J. OPERATOR THEORY
34(1995), 239–249

WHEN THE ALGEBRA GENERATED BY AN OPERATOR IS AMENABLE

G.A. WILLIS

Communicated by William B. Arveson

ABSTRACT. It is shown that, if the algebra generated by a compact operator on Hilbert space is amenable, then the operator is similar to a normal operator. Problems arise with attempts to extend this to Banach spaces other than Hilbert space, for example it cannot even be shown that the operator is not quasinilpotent. The approximation property appears to be implicated in these problems.

KEYWORDS: *Amenable, compact operator, normal operator, approximation property.*

AMS SUBJECT CLASSIFICATION: Primary 46H35; Secondary 46J45, 46M99, 47A15, 46B28.

Many properties of an operator, T, on some Banach space, X, are related to properties of the Banach algebra, $\mathcal{A}_T$, which it generates. For example, T is quasinilpotent if and only if $\mathcal{A}_T$ is radical. For another example, there is a non-trivial subspace hyperinvariant for T whenever $\mathcal{A}_T$ is not an integral domain because, if A and B in $\mathcal{A}_T$ are not zero and satisfy $AB = 0$, then $(AX)^-$ is such a subspace. Amenability is a finiteness condition for Banach algebras which in many cases has strong consequences for the algebra. Some instances of this are discussed below. The aim of this paper is to investigate the consequences for the operator T which follow from the amenability of $\mathcal{A}_T$. It is shown that, if T is a compact operator on Hilbert space and $\mathcal{A}_T$ is amenable, then T is similar to a normal operator. The properties of Hilbert space are used in an essential way at several points in the proof and it is an intriguing problem whether an analogous statement holds for operators on other Banach spaces.

A Banach algebra, $\mathcal{A}$, is said to be *amenable* if, for every Banach $\mathcal{A}$-bimodule X, every derivation, $D : \mathcal{A} \to X^*$, is inner. Here a *Banach $\mathcal{A}$-bimodule* is an $\mathcal{A}$-bimodule, X, which is a Banach space such that the bimodule maps $(a, x) \mapsto x \cdot a$ and $(a, x) \mapsto a \cdot x$ are jointly continuous from $\mathcal{A} \times X$ to X. If X is a Banach $\mathcal{A}$-bimodule, then the dual space, X^*, is also, with the $\mathcal{A}$-actions defined by $\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle$ and $\langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle$, $(a \in \mathcal{A};\ x \in X;$ and $x^* \in X^*)$. A *derivation* is a linear map, $D : \mathcal{A} \to X$, such that $D(ab) = a \cdot D(b) + D(a) \cdot b$, $(a, b \in \mathcal{A})$. For each x in X, the map $D_x : a \mapsto a \cdot x - x \cdot a$, is a derivation and such derivations are called *inner*. These notions are explained in more detail in [1] and [15].

There are many other characterisations of amenable Banach algebras, see [15], Chapter VII, Section 2 or [1], Section 43. However, the above is the original definition and is the only characterisation which is required here. The class of amenable Banach algebras is stable under several operations which construct new algebras from old. (Although not under passing to subalgebras, for example, $\ell^1(\mathbf{Z})$, with convolution product, is amenable but $\ell^1(\mathbf{N})$ is not.) Two such stability properties will be needed in the following. The first is that, if $\mathcal{A}$ is amenable and $\Theta : \mathcal{A} \to \mathcal{B}$ is a homomorphism with $\Theta(\mathcal{A})$ dense in $\mathcal{B}$, then $\mathcal{B}$ is also amenable. This fact, which is in Proposition 5.3 in [17], follows immediately from the above definition. The second is that, if $\mathcal{A}$ is amenable and I is a two-sided ideal in $\mathcal{A}$ with a bounded approximate identity, then I is an amenable Banach algebra, see [17], Proposition 5.1 and [12], Theorem 5.1. We shall also need the fact that each amenable Banach algebra has a bounded approximate identity, see [17], Proposition 1.6.

The name of "amenability" for this cohomological property of Banach algebras comes from the theory of locally compact groups. It is a theorem, due to B.E. Johnson, that a locally compact group, G, is amenable if and only if the Banach algebra $L^1(G)$ is amenable, see [17], Theorem 2.5, [1], Proposition 43.3 or [15], Theorem VI.2.33. Amenability is very important for many aspects of group theory and in particular for the study of representations. For example, if G is amenable, then each bounded representation of G on a Hilbert space is equivalent to a unitary representation, ([25]) see [10], Theorem 3.4.1. Also, G is amenable if and only if each of its irreducible, unitary-representations is weakly contained in the regular representation on $L^2(G)$, ([16]) see [10], Theorem 3.5.2.

