J. OPERATOR THEORY
34(1995), 307–314



# HILBERT-SCHMIDT AND FINITE RANK OPERATORS IN CSL ALGEBRAS

M. ANOUSSIS and E.G. KATSOULIS

*Communicated by William B. Arveson*

ABSTRACT. In this note certain results on the structure of the finite rank subalgebra of a CSL algebra are proved. These enable us to show that there are CSLs not generated by any finite family of CD CSLs.

KEYWORDS: CSL *algebra, complete distributivity, Hilbert-Schmidt operator, finite rank operator.*

AMS SUBJECT CLASSIFICATION: Primary 47D25; Secondary 47C05.

Given subspace lattices $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$, it is of interest to know how certain properties of $\operatorname{Alg}\mathcal{L}_i$, $i = 1, 2, \ldots, n$, reflect on the structure of $\operatorname{Alg}\Big(\bigvee\limits_{i=1}^{n} \mathcal{L}_i\Big) = \bigcap\limits_{i=1}^{n} \operatorname{Alg}\mathcal{L}_i$. This is a direction taken by A. Hopenwasser and R.L. Moore in investigating the structure of the rank one subalgebra of a commutative subspace lattice. In [5], their main theorem shows that if $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ are mutually commuting CSLs and $F$ a finite rank operator which, for each $i = 1, 2, \ldots, n$, is the sum of rank one operators from $\operatorname{Alg}\mathcal{L}_i$, then there are finite many rank one operators in $\operatorname{Alg}\Big(\bigvee\limits_{i=1}^{n} \mathcal{L}_i\Big)$, whose sum equals $F$. Their result inspired subsequent work by Froelich ([4]), Katsoulis and Moore ([6]), Rosenoer ([10]) and others; the main object of investigation in these papers was the rank one subalgebra and its relation to certain classes of compact operators.

The main result of this note is an analytic variant of the result of A. Hopenwasser and R. Moore mentioned above; we show that if $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ are mutually commuting CSLs and $T$ a Hilbert-Schmidt operator which, for each $i = 1, 2, \dots, n$, is the $\|\cdot\|_2$ limit of finite rank operators in $\operatorname{Alg}\mathcal{L}_i$, then there exists a sequence of finite rank operators in $\operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ converging to $T$.

For lattices generated by finitely many CD CSLs, our result shows that algebras of such lattices contain non-zero Hilbert-Schmidt operators iff they contain non-zero rank one operators. In addition, we prove that algebras of lattices with prime elements must contain non-zero rank one operators. These results enable us to answer the question of whether every CSL is generated by finitely many CD CSLs. This question was raised to the second named author by D.R. Larson. We give examples to show that the answer is no.

Let us establish some notation and terminology. A *commutative subspace lattice* (abbr. CSL) is a complete lattice of selfadjoint projections acting on a separable Hilbert space $\mathfrak{H}$. A CSL is said to be *completely distributive* (abbr. CD CSL) if and only if it satisfies any infinite distributive law (see [8] for a precise definition and properties of such lattices). If $\{\mathcal{L}_i\}_{i=1}^{n}$ is a finite family of mutually commuting CSL's then $\bigvee_{i=1}^{n} \mathcal{L}_i$ is the complete lattice generated by $\{\mathcal{L}_i\}_{i=1}^{n}$. A lattice generated by finitely many commuting nests is said to be of finite width. Lattices of infinite width have already appeared in the literature and the reader will not find it difficult to construct one. Lattices which are not generated by finitely many commuting CD CSLs are less obvious.

If $\mathcal{L}$ is a CSL then $\operatorname{Alg}\mathcal{L}$ denotes the algebra of all bounded operators which leave invariant every element of $\mathcal{L}$. The algebra generated by all finite rank (resp. rank one) operators belonging to $\operatorname{Alg}\mathcal{L}$ will be denoted by $\mathfrak{F}(\mathcal{L})$ (resp. $\mathfrak{R}(\mathcal{L})$). In general $\mathfrak{R}(\mathcal{L})$ is strictly contained in $\mathfrak{F}(\mathcal{L})$. But, the norm closure of $\mathfrak{R}(\mathcal{L})$ always contains $\mathfrak{F}(\mathcal{L})$ (see [3], Theorem 23.16).

