invertible*) if there exists $\tilde{a} \in \mathcal{A}$ such that

$$a\tilde{a}a = a \quad \text{and} \quad \tilde{a}a\tilde{a} = \tilde{a}.$$

$\mathcal{R}_a$ denotes the set of all such $\tilde{a}$ and $\mathcal{R}(\mathcal{A})$ the set of all regular elements in $\mathcal{A}$. $\mathcal{R}(L(H))$ is the set of operators with closed range. For $a \in \mathcal{R}(\mathcal{A})$, $\tilde{a} \in \mathcal{R}_a$ we put

$$u(b) := u_{a,\tilde{a}}(b) := \tilde{a}(e + (b - a)\tilde{a})^{-1} ,$$

defined in a neighborhood $W(a) = W_{\tilde{a}}(a)$ of a ($\mathcal{A}^{-1}$ is open).

We state the following propositions, which are by now basic facts in Functional Analysis when dealing with regular elements in operator algebras:

PROPOSITION 1.6. *Let $b \in \mathcal{A}$ such that $u(b)$ exists ($a, \tilde{a}$ fixed). Then*
(i) $u(b)bu(b) = u(b); (u(b)b)^2 = u(b)b; (bu(b))^2 = bu(b)$;
(ii) $u(b) = \tilde{a}(e + (b - a)\tilde{a})^{-1} = (e + \tilde{a}(b - a))^{-1}\tilde{a}$;
(iii) $u(b)b = e - (e + \tilde{a}(b - a))^{-1}(e - \tilde{a}a)$;
(iv) $u(b)b\tilde{a}a = \tilde{a}a$; $\tilde{a}au(b)b = u(b)b$.

Proof. See [13], Definition 4.1. ∎

THEOREM 1.7. *Let $\mathcal{A} \subseteq L(H)$ with* (i) *and* (iii) *of Notation* 1.1. *The set $\mathcal{R} := \mathcal{R}(\mathcal{A})$ is a homogeneous space under the group action*

$$\pi : \mathcal{A}^{-1} \times \mathcal{A}^{-1} \times \mathcal{R} \longrightarrow \mathcal{R}, \qquad \pi(g, \dot{g}, a) := ga\dot{g}^{-1}$$

(note that $\tilde{a} \in \mathcal{R}_a \Longrightarrow \dot{g}\tilde{s}g^{-1} \in \mathcal{R}_{ga\dot{g}^{-1}}$). $\mathcal{R}$ carries a natural topology $\tau(\mathcal{R})$, finer than $\tau(\mathcal{A})$, which can be described in three equivalent ways:
(i) *the homogeneous topology defined by the above group action.*
(ii) *the topology given by the following system of neighborhoods of $a \in \mathcal{R}$:*

$$V_{\tilde{a}}(a) := \{b \in W_{\tilde{a}} : b = bu_{a,\tilde{a}}(b)b\}, \quad \tilde{a} \in \mathcal{R}_a.$$

(iii) *the coarsest topology on $\mathcal{R}$, such that the maps*

$$\mathcal{R} \ni b \longmapsto b \in (\mathcal{A}, \tau(\mathcal{A})) \quad \text{and} \quad \mathcal{R} \ni b \longmapsto \ker b := X_{e-\tilde{b}b} \in (\Gamma(\mathcal{A}), \tau(\Gamma(\mathcal{A})))$$

are continuous (note that $\ker b = X_{e-\tilde{b}b}$ *is independent on $\tilde{b} \in \mathcal{R}_b$).*

Proof. See [13], 4.2–4.7. ∎

REMARK 1.8. Let $\mathcal{A} \subseteq L(H)$ with (i), (ii), (iii) of Notation 1.1 and holomorphic functional calculus. For $b \in \mathcal{A} \bigcap \mathcal{R}(L(H))$ one has the unique orthogonal pseudo-inverse $\tilde{b}$ of b (i.e. $b\tilde{b}$ and $e - \tilde{b}b$ are orthogonal projections) within the algebra $\mathcal{A}$. It is given by the formula

$$\tilde{b} = (p + b^*b)^{-1}b^*,$$

where $p \in \mathcal{A}$ is the orthogonal projection onto $N(b) = N(b^*b)$, which can be expressed as a Cauchy integral in $\mathcal{A}$. So

$$\mathcal{A} \bigcap \mathcal{R}(L(H)) = \mathcal{R}(\mathcal{A}),$$

and this relation also holds topologically. For reference see ([13], 5.7; [27], 1.5, 3.3, 3.5, 3.8, 3.9 and 1.9).

PROPOSITION 1.9. *The topology $\tau(\mathcal{R}(\mathcal{A}))$ is equivalent to $\tau(\mathcal{A})$ together with the restriction of $\tau(\mathcal{R}(L(H)))$ on $\mathcal{R}(\mathcal{A})$.*

Proof. We have to show that $\tau(\mathcal{A}) \bigcap \tau(\mathcal{R}(L(H)))|_{\mathcal{R}(\mathcal{A})}$ is finer than $\tau(\mathcal{R}(\mathcal{A}))$. To do so we fix an $a \in \mathcal{A}$, $\tilde{a} \in \mathcal{R}(\mathcal{A})_a$ and consider the function $u_{a,\tilde{a}}$, defined in a $L(H)$-neighborhood of a. For b in this neighborhood intersected with $\mathcal{A}$ the function u takes values in $\mathcal{A}$ because of spectral invariance. Now the topology $\tau(\mathcal{R}(L(H)))|_{\mathcal{R}(\mathcal{A})}$ forces $b = bu(b)b$ locally and this algebraic relation also holds within $\mathcal{A}$. But this gives the topology $\tau(\mathcal{R}(\mathcal{A}))$, since u is continuous with respect to $\tau(\mathcal{A})$ and $\tau(\mathcal{A})$ is assumed to be contained in the topology on $\mathcal{R}(\mathcal{A})$ (see Theorem 1.7). ∎

LEMMA 1.10. *Let $\mathcal{A} \subset L(H)$ be a Ψ^*-algebra as in Notation 1.1. Then for $a \in \mathcal{R}(\mathcal{A})$ the unique orthogonal pseudo-inverse of a is in $\mathcal{A}$.*

Proof. Let $a \in \mathcal{R}(\mathcal{A})$, $\tilde{a} \in \mathcal{R}(\mathcal{A})_a$. Put

$$\tilde{\tilde{a}} := (e - (e - \tilde{a}a)_\perp) \cdot \tilde{a} \cdot (a\tilde{a})_\perp \;\in \mathcal{A}$$

(see [14], 4.5, [18], 6). Then

$$\begin{aligned} a\tilde{\tilde{a}} &= a\tilde{a}(a\tilde{a})_\perp - \underbrace{a(e - \tilde{a}a)_\perp}_{=a(e-\tilde{a}a)(e-\tilde{a}a)_\perp=0} \tilde{a}(a\tilde{a})_\perp \\ &= (a\tilde{a})_\perp \in \mathcal{P}_\perp(\mathcal{A}) \end{aligned}$$

and

$$\begin{aligned} e - \tilde{\tilde{a}}a &= e - (e - (e - \tilde{a}a)_\perp)\tilde{a}\underbrace{(a\tilde{a})_\perp a}_{=a} \\ &= e - \underbrace{= (e - (e - \tilde{a}a)_\perp) \cdot (e - (e - \tilde{a}a))}_{(e-(e-\tilde{a}a)_\perp)} \\ &= (e - \tilde{a}a)_\perp \in \mathcal{P}_\perp(\mathcal{A}). \end{aligned}$$

From this we obtain $a\tilde{\tilde{a}}a = (a\tilde{a})_\perp a = a$ and $\tilde{\tilde{a}}a\tilde{\tilde{a}} = \tilde{\tilde{a}}(a\tilde{a})_\perp = \tilde{\tilde{a}}$. ∎

2. CONSTRUCTION OF JORDAN NORMAL FORMS IN THE NILPOTENT CASE

In this section we show the following

THEOREM 2.1. *Let $\mathcal{A} \subseteq L(H)$ be a Ψ^*-algebra and let $b \in \mathcal{A}$ nilpotent of order n. Then b is a Jordan element in $\mathcal{A}$ iff $R(b^\nu)$ is closed for all $1 \leqslant \nu \leqslant n-1$.*

We first give a short outline of the proof of Theorem 2.1:

The implication that Jordan nilpotents $b \in \mathcal{A}$ have closed ranges $R(b^\nu)$ $(1 \leqslant \nu \leqslant n-1)$ is obvious and also follows directly from the known result of D.A. Herrero on the general Hilbert space situation ([19], 7.11). The crucial point is to prove the reverse implication and for this to construct projections within the algebra $\mathcal{A}$ for the occuring Jordan reducing subspaces of H. To begin, we take the orthogonal pseudo-inverses $\widetilde{b^\nu} \in \mathcal{A}$ of the powers b^ν for all $1 \leqslant \nu \leqslant n-1$. With these pseudo-inverses we have the orthogonal projections on $N(b^\nu)$ and $N(b^\nu) \ominus N(b^{\nu-1})$ in the algebra $\mathcal{A}$ and we have the natural triangular decomposition of the nilpotent b within the algebra. So we write

$$b = \begin{pmatrix} 0 & b_{12} & \dots & \dots & b_{1n} \\ 0 & 0 & b_{23} & \dots & b_{2n} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & b_{n-1,n} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \qquad b_{ij} = Q_i b Q_j,$$

where $Q_i \longrightarrow N(b^i) \ominus N(b^{i-1})$, $Q_j \longrightarrow N(b^j) \ominus N(B^{j-1})$.

To construct now a Jordan normal form for b, we proceed as in finite dimensional linear algebra (see for example [5], Section 2.9), and consider the injective induced mappings

$$H/N(b^{n-1}) \xrightarrow{\bar{b}_{n-1}} N(b^{n-1})/N(b^{n-2}) \xrightarrow{\bar{b}_{n-2}} \cdots \xrightarrow{\bar{b}_2} N(b_2)/N(b) \xrightarrow{\bar{b}_1} N(b) \longrightarrow 0,$$

defined by

$$\bar{b}_\nu(x + N(b^\nu)) := bx + N(b^{\nu-1}), \quad x \in N(b^{\nu+1}), \quad 1 \leqslant \nu \leqslant n-1.$$

These induced mappings are in fact completely determined by the first upper diagonal entries of the triangular decomposition of b.

To obtain a Jordan normal form one constructs a projected complement of the image of $\bar{b}_\nu$ in each quotient space $N(b^\nu)/N(b^{\nu-1})$. From this complement one passes to a maximal linear independent system of generating vectors in $N(b^\nu)$.

This generating system (which, as it turns out, can be chosen to be projected in $\mathcal{A}$), together with the iterated applications of b, form Jordan chains of length ν. Combining all the possible length ($\leqslant n$) of Jordan chains one obtains a Jordan normal form of b.

The critical point in construction is that we require every occuring subspace related to the operator b to be projected in $\mathcal{A}$. This will lead at the end to a Jordan decomposition of b within the algebra $\mathcal{A}$.

The whole proof is constructive.

Now we pass to the proof of Theorem 2.1:

LEMMA 2.2. *Under the assumptions of Theorem* 2.1 *there exists for every* $\nu \in \{1, \ldots, n-1\}$ *a pseudo-inverse* $\widetilde{b^\nu} \in \mathcal{A}$ *for* b^ν *such that*

(i) $b^\nu \widetilde{b^\nu}$ *is the orthogonal projection onto* $R(b^\nu)$, *and*

(ii) $e - \widetilde{b^\nu} b^\nu$ *is the orthogonal projection onto* $N(b^\nu)$.

For $\nu = 0$, $b^0 = e$ *we put* $\widetilde{b^0} := e$; *and for* $\nu = n$, $b^n = 0$ *let* $\widetilde{b^n} := 0$.

Proof. This is a consequence of Lemma 1.10. ∎

REMARK 2.3. For $\mu \leqslant \nu$ we have $(b^\mu \widetilde{b^\mu})(b^\nu \widetilde{b^\nu}) = b^\nu \widetilde{b^\nu}$ and from the orthogonality of these projections we also get (take *) $(b^\nu \widetilde{b^\nu})(b^\mu \widetilde{b^\mu}) = b^\nu \widetilde{b^\nu}$. Therefore we have for all $\mu, \nu \leqslant n$

$$(b^\mu \widetilde{b^\mu})(b^\nu \widetilde{b^\nu}) = b^\varrho \widetilde{b^\varrho}, \quad \varrho = \max(\mu, \nu).$$

Similarly one obtains

$$(e - \widetilde{b^\mu} b^\mu)(e - \widetilde{b^\nu} b^\nu) = e - \widetilde{b^\varrho} b^\varrho, \quad \varrho = \min(\mu, \nu).$$

DEFINITION 2.4. Let $P_j := e - \widetilde{b^j} b^j$, $1 \leqslant j \leqslant n-1$, $P_n := e$.

By Remark 2.3 we have $P_i P_j = P_{\min(i,j)} \forall i, j \leqslant n$.

Further define $Q_1 := P_1$ and $Q_\nu := P_\nu - P_{\nu-1}$, $2 \leqslant \nu \leqslant n$. Then $Q_\nu = Q_\nu^2 = Q_\nu^* \in \mathcal{A}$, $Q_\nu Q_\mu = \delta_{\nu\mu} Q_\nu$ and $\sum_{\nu=1}^{n} Q_\nu = e$.

LEMMA 2.5.

(i) $bP_1 = 0$, $bP_j = P_{j-1} b P_j$ *for* $2 \leqslant j \leqslant n$;

(ii) $Q_i b Q_j = 0$ *for* $i \geqslant j$.

Proof. (i) $bP_1 = b(e - \tilde{b} b) = 0$. Now let $j \in \{2, \ldots, n\}$

$$\begin{aligned} P_{j-1} b P_j &= (e - \widetilde{b^{j-1}} b^{j-1}) b (e - \widetilde{b^j} b^j) = (b - \widetilde{b^{j-1}} b^j)(e - \widetilde{b^j} b^j) \\ &= b - b \widetilde{b^j} b^j = b(e - \widetilde{b^j} b^j) = bP_j. \end{aligned}$$

(ii) Let $i \geqslant j$. For $j = 1$ we have for arbitrary i: $Q_i b Q_1 = Q_i b P_1 = 0$. Now let $j \geqslant 2$ (then $i \geqslant 2$)

$$\begin{aligned} Q_i b Q_j &= (P_i - P_{i-1}) b (P_j - P_{j-1}) \\ &= P_i b P_j - P_i b P_{j-1} - P_{i-1} b P_j + P_{i-1} b P_{j-1} \\ &= \underbrace{P_i P_{j-1}}_{=P_{j-1}} b P_j - \underbrace{P_i P_{j-2}}_{=P_{j-2}} b P_{j-1} - \underbrace{P_{i-1} P_{j-1}}_{=P_{j-1}} b P_j + \underbrace{P_{i-1} P_{j-2}}_{=P_{j-2}} b P_{j-1} \\ &= 0. \quad \blacksquare \end{aligned}$$

LEMMA 2.6. *Let $1 \leqslant j \leqslant n-1$. Then the induced map*

$$\bar{b}_j : N(b^{j+1})/N(b^j) \longrightarrow N(b^j)/N(b^{j-1})$$

is given by $Q_j b Q_{j+1}$.