These theorems for amenable groups generalise basic results about finite groups and thus show one aspect of amenability as a finiteness condition.

Amenability of Banach algebras is not well understood in general but has good characterisations in particular classes of Banach algebras other than the group algebras. First, it follows easily from the structure theory of finite dimensional algebras that such an algebra is amenable if and only if it is isomorphic to a direct sum of full matrix algebras. Also, it is one of the deep theorems about C^*-algebras that a C^*-algebra is amenable if and only if it is nuclear, see [5] and [13]. This is another aspect of amenability as a finiteness condition. Finally, it is a beautiful theorem of Sheinberg, see [23], that a uniform algebra is amenable if and only if it is $C(X)$.

These examples suggest that amenability of $\mathcal{A}_T$ could be important for the analysis of the operator T. In the case when T is a finite rank operator we can say exactly what amenability means. In that case $\mathcal{A}_T$ is finite dimensional, commutative and amenable and hence isomorphic to the direct sum of copies of $\mathbb{C}$. Therefore, for finite rank operators, amenability of $\mathcal{A}_T$ is equivalent to T being diagonable. Perhaps amenability of $\mathcal{A}_T$ is equivalent to some good generalisation of diagonability of T.

There are no general theorems about amenable Banach algebras which allow us to make some immediate conclusions about the structure of T, other than the obvious remark that, since $\mathcal{A}_T$ has an approximate identity, T cannot be nilpotent. It is not known, for example, whether an amenable, commutative Banach algebra can be radical or, if it is not $\mathbb{C}$, an integral domain. Hence we cannot yet answer the questions which are usually the first to be asked about operators. Indeed, in their full generality, these questions about the structures of operators are almost as difficult as the questions about amenable algebras because each singly generated, amenable Banach algebra is of the form $\mathcal{A}_T$ where T is the generator of the algebra. (This follows from the fact that each amenable Banach algebra has a bounded approximate identity.) However it is possible to say more by considering operators satisfying further conditions.

The following theorem extends the remarks about finite rank operators made above. It is really a theorem about commutative subalgebras of $\mathcal{K}(\mathcal{H})$ and so may also be thought of as a relative of Sheinberg's theorem.

THEOREM. *Let T be a compact operator on Hilbert space, $\mathcal{H}$, and suppose that $\mathcal{A}_T$ is amenable. Then T is similar to a normal operator.*

Proof. We first reduce to the case where T has trivial kernel. Since the unit ball in $\mathcal{B}(\mathcal{H})$ is compact in the weak operator topology, it follows that the bounded approximate identity in $\mathcal{A}_T$ converges, with respect to this topology, to an idempotent operator, P, with range$(P) = (\mathcal{A}_T\mathcal{H})^-$ and kernel$(P) =$ kernel(T). Hence $\mathcal{H} = (\mathcal{A}_T\mathcal{H})^- \oplus$ kernel(T). Clearly T is similar to a normal operator if the restriction of T to $(\mathcal{A}_T\mathcal{H})^-$ is, and so it suffices to prove the result in the case when kernel$(T) = (0)$. Note that, in this case, we also have that range(T) is dense in $\mathcal{H}$ and that, since T is compact, the Hilbert space $\mathcal{H}$ is separable.

There are three further steps in the proof. The first step is to show that T has a complete set of eigenvectors, and this step shows incidentally that $\mathcal{A}_T$ is not a radical algebra. The second step is to show that the set of eigenvectors is in fact an unconditional basis for $\mathcal{H}$. For the third step we appeal to the theorem, due to Köthe and Lorch, that each unconditional basis for $\mathcal{H}$ is equivalent to an orthonormal basis, see [18], Proposition 2.b.9. The operator implementing the equivalence between bases implements the similarity between T and a normal operator.