In this article we shall need to make use of Arveson's spectral representation theorem for commutative subspace lattices. This theorem states that the following scheme for constructing examples of commutative subspace lattices yields, up to unitary equivalence, all commutative subspace lattices. Let $\mathfrak{X}$ be a compact metric space, let $\leqslant$ be a reflexive and transitive relation on $\mathfrak{X}$ whose graph $G(\mathfrak{X}, \leqslant)$ is a closed subset of $\mathfrak{X} \times \mathfrak{X}$ and let $\mu$ be a finite Borel measure on $\mathfrak{X}$. A Borel subset $S \subseteq \mathfrak{X}$ is said to be increasing if $x \in S$ and $x \leqslant y$ imply $y \in S$. For each Borel subset $S$ of $\mathfrak{X}$ let $P(S)$ denote the corresponding orthogonal projection acting on the Hilbert space $L^2(\mathfrak{X}, \mu)$, i.e. $P(S)$ is the multiplication operator obtained from the characteristic function of $S$. Let $\mathcal{L}(\mathfrak{X}, \leqslant, \mu) = \{P(S) \mid S \text{ is an increasing Borel subset of } \mathfrak{X}\}$. Arveson's Theorem ([2], Theorem 1.3.1) asserts that every CSL is unitarily equivalent to some $\mathcal{L}(\mathfrak{X}, \leqslant, \mu)$.

We start with the main result (observe that part (ii) is Theorem 5 in [5]). Its proof depends on Arveson's spectral representation theorem and the following well-known facts. If $T$ is a Hilbert-Schmidt operator acting on $L^2(\mathfrak{X},\mu)$ then there exists a function $T \in L^2(\mathfrak{X}\times\mathfrak{X},\mu\times\mu)$, the *kernel function* for $T$, so that $Tf(x) = \int T(x,y)f(y)\,\mathrm{d}\mu(y)$, for any $f \in L^2(\mathfrak{X},\mu)$. If $T$ is a rank one operator then its kernel is of the form $T(x,y) = g(x)h(y)$, for suitable $g$, $h$ in $L^2(\mathfrak{X},\mu)$; such kernels will be called *elementary*.

THEOREM 1. *If $\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n$ are mutually commuting* CSLs *then,*
(i) $\mathrm{cl}_2\,\mathfrak{R}\Big(\bigvee\limits_{i=1}^{n}\mathcal{L}_i\Big) = \bigcap\limits_{i=1}^{n}\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}_i) = \mathrm{cl}_2\,\mathfrak{F}\Big(\bigvee\limits_{i=1}^{n}\mathcal{L}_i\Big)$;
(ii) $\mathfrak{R}\Big(\bigvee\limits_{i=1}^{n}\mathcal{L}_i\Big) = \bigcap\limits_{i=1}^{n}\mathfrak{R}(\mathcal{L}_i)$,
*where* $\mathrm{cl}_2$ *denotes the closure with respect to the Hilbert-Schmidt norm.*

*Proof.* It is enough to prove the theorem in the case where $n = 2$. The general case follows by simple induction.

We start by providing a suitable Arveson model for the lattice $\mathcal{L} = \mathcal{L}_1 \vee \mathcal{L}_2$. Let $\{P_1^{(i)}, P_2^{(i)},\ldots\}$ be a strongly dense subset of $\mathcal{L}_i$, $i = 1,2$. Now, imbed $\mathcal{L}$ in a maximal abelian von Neumann algebra $\mathfrak{R}$ and choose countable strongly dense subset $\{Q_1,Q_2,\ldots\}$ of the projection lattice of $\mathfrak{R}$. Let $\mathfrak{U}$ be the $C^*$-algebra generated by $\{P_k^{(i)} \mid k\in\mathbf{N}, i=1,2\}$ and $\{Q_k\}_{k=1}^{\infty}$ and let $\mathfrak{X}$ be the spectrum of $\mathfrak{U}$.

Arguing as in the proof of Theorem 1.3.1 in [2], we obtain a probability measure $\mu$ and a partial order $\leqslant$ on $\mathfrak{X}$ such that $U\mathcal{L}(\mathfrak{X},\leqslant,\mu)U^{-1} = \mathcal{L}$, for a suitable unitary $U$. In particular $x \leqslant y$ if and only if $\chi_{E_k}^{(i)}(x) \leqslant \chi_{E_k}^{(i)}(y)$, for all $i$ and $k$, where $E_k^{(i)}$ is the increasing set corresponding to $U^{-1}P_k^{(i)}U$. Define partial orders $\leqslant_i$, $i = 1,2$ on $\mathfrak{X}$ to mean $x \leqslant_i y$ if and only if $\chi_{E_k}^{(i)}(x) \leqslant \chi_{E_k}^{(i)}(y)$, for all $k$. Using Theorem 1.2.2 in [2], one can show that $U\cdot U^{-1}$ carries $\mathcal{L}(\mathfrak{X},\leqslant_i,\mu)$ onto $\mathcal{L}_i$, $i = 1,2$.