Proof. Let $x \in N(b^{j+1})$, that is $x = P_{j+1}x$. Then $x + N(b^j) = P_{j+1}x + N(b^j) = Q_{j+1}x + \underbrace{P_j x}_{\in N(b^j)} + N(b^j) = Q_{j+1}x + N(b^j)$. Therefore

$$\begin{aligned} \bar{b}_j(x + N(b^j)) &= \bar{b}_j(Q_{j+1}x + N(b^j)) = bQ_{j+1}x + N(b^{j-1}) \\ &= Q_j b Q_{j+1} x + \underbrace{P_{j-1} b Q_{j+1} x}_{\in N(b^{j-1})} + N(b^{j-1}) \\ &= (Q_j b Q_{j+1})x + N(b^{j-1}). \quad \blacksquare \end{aligned}$$

LEMMA 2.7. *Let $1 \leqslant \nu \leqslant n-1$. There exists $(Q_\nu b Q_{\nu+1})^\sim \in Q_{\nu+1} \mathcal{A} Q_\nu \cap \mathcal{R}_{Q_\nu b Q_{\nu+1}}$ such that $(Q_\nu b Q_{\nu+1})^\sim (Q_\nu b Q_{\nu+1}) = Q_{\nu+1}$.*

Proof. We have

$$\begin{aligned} Q_\nu b Q_{\nu+1} &= (\widetilde{b^{\nu-1}} b^{\nu-1} - \widetilde{b^\nu} b^\nu) b (\widetilde{b^\nu} b^\nu - \widetilde{b^{\nu+1}} b^{\nu+1}) \\ &= (\widetilde{b^{\nu-1}} b^\nu - \widetilde{b^\nu} b^{\nu+1})(\widetilde{b^\nu} b^\nu - \widetilde{b^{\nu+1}} b^{\nu+1}) \\ &= \widetilde{b^{\nu-1}} b^\nu - \widetilde{b^{\nu-1}} b^\nu \widetilde{b^{\nu+1}} b^{\nu+1} - \widetilde{b^\nu} b^{\nu+1} + \widetilde{b^\nu} b^{\nu+1} \\ &= \widetilde{b^{\nu-1}} b^\nu (e - \widetilde{b^{\nu+1}} b^{\nu+1}) = \widetilde{b^{\nu-1}} b^\nu P_{\nu+1}. \end{aligned}$$

Put $(Q_\nu b Q_{\nu+1})^\sim := P_{\nu+1} \widetilde{b^\nu} b^{\nu-1} Q_\nu = \underbrace{P_\nu \widetilde{b^\nu}}_{=0} b^{\nu-1} Q_\nu + Q_{\nu+1} \widetilde{b^\nu} b^{\nu-1} Q_\nu \in Q_{\nu+1} \mathcal{A} Q_\nu$.

Then

$$\begin{aligned} (Q_\nu b Q_{\nu+1})^\sim (Q_\nu b Q_{\nu+1}) &= P_{\nu+1} \widetilde{b^\nu} b^{\nu-1} \underbrace{Q_\nu \widetilde{b^{\nu-1}} b^\nu P_{\nu+1}}_{=\widetilde{b^{\nu-1}} b^\nu P_{\nu+1}} \\ &= P_{\nu+1} \widetilde{b^\nu} \underbrace{b^{\nu-1} \widetilde{b^{\nu-1}} b^\nu}_{=b^\nu} P_{\nu+1} \\ &= \underbrace{(e - \widetilde{b^{\nu+1}} b^{\nu+1}) \widetilde{b^\nu} b^\nu}_{=Q_{\nu+1}} P_{\nu+1} = Q_{\nu+1}. \end{aligned}$$

From this we get

$$(Q_\nu bQ_{\nu+1})\cdot(Q_\nu bQ_{\nu+1})^\sim\cdot(Q_\nu bQ_{\nu+1})=(Q_\nu bQ_{\nu+1})$$

and

$$(Q_\nu bQ_{\nu+1})^\sim\cdot(Q_\nu bQ_{\nu+1})\cdot(Q_\nu bQ_{\nu+1})^\sim=(Q_\nu bQ_{\nu+1})^\sim. \quad \blacksquare$$

DEFINITION 2.8. For $j\in\{1,\ldots,n-1\}$ we define

$$R_j:=Q_j-(Q_jbQ_{j+1})(Q_jbQ_{j+1})^\sim,$$

and

$$R_n:=Q_n.$$

From $Q_\nu Q_\mu=\delta_{\nu\mu}Q_\nu$ we get $R_\nu R_\mu=\delta_{\nu\mu}R_\nu$ for all $\nu,\mu\leqslant n$.

Now we look at the iterated applications of b on R_j : $b^\nu R_j$, $\nu=1,\ldots,j-1$. For $\nu=j$ we have $b^jR_j=b^jQ_jR_j=0$.

LEMMA 2.9. *Let* $j\in\{2,\ldots,n\}$, $\nu\in\{1,\ldots,j-1\}$. *Then*

$$(R_j\widetilde{b^\nu})\cdot(b^\nu R_j)=R_j.$$

In particular $R_j\widetilde{b^\nu}\in\mathcal{R}_{b^\nu R_j}$.

Proof. $\nu\leqslant j-1\Longrightarrow\widetilde{b^\nu}b^\nu Q_j=\widetilde{b^\nu}b^\nu(\widetilde{b^{j-1}}b^{j-1}-\widetilde{b^j}b^j)=Q_j.$

$$\Longrightarrow R_j\widetilde{b^\nu}b^\nu R_j=R_j\widetilde{b^\nu}b^\nu Q_jR_j=R_jQ_jR_j=R_j. \quad \blacksquare$$

DEFINITION 2.10. For $j\in\{1,\ldots,n\}$ we define

$$S_\nu^{(j)}:=b^{j-\nu}R_j\widetilde{b^{j-\nu}},\quad \nu=1,\ldots,j.$$

Then $S_j^{(j)}=R_j$ and from Lemma 2.9 we get that all $S_\nu^{(j)}$ are idempotents in $\mathcal{A}$.

LEMMA 2.11. *We have*
(i) $bS_1^{(j)}=0$ *for all* $1\leqslant j\leqslant n$.
(ii) $bS_\nu^{(j)}=S_{\nu-1}^{(j)}bS_\nu^{(j)}$ *for all* $2\leqslant j\leqslant n$ *and* $2\leqslant\nu\leqslant j$.

Proof. (i) $bS_1^{(j)}=bb^{j-1}R_j\widetilde{b^{j-1}}=\underbrace{b^jR_j}_{=0}\widetilde{b^{j-1}}=0.$

(ii)
$$S^{(j)}_{\nu-1} b S^{(j)}_\nu = b^{j-\nu+1} R_j \widetilde{b^{j-\nu+1}} b b^{j-\nu} R_j \widetilde{b^{j-\nu}}$$
$$= b^{j-\nu+1} R_j \underbrace{\widetilde{b^{j-\nu+1}} b^{j-\nu+1} Q_j}_{=Q_j} R_j \widetilde{b^{j-\nu}}$$

(since $j-\nu+1 \leqslant j-1$ because of $\nu \geqslant 2$),

$$= b \cdot b^{j-\nu} R_j \widetilde{b^{j-\nu}} = b \cdot S^{(j)}_\nu. \quad \blacksquare$$

LEMMA 2.12. *Let $2 \leqslant j \leqslant n$ and $2 \leqslant \nu \leqslant j$. Then*

$$S^{(j)}_{\nu-1} b S^{(j)}_\nu : S^{(j)}_\nu(H) \longrightarrow S^{(j)}_{\nu-1}(H) \quad \text{is bijective.}$$

Proof. (i) Injectivity:
Let $S^{(j)}_{\nu-1} b S^{(j)}_\nu x = 0, x = S^{(j)}_\nu x \in S^{(j)}_\nu(H)$. Then $bS^{(j)}_\nu x = 0$ by Lemma 2.11. Now
$$0 = bS^{(j)}_\nu x = b b^{j-\nu} \underbrace{R_j \widetilde{b^{j-\nu}} x}_{=: y \in R_j(H) \subseteq Q_j(H)}$$
$$= b^{j-\nu+1} y.$$

Since $j-\nu+1 \leqslant j-1$ and $y \in Q_j(H) = N(b^j) \ominus N(b^{j-1})$ it follows $y = 0$ and therefore $S^{(j)}_\nu x = b^{j-\nu} y = 0$.

(ii) Surjectivity:
Let $y \in S^{(j)}_{\nu-1}(H)$ be given. Then $y = S^{(j)}_{\nu-1} y = b^{j-\nu+1} R_j \widetilde{b^{j-\nu+1}} y$. Put $x := b^{j-\nu} R_j \widetilde{b^{j-\nu+1}} \overset{2.9.}{=} b^{j-\nu} R_j \widetilde{b^{j-\nu}} b^{j-\nu} R_j \widetilde{b^{j-\nu+1}} y \in S^{(j)}_\nu(H)$.
Then $bx = b^{j-\nu+1} R_j \widetilde{b^{j-\nu-1}} y = S^{(j)}_{\nu-1} y = y$. $\blacksquare$

LEMMA 2.13. *We have*
(i) $H/N(b^{n-1}) = (R_n(H) + N(b^{n-1}))/N(b^{n-1})$,
(ii) $N(b^j)/N(b^{j-1}) = (R_j(H) + N(b^{j-1}))/N(b^{j-1}) \dot{+} \bar{b}_j(N(b^{j+1})/N(b^j))$ *for* $1 \leqslant j \leqslant n-1$.

Proof. (i) This is obvious, since $R_n = Q_n$ by definition and $Q_n(H) + N(b^{n-1}) = H$.

(ii)

(a) Let $y + N(b^{j-1}) \in N(b^j)/N(b^{j-1})$ be given, $y = Q_j y$ without loss of

generality.
$$\Longrightarrow y = Q_j y = R_j y + (Q_j b Q_{j+1}) \underbrace{(Q_j b Q_{j+1})^{\sim} y}_{=: x \in Q_{j+1}(H) \subseteq N(b^{j+1})}$$

$$\Longrightarrow y + N(b^{j-1}) = (R_j y + N(b^{j-1})) + ((Q_j b Q_{j+1})x + N(b^{j-1}))$$
$$\overset{2.6}{=} (R_j y + N(b^{j-1})) + \bar{b}_j(x + N(b^j)).$$

(b) Let $R_j x + N(b^{j-1}) = \bar{b}_j(y + N(b^j)) = Q_j b Q_{j+1} y + N(b^{j-1})$

$$\Longrightarrow z := R_j x - Q_j b Q_{j+1} y \in N(b^{j-1}) = P_{j-1}(H)$$
$$\Longrightarrow z = P_{j-1} z = \underbrace{P_{j-1} R_j}_{=0} x - \underbrace{P_{j-1} Q_j}_{=0} b Q_{j+1} y = 0$$
$$\Longrightarrow R_j x = Q_j b Q_{j+1} y$$

multiplication by $R_j = Q_j - (Q_j b Q_{j+1})(Q_j b Q_{j+1})^{\sim}$ from the left yields

$$R_j x = 0 = Q_j b Q_{j+1}. \quad \blacksquare$$

LEMMA 2.14.
$$H = \dot{+}_{j=1}^{n} \dot{+}_{\nu=1}^{j} S_\nu^{(j)}(H).$$

Proof. (i) We first show linear independence of the spaces $S_\nu^{(j)}(H)$:
We are in the following situation:

$$\begin{array}{llllll}
b^{n-1}R_n(H) & & & & & \\
\vdots & b^{n-2}R_{n-1}(H) & & & & \\
\vdots & \vdots & \ddots & & & \\
\vdots & \vdots & \ddots & & & \\
b^2 R_n(H) & b^2 R_{n-1}(H) & \cdots\ \cdots & b^2 R_3(H) & & \\
bR_n(H) & bR_{n-1}(H) & \cdots\ \cdots & bR_3(H) & bR_2(H) & \\
R_n(H) & R_{n-1}(H) & \cdots\ \cdots & R_3(H) & R_2(H) & R_1(H),
\end{array}$$

where $b^{j-\nu}R_j(H) = S_\nu^{(j)}(H)$, since by construction $S_\nu^{(j)}$ is a projection onto $R(b^{j-\nu}R_j)$ $(1 \leqslant j \leqslant n, 1 \leqslant \nu \leqslant j)$.