For the first step, let $\mathcal{K}$ be the space of all compact operators on $\mathcal{H}$ and $\mathcal{N}$ the ideal of all nuclear, or trace class, operators. Then $\mathcal{N}$ may be identified with dual space of $\mathcal{K}$ by

$$\langle K, N\rangle = \mathrm{trace}(KN), \quad (K \in \mathcal{K}; N \in \mathcal{N}).$$

Let N be in $\mathcal{N}$. Then, for each K and L in $\mathcal{A}_T$,

$$\langle K, LN - NL\rangle = \langle KL - LK, N\rangle = 0$$

because $\mathcal{A}_T$ is commutative. Hence $D_N : L \mapsto LN - NL$, is a derivation from $\mathcal{A}_T$ to $\mathcal{N}$ with range contained in $\mathcal{A}_T^\perp$, the annihilator of $\mathcal{A}_T$ in $\mathcal{N}$. Now $\mathcal{A}_T^\perp$ is weak*-closed and is invariant under the $\mathcal{A}_T$-module actions. Hence it is a dual $\mathcal{A}_T$-bimodule and, in fact, it is isomorphic to $(\mathcal{K}/\mathcal{A}_T)^*$. Since $\mathcal{A}_T$ is amenable, $D_N : A \to \mathcal{A}_T^\perp$ is inner and so there is some N' in $\mathcal{A}_T^\perp$ such that $D_N = D_{N'}$. Then $N - N'$ belongs to

$$\mathcal{C}_T \equiv \{M \in \mathcal{N} : ML = LM,\ L \in \mathcal{A}_T\}.$$

Since N was arbitrary, it follows that $\mathcal{N} = \mathcal{C}_T + \mathcal{A}_T^\perp$.

Since, as we obviously may suppose, T is not zero, there is, by the Hahn-Banach theorem, an operator N in $\mathcal{N}$ such that

$$\mathrm{trace}(TN) = \langle T, N\rangle \neq 0.$$

The argument in the last paragraph shows that there is Z in $\mathcal{C}_T$ such that trace(TZ) = trace$(TN) \neq 0$. Now the trace of a nuclear operator on Hilbert space is the sum of its eigenvalues, see [8], Theorem XI.9.19, and so TZ has a non-zero eigenvalue. Since T commutes with Z, it follows that T has a non-zero eigenvalue, λ say. Since T is compact, the functional calculus implies that $\mathcal{A}_T$ contains a minimal, finite rank projection, P, whose range contains an eigenvector corresponding to λ.

For each minimal, finite rank projection, P, in $\mathcal{A}_T$, the space $\mathcal{A}_T P$ is an ideal in $\mathcal{A}_T$ with identity P. Hence $\mathcal{A}_T P$ is a finite dimensional, amenable algebra and is consequently isomorphic to a direct sum of copies of $\mathbb{C}$. Since P is also a minimal idempotent in $\mathcal{A}_T P$, it follows that $\mathcal{A}_T P$ is one dimensional and that P is a projection onto an eigenspace of T.

Now let $\mathcal{P} = \{P_1, P_2, \ldots\}$ be a family of minimal idempotents in $\mathcal{A}_T$. Then the restriction of T to $\bigcap_{P \in \mathcal{P}} \text{kernel}(P)$ is a compact operator on this Hilbert space. Furthermore, the algebra generated by the restriction of T is amenable because the restriction map is a homomorphism from $\mathcal{A}_T$ having dense range in this algebra. If $\bigcap_{P \in \mathcal{P}} \text{kernel}(P) \neq (0)$, then the above argument shows that there is a minimal idempotent in $\mathcal{A}_T$ which does not belong to $\mathcal{P}$.

The closed subspace, $\mathcal{J}$, of $\mathcal{H}$ generated by the ranges of the projections in $\mathcal{P}$ is also invariant under T and so T induces a compact operator on the quotient Hilbert space $\mathcal{H}/\mathcal{J}$. The algebra generated by this induced operator is amenable and so, if $\mathcal{J} \neq \mathcal{H}$, the above argument shows once again that there is a minimal idempotent in $\mathcal{A}_T$ which does not belong to $\mathcal{P}$.

Therefore, if we now take $\mathcal{P} = \{P_1, P_2, \ldots\}$ to be the set of all minimal idempotents in $\mathcal{A}_T$, then

$$(1) \qquad \bigcap_{P \in \mathcal{P}} \text{kernel}(P) = (0) \quad \text{and} \quad \mathcal{J} = \mathcal{H}.$$

Hence T has a complete set of eigenvectors.