Now let

$$\sum_{j=1}^{n}\sum_{i=0}^{j-1} \lambda_{i,j} b^i x_{i,j} = 0, \tag{1}$$

where $x_{i,j} \in R_j(H) \setminus \{0\}$ and therefore $b^\nu x_{i,j} \neq 0\ \forall \nu \leqslant j-1$. We have to show that all $\lambda_{i,j}$ vanish. First, we change order of summation in (1), and we get

$$0 = (1) = \sum_{l=1}^{n}\sum_{j=l}^{n} \lambda_{j-l,j} b^{j-l} x_{j-l,j}. \tag{2}$$

We show per introduction on l

$$\lambda_{j-l,j} = 0 \qquad \forall j \in \{l, \ldots, n\} \tag{$*$}$$

for $l = n, n-1, \ldots, 1$.
l=n: The application of b^{n-1} to (2) yields

$$\lambda_{0,n} \cdot \underbrace{b^{n-1}x_{0,n}}_{\neq 0} = 0 \quad \Longrightarrow \quad \lambda_{0,n} = 0.$$

Now let $(*)$ hold for $l = n, n-1, \ldots, \nu+1$ $(1 \leqslant \nu \leqslant n-1)$. Then (2) becomes

$$\sum_{l=1}^{\nu}\sum_{j=l}^{n} \lambda_{j-l,j} b^{j-l} x_{j-l,j} = 0. \tag{3}$$

Application of $b^{\nu-1}$ to (3) yields

$$\sum_{l=1}^{\nu}\sum_{j=l}^{n} \lambda_{j-l,j} \underbrace{b^{j-l+\nu-1}x_{j-l,j}}_{=0 \text{ for } j-l+\nu-1 \geqslant j \Leftrightarrow \nu-1 \geqslant l} = 0$$

$$\Longrightarrow \sum_{j=\nu}^{n} \lambda_{j-\nu,j} b^{j-1} x_{j-\nu,j} = 0 \tag{4}$$

$$\Longrightarrow b^{\nu-1}\left(\sum_{j=\nu}^{n} \lambda_{j-\nu,j} b^{j-\nu} x_{j-\nu,j}\right) = 0$$

$$\Longrightarrow \sum_{j=\nu}^{n} \lambda_{j-\nu,j} b^{j-\nu} x_{j-\nu,j} \in N(b^{\nu-1}) \tag{5}$$

$$\Longrightarrow \sum_{j=\nu}^{n} \lambda_{j-\nu,j} (\underbrace{b^{j-\nu}x_{j-\nu,j}}_{\in N(b^{\nu})} + N(b^{\nu-1})) = \bar{0} \in N(b^{\nu})/N(b^{\nu-1})$$

$$\Longrightarrow \bar{0} = \lambda_{0,\nu}(x_{0,\nu} + N(b^{\nu-1})) + \sum_{j=\nu+1}^{n} \lambda_{j-\nu,j}(b^{j-\nu}x_{j-\nu,j} + N(b^{\nu-1}))$$

$$= \lambda_{0,\nu}(x_{0,\nu} + N(b^{\nu-1})) + \bar{b}_{\nu}\underbrace{\left(\sum_{j=\nu+1}^{n} \lambda_{j-\nu,j} b^{j-\nu-1} x_{j-\nu,j} + N(b^{\nu})\right)}_{\in N(b^{\nu+1})}.$$

From Lemma 2.13 we obtain $\lambda_{0,\nu} = 0$ and

$$\bar{b}_{\nu}\left(\sum_{j=\nu+1}^{n} \lambda_{j-\nu,j} b^{j-\nu-1} x_{j-\nu,j} + N(b^{\nu})\right) = \bar{0}.$$

From the injectivity of $\bar{b}_{\nu}$ we deduce

$$\sum_{j=\nu+1}^{n} \lambda_{j-\nu,j} b^{j-\nu-1} x_{j-\nu,j} \in N(b^{\nu}). \tag{6}$$

As in equation (5) it follows from equation (6) that $\lambda_{1,\nu+1} = 0$ and

$$\sum_{j=\nu+2}^{n} \lambda_{j-\nu,j} b^{j-\nu-2} x_{j-\nu,j} \in N(b^{\nu+1}).$$

Iterating this argument yields $\lambda_{2,\nu+2} = \cdots = \lambda_{n-\nu,\nu} = 0$. This given $(*)$ for $l = \nu$.

(ii) We show per induction on ν $(1 \leqslant \nu \leqslant n)$

$$N(b^{\nu}) = \dot{+}_{i=1}^{\nu} \dot{+}_{j=i}^{n} S_{\nu}^{(j)}(H).$$

For $\nu = n$ this will give the Lemma 2.14.

$\underline{\nu = 1}$: $N(b) = \dot{+}_{j=1}^{n} S_1^{(j)}(H)$:

The inclusion '$\supseteq$' is obvious. Now let $x = Q_1 x \in N(b)$. We show per induction on $l = 1, \ldots, n-1$ that x has a representation as

$$(**) \qquad x = \sum_{\alpha=1}^{l} b^{\alpha-1} R_\alpha x_\alpha + b^l Q_{l+1} x_{l+1} \quad \forall 1 \leqslant l \leqslant n-1.$$

$l = 1:$

$$\begin{aligned} x = Q_1 x &= R_1 x + (Q_1 b Q_2) \underbrace{(Q_1 b Q_2)^{\sim} x}_{=:x^2} \\ &= R_1 x + Q_1 b Q_2 x_2 = R_1 x + b Q_2 x_2. \end{aligned}$$

Now let $x = \sum\limits_{\alpha=1}^{l} b^{\alpha-1} R_\alpha x_\alpha + b^l Q_{l+1} x_{l+1}$, $1 \leqslant l \leqslant n-2$. Then

$$\begin{aligned} b^l Q_{l+1} x_{l+1} &= b^l (R_{l+1} x_{l+1} + (Q_{l+1} b Q_{l+2}) \underbrace{(Q_{l+1} b Q_{l+2})^{\sim} x_{l+1})}_{=:x_{l+2} = Q_{l+2} x_{l+2}} \\ &= b^l R_{l+1} x_{l+1} + \underbrace{b^l Q_{l+1} b Q_{l+2}}_{= b^{l+1}} x_{l+2} \\ &= b^l R_{l+1} x_{l+1} + b^{l+1} Q_{l+2} x_{l+2} \\ \Longrightarrow x &= \sum_{\alpha=1}^{l+1} b^{\alpha-1} R_\alpha x_\alpha + b^{l+1} Q_{l+2} x_{l+2} \\ \Longrightarrow &\qquad (**) \text{ for } l+1. \end{aligned}$$

From $(**)$ for $l = n-1$ we obtain

$$\begin{aligned} x &= \sum_{\alpha=1}^{n-1} b^{\alpha-1} R_\alpha x_\alpha + b^{n-1} \underbrace{Q_n}_{=R_n} x_n \\ &= \sum_{\alpha=1}^{n} b^{\alpha-1} R_\alpha x_\alpha \in \dot{+}_{j=1}^{n} S_1^{(j)}(H). \end{aligned}$$

Now let $\underline{1 \leqslant \nu \leqslant n-1}$ and assume $N(b^\nu) = \dot{+}_{i=1}^{\nu} \dot{+}_{j=i}^{n} S_i^{(j)}(H)$. If $\nu = n-1$ then

$$\begin{aligned} H = N(b^n) &= [N(b^n) \ominus N(b^{n-1})] \oplus N(b^{n-1}) \\ &= Q_n(H) \oplus N(b^{n-1}) = R_n(H) \oplus N(b^{n-1}) \\ &= R_n(H) \oplus [\dot{+}_{i=1}^{n-1} \dot{+}_{j=i}^{n} S_i^{(j)}(H)] = \dot{+}_{i=1}^{n} \dot{+}_{j=i}^{n} S_i^{(j)}(H), \end{aligned}$$

since $R_n = S_n^{(n)}$ by definition and the proof is complete. So assume $\nu < n-1$ and show $N(b^{\nu+1}) = \dot{+}_{i=1}^{\nu+1} \dot{+}_{j=i}^{n} S_i^{(j)}(H)$:

The inclusion '$\supseteq$' is clear. Now let $x = P_{\nu+1}x \in N(b^{\nu+1})$. We show that for every $l \in \{1, \ldots, n-\nu-1\}$ x can be represented as

$$(***) \qquad x = \sum_{\alpha=1}^{l} b^{\alpha-1} R_{\nu+\alpha} x_\alpha + b^l Q_{\nu+l+1} x_{l+1} + P_\nu y_l,$$

with $x_1, \ldots, x_{l+1}, y_l \in H$.

$l = 1$:

$$\begin{aligned} x &= P_{\nu+1}x = Q_{\nu+1}x + P_\nu x \\ &= R_{\nu+1}x + (Q_{\nu+1}bQ_{\nu+2}) \underbrace{(Q_{\nu+1}bQ_{\nu+2})^\sim x}_{=:x_1 = Q_{\nu+2}x} + P_\nu x \\ &= R_{\nu+1}x + (Q_{\nu+1}bQ_{\nu+2})x_1 + P_\nu x \\ &= R_{\nu+1}x + bQ_{\nu+2}x_1 \underbrace{- P_\nu bQ_{\nu+2}x_1 + P_\nu x}_{=:y_1 = P_\nu y_1} \\ &= R_{\nu+1}x + bQ_{\nu+2}x_1 + P_\nu y_1 \\ \Longrightarrow &\quad (***) \text{ for } l = 1. \end{aligned}$$

Now let

$$\begin{aligned} x &= \sum_{\alpha=1}^{l} b^{\alpha-1} R_{\nu+\alpha} x_\alpha + b^l Q_{\nu+l+1} x_{l+1} + P_\nu y_l, \\ \Longrightarrow b^l Q_{\nu+l+1} x_{l+1} &= b^l R_{\nu+l+1} x_{l+1} + b^l (Q_{\nu+l+1} b Q_{\nu+l+2}) \underbrace{(Q_{\nu+l+1} b Q_{\nu+l+2})^\sim x_{l+1}}_{=:x_{l+2} = Q_{\nu+l+2} x_{l+2}} \\ &= b^l R_{\nu+l+1} x_{l+1} + \underbrace{b^l Q_{\nu+l+1}}_{= P_{\nu+1} b^l Q_{\nu+l+1}} b Q_{\nu+l+2} x_{l+2} \end{aligned}$$

$$
\begin{aligned}
&= b^l R_{\nu+l+1} x_{l+1} + P_{\nu+1} b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2} \\
&= b^l R_{\nu+l+1} x_{l+1} + Q_{\nu+1} b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2} \\
&\quad - P_\nu b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2} \\
&= b^l R_{\nu+l+1} x_{l+1} + Q_{\nu+1} b^{l+1} Q_{\nu+l+2} x_{l+2} \\
&\quad - P_\nu b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2} \\
&= b^l R_{\nu+l+1} x_{l+1} + Q_{\nu+1} b^{l+1} Q_{\nu+l+2} x_{l+2} \\
&\quad - P_\nu b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2} \\
&= b^l R_{\nu+l+1} x_{l+1} + b^{l+1} Q_{\nu+l+2} x_{l+2} \\
&\quad \underbrace{- P_\nu b^{l+1} Q_{\nu+l+2} x_{l+2} - P_\nu b^l Q_{\nu+l+1} b Q_{\nu+l+2} x_{l+2}}_{=: y_{l+1} \in N(b^\nu)}
\end{aligned}
$$

$$
\Longrightarrow x = \sum_{\alpha=1}^{l+1} b^{\alpha-1} R_{\nu+\alpha} x_\alpha + b^{l+1} Q_{\nu+l+2} x_{l+2} + P_\nu y_{l+1},
$$

$$
\Longrightarrow \qquad (***) \text{ for } l+1.
$$

Now $(***)$ gives for $l = n - \nu - 1$

$$
\begin{aligned}
x &= \sum_{\alpha=1}^{n-\nu-1} b^{\alpha-1} R_{\nu+\alpha} x_\alpha + b^{n-\nu-1} \underbrace{Q_n}_{=R_n} x_{n-\nu} + P_\nu y_{n-\nu-1} \\
&= \sum_{\alpha=1}^{n-\nu} \underbrace{b^{\alpha-1} R_{\nu+\alpha} x_\alpha}_{\in S_{\nu+1}^{(\nu+\alpha)}(H)} + \underbrace{P_\nu y_{n-\nu-1}}_{\in N(b^\nu)} .
\end{aligned}
$$

This shows $x \in \dot{+}_{i=1}^{\nu+1} \dot{+}_{j=i}^{n} S_i^{(j)}(H)$ and finally completes the proof of the Lemma 2.14. ∎

LEMMA 2.15. *Let $b \in \mathcal{A}$; $p, q \in \mathcal{P}(\mathcal{A})$.*

(i) *If $a := pbq : R(q) \longrightarrow R(p)$ has closed range, then there exists $\tilde{a} \in q\mathcal{A}p \bigcap \mathcal{R}_a$.*

(ii) *If $a : R(q) \longrightarrow R(p)$ is bijective, then $\tilde{a}$ from* (i) *fulfills*

$$
a\tilde{a} = p \qquad , \qquad \tilde{a}a = q.
$$

(iii) *If $bq = pbq$ then also $bq' = p'bq'$ for all $p' \sim p$ and $q' \sim q$.*

Proof. (i) By assumption the orthogonal pseudo-inverse $\tilde{a}$ of a lies in $\mathcal{A}$. Put

$$
\tilde{\tilde{a}} := q\tilde{a}p \in q\mathcal{A}p
$$

$$
\Longrightarrow \qquad a\tilde{\tilde{a}}a = \underbrace{aq}_{=a} \tilde{a} \underbrace{pa}_{=a} = a
$$

and

$$\tilde{\tilde{a}}a\tilde{\tilde{a}} = q\tilde{a}\underbrace{paq}_{=a}\tilde{a}p = q\tilde{a}p = \tilde{\tilde{a}}.$$

(ii) Let $x = qx \in R(q)$, put $y := \tilde{a}ax \in R(q)$. Then $ay = a\tilde{a}ax = ax$ and the injectivity of a on $R(q)$ gives $y = x$. Therefore $\tilde{a}ax = x$ and this implies $\tilde{a}a = q$ since $\tilde{a}a \in q\mathcal{A}q$.

Let $y = py \in R(p)$. Then there exists an $x = qx \in R(q)$ such that $ax = y$. This implies $a\tilde{a}y = a\tilde{a}ax = ax = y$. Therefore $a\tilde{a} = p$ since $a\tilde{a} \in p\mathcal{A}p$.

(iii) $bq = pbq \Longrightarrow p'bq' = p'(bq)q' = (p'p)bqq' = (pbq)q' = bqq' = bq'$. ∎

PROPOSITION 2.16. *Let $p, q \in \mathcal{P}(\mathcal{A})$ with $R(p)\dot{+}R(q) = H$. Then there exist projections $p' \sim p$ and $q' \sim q$ in $\mathcal{A}$ such that $e - p' = q'$. In particular $p'q' = 0 = q'p'$.*

Proof. **First step**: Replace p and q by the orthogonal projections on $X_1 := R(p)$ and $X_2 := R(q)$. These projections are in $\mathcal{A}$ by 1.2 and fulfill the same assumptions as p and q. So without loss of generality $p = p^*$ and $q = q^*$.

Second step: We show that $(e-p)q(e-p) \in (e-p)\mathcal{A}(e-p)$ is invertible in $(e-p)L(H)(e-p)$ and therefore also in $(e-p)\mathcal{A}(e-p)$ because of spectral invariance.

Since $H = X_1\dot{+}X_2$ we also have $H = X_1^\perp\dot{+}X_2^\perp$ (because X_1, X_2 and $X_1\dot{+}X_2$ are closed, the projection P onto X_1 with kernel X_2 is continuous and the continuous projection P^* has range $X_2^\perp$ and kernel $X_1^\perp$). Let

$$\begin{aligned} & (e-p)q(e-p)x = 0 \\ \Longrightarrow\quad & q(e-p)x \in N(e-p)\bigcap R(q) = R(p)\bigcap R(q) = \{0\} \\ \Longrightarrow\quad & q(e-p)x = 0 \\ \Longrightarrow\quad & (e-p)x \in N(q)\bigcap R(e-p) = X_2^\perp\bigcap X_1^\perp = \{0\} \\ \Longrightarrow\quad & (e-p)x = 0 \\ \Longrightarrow\quad & (e-p)q(e-p) \text{ is injective (on } R(e-p)). \end{aligned}$$

Let $x = (e-p)x \in R(e-p)$

$$\begin{aligned} \Longrightarrow\quad & x = x_1 + x_2, \text{ where } x_1 = px_1 \in X_1, x_2 = qx_2 \in X_2 \\ \Longrightarrow\quad & x = (e-p)x = (e-p)(x_1+x_2) = (e-p)x_2 = (e-p)qx_2. \end{aligned}$$

Now $x_2 = y_1 + y_2$, where $y_1 = (e-p)y_1 \in X_1^\perp, y_2 = (e-p)y_2 \in X_2^\perp$

$$\begin{aligned} \Longrightarrow\quad & x_2 = qx_2 = q(y_1+y_2) = qy_1 = q(e-p)y_1 \\ \Longrightarrow\quad & x = (e-p)qx_2 = (e-p)q(e-p)y_1 \\ \Longrightarrow\quad & \text{surjectivity.} \end{aligned}$$

Third step: Construction of a similarity: Put $g := p + q(e-p) \in \mathcal{A}$. In p-coordinates we can write

$$\begin{aligned} g &= \begin{pmatrix} p & pq(e-p) \\ 0 & (e-p)q(e-p) \end{pmatrix} \\ &= \begin{pmatrix} p & pq(e-p)[(e-p)q(e-p)]^{-1} \\ 0 & e-p \end{pmatrix} \cdot \begin{pmatrix} p & 0 \\ 0 & (e-p)q(e-p) \end{pmatrix} \\ &=: g_1 \cdot g_2 \end{aligned}$$

g_1, g_2 are invertible, so g is invertible

$$g_1^{-1} = \begin{pmatrix} p & -pq(e-p)[(e-p)q(e-p)]^{-1} \\ 0 & e-p \end{pmatrix}, \quad g_2^{-1} = \begin{pmatrix} p & 0 \\ 0 & [(e-p)q(e-p)]^{-1} \end{pmatrix}.$$

Define $p' := gpg^{-1} = (p + q(e-p))pg^{-1} = pg^{-1}$. This directly gives $pp' = p'$. On the other hand

$$p'p = \underbrace{g_1 g_2 p}_{=p} \underbrace{g_2^{-1} g_1^{-1} p}_{=p} = p \implies p' \sim p.$$

Now $q' := e - p' = g(e-p)g^{-1} = (p + q(e-p))(e-p)g^{-1} = q(e-p)g^{-1}$

$$\implies qq' = q' \implies R(q') \subseteq R(q).$$

But $H = R(p') \dot{+} R(e-p') = R(p) \dot{+} R(q')$ and $H = R(p) \dot{+} R(q)$. Since $R(q') \subseteq R(q)$ we must have $R(q') = R(q)$ and therefore $q' \sim q$. ∎

PROPOSITION 2.17. *Let $P_1, P_2 \in \mathcal{P}_\perp(\mathcal{A})$, $R(P_1) \bigcap R(P_2) = \{0\}$ and $R(P_1) + R(P_2)$ closed. Then*

(i) $R(P_1 + P_2) = R(P_1) \dot{+} R(P_2)$.

(ii) *The orthogonal projection P on $R(P_1) \dot{+} R(P_2)$ is in $\mathcal{A}$.*

(iii) *There exist projections $P_1' \sim P_1, P_2' \sim P_2$ in $P\mathcal{A}P$ such that*

$$P_1'P_2' = 0 = P_2'P_1' \quad \textit{and} \quad P_1' + P_2' = P.$$

Proof. (i) The inclusion '$\subseteq$' is clear. Show $R(P_1) \subseteq R(P_1 + P_2)$. Let $X_1 := R(P_1), X_2 := R(P_2)$. Since $X_1 \bigcap X_2 = \{0\}$ and $X_1 + X_2$ is closed, we have $X_1^\perp + X_2^\perp = H$. But P_1 doesn't act on $X_1^\perp$, so $P_1|_{X_2^\perp} : X_2^\perp \to R(P_1)$ is injective. Therefore we can find for $x = P_1 x \in R(P_1)$ and $y = (e - P_2)y \in X_2^\perp$ such that $P_1 x = y$. This gives $(P_1 + P_2)y = P_1 y + P_2 y = P_1 y = x$. Similarly $R(P_2) \subseteq R(P_1 + P_2)$.

(ii) $P_1 + P_2 \in \mathcal{A}$ has by assumption the closed range $R(P_1) \dot{+} R(P_2)$ and the orthogonal projection P on $R(P_1 + P_2) = R(P_1) \dot{+} R(P_2)$ is in $\mathcal{A}$ because $\mathcal{A}$ is Ψ^*.

(iii) Let P be the orthogonal projection onto $R(P_1) \dot{+} R(P_2)$ constructed in (ii). Then $R(P_i) \subseteq R(P)$, so $PP_i = P_i$ and taking * we have $P_i P = P_i$ $(i = 1, 2)$. Now we apply Proposition 2.16 on the Ψ^*-algebra $P\mathcal{A}P \subseteq PL(H)P, P(H) = P_1(H) \dot{+} P_2(H)$ and we get the desired projections. ∎

PROPOSITION 2.18. *Let $p_1, \dots, p_n \in \mathcal{P}(\mathcal{A})$ such that the sum $p_1(H) + \cdots + p_n(H)$ is direct (some $p_j = 0$ possible) and $p_1(H) + \cdots + p_i(H)$ are closed for $i = 2, \dots, n$. Then the orthogonal projection P on $p_1(H) \dot{+} \cdots \dot{+} p_n(H)$ is in $\mathcal{A}$ and there exist projections $p_j' \sim p_j$ in $P\mathcal{A}P$ $(1 \leqslant j \leqslant n)$ such that*

$$p_i' p_j' = \delta_{i,j} p_i' \quad \text{and} \quad \sum_{j=1}^{n} p_j' = P.$$

In particular, if in this situation $p_1(H) \dot{+} \cdots \dot{+} p_n(H) = H$, then there exist projections $p_j' \sim p_j$ in $\mathcal{A}$ $(1 \leqslant j \leqslant n)$ such that

$$p_i' p_j' = \delta_{ij} p_i' \quad \text{and} \quad \sum_{j=1}^{n} p_j' = e.$$

Proof. As a first step we choose all $p_j \in \mathcal{A}$ to be the orthogonal projection on its range by Remark 1.2. Now we start an iterative construction process and apply Proposition 2.17 to the orthogonal projections p_1 and p_2. This gives the orthogonal projection $T_2 \in \mathcal{A}$ on $p_1(H) \dot{+} p_2(H)$ and projections $p_1' \sim p_1$, $p_2' \sim p_2$ in $T_2 \mathcal{A} T_2$ such that $p_1' + p_2' = T_2$ and in particular $p_1' p_2' = 0 = p_2' p_1'$.

Now let us assume that we already have constructed the orthogonal projection $T_{j-1} \in \mathcal{A}$ on $p_1(H) \dot{+} \cdots \dot{+} p_{j-1}(H)$ and projections $p_1', \dots, p_{j-1}' \in T_{j-1} \mathcal{A} T_{j-1}$ with the properties

(i) $p_1' \sim p_1, \dots, p_{j-1}' \sim p_{j-1}$
(ii) $p_1' + \cdots + p_{j-1}' = T_{j-1}$
(iii) $p_\nu' p_\mu' = \delta_{\nu\mu} p_\nu'$ $\forall \nu, \mu \leqslant j-1$.

Then we apply Proposition 2.17 to the orthogonal projections p_j and $T_{j-1} \in \mathcal{A}$ and obtain the orthogonal projection $T_j \in \mathcal{A}$ on $T_{j-1}(H) \dot{+} p_i(H) = p_1(H) \dot{+} \cdots \dot{+} p_j(H)$, together with idempotents $T_{j-1}' \sim T_{j-1}, p_j' \sim p_j$ in $T_j \mathcal{A} T_j$ such that $T_{j-1}' + p_j' = T_j$, in particular $T_{j-1}' p_j' = 0 = p_j' T_{j-1}'$. Now we put

$$p_1'' := p_1' T_{j-1}', \dots,\ p_{j-1}'' := p_{j-1}' T_{j-1}',\ p_j'' := p_j' \in \mathcal{A}.$$

Since $p_i'(H) = p_i(H) \subseteq T_{j-1}(H) = T_{j-1}'(H)$, we have $T_{j-1}' p_i' = p_i'$ for $1 \leqslant i \leqslant j-1$ and $p_1'', \dots, p_{j-1}'', p_j''$ are projections in $T_j \mathcal{A} T_j$. Now we show (i)-(iii):

(i) $p_j'' := p_j' \sim p_j$ by Proposition 2.17. Let $i \in \{1, \dots, j-1\}$. Then

$$p_i' p_i'' = p_i' p_i' T_{j-1}' = p_i'' \quad \text{and} \quad p_i'' p_i' = p_i' \underbrace{T_{j-1}' p_i'}_{=p_i'} = p_i'$$

$$\Longrightarrow p_i'' \sim p_i' \sim p_i.$$

(ii)

$$\sum_{i=1}^{j-1} p_i'' = \left(\sum_{i=1}^{j-1} p_i'\right) T_{j-1}' = T_{j-1} T_{j-1}' = T_{j-1}'$$

$$\Longrightarrow \quad \sum_{i=1}^{j} p_i'' = T_{j-1}' + p_j' = T_j.$$

(iii) For $\nu, \mu \leqslant j-1$ we already have $p_\nu' p_\mu' = \delta_{\nu\mu} p_\nu'$

$$\Longrightarrow \quad p_\nu'' p_\mu'' = p_\nu' \underbrace{T_{j-1}' p_\mu'}_{=p_\mu'} T_{j-1}' = \delta_{\nu\mu} p_\nu' T_{j-1}' = \delta_{\nu\mu} p_\nu''$$

and

$$p_j'' p_\nu'' = p_j' \underbrace{p_\nu''}_{=T_{j-1}' p_\nu''} = p_j' T_{j-1}' p_\nu'' = 0,$$

$$p_\nu'' p_j'' = p_\nu' T_{j-1}' p_j' = 0.$$

After $j = n-1$ steps the process finished with $P = T_n \in \mathcal{A}$. ∎

LEMMA 2.19. *There is an enumeration of the subspaces $S_\nu^{(j)}(H)$ $(1 \leqslant \nu \leqslant j, 1 \leqslant j \leqslant n)$, such that the iterated direct sums are closed.*

Proof. We chose the enumeration of the subspaces as follows: In the picture of the proof of Lemma 2.14 we start with the top line until the bottom line and within each line we pass from the left to the right.

(i) First, we show that within each line the iterated direct sums are closed: Let $i, j \in \{1, \ldots, n\}, i \neq j, 0 \leqslant \nu < \min\{i, j\}$. Then

$$Q_i \widetilde{b^\nu} b^\nu = (\widetilde{b^{i-1}} b^{i-1} - \widetilde{b^i} b^i) \widetilde{b^\nu} b^\nu = ((\widetilde{b^{i-1}} b^{i-1} - \widetilde{b^i} b^i) = Q_i$$

$$\begin{aligned} \Longrightarrow \quad S_{i-\nu}^{(i)} \cdot S_{j-\nu}^{(j)} &= b^\nu R_i \widetilde{b^\nu} b^\nu R_j \widetilde{b^\nu} \\ &= b^\nu R_i Q_i \widetilde{b^\nu} b^\nu R_j \widetilde{b^\nu} \\ &= b^\nu R_i \underbrace{Q_i R_j}_{=0} \widetilde{b^\nu} = 0. \end{aligned}$$

Therefore, all possible combinations of these subspaces have a positive angle.

(ii) Next, let

$$L_i := \sum_{j=n-i+1}^{n} S_{j-(n-i)}^{(j)}(H)$$

be the subspace determined by the i-th line of the scheme, $1 \leqslant i \leqslant n$, which is closed by (i). Because of $R(b^{n-i}) = L_1 \dot{+} \cdots \dot{+} L_i$, we see that $L_1 \dot{+} \cdots \dot{+} L_i$ is closed. ∎

Final proof of Theorem 2.1. We have by Lemma 2.14

$$H = \dot{+}_{j=1}^{n} \dot{+}_{\nu=1}^{j} S_\nu^{(j)}(H), \quad S_\nu^{(j)} \in \mathcal{A}.$$

From Lemma 2.18 we get the existence of projections $S_\nu^{(j)'} \sim S_\nu^{(j)}, S_\nu^{(j)'} \in \mathcal{A}$ with the properties

$$S_\nu^{(j)'} S_\mu^{(j)'} = \delta_{ij}\delta_{\nu\mu} S_\nu^{(j)'} \quad \text{and} \quad \sum_{j=1}^{n}\sum_{\nu=1}^{j} S_\nu^{(j)'} = e.$$

Lemma 2.15, (iii) together with 2.11 gives

$$\begin{aligned} bS_1^{(j)'} &= 0 \quad \forall 1 \leqslant j \leqslant n \\ bS_\nu^{(j)'} &= S_{\nu-1}^{(j)'} b S_\nu^{(j)'} \quad \forall 2 \leqslant j \leqslant n, \quad 2 \leqslant \nu \leqslant j. \end{aligned}$$

From this we obtain a Jordan decomposition of b:

$$\begin{aligned} b = b \cdot e = b\left(\sum_{j=1}^{n}\sum_{\nu=1}^{j} S_\nu^{(j)'}\right) &= \sum_{j=1}^{n}\sum_{\nu=1}^{j} bS_\nu^{(j)'} \\ = \sum_{j=2}^{n}\sum_{\nu=2}^{j} bS_\nu^{(j)'} &= \sum_{j=2}^{n}\sum_{\nu=2}^{j} S_{\nu-1}^{(j)'} b S_\nu^{(j)'}, \end{aligned}$$

where $S_{\nu-1}^{(j)'} b S_\nu^{(j)'} : R(S_\nu^{(j)'}) \to R(S_{\nu-1}^{(j)'})$ is bijective since $S_{\nu-1}^{(j)} b S_\nu^{(j)} : R(S_\nu^{(j)}) \to R(S_{\nu-1}^{(j)})$ is bijective by Lemma 2.12 and $R(S_{\nu-1}^{(j)'}) = R(S_{\nu-1}^{(j)})$, $R(S_\nu^{(j)'}) = R(S_\nu^{(j)})$ $(j \geqslant 2, 2 \leqslant \nu \leqslant j)$.