Next we show that these eigenvectors form an unconditional basis for $\mathcal{H}$. For this we identify $\mathcal{B}(\mathcal{H})$ with the dual space of $\mathcal{N}$ by

$$\langle N, S \rangle = \text{trace}(NS), \quad (N \in \mathcal{N};\ S \in \mathcal{B}(\mathcal{H})).$$

Define, for each S in $\mathcal{B}(\mathcal{H})$,

$$S_{m,n} = P_m S P_n, \quad (m, n = 1, 2, 3, \ldots).$$

Then it follows from (1) that S is the zero operator if and only if $S_{m,n} = 0$ for each m and n.

Put $\mathcal{D}_T \equiv \{S \in \mathcal{B}(\mathcal{H}) : S_{m,n} = 0 \text{ if } m \neq n,\ m,n = 1,2,3,\ldots\}$ and $\mathcal{E}_T \equiv \{S \in \mathcal{B}(\mathcal{H}) : S_{n,n} = 0,\ n = 1,2,3,\ldots\}$. Then $\mathcal{D}_T$ and $\mathcal{E}_T$ are closed subspaces of $\mathcal{B}(\mathcal{H})$ and, by the previous paragraph,

$$\mathcal{D}_T \cap \mathcal{E}_T = (0). \tag{2}$$

If S in $\mathcal{B}(\mathcal{H})$ satisfies $ST = TS$, then $SP_n = P_nS$ for each n and so $S_{m,n} = P_mSP_n = SP_mP_n = 0$ if $m \neq n$. Hence we have

$$\{S \in \mathcal{B}(\mathcal{H}) : ST = TS\} \subseteq \mathcal{D}_T. \tag{3}$$

Furthermore, since $TP_n = \lambda_n P_n = P_nT$, for some λ_n in $\mathbf{C}$, P_nNP_n belongs to $\mathcal{C}_T$ for each N in $\mathcal{N}$ and each n. Hence, if S in $\mathcal{B}(\mathcal{H})$ belongs to $\mathcal{C}_T^{\perp}$, then $0 = \langle P_nNP_n, S\rangle = \langle N, P_nSP_n\rangle$ for each N in $\mathcal{N}$ and so $P_nSP_n = 0$ for each n. It follows that

$$\mathcal{C}_T^{\perp} \subseteq \mathcal{E}_T. \tag{4}$$

Now, for each S in $\mathcal{B}(\mathcal{H})$ define a derivation $D_S : \mathcal{A}_T \to \mathcal{B}(\mathcal{H})$ by $D_S(L) = LS - SL$. Then the range of D_S is contained in the weak*-closed, $\mathcal{A}_T$-submodule $\mathcal{C}_T^{\perp}$. Therefore, since $\mathcal{A}_T$ is amenable, there is S' in $\mathcal{C}_T^{\perp}$ such that $D_{S'} = D_S$. We have then that $S = S' + (S - S')$, where S' belongs to $\mathcal{C}_T^{\perp}$ and $S - S'$ to $\{S \in \mathcal{B}(\mathcal{H}) : ST = TS\}$. It follows, by (2), (3) and (4), that

$$\mathcal{B}(\mathcal{H}) = \mathcal{D}_T \oplus \mathcal{E}_T.$$

Let $Q : \mathcal{B}(\mathcal{H}) \to \mathcal{D}_T$ be the projection with kernel $\mathcal{E}_T$. Then, by the open mapping theorem, Q is bounded. Furthermore, for each S in $\mathcal{B}(\mathcal{H})$, the operator $Q(S)$ satisfies,

$$P_mQ(S)P_n = \begin{cases} S_{n,n} & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}$$

For each P_n in $\mathcal{P}$, choose an orthonormal basis for the range of P_n and then let $x_1, x_2, x_3, x_4 \ldots$ be an enumeration of all these vectors so that the basis for the range of P_1 is listed first, followed by the basis for the range of P_2 and so on. Let $e_1, e_2, e_3, e_4, \ldots$ be the Gram-Schmidt orthonormalisation of this sequence of basis vectors.