Now let $j \in \{2, \ldots, n\}$ be fixed. To complete the proof of the theorem it remains to show the existence of operators

$$I_{\alpha,\beta}^{(j)} \in S_\alpha^{(j)'} \mathcal{A} S_\beta^{(j)'}; \quad \alpha, \beta \in \{1, \ldots, j\}$$

with the properties

(i) $I_{\alpha,\alpha}^{(j)} = S_\alpha^{(j)'}, \quad 1 \leqslant \alpha \leqslant j$

(ii) $I_{\alpha,\alpha+1}^{(j)} = S_\alpha^{(j)'} b S_{\alpha+1}^{(j)'}, \quad 1 \leqslant \alpha \leqslant j-1$

(iii) $I_{\alpha,\beta}^{(j)} I_{\beta,\gamma}^{(j)} = I_{\alpha,\gamma}^{(j)} \quad \forall \alpha, \beta, \gamma \in \{1, \ldots, j\}$

The operators $I_{\alpha,\alpha}^{(j)}, I_{\alpha,\alpha+1}^{(j)}$ we define by (i) and (ii) respectively. Next we define for $\alpha \in \{1, \ldots, j-1\}$, $I_{\alpha+1,\alpha}^{(j)} := \widetilde{I_{\alpha,\alpha+1}^{(j)}} \in S_{\alpha+1}^{(j)'} \mathcal{A} S_\alpha^{(j)'}$ by Lemma 2.15;

$$\Longrightarrow \quad I_{\alpha,\alpha+1}^{(j)} I_{\alpha+1,\alpha}^{(j)} = S_\alpha^{(j)'} \quad \text{and} \quad I_{\alpha+1,\alpha}^{(j)} I_{\alpha,\alpha+1}^{(j)} = S_{\alpha+1}^{(j)'}.$$

We further put

$$I_{\alpha,\beta}^{(j)} := \begin{cases} I_{\alpha,\alpha+1}^{(j)} & \cdots & I_{\beta-1,\beta}^{(j)} & \text{for} \quad \alpha < \beta \\ I_{\alpha,\alpha-1}^{(j)} & \cdots & I_{\beta+1,\beta}^{(j)} & \text{for} \quad \alpha > \beta \end{cases} .$$

This directly implies

$$I_{\alpha,\beta}^{(j)} I_{\beta,\alpha}^{(j)} = S_{\alpha}^{(j)'} \quad \forall 1 \leqslant \alpha, \beta \leqslant j.$$

As a last step one easily checks the relations

$$\begin{aligned} I_{\alpha,\beta}^{(j)} I_{\beta,\beta+1}^{(j)} &= I_{\alpha,\beta+1}^{(j)} \quad (\beta < j) \\ I_{\alpha,\beta}^{(j)} I_{\beta,\beta-1}^{(j)} &= I_{\alpha,\beta-1}^{(j)} \quad (\beta > 1) \\ I_{\alpha-1,\alpha}^{(j)} I_{\alpha,\beta}^{(j)} &= I_{\alpha-1,\beta}^{(j)} \quad (\alpha > 1) \\ I_{\alpha+1,\alpha}^{(j)} I_{\alpha,\beta}^{(j)} &= I_{\alpha+1,\beta}^{(j)} \quad (\alpha < j) \end{aligned}$$

for the possible cases $\alpha < \beta, \alpha = \beta, \alpha > \beta$ and this implies (iii).

Now the proof of Theorem 2.1 is complete. ∎

To illustrate the calculations of the proof of Theorem 2.1, we give the following:

EXAMPLE 2.20. Let $S \subset \mathbf{R}^N$ be a compact identification set of complex open neighborhoods of S, which is a real smooth compact manifold at the same time (for example $S = [0,1] \subset \mathbf{C}$). We consider the $*$-algebras $\mathcal{A} = C(S, L(\mathbf{C}^n))$, $C^k(S, L(\mathbf{C}^n))$, $C^\infty(S, L(\mathbf{C}^n))$, $\mathcal{H}(S) \underset{\varepsilon}{\otimes} L(\mathbf{C}^n)$, $R(S, L(\mathbf{C}^n))$ (the algebra of rational matrix functions with poles outside S endowed with the topology of $\mathcal{H}(S) \underset{\varepsilon}{\otimes} L(\mathbf{C}^n)$), all within the C^*-algebra $C(S, L(\mathbf{C}^n))$.

Or take $S = S^1 \subset \mathbf{C}$ and consider $\mathcal{A} = \mathcal{H}(S) \underset{\varepsilon}{\otimes} L(\mathbf{C})^n)$ or $R(S, L(\mathbf{C}))$ with the adjoint operation $f^\sharp(z) := (f(1/\bar{z}))^*$ on holomorphic germs of functions in a neighborhood of S not containing 0. Then $f^\sharp(z) = (f(z))^*$ for $z \in S$ and $1 + f^\sharp f$ is always invertible in $\mathcal{A}$.

These algebras have properties (i), (ii) and (iii) of Notation 1.1 (see [31] for the holomorphic case). Now for $a \in C(S, L(\mathbf{C}^n))$ having a pseudo-inverse $\tilde{a} \in C(S, L(\mathbf{C}^n))$ is equivalent to the property rank $a(x) =$ const on S, and this also characterizes the regular elements in the other algebras (for $R(S, L(\mathbf{C}^n))$ see [26], 6.0.1). In all the cases the unique orthogonal pseudo-inverse is within the algebra (so (iv)) of Notation 1.1 is fulfilled). These results can be obtained constructing the pointwise orthogonal pseudo-inverse of a matrix function with the function $u(\cdot)$ of Definition 1.5 together with the definition of Lemma 1.10 and using the fact that S is an identification set (see [26], 6.0.1). The constructions of this section can be applied to nilpotent matrix functions $f : S \to L(\mathbf{C}^n)$ and the question of $\mathcal{A}$-morphic possible choices of Jordan normal forms to f. For example, a nilpotent $b \in C(S, L(\mathbf{C}^n))$ is a Jordan nilpotent in $C(S, L(\mathbf{C}^n))$ iff there is a continuous choice of Jordan normal forms for b. Similar interpretations holds in the other algebras for a nilpotent $b \in \mathcal{A}$ being a Jordan nilpotent within $\mathcal{A}$. Now under the assumptions of Theorem 2.1 the constructed Jordan normal form of b is contained within the considered algebras. In this context we can read 2.1 and the constructions of this section as follows:

Let $b \in \mathcal{A}$ be nilpotent of order m. Then

(i) b is a Jordan nilpotent within $\mathcal{A}$ iff $\operatorname{rank} b^\nu$ is constant $\forall 1 \leqslant \nu \leqslant m-1$.

(ii) If there exists a continuous choice of Jordan normal forms for b then there exists such a choice within $\mathcal{A}$.

See for example [5], [11], [12] for related results.

3. FURTHER CONSTRUCTIONS

First, we generalize Theorem 2.1 to the situation of algebraic operators:

LEMMA 3.1. *Let $\mathcal{A} \subseteq L(H)$ be a Ψ^*-algebra and $J \in \mathcal{A}$ a Jordan operator in $L(H)$. Then the spectral projections*

$$Q_{\lambda_j}(J) := \frac{1}{2\pi i} \oint_{|\lambda-\lambda_j|<\epsilon_j} (\lambda e - J)^{-1}\,d\lambda \quad (\lambda_j \in \sigma(J))$$

are in $\mathcal{A}$.

Proof. Since J is a Jordan operator in $L(H)$, $R(Q_{\lambda_j}(J)) = R((\lambda_j e - J)^{m_j})$ is closed and we get a projection $P_j \in \mathcal{A}$ on $N((\lambda_j e - J)^{m_j})$ (using for example the orthogonal pseudo-inverse of $(\lambda_j e - J)^{m_j}$, which is in $\mathcal{A}$), $j = 1, \ldots, n$. Now $H = \dot{+}_{j=1}^n P_j(H)$, $P_1(H) \dot{+} \cdots \dot{+} P_i(H)$ is closed for $1 \leqslant i \leqslant n$ by the functional calculus of $L(H)$, and so we can apply the constructions of Propositions 2.16 - 2.18 to obtain within $\mathcal{A}$ the unique set of projections $Q_1, \ldots, Q_n$ such that $R(Q_i) = R(P_j)$ and $Q_i Q_j = \delta_{ij} Q_i$ $(i, j = 1, \ldots, n)$. Since the spectral projections fulfill the same, we must have $Q_{\lambda_j}(J) = Q_j \in \mathcal{A}, j = 1, \ldots, n$. ∎

THEOREM 3.2. *Let $\mathcal{A} \subseteq L(H)$ a Ψ^*-algebra and $J \in \mathcal{A}$ an algebraic operator, $p(J) = 0$ for $p \in \mathbb{C}[z]$. Then J is a Jordan element in $\mathcal{A}$ iff $R(q(J))$ is closed for every polynomial $q \in \mathbb{C}[z]$ dividing p.*

Proof. If J is a Jordan element in $\mathcal{A}$, then J is also a Jordan operator in $L(H)$ and $R(q(J))$ is closed for every polynomial q dividing p by ([19], 7.13). Now we prove the converse. Let $p(z) = \prod_{j=1}^{n}(z - \lambda_j)^{m_j}$. Then because of spectral invariance we have $\sigma_{\mathcal{A}}(J) = \sigma_{L(H)}(J) = \{\lambda_1, \ldots, \lambda_m\}$ and the spectral projections $p^{(j)} := Q_{\lambda_j}(J) \in \mathcal{A}$ reduce J, i.e.

$$Jp^{(j)} = p^{(j)}Jp^{(j)} = p^{(j)}J, \quad p^{(i)}p^{(j)} = \delta_{ij}p^{(i)}, \quad \sum_{j=1}^{n} p^{(j)} = e.$$

Now we consider for $1 \leqslant j \leqslant n$ the nilpotent operators

$$N_j := p^{(j)}Jp^{(j)} - \lambda_j p^{(j)} \in p^{(j)}\mathcal{A}p^{(j)}$$

and show that $R(N_j^l)$ is closed for all $1 \leqslant l \leqslant m_j - 1$. To do so, we define $q \in \mathbb{C}[z]$ by $q(z) := (z - \lambda_j)^l$. Then q/p and

$$q(J) = (J - \lambda_j e)^l = \sum_{\nu \neq j}(p^{(\nu)}Jp^{(\nu)} - \lambda_j p^{(\nu)})^l + N_j^l .$$

For $\nu \neq j$ $(p^{(\nu)}Jp^{(\nu)} - \lambda_j p^{(\nu)})^l : p^{(\nu)}(H) \to p^{(\nu)}(H)$ is invertible and therefore

$$R(q(J)) = \left[\dot{+}_{\nu \neq j} p^{(\nu)}(H)\right] \dot{+} R(N_j^l)$$

and $R(N_j^l) \subseteq p^{(j)}(H)$ has to be closed.

Let $j \in \{1, \ldots, n\}$ be fixed and put $p := p^{(j)}$ and $N := N_j$. Then the orthogonal projection $p_\perp$ on $R(p)$ is also in $\mathcal{A}$ and we define $N' := Np_\perp \in p_\perp \mathcal{A} p_\perp$. Then

$$(N')^l = N^l p_\perp \quad \text{and} \quad (N')^l p = N^l \quad \forall l \in \mathbb{N}$$

and so it is easy to see that N' is nilpotent of order m and $(R(N')^l)$ is closed for all $1 \leqslant l \leqslant m - 1$. By Theorem 2.1 N' is a Jordan nilpotent within the Ψ^*-algebra $p_\perp \mathcal{A} p_\perp \subseteq p_\perp L(H) p_\perp$. Now, multiplying the Jordan decomposition of N' and the corresponding projections by p from the right we conclude that N has a Jordan decomposition within $p\mathcal{A}p$. This completes the proof. ∎

Our next aim is to show that the similarity orbits of Jordan nilpotents in Ψ^*-algebras $\mathcal{A}$ are invariant under the $*$-operation. We do this in the following lemmata:

LEMMA 3.3. *Let* $b = \sum_{j=1}^{n}\sum_{i=1}^{j-1} I_{i,i+1}^{(j)} \in JN_n(\mathcal{A})$. *Put* $\widehat{b} := \sum_{j=1}^{n}\sum_{i=1}^{j-1} I_{i+1,i}^{(j)} \in \mathcal{A}$. *Then*

(i) $\widehat{b}^l \in \mathcal{R}_{b^l} \quad \forall l$,

(ii) $\widehat{b} \in S_{\mathcal{A}^{-1}}(b)$.

Proof. (i) This is direct computation and can be found in ([26], Anhang B).

(ii) Let $T_j := \sum_{i=1}^{j} I_{i,i}^{(j)} = \sum_{i=1}^{j} S_i^{(j)} \in \mathcal{P}(\mathcal{A})$. Put $\mathcal{A}_j := T_j \mathcal{A} T_j$.

Then

$$b_j := bT_j = T_j b T_j = \sum_{i=1}^{j-1} I_{i,i+1}^{(j)}, \qquad \widehat{b}_j := \widehat{b}T_j = T_j \widehat{b} T_j = \sum_{i=1}^{j-1} I_{i+1,i}^{(j)}.$$

We construct $g_j \in (\mathcal{A}_j)^{-1}$ such that $\widehat{b}_j = g_j b_j g_j^{-1}$: Define

$$g_j := \sum_{i=1}^{j} I_{i,j-i+1}^{(j)} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & I_{1,n}^{(j)} \\ \vdots & & & I_{2,n-1}^{(j)} & 0 \\ \vdots & & \cdot & & \vdots \\ 0 & I_{n-1,2}^{(j)} & & & \vdots \\ I_{n,1}^{(j)} & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathcal{A}_j .$$

Then $g_j^2 = T_j$, so $g_j \in (\mathcal{A})^{-1}$ and

$$\begin{aligned} g_j b_j g_j^{-1} &= g_j b_j g_j = g_j \left(\sum_{i=1}^{j-1} I_{i,i+1}^{(j)}\right)\left(\sum_{k=1}^{j} I_{k,j-k+1}^{(j)}\right) \\ &= g_j \left(\sum_{i=1}^{j-1}\sum_{k=1}^{j} \underbrace{I_{i,i+1}^{(j)} I_{k,j-k+1}^{(j)}}_{=\delta_{i+1,k} I_{i,j-i}^{(j)}}\right) = \left(\sum_{l=1}^{j} I_{l,j-l+1}^{(j)}\right)\left(\sum_{i=1}^{j-1} I_{i,j-i}^{(j)}\right) \\ &= \sum_{l=1}^{j}\sum_{i=1}^{j-1} \underbrace{I_{l,j-l+1}^{(j)} I_{i,j-i}^{(j)}}_{=\delta_{j-l+1,i} I_{j-i+1,j-i}^{(j)}} \overset{k:=j-i}{=} \sum_{k=1}^{j-1} I_{k+1,k}^{(j)} = \widehat{b}_j . \end{aligned}$$

Finally define $g := g_1 + \cdots + g_n$. Then $g^2 = e$ and $gbg^{-1} = gbg = \widehat{b}$. ∎

LEMMA 3.4. *Let* $b = \sum_{j=1}^{n}\sum_{i=1}^{j-1} I_{i,i+1}^{(j)} \in JN_n(\mathcal{A})$ *such that* $I_{i,i}^{(j)*} = I_{i,i}^{(j)}$ $\forall j, i$ *(i.e. the Jordan reducing projections are orthogonal). Then* $b^* \in S_{\mathcal{A}^{-1}}(b)$.

Proof. Let $j \in \{1, \ldots, n\}$ fixed. We have

$$b_j^* := b^* T_j = T_j b^* T_j = \sum_{i=1}^{j-1} I_{i,i+1}^{(j)^*} \quad \text{where} \quad I_{i,i+1}^{(j)^*} \in S_{i+1}^{(j)} \mathcal{A} S_i^{(j)}$$

by the orthogonality of the occuring projections. We will show that b_j^* is similar to $\widehat{b}_j$ (defined as in Lemma 3.3) in $\mathcal{A}_j$. For $\nu = 1, \ldots, j-1$ we define

$$b_{j,\nu} := I_{1,2}^{(j)} \cdots I_{\nu,\nu+1}^{(j)} \in S_1^{(j)} \mathcal{A} S_{\nu+1}^{(j)} \subseteq \mathcal{A}_j$$

and

$$\widehat{b}_{j,\nu} := I_{\nu+1,\nu}^{(j)} \cdots I_{2,1}^{(j)} \in S_{\nu+1}^{(j)} \mathcal{A} S_1^{(j)} \subseteq \mathcal{A}_j \ .$$

Then $b_{j,\nu}\widehat{b}_{j,\nu} = S_1^{(j)}$, $\widehat{b}_{j,\nu} b_{j,\nu} = S_{\nu+1}^{(j)}$, so $\widehat{b}_{j,\nu} \in \mathcal{R}_{b_{j,\nu}}$. Moreover, since $T_j - \widehat{b}_{j,\nu} b_{j,\nu} = T_j - S_{\nu+1}^{(j)}$ and $b_{j,\nu}\widehat{b}_{j,\nu} = S_1^{(j)}$ are orthogonal, $\widehat{b}_{j,\nu}$ is the unique orthogonal pseudo-inverse of $b_{j,\nu}$ in $\mathcal{A}_j$ and is thus given by

$$\widehat{b}_{j,\nu} := (P_{N(b_{j,\nu})} + b_{j,\nu}^* b_{j,\nu})^{-1} b_{j,\nu}^* \tag{7}$$

where $P_{N(b_{j,\nu})} = T_j - S_{\nu+1}^{(j)}$ is the orthogonal projection on $N(b_{j,\nu})$. Put

$$g_{j,\nu} := ((T_j - S_{\nu+1}^{(j)}) + b_{j,\nu}^* b_{j,\nu})^{-1} \in \mathcal{A}_j^{-1}, \quad \nu = 1, \ldots, j-1$$

and

$$g_j := g_{j,j-1} \cdots g_{j,1} \in \mathcal{A}_j^{-1} \ .$$

We have $g_{j,\nu}^{-1} = \operatorname{diag}\left(S_1^{(j)}, \ldots, S_\nu^{(j)}, \left[b_{j,\nu}^* b_{j,\nu}\right], S_{\nu+2}^{(j)}, \ldots, S_j^{(j)}\right)$ and $g_{j,\nu} = \operatorname{diag}\left(S_1^{(j)}, \ldots, S_\nu^{(j)}, \left[b_{j,\nu}^* b_{j,\nu}\right]^{-1}, S_{\nu+2}^{(j)}, \ldots, S_j^{(j)}\right)$, so

$$g_j^{-1} = g_{j,1}^{-1} \cdots g_{j,j-1}^{-1} = \operatorname{diag}\left(S_1^{(j)}, \left[b_{j,1}^* b_{j,1}\right], \ldots, \left[b_{j,j-1}^* b_{j,j-1}\right]\right)$$

and

$$g_j = g_{j,j-1} \cdots g_{j,1} = \operatorname{diag}\left(S_1^{(j)}, \left[b_{j,1}^* b_{j,1}\right]^{-1}, \ldots, \left[b_{j,j-1}^* b_{j,j-1}\right]^{-1}\right) \ .$$

Now consider $g_j b_j^* g_j^{-1}$

$$= g_j \cdot \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ I_{1,2}^{(j)^*} & 0 & & & \vdots \\ & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & I_{j-1,j}^{(j)^*} & 0 \end{pmatrix} \cdot \operatorname{diag}\left(S_1^{(j)}, [b_{j,1}^* b_{j,1}], \ldots, [b_{j,j-1}^* b_{j,j-1}]\right)$$

$$= g_j \cdot \left(I_{1,2}^{(j)^*} + \underbrace{I_{2,3}^{(j)^*}[b_{j,1}^*}_{=b_{j,2}^*} b_{j,1}] + \cdots + \underbrace{I_{j-1,j}^{(j)^*}[b_{j,j-2}^*}_{b_{j,j-1}^*} b_{j,j-2}] \right)$$

$$= g_j \cdot \left(b_{j,1}^* b_{j,2}^* b_{j,1} + \cdots + b_{j,j-1}^* b_{j,j-2}\right)$$

$$= \operatorname{diag}\left(S_1^{(j)}, [b_{j,1}^* b_{j,1}]^{-1}, \ldots, [b_{j,j-1}^* b_{j,j-1}]^{-1}\right) \times$$

$$\times \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ b_{j,1}^* & 0 & & & \vdots \\ 0 & b_{j,2}^* b_{j,1} & 0 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & b_{j,j-1}^* b_{j,j-2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ [b_{j,1}^* b_{j,1}]^{-1} b_{j,1}^* & 0 & & & \vdots \\ 0 & [b_{j,2}^* b_{j,2}]^{-1} b_{j,2}^* b_{j,1} & 0 & & \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & [b_{j,j-1}^* b_{j,j-1}]^{-1} b_{j,j-1}^* b_{j,j-2} & 0 \end{pmatrix}$$

$$= [b_{j,1}^* b_{j,1}]^{-1} b_{j,1}^* + [b_{j,2}^* b_{j,2}]^{-1} b_{j,2}^* b_{j,1} + \cdots + [b_{j,j-1}^* b_{j,j-1}]^{-1} b_{j,j-1}^* b_{j,j-2}$$

$$= g_{j,1} \cdot b_{j,1}^* + g_{j,2} \cdot b_{j,2}^* b_{j,1} + \cdots + g_{j,j-1} \cdot b_{j,j-1}^* b_{j,j-2}$$

$$\overset{(7)}{=} \widehat{b}_{j,1} + \widehat{b}_{j,2} b_{j,1} + \cdots + \widehat{b}_{j,j-1} b_{j,j-2}$$

$$= I_{2,1}^{(j)} + I_{3,2}^{(j)} I_{2,1}^{(j)} I_{1,2}^{(j)} + \cdots + (I_{j,j-1}^{(j)} \cdots I_{2,1}^{(j)})(I_{1,2}^{(j)} \cdots I_{j-2,j-1}^{(j)})$$

$$= I_{2,1}^{(j)} + \cdots + I_{j,j-1}^{(j)} = \widehat{b}_j \ .$$

Finally define $g := g_1 + \cdots + g_n \in \mathcal{A}^{-1}$. Then $\widehat{b} = g b^* g^{-1}$. From Lemma 3.3 we deduce $b^* \in S_{\mathcal{A}^{-1}}(b)$. ∎

DEFINITION 3.5. Gram-Schmidt orthogonalizing process for systems of idempotents: Let $N \in \mathbf{N}$ and

$$\mathcal{P}^N(\mathcal{A}) := \Big\{(p_1,\dots,p_N) \in \mathcal{P}(\mathcal{A})^N : p_ip_j = \delta_{ij}p_i, \sum_{i=1}^N p_i = e\Big\},$$

$$\mathcal{P}_\perp^N(\mathcal{A}) := \{(p_1,\dots,p_N) \in \mathcal{P}^N(\mathcal{A}) : p_i = p_i^* \ \forall i\}.$$

For $p \in \mathcal{P}(\mathcal{A})$ we define

$$GS_2(p) := p_\perp := pp^*(e-(p-p^*)^2)^{-1} \in \mathcal{A} \quad \text{(see Remark 1.2)}$$

the orthogonal projection on $R(p)$. We further define $GS_N : \mathcal{P}^N(\mathcal{A}) \longrightarrow \mathcal{P}_\perp^N(\mathcal{A})$ by

$$\begin{aligned} GS_N(p_1,\dots,p_N) &:= ([GS_N(p_1,\dots,p_N)]_1,\dots,([GS_N(p_1,\dots,p_N)]_N) \\ [GS_N(p_1,\dots,p_N)]_1 &:= GS_2(p_1) \\ [GS_N(p_1,\dots,p_N)]_j &:= GS_2\left(\sum_{i=1}^j p_i\right) - GS_2\left(\sum_{i=1}^{j-1} p_i\right), \quad 1 < j \leqslant N. \end{aligned}$$

PROPOSITION 3.6. *We have (see* [3], 1.3, 1.4*)*

(i) $R(p_1+\dots+p_j) = R([GS_N(p_1,\dots,p_N)]_1+\dots+[GS_N(p_1,\dots,p_N)]_j)$, $1 \leqslant j \leqslant N$.

(ii) $U := \sum_{j=1}^N [GS_N(p_1,\dots,p_N)]_j \cdot p_j \in \mathcal{A}^{-1}$ *and* $[GS_N(p_1,\dots,p_N)]_j = U \cdot p_jU^{-1}$ $\forall 1 \leqslant j \leqslant N$. *The proof of* [3] *even works for all unital $*$-algebras $\mathcal{A}$ such that $\{e + a^*a : a \in \mathcal{A}\} \subseteq \mathcal{A}^{-1}$.*

COROLLARY 3.7. *Let $J \in J\,\mathrm{Alg}(\mathcal{A})$ be a Jordan operator. Then there exists $J' \in S_{\mathcal{A}^{-1}}(J)$ such that the projections associated to the Jordan decomposition of J' are all orthogonal.*

Proof. Let $J = \sum_{j=1}^n \Big[\lambda_j p^{(j)} + \sum_{i=1}^{k_j} \sum_{r=1}^{n_i^{(j)}-1} I_{r,r+1}^{j,i}\Big]$. Consider the N-tuple

$$(q_1,\dots,q_N) := (p_{i,r}^{(j)} : 1 \leqslant j \leqslant n, 1 \leqslant i \leqslant k_j, 1 \leqslant r \leqslant n_i^{(j)}), \qquad N = \sum_{j=1}^n \sum_{i=1}^{k_j} n_i^{(j)},$$

endowed with an ordering. Then $J' := UJU^{-1}$, $U := \sum_{j=1}^N [GS_N(q_1,\dots,q_N)]_j \cdot q_j$, has $GS_N(q_1,\dots,q_N) \in \mathcal{P}_\perp^N(\mathcal{A})$ as a set of Jordan reducing projections. ∎

THEOREM 3.8. *Let $b \in JN(\mathcal{A})$. Then $b^* \in S_{\mathcal{A}^{-1}}(b)$.*

Proof. Let $b' = UbU^{-1} \in S_{\mathcal{A}^{-1}}(b)$ as in Corollary 3.7 with orthogonal decomposition. By Lemma 3.4 $(b')^* \in S_{\mathcal{A}^{-1}}(b') = S_{\mathcal{A}^{-1}}(b)$, so $(b')^* = gbg^{-1}, g \in \mathcal{A}^{-1}$. Now: $b = U^{-1}b'U \Longrightarrow b^* = U^*(b')^*(U^*)^{-1} = U^*gbg^{-1}(U^*)^{-1} = [U^*g]b[U^*g]^{-1} \in S_{\mathcal{A}^{-1}}(b)$. ∎

COROLLARY 3.9. *Let $J \in \sum_{j=1}^{n}(\lambda_j p^{(j)} + N_j) \in J\,\mathrm{Alg}(\mathcal{A})$ with $\lambda_j \in \mathbf{R}$. Then $J^* \in S_{\mathcal{A}^{-1}}(J)$.*

Proof. Let $J' = UJU^{-1} = \sum_{j=1}(\lambda_j p^{(j)'} + N_j')$ with orthogonal decomposition $e = \sum_{j=1}^{n} p^{(j)'}$. Then

$$\begin{aligned}
&(N_j')^* = g_j N_j' g_j^{-1}, \quad g_j \in (p^{(j)'}\mathcal{A}p^{(j)'})^{-1}, \quad 1 \leqslant j \leqslant n \\
\Longrightarrow\ &(\lambda_j p^{(j)'} + N_j')^* = g_j(\lambda_j p^{(j)'} + N_j')g_j^{-1}, \quad 1 \leqslant j \leqslant n \\
\Longrightarrow\ &(J')^* = gJ'g^{-1}, \quad g := g_1 + \cdots + g_n \in \mathcal{A}^{-1} \\
\Longrightarrow\ &J^* = (U^{-1}J'U)^* = U^*(J')^*(U^*)^{-1} = U^*gJ'g^{-1}(U^*)^{-1} \\
&\quad = [U^*gU]J[U^*gU]^{-1}.
\end{aligned}$$ ∎

THEOREM 3.10. *Let $\mathcal{A} \subseteq L(H)$ be a Ψ^*-algebra.*

(i) *The set $JN(\mathcal{A})$ of nilpotent Jordan operators is the union of similarity orbits, each orbit being a locally-$\mathcal{A}$-rational, $*$-invariant manifold in the homogeneous topology.*

(ii) *The set $J\,\mathrm{Alg}(\mathcal{A})$ of all Jordan operators is the union of similarity orbits and these orbits are locally-$\mathcal{A}$-rational manifolds in their homogeneous topologies. If $\sigma(J) \subset \mathbf{R}$ for $J \in J\,\mathrm{Alg}(\mathcal{A})$, then the similarity orbit of J is $*$-invariant.*

Proof. This is now a combination of the results of [24], 1.7 (see also [23], 1.7) and 3.2, 3.8, 3.9). For the notion of locally-$\mathcal{A}$-rational manifolds see ([13], Section 1). It essentially means that changes of coordinates are given by purely algebraic operations. ∎

4. CHARACTERIZATION OF THE HOMOGENEOUS TOPOLOGY ON THE SIMILARITY ORBITS

In this section we give functional analytic description of the homogeneous topology of the similarity orbit $S_{\mathcal{A}^{-1}}(J)$ of a Jordan operator J in a Ψ^*-algebra $\mathcal{A}$.