Each bounded sequence $\mu_1, \mu_2, \mu_3, \ldots$ of complex numbers determines a multiplication operator, M, on $\mathcal{H}$ by $M(e_n) = \mu_n e_n$ and $\|M\| = \sup\{|\mu_1|, |\mu_2|, |\mu_3|, \ldots\}$. Since $e_1, e_2, e_3, \ldots$ is the Gram-Schmidt orthonormalisation of $x_1, x_2, x_3 \ldots,$

$M_{m,n} = P_m M P_n = 0$ whenever $m > n$, that is, $\{M_{m,n}\}$ is an upper triangular matrix. Furthermore, for each n, $M_{n,n}$ is an operator on the range of P_n and, since the basis for this space, $\{x_r, \ldots, x_s\}$ say, was chosen to be already orthonormal, $M_{n,n}x_t = \mu_t x_t$ for $r < t < s$. Therefore we have

$$Q(M)x_t = \mu_t x_t, \quad t = 1, 2, 3, \ldots . \tag{5}$$

For each positive integer r let M_r be the multiplication operator determined by the sequence $\mu_t = 1$, if $1 \leqslant t \leqslant r$ and $\mu_t = 0$ if $t > r$. Then, for each r, $\|Q(M_r)\| \leqslant \|Q\|$ and, by (5),

$$Q(M_r)x_t = \begin{cases} x_t & \text{if } 1 \leqslant t \leqslant r \\ 0 & \text{otherwise.} \end{cases}$$

Since the set of eigenvectors $\{x_1, x_2, x_3, \ldots\}$ spans a dense subspace of $\mathcal{H}$, it follows that $\lim_{r\to\infty} Q(M_r)x = x$, for each x in $\mathcal{H}$ and hence that $x_1, x_2, x_3, \ldots$ is a Schauder basis for $\mathcal{H}$. Reverting now to an arbitrary bounded sequence $\mu_1, \mu_2, \mu_3, \ldots$, the operator $Q(M)$ satisfies (5) and $\|Q(M)\| \leqslant \|Q\|\,\|M\|$. It follows that the sequence of eigenvectors is an unconditional basis for $\mathcal{H}$.

As already indicated, each unconditional basis for $\mathcal{H}$ is equivalent to an orthonormal basis and hence T is similar to a normal operator. ∎

SOME REMARKS ON THE USE OF AMENABILITY IN THE PROOFS OF BANACH SPACE THEOREMS

The theorem of Köthe and Lorch which is used at the end of the proof also involves an application of amenability. To see this, let $x_1, x_2, x_3, \ldots$ be an unconditional basis for the Hilbert space $\mathcal{H}$ and let G be the abelian group which is the direct sum of infinitely many copies of the multiplicative group $\{-1, 1\}$. Denote the elements of G as sequences $\hat{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$, where $\varepsilon_n = \pm 1$ and equals -1 only finitely often. Then, since the basis is unconditional, the representation, ρ, of G on $\mathcal{H}$ determined by $\rho(\hat{\varepsilon})x_n = \varepsilon_n x_n$ is bounded. Hence ρ is a bounded representation of the amenable group G on $\mathcal{H}$. Now the theorem of Sz.-Nagy, [10], Theorem 3.4.1, implies that ρ is equivalent to a unitary representation, ρ' say, and it follows that $x_1, x_2, x_3, \ldots$ is equivalent to the orthonormal basis of common eigenvectors of the unitary operators $\rho'(\hat{\varepsilon})$. The proof of the Köthe-Lorch theorem given in [18] also introduces the bounded representation of the group G, without mentioning groups explicitly, and then uses an averaging argument to deduce the result. In fact, this averaging argument is just using the amenability of G and is proving a special case of Sz.-Nagy's theorem.

There is another Banach space theorem concerning unconditional bases whose proof is an amenability argument. If X and Y are Banach spaces with bases $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ respectively, then each operator $T : X \to Y$ has a "matrix", $\{T_{m,n}\}$, where $T_{m,n} = \langle Tx_n, y_m^*\rangle$, $\{y_m^*\}_{n=1}^{\infty}$ being the sequence biorthogonal to $\{y_n\}_{n=1}^{\infty}$. If the bases are unconditional, then it may be shown that the diagonal of this matrix also determines an operator from X to Y, that is, that $x_n \mapsto T_{n,n}y_n$ extends to a bounded linear operator. The proof of this fact given in [18], Proposition 1.c.8, uses multiplication of basis vectors by ± 1 followed by averaging and is thus also using amenability of the group G which was defined in the previous paragraph.

CAN THE HYPOTHESES ON T BE WEAKENED?

The proof of the theorem relies very heavily on T being a compact operator on Hilbert space. It is an interesting question whether the hypothesis that T be a compact operator can be weakened. An extension of the theorem to general bounded operators on Hilbert space presumably could not use an eigenvalue and eigenvector argument as in the above proof.

Now consider the case when T is a compact operator on some Banach space other than a Hilbert space. The argument breaks down when we try to find an eigenvector for T because, for nuclear operators on spaces other than Hilbert space, it is not true in general that the trace is the sum of the eigenvalues. It is not so for the space ℓ_p, $p \neq 2$, for example, see [18], Theorem 2.d.3 and this is equivalent to the existence of subspaces of ℓ_p which do not have the approximation property. The possibility remains open therefore that there is a compact, quasinilpotent operator on ℓ_p, where $p \neq 2$, such that $\mathcal{A}_T$ is amenable. This possibility suggests that there could be a deep connection between the existence of radical, amenable algebras of compact operators and the approximation property in general Banach spaces.

OTHER HOMOLOGICAL CONDITIONS

The definition of amenability given above is simply another way of stating that the continuous cohomology groups $\mathrm{H}^1(\mathcal{A}, X^*)$ vanish for each Banach $\mathcal{A}$-bimodule X, see [15], Chapter I.2. There are other homological properties of Banach algebras which could also be relevant to operator theory.

One such condition is biprojectivity, where $\mathcal{A}$ is said to be *biprojective* if the $\mathcal{A}$-bimodule homomorphism,

$$\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$$

determined by the product map $\pi(a \otimes b) = ab$, has an $\mathcal{A}$-bimodule right inverse, see [20] and [21]. For algebras having a bounded approximate identity, biprojectivity is a stronger condition than amenability, see Theorem VII.2.18 in [15].

Another homological property stronger than amenability is that all the continuous cohomology groups $\mathrm{H}^1(\mathcal{A}, X)$ should vanish. It was shown by Taylor ([26]), that if $\mathcal{A}$ has the compact approximation property and satisfies this condition, then $\mathcal{A}$ is finite dimensional. This condition is thus too strong to distinguish an interesting class among the familiar classes of Banach algebras but does provide another indication that amenability is a finiteness condition.

A homological condition on Banach algebra $\mathcal{A}$ which is not as strong as amenability is that of being *weakly amenable,* that is, satisfying the condition $\mathrm{H}^1(\mathcal{A}, \mathcal{A}^*) = (0)$, see [2] and [11]. P.C. Curtis has discovered an example of a commutative, radical weakly amenable Banach algebra, see [6]. There are many weakly amenable algebras which are not amenable and Curtis' algebra appears to be another such example. The algebras generated by weighted shifts and integral operators are not weakly amenable, see [9].

AMENABILITY AND APPROXIMATION PROPERTIES

It has already been remarked that the fact that Hilbert space and all of its subspaces have the approximation property is used several times in the proof of the above theorem. Many other homological results for Banach algebras have involved the approximation property. The theorem of Taylor mentioned before is one example. The approximation property and compact approximation property are also important in the work of Selivanov on cohomology of Banach algebras. In particular, in [22] he gives some homological characterisations of the approximation property. For many of these results it remains an open question whether the approximation property or the compact approximation property can be dispensed with.

Another indication of a strong connection between amenability and the approximation property occurs in the theory of C^*-algebras. The work of Connes, Haagerup and Choi and Effros, see [13], [5] and [4], implies that, if a C^*-algebra is amenable, then it has the approximation property. It was shown by Szankowski ([24]) that the C^*-algebra $\mathcal{B}(\mathcal{H})$ does not have the approximation property and so amenability does seem to be important.

Amenability might also be used to describe new approximation properties for Banach spaces. In [12] the question of when the algebras of approximable operators and of compact operators are amenable is studied. The results there indicate that amenability of these algebras seems to be equivalent to "basis free" versions of the notion of a shrinking, subsymmetric basis. In a similar way, the existence of a commutative, amenable algebra, $\mathcal{A}$, of compact operators on X such that $X = (\mathcal{A}X)^-$ could be equivalent to an approximation property for X between the commuting, bounded approximation property and the existence of a symmetric basis. See [3] and [19] for some recent results on various approximation properties.

Acknowledgements. It is a pleasure to thank Ernst Albrecht, Barry Johnson, Andrew Kepert, Rick Loy and Heydar Radjavi for the stimulating discussions which led to the proof of the above theorem. These discussions were made possible by support from the Australian Research Council and the Australian National University.