DEFINITION 4.1. Let $J = \sum_{j=1}^{n}\left[\lambda_j p^{(j)} + \sum_{i=1}^{k_j}\sum_{r=1}^{n_i^{(j)}-1} I_{r,r+1}^{j,i}\right] \in \mathcal{A}$ be a Jordan operator. Put $m_j := n_{k_j}^{(j)}$ (= order of nilpotency of $(\lambda_j e - J)$ restricted to $p^{(j)}(H)$). We define τ_1 to be the homogeneous topology on $S_{\mathcal{A}^{-1}}(J)$, induced by $\mathcal{A}^{-1}$. Second, we define τ_2 to be the coarsest topology on $S_{\mathcal{A}^{-1}}(J)$, such that the finite number of mappings $(S_{\mathcal{A}^{-1}}(J),\tau_2) \ni b \longmapsto b \in (\mathcal{A},\tau(\mathcal{A})$ and $(S_{\mathcal{A}^{-1}}(J),\tau_2) \ni b \longmapsto \ker(\lambda_j e - b)^l = X_{e-((\widetilde{\lambda_j e - b})^l)\cdot(\lambda_j e - b)^l} \in (\Gamma(\mathcal{A}),\tau(\Gamma(\mathcal{A})))$, $j \in \{1,\ldots,n\}$, $l \in \{1,\ldots,m_j-1\}$ are continuous.

The aim is now to show

THEOREM 4.2. *$\tau_1 = \tau_2$ on $S_{\mathcal{A}^{-1}}(J)$.*

It is easily seen that τ_1 is finer than τ_2 (see [26], 3.1.5). The converse is done in the following construction, which leads to explicit formulas for local similarity cross sections.

Let $j \in \{1,\ldots,n\}$ be fixed. For $(\lambda_j e - J)$ we choose a pseudo-inverse $(\widetilde{\lambda_j e - J})$ as in Lemma 3.3. For $j \in \{1,\ldots,n\}$ and $l \in \{1,\ldots,m_j\}$ (fixed) we locally define $u_{j,l}(b) := u_{a,\tilde a}((\lambda_j e - b)^l)$, where $a = (\lambda_j e - J)^l$, $\tilde a = \left[(\widetilde{\lambda_j e - J})\right]^l$; i.e.

$$(8)\qquad u_{j,l}(b) = \left[(\widetilde{\lambda_j e - J})\right]^l \cdot \left[e + ((\lambda_j e - b)^l - (\lambda_j e - J)^l)\cdot\left[(\widetilde{\lambda_j e - J})\right]^l\right]^{-1}.$$

We get an $\mathcal{A}$-neighborhood W_J of J, such that all $u_{j,l}$ are defined on W_J. We further define for $b \in W_J$

$$P_{j,l}(b) := e - u_{j,l}(b)\cdot(\lambda_j e - b)^l$$

$(j \in \{1,\ldots,n\}, l \in \{1,\ldots,m_j\})$. These are rational functions of $b \in W_J$.

LEMMA 4.3. *There exists a τ_2-neighborhood W'_J of J in $S_{\mathcal{A}^{-1}}(J)$ ($W'_J \subseteq W_J \bigcap S_{\mathcal{A}^{-1}}(J)$), such that for all $b \in W'_J$ and $j \in \{1,\ldots,n\}, l \in \{1,\ldots,m_j\}$ we have*

$$(\lambda_j e - b)^l \cdot u_{j,l}(b)\cdot(\lambda_j e - b)^l = (\lambda_j e - b)^l .$$

In particular

$$P_{j,l}(b) \in \mathcal{P}(\mathcal{A}) \qquad \textit{and} \qquad P_{j,l}(b) \in \ker(\lambda_j e - b)^l$$

for all $b \in W_J'$ and all j, l by Proposition 1.6.(i)*).*

Proof. We consider $S_{\mathcal{A}^{-1}}(J)$ with the topology τ_2. Let $j \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m_j\}$ be fixed and put

$$M := \{(\lambda_j e - b)^l : b \in S_{\mathcal{A}^{-1}}(J)\} = S_{\mathcal{A}^{-1}}((\lambda_j e - J)^l) \subseteq \mathcal{R}(\mathcal{A}).$$

We have a natural surjective mapping $\beta : S(J) \to M$, given by $\beta(b) := (\lambda_j e - b)^l$. We consider on M the final topology τ such that $\beta : (S_{\mathcal{A}^{-1}}(J), \tau_2) \to (M, \tau)$ is continuous. Now we show τ is finer than $\tau(\mathcal{R}(\mathcal{A}))$ on M. To do so, we have to show the continuity of the maps

$$(M, \tau) \ni m \longmapsto m \in (\mathcal{A}, \tau(\mathcal{A}))$$

and

$$(M, \tau) \ni m \longmapsto \ker m \in (\Gamma(\mathcal{A}), \tau(\Gamma(\mathcal{A})))$$

by Theorem 1.7. The first one is easily seen to be continuous, simply because β is continuous. The second one is continuous since

$$\ker \circ \beta(S_{\mathcal{A}^{-1}}(J), \tau_2) \ni b \longmapsto \ker(\beta(b)) = \ker(\lambda_j e - b)^l \in (\Gamma(\mathcal{A}), \tau(\Gamma(\mathcal{A})))$$

is continuous by the assumptions on τ_2. For $l = m_j$ this map is continuous into $(\Gamma(\mathcal{A}), \tau(\Gamma(L(H))))$ by the Functional Calculus in $L(H)$, since the spectral projections $Q_{\lambda_j}(b) = \frac{1}{2\pi i} \oint\limits_{|\lambda - \lambda_j| = \epsilon_j} (\lambda e - b)^{-1} \, d\lambda \in \ker(\lambda_j e - b)^{m_j}$ for $b \in S_{\mathcal{A}^{-1}}(J)$. Since $\tau(\mathcal{A})$ is contained in τ this map is also continuous into $(\Gamma(\mathcal{A}), \tau(\Gamma(\mathcal{A})))$ by Proposition 1.9.

Now put $a := (\lambda_j e - J)^l$ and $\tilde{a}$ a pseudo-inverse as in Lemma 3.3. By Theorem 1.7 there is a $\tau(\mathcal{R}(\mathcal{A}))$-neighborhood W of a in M, such that

$$m \cdot u_{a,\tilde{a}}(m) \cdot m = m$$

for $m \in W$. Since τ is finer than $\tau(\mathcal{R}(\mathcal{A}))$, W is also open in τ. Put $W_J^{j,l} := \beta^{-1}(W)$. Then $W_J^{j,l}$ is a τ_2-neighborhood of J in $S_{\mathcal{A}^{-1}}(J)$ because of the continuity of β. Now for $b \in W_J^{j,l}$ we have the construction

$$\beta(b) \cdot u_{a,\tilde{a}}(\beta(b)) \cdot \beta(b) = \beta(b),$$

and this is what we had to show since $u_{j,l}(b) = u_{a,\tilde{a}}(\beta(b))$. Finally take W_J' as the intersection of the $W_J^{j,l}$, and we are done. ∎

LEMMA 4.4. *Let $j \in \{1,\ldots,n\}$ and $l,k \in \{1,\ldots,m_j\}$ be fixed. Then we have for all $b \in W_J'$ (constructed as in Lemma 4.3.)*

$$P_{j,l}(b)\cdot P_{j,k}(b) = P_{j,k}(b)\cdot P_{j,l}(b) = P_{j,\min(l,k)}(b).$$

Proof. Let $l \leqslant k$ without loss of generality. Since $N((\lambda_j e - b)^l) \subseteq N((\lambda_j e - b)^k)$ we must have $P_{j,k}\cdot P_{j,l} = P_{j,l}$. On the other hand

$$\begin{aligned}
& u_{j,l}(b)(\lambda_j e - b)^l u_{j,k}(b)(\lambda_j e - b)^k \\
\overset{(8)}{=}\ & u_{j,l}(b)(\lambda_j e - b)^l \\
& \cdot \left[\widetilde{(\lambda_j e - J)}\right]^k \cdot \left(e + ((\lambda_j e - b)^k - (\lambda_j e - J)^k)\left[\widetilde{(\lambda_j e - J)}\right]^k\right)^{-1} \cdot (\lambda_j e - b)^k \\
=\ & u_{j,l}(b)(\lambda_j e - b)^l \cdot \left[\widetilde{(\lambda_j e - J)}\right]^l \cdot \left[\widetilde{(\lambda_j e - J)}\right]^{k-l} \\
& \left(e + ((\lambda_j e - b)^k - (\lambda_j e - J)^k)\left[\widetilde{(\lambda_j e - J)}\right]^k\right)^{-1} \cdot (\lambda_j e - b)^k \\
\overset{(1.6,(iii))}{=}\ & \left\{ e - \left[e + \left[\widetilde{(\lambda_j e - J)}\right]^l \cdot ((\lambda_j e - b)^l - (\lambda_j e - J)^l)\right]^{-1} \right. \\
& \left. \cdot \left(e - \left[\widetilde{(\lambda_j e - J)}\right]^l (\lambda_j e - J)^l\right)\right\} \cdot \left[\widetilde{(\lambda_j e - J)}\right]^l \cdot \left[\widetilde{(\lambda_j e - J)}\right]^{k-l} \\
& \left(e + ((\lambda_j e - b)^k - (\lambda_j e - J)^k)\left[\widetilde{(\lambda_j e - J)}\right]^k\right)^{-1} \cdot (\lambda_j e - b)^k \\
=\ & \left[\widetilde{(\lambda_j e - J)}\right]^k \cdot \left(e + ((\lambda_j e - b)^k - (\lambda_j e - J)^k)\left[\widetilde{(\lambda_j e - J)}\right]^k\right)^{-1} \cdot (\lambda_j e - b)^k \\
=\ & u_{j,k}(b)(\lambda_j e - b)^k .
\end{aligned}$$

Therefore

$$\begin{aligned}
P_{j,l}(b)\cdot P_{j,k}(b) &= (e - u_{j,l}(b)(\lambda_j e - b)^l)\cdot(e - u_{j,k}(b)(\lambda_j e - b)^k) \\
&= (e - u_{j,k}(b)(\lambda_j e - b)^k) \\
&= P_{j,l}(b). \quad \blacksquare
\end{aligned}$$

LEMMA 4.5. *Let J, W_J as above. There exists a neighborhood $\mathcal{W}_J (\subseteq W_J)$ of J and $*$-rational functions $\mathcal{Q}_{\lambda_1},\ldots,\mathcal{Q}_{\lambda_n}$ defined on $\mathcal{W}_J$ with values in $\mathcal{A}$ such that*

$$\mathcal{Q}_{\lambda_j}(b) = Q_{\lambda_j}(b) \quad for \quad b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J), \quad j = 1,\ldots,n.$$

Proof. **First step:**

For $b \in W_J$, the rational expressions $P_j(b) := P_{j,m_j}(b)$ are defined as above. Using the holomorphic functional calculus of $L(H)$ and arguing as in the proof of Lemma 4.3 yield that $P_j(b) \in \mathcal{P}(\mathcal{A})$ and $R(P_j(b)) = N((\lambda_j e - b)^{m_j})$ for all j and all $b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J), \mathcal{W}_j \subseteq W_J$ a possibly smaller $\mathcal{A}$-neighborhood of J. Furthermore we have $R(P_j(b)) = R(Q_{\lambda_j}(b))$ for all $b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J)$, since b is a Jordan operator of $L(H)$ in this situation.

Second step: Local $*$-rational construction of the orthogonal projection on sums of spectral subspaces:

Take $i, j \in \{1, \ldots, n\}$. Consider $P_i(b)$ and $P_j(b)$ for $b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J)$. Then $R(P_i(b)) \bigcap R(P_j(b)) = \{0\}$ and $R([P_i(b)]_\perp) \dot{+} R([P_j(b)]_\perp) = R([P_i(b)]_\perp) + R([P_j(b)]_\perp)$ by Proposition 2.17. Therefore $a_{ij}(b) := [P_i(b)]_\perp + [P_j(b)]_\perp \in \mathcal{R}(\mathcal{A})$ and $a_{ij}(b)$ depends continuously on b with respect to $\tau(\mathcal{A})$. Moreover the images of $a_{ij}(b)$ vary continuously with respect to $\tau(L(H))$, because continuously depending projections of $L(H)$ on these images can be constructed using the holomorphic functional calculus of $b \in L(H)$ with the locally holomorphic scalar function $\alpha_{ij}(z) := \begin{cases} 1 & \text{on} \quad \{\lambda_i\} \bigcap \{\lambda_j\} \\ 0 & \text{on} \quad \sigma(b) \setminus \{\lambda_i, \lambda_j\} \end{cases}$ on the compact set $\sigma(b) = \sigma(J)$ (note that $R(a_{ij}(b)) = R(Q_{\lambda_i}(b) + Q_{\lambda_j}(b))$). This implies that the assignment

$$\mathcal{W}_J \cap S_{\mathcal{A}^{-1}}(J) \ni b \mapsto a_{ij}(b) \in (\mathcal{R}(\mathcal{A}), \tau(\mathcal{A}) \cap \tau(\mathcal{R}(L(H)))|_{\mathcal{R}(\mathcal{A})}) = (\mathcal{R}(\mathcal{A}), \tau(\mathcal{R}(\mathcal{A}))),$$

is continuous and therefore that

$$u_{a_{i,j}(J), a_{ij}(J)^\sim}(a_{ij}(b)) \in \mathcal{R}(\mathcal{A})_{a_{ij}(b)} \quad \text{for} \quad b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J),$$

$a_{ij}(J)^\sim \in \mathcal{R}(\mathcal{A})_{a_{ij}(J)}$ fixed and $\mathcal{W}_J$ a possible smaller $\mathcal{A}$-neighborhood of J (this can be seen similar to considerations of the proof of ([13], 4.7 - 4.9). Thus

$$Q_{ij}(b) := \left[a_{ij}(b) \cdot u_{a_{ij}(J), a_{ij}(J)^\sim}(a_{ij}(b))\right]_\perp \in \mathcal{A}$$

is the orthogonal projection on $R(a_{ij}(b)) = R(Q_{\lambda_i}(b) + Q_{\lambda_j}(b))$ for $b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J)$. $Q_{ij}(b)$ is defined in a possibly smaller $\mathcal{A}$-neighborhood of J and is a $*$-rational function.

Third step: Let $j \in \{1, \ldots, n\}$ fixed.

Iterating the process of the second step we construct the orthogonal projection on $\dot{+}_{\nu \neq j} R(P_\nu(b)) = R\Big(\sum_{\nu \neq j} Q_{\lambda_\nu}(b)\Big)$, using at each step Lemma 2.17, the holomorphic functional calculus of $L(H)$ and the function $u(\cdot)$ to obtain locally pseudo-inverses. At the end we can find a possibly smaller $\mathcal{A}$-neighborhood $\mathcal{W}_J$ of J and a $*$-rational function $T_j : \mathcal{W}_J \to \mathcal{A}$ such that $T_j(b)$ is the orthogonal projection on $\dot{+}_{\nu \neq j} R(P_\nu(b))$ for $b \in \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J)$.