REFERENCES

1. F.F. BONSALL, J. DUNCAN, *Complete Normed Algebras*, Springer-Verlag, Berlin–Heidelberg–New York, 1973.
2. W.G. BADE, P.C. CURTIS JR., H.G. DALES, Amenability and weak amenability in Beurling and Lipschitz algebras, *Proc. London Math. Soc. (3)* 55(1987), 359–377.
3. P.G. CASAZZA, N.J. KALTON, Notes on approximation properties in separable Banach spaces, preprint.
4. M.-D. CHOI, E.G. EFFROS, Nuclear C^*-algebras and the approximation property, *Amer. J. Math.* 100(1978), 61–79.
5. A. CONNES, On the cohomology of operator algebras, *J. Funct. Anal.* 28(1978), 248–253.
6. P.C. CURTIS JR, *Proceedings of the Second Conference on Function Spaces*, Southern Illinois University, Edwardsville, 1994.
7. P.C. CURTIS JR, R.J. LOY, The structure of amenable Banach algebras, *J. London Math. Soc. (2)* 40(1989), 89–104.
8. N. DUNFORD, J.T. SCHWARTZ, *Linear Operators, Part II: Spectral Theory*, Interscience Publishers, New York, 1958–1971.

9. F. GHAHRAMANI, Homomorphisms and derivations on weighted convolution algebras, *J. London Math. Soc. (2)* **21**(1980), 149–161.
10. F.P. GREENLEAF, *Invariant Means on Topological Groups*, van Nostrand, New York 1969.
11. N. GRØNBÆK, Constructions preserving weak amenability, *Proc. Centre Math. Anal. Austral. Nat. Univ.* **21**(1989), pp. 186–202.
12. N. GRØNBÆK, B.E. JOHNSON, G.A. WILLIS, Amenability of Banach algebras of compact operators, *Israel J. Math.*, to appear.
13. U. HAAGERUP, All nuclear C^*-algebras are amenable, *Invent. Math.* **74**(1983), 305–319.
14. A.YA. HELEMSKII, Flat Banach modules and amenable algebras, *Trans. Moscow Math. Soc.* **47**(1984), 179–218; *Amer. Math. Soc. Transl. Ser. 2* (1985), pp. 199–224.
15. A.YA. HELEMSKII, *The Homology of Banach and Topological Algebras*, Kluwer, Dordrecht, 1989. (Russian original 1986.)
16. A. HULANICKI, Means and Følner conditions on locally compact groups, *Studia Math.* **27**(1966), 87–104.
17. B.E. JOHNSON, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.* **127**(1972).
18. J. LINDENSTRAUSS, L. TZAFRIRI, *Classical Banach Spaces. I*, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
19. C.J. READ, Different forms of the approximation property, preprint.
20. YU.V. SELIVANOV, Biprojective Banach algebras, *Izv. Akad. Nauk S.S.S.R. Ser. Mat.* **43**(1979), 1159–1174; Engl. transl., *Amer. Math. Soc. Transl. Ser. 2* **15**(1980), pp. 387–399.
21. YU.V. SELIVANOV, Biprojective Banach algebras and their structure, cohomology and relation with nuclear operators, *Funktsional Anal. i Prilozhen.* **10**(1976), 89–90.
22. YU.V. SELIVANOV, Homological characterisations of the approximation property for Banach spaces, *Glasgow Math. J.* **34**(1992), 229–239.
23. M.V. SHEINBERG, A characterisation of the algebra $C(\Omega)$ in terms of cohomology groups, *Uspekhi Matem. Nauk* **32**(1977), 203–204.
24. A. SZANKOWSKI, $\mathcal{B}(\mathcal{H})$ does not have the approximation property, *Acta Math.* **147** (1981), 89–108.
25. B. SZ.-NAGY, Uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math. (Szeged)* **11**(1947), 152–157.
26. J.L. TAYLOR, Homology and cohomolgy for topological algebras, *Adv. Math.* **9**(1972) 137–252.
27. G.A. WILLIS, The compact approximation property does not imply the approximation property, *Studia Math.* **103**(1992), 99–108.

G.A. WILLIS
Department of Mathematics
University of Newcastle
University Drive, Callaghan
New South Wales, 2308
AUSTRALIA

Received March 15, 1994; revised September 14, 1994 and June 30, 1995.