Fourth step: We apply Lemma 2.16 to the situation $H=R(P_j(b))\dot{+}R(T_j(b))$ and get the projection $\mathcal{Q}_{\lambda_j}(b)\in\mathcal{A}$ such that $R(\mathcal{Q}_{\lambda_j}(b))=R(P_j(b))$ and $N(\mathcal{Q}_{\lambda_j}(b)) = R(T_j(b))$ as a rational expression of $P_j(b)$ and $T_j(b)$ and hence as a $*$-rational function of b. This $*$-rational expression $\mathcal{Q}_{\lambda_j}(b)$ is defined in a possibly smaller $\mathcal{A}$-neighborhood $\mathcal{W}$ of J and we have by construction

$$(*) \qquad \mathcal{Q}_{\lambda_j}(b)=Q_{\lambda_j}(b) \quad \text{for} \quad b\in\mathcal{W}_J\bigcap S_{\mathcal{A}^{-1}}(J).$$

Fifth step: Choosing $\mathcal{W}_J$ one again smaller, we can get $(*)$ for every $j\in\{1,\ldots,n\}$. ∎

DEFINITION 4.6. Let $J,\mathcal{W}_J$ as above. We define

$$\mathcal{P}_{j,l}(b):=P_{j,l}(b)\cdot\mathcal{Q}_{\lambda_j}(b) \quad (j=1,\ldots,n;\ l=1,\ldots,m_j),$$

as well as $\mathcal{P}_{j,0}(b):=0$ for $b\in\mathcal{W}_J$. The $\mathcal{P}_{j,l}$ are $*$-rational functions of $b\in\mathcal{W}_J$. Put $\mathcal{W}'_J:=\mathcal{W}_J\bigcap\mathcal{W}'_J\subseteq S_{\mathcal{A}^{-1}}(J)$ (see Lemma 4.3). Then we have for all $b\in\mathcal{W}'_J$:

(i) $\mathcal{P}_{j,l}\in\ker(\lambda_j e-b)^l\ \forall j,l$

(ii) $\mathcal{P}_{j_1,l_1}(b)\cdot\mathcal{P}_{j_2,l_2}(b)=\delta_{j_1,j_2}\cdot\mathcal{P}_{j_1,\min(l_1,l_2)}(b)(j_i\in\{1,\ldots,n\},l_i\in 0,\ldots,m_j\}$, $i=1,2)$,

which are consequences of Lemmas 4.3, 4.4 and 4.5. For further details see ([25], 3.6) and ([26], 3.1.16).

REMARK 4.7. If $\mathcal{A}$ has holomorphic functional calculus in one variable then we can choose $\mathcal{W}_J$ small enough such that the

$$Q_{\lambda_j}(b)=\frac{1}{2\pi\mathrm{i}}\oint_{|\lambda-\lambda_j|=\varepsilon_j}(\lambda e-b)^{-1}\,\mathrm{d}\lambda\in\mathcal{A} \qquad (\varepsilon_j>0 \text{ fixed})$$

form a resolution of the identity for all $b\in\mathcal{W}_J$. In this case we can define

$$\mathcal{P}'_{j,l}(b):=P_{j,l}(b)\cdot Q_{\lambda_j}(b) \quad \text{and} \quad P'_{j,0}(b):=0.$$

The $\mathcal{P}'_{j,l}$ are holomorphic functions of $b\in\mathcal{W}_J$. For $b\in\mathcal{W}'_J:=\mathcal{W}_J\bigcap\mathcal{W}'_J\subseteq \mathcal{W}_J\bigcap S_{\mathcal{A}^{-1}}(J)$ we have $\mathcal{P}'_{j,l}(b):=P_{j,l}(b)$. Thus the conclusions of Definition 4.6 hold also in this case (see [25], 3.6).

LEMMA 4.8. (see also [4], p. 358-361). *Let for* $b\in\mathcal{W}_J$

$$\alpha(b):=\sum_{j=1}^{n}\sum_{i=1}^{m_j}(\mathcal{P}_{j,i}(b)-\mathcal{P}_{j,i-1}(b))\cdot(\mathcal{P}_{j,i}(J)-\mathcal{P}_{j,i-1}(J)).$$

We further put for $S \in \mathcal{A}$:

$$\mathcal{U}(S) := \sum_{j=1}^{n} \sum_{i=1}^{k_j} \sum_{l=0}^{n_i^{(j)}-1} (\lambda_j e - S)^l \cdot p^{(j)}_{i,n_i^{(j)}} \cdot \left[\widetilde{(\lambda_j e - J)}\right]^l .$$

Then we can take $\mathcal{W}_J$ *smaller such that for all* $b \in \mathcal{W}_J$

$$\omega : V(J) \longrightarrow \mathcal{A}^{-1}, \quad \omega(b) := \alpha(b) \cdot \mathcal{U}(\alpha(b)^{-1} b \alpha(b))$$

is defined. By construction ω *is a* $*$*-rational function of* $b \in \mathcal{W}_J$ *and is therefore continuous. The restriction of* ω *to* $\mathcal{W}'_J \subseteq \mathcal{W}_J S_{\mathcal{A}^{-1}}(J)$ *is a local cross section of* π^J. *If* $\mathcal{A}$ *is with holomorphic functional calculus we can take*

$$\alpha'(b) := \sum_{j=1}^{n} \sum_{i=1}^{m_j} (\mathcal{P}'_{j,i}(b) - \mathcal{P}'_{j,i-1}(b)) \cdot (\mathcal{P}'_{j,i}(J) - \mathcal{P}'_{j,i-1}(J)),$$

$$\text{as well as } \omega'(b) := \alpha'(b) \cdot \mathcal{U}(\alpha'(b)^{-1} b \alpha'(b))$$

for $b \in \mathcal{W}_J, \mathcal{W}_J$ *small enough.* ω' *is a holomorphic function of* $b \in \mathcal{W}_J$ *and the restriction of* ω' *to* $\mathcal{W}'_J \subseteq \mathcal{W}_J \bigcap S_{\mathcal{A}^{-1}}(J)$ *gives the same as* ω *restricted to* $\mathcal{W}'_J$ *and is therefore also a local similarity cross section.*

Proof. We easily compute $\alpha(J) = e = \mathcal{U}(J)$, so $\omega(J) = e$. Now the existence of $\mathcal{W}_J$ is a consequence of the property that $\mathcal{A}^{-1}$ is open and inversion is continuous within $\mathcal{A}$. The fact that the restriction of ω to $\mathcal{W}'_J$ is a similarity cross section has to be calculated as in ([25], 3.8, 3.11, 3.13) or ([26], 3.1.8, 3.1.18, 3.1.20). ∎

Now Lemma 4.8 gives directly the implication τ_2 is finer than τ_1 on $S_{\mathcal{A}^{-1}}(J)$ and the proof of Theorem 4.2 is complete.

COROLLARY 4.9. *Let* $J = \sum_{j=1}^{n} \left[\lambda_j p^{(j)} + \sum_{i=1}^{k_j} \sum_{r=1}^{n_i^{(j)}-1} I^{j,i}_{r,r+1}\right]$ *be a Jordan operator in* $L(H)$. *For* $S, T \in S_{L(H)^{-1}}(J)$ *define*

$$d(S,T) := \|S - T\| + \sum_{j=1}^{n} \sum_{l=1}^{m_j - 1} \|P_{\ker(\lambda_j e - S)^l} - P_{\ker(\lambda_j e - T)^l}\|,$$

where P_X *denotes the orthogonal projection on* X *(*$X \subseteq H$ *closed subspace). Let* τ *denote the topology induced by the metric* d.

Then τ *is equivalent to the homogeneous topology on* $S_{L(H)^{-1}}(J)$.

Proof. This follows from Theorem 4.2. See also ([25], 3.3). ∎

DEFINITION 4.10. Let J be a Jordan operator in $\mathcal{A}$. We define τ_3 to be the topology on $S_{\mathcal{A}^{-1}}(J)$ given by $\tau(\mathcal{A})$ together with the metric d of Corollary 4.9.

THEOREM 4.11. *We have* $\tau_1 = \tau_3$ *on* $S_{\mathcal{A}^{-1}}(J)$.

Proof. By Theorem 4.2 it remains to show the continuity of

$$(S_{\mathcal{A}^{-1}}(J), \tau_3) \ni T \longmapsto \ker(\lambda_j e - T)^l \in (\Gamma(\mathcal{A}), \tau(\Gamma(\mathcal{A}))) \tag{9}$$

for fixed $1 \leqslant j \leqslant n, 1 \leqslant l \leqslant m_j - 1$. To do so we consider

$$M := S_{\mathcal{A}^{-1}}((\lambda_j e - J)^l) = \{g(\lambda_j e - J)^l g^{-1} : g \in \mathcal{A}^{-1}\} \subseteq \mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(L(H)),$$

together with the final topology τ, such that

$$(S_{\mathcal{A}^{-1}}(J), \tau_3) \ni T \longrightarrow \beta(T) := (\lambda_j e - T)^l \in (M, \tau)$$

becomes continuous. Now the inclusion $(M, \tau) \hookrightarrow \mathcal{R}(L(H))$ is assumed to be continuous as well as $(M, \tau) \hookrightarrow \mathcal{A}$. From Proposition 1.9 we see (M, τ) is continuously embedded in $\mathcal{R}(\mathcal{A})$ and this gives the continuity of (9). ∎

THEOREM 4.12. *Let* $\mathcal{A} \subseteq L(H)$ *be a* Ψ^**-algebra. Suppose* $J \in \mathcal{A}$ *has the property that the similarity orbit* $S_{L(H)^{-1}}(J) \subseteq L(H)$ *admits norm-continuous local similarity cross sections into the group* $L(H)^{-1}$*. Then the* $\mathcal{A}$*-orbit* $S_{\mathcal{A}^{-1}}(J) \subseteq \mathcal{A}$ *has* $\mathcal{A}$*-continuous local similarity cross sections into* $\mathcal{A}^{-1}$*, which can be chosen to be restrictions of* $*$*-rational functions defined in a* $\mathcal{A}$*-neighborhood of* J *to the similarity orbit.*

Proof. From the theorem of D. A. Herrero ([2], 16.1) we know that J is a nice Jordan operator in $L(H)$. By Theorem 3.2 J is also a Jordan operator in $\mathcal{A}$. Since J is nice, the homogeneous topology on $S_{L(H)^{-1}}(J)$ is the same as the norm topology and by Theorem 4.11 the homogeneous topology on $S_{\mathcal{A}^{-1}}(J)$ coincides with $\tau(\mathcal{A})$. Therefore the function ω of Lemma 4.8 is a similarity cross section on a $\mathcal{A}$-neighborhood of J in $S_{\mathcal{A}^{-1}}(J)$ and has the desired properties. ∎

THEOREM 4.13. *Let* J *be a nice Jordan operator in the* Ψ^**-algebra* $\mathcal{A}$ *with* $\sigma(J) = \{\lambda_0\}$ *(for example* J *nilpotent). Then* $\pi^J : \mathcal{A}^{-1} \longrightarrow S_{\mathcal{A}^{-1}}(J)$ *has rational local cross sections.*

Proof. In this situation, $P_{\lambda_0, m_{\lambda_0}}(b) = e$ for b in an open neighborhood of J intersected with the similarity orbit. The function ω of Lemma 4.8 is a rational local similarity cross section in this case. ∎

REMARK 4.14.

(i) For each Jordan operator $J \in \mathcal{A}$ the similarity orbit $S_{\mathcal{A}^{-1}}(J)$ carries a local structure that makes it a locally-$\mathcal{A}$-rational manifold (see [23], 1.7 and [24], 1.7). In the chart $\varphi_J : U_J \longrightarrow \varphi_J(U_J) \subseteq T_J$ (see [24], 2.7) the map $\varphi_J(U_J) \ni x \longmapsto e + x \in \mathcal{A}^{-1}$ is a local cross section to π^J. Since changes of coordinates are $\mathcal{A}$-rational ([24], 2.8), this map defines a local cross section which is a rational morphism from the $\mathcal{A}$-rational local structure of $S_{\mathcal{A}^{-1}}(J)$ into the group $\mathcal{A}^{-1}$.

(ii) We can also consider two sided continuously embedded topological ideals $\mathcal{I}$ within $\mathcal{A}$ ($\mathcal{I}$ for example locally pseudo convex). In this case we let the group $G :=$ connected component of e in $\{e + y \in \mathcal{A}^{-1} : y \in \mathcal{I}\}$ with respect to $\tau(\mathcal{I})$ operate on Jordan operators in $\mathcal{A}$ via similarity and look at the orbits $S_G(J) := \{gJg^{-1} : g \in G\} \subseteq J + \mathcal{I}$. Then the results of Theorems 4.2 and 4.11 remain true (where the homogeneous topology is now determined by $\tau(\mathcal{I})$ via G) if we only additionally assume for τ_2 and τ_3 the continuity of

$$(S_G(J), \tau_i) \ni T \longmapsto T \in (J + \mathcal{I}, \tau(\mathcal{I})), \quad i = 2, 3.$$

The function ω of Lemma 4.8 takes values in G in that case (see [26], Chapter 3).

EXAMPLE 4.15. (Continuation of 2.20). Theorems 3.2, 3.8, 4.2, 4.11, 4.12 have obvious applications to the matrix algebras $\mathcal{A}$ of 2.20. We note that in these situations the homogeneous topology on the similarity orbit of a Jordan element is always equivalent to $\tau(\mathcal{A})$, which is a consequence of pointwise finite dimension arguments (see [26], 6.0.3).

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Received July 12, 1993; revised August 24, 1994.

Note added in proof. 4.16 Meanwhile the author proved the following sharpening of the Theorems 0.4, 0.5, 4.12, 4.13:

THEOREM. *Let $\mathcal{A} \subseteq L(H)$ be a Ψ^*-algebra, $J \in \mathcal{A}$ a Jordan operator of $L(H)$. Then there exist*

1. *an open neighborhood W_J of J in $(\mathcal{A}, \tau(\mathcal{A}))$,*
2. *a rational function $\omega : W_J \to \mathcal{A}$,*
3. *a neighborhood W'_J of J in $S_{\mathcal{A}^{-1}}(J)$, open with respect to the hologeneous topology of the orbit, $W'_J \subseteq S_{\mathcal{A}^{-1}}(J) \bigcap W_J$*

such that $\omega|_{W'_J} : W'_J \to \mathcal{A}^{-1}$ is a local cross section of π^J, $\omega(J) = e$.