(i) It is enough to show that $\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}_1)\cap\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}_2)$ is contained in $\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}_1 \vee \mathcal{L}_2)$. Let $T$ be a Hilbert-Schmidt operator in

$$\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}(\mathfrak{X},\leqslant_1,\mu))\cap\mathrm{cl}_2\,\mathfrak{R}(\mathcal{L}(\mathfrak{X},\leqslant_2,\mu))$$

and let $T(\cdot\,,\cdot)$ be its kernel function, i.e., $Tf(x) = \int T(x,y)f(y)\,\mathrm{d}\mu(y)$, $\forall f \in L^2(\mathfrak{X},\mu)$. Since $T$ belongs to $\mathrm{cl}_2\,(\mathfrak{R}(\mathcal{L}(\mathfrak{X},\leqslant_i,\mu)))$ there exist sequences $\{g_j^{(i)}\}_{j=1}^{\infty}$, where

$$g_j^{(i)}(x,y) = \sum_{k=1}^{n_j} e_{j,k}^{(i)}(x) f_{j,k}^{(i)}(y)$$

and

$$\Omega_{j,k}^{(i)} \triangleq (\operatorname{supp} e_{j,k}^{(i)}) \times (\operatorname{supp} f_{j,k}^{(i)}) \subseteq G(\mathfrak{X},\leqslant_i),$$

such that $T(\cdot\,,\cdot)$ is the $L^2$-limit of $\{g_j^{(i)}\}_{j=1}^{\infty}$, $i=1,2$. Rearrange the rectangles $\Omega_{j,k}^{(i)}$ so that they form a sequence of the form $\{\Omega_n^{(i)}\}_{n=1}^{\infty}$: clearly, $T(\cdot\,,\cdot)$ lives $\mu\times\mu$-almost everywhere on

$$\bigcup_{n=1}^{\infty}\Omega_n^{(i)}, \qquad i=1,2$$

and thus it lives $\mu\times\mu$-almost everywhere on

$$\Big(\bigcup_{n=1}^{\infty}\Omega_n^{(1)}\Big)\cap\Big(\bigcup_{m=1}^{\infty}\Omega_m^{(2)}\Big)=\bigcup_{m,n\in\mathbf{N}}(\Omega_n^{(1)}\cap\Omega_m^{(2)})\subseteq G(\mathfrak{X},\leqslant_1)\cap G(\mathfrak{X},\leqslant_2)=G(\mathfrak{X},\leqslant).$$

Since every $\Omega_n^{(1)}\cap\Omega_m^{(2)}$ is a subrectangle of $G(\mathfrak{X},\leqslant)$, a standard application of the Stone-Weierstrass Theorem shows that the Hilbert-Schmidt operator with kernel function $T(\cdot\,,\cdot)\chi_{\Omega_n^{(1)}\cap\Omega_m^{(2)}}(\cdot\,,\cdot)$ can be approximated by elements of $\mathfrak{R}(\mathcal{L}_1\vee\mathcal{L}_2)$. An application of Lebesgue Convergence Theorem shows that the same is true for $T$.

(ii) It is enough to show that $\mathfrak{R}(\mathcal{L}_1)\cap\mathfrak{R}(\mathcal{L}_2)\subseteq\mathfrak{R}(\mathcal{L}_1\vee\mathcal{L}_2)$. If $T$ belongs to $\mathfrak{R}(\mathcal{L}_1)\cap\mathfrak{R}(\mathcal{L}_2)$ then, arguing as in case (i), we generate sequences $\{\Omega_n^{(1)}\}_{n\in\mathbf{N}}$, $\{\Omega_m^{(2)}\}_{m\in\mathbf{N}}$ of rectangles contained in $G(\mathfrak{X},\leqslant_1)$ and $G(\mathfrak{X},\leqslant_2)$ respectively. However, the assumption $T\in\mathfrak{R}(\mathcal{L}_i)$, $i=1,2$, implies that these sequences can be chosen finite, of length, say, $k$. Then,

$$T(\cdot\,,\cdot)=\sum_{n,m=1}^{k}T(\cdot\,,\cdot)\chi_{\Omega_n^{(1)}\cap\Omega_m^{(2)}}(\cdot\,,\cdot).$$

For all $m,n,\Omega_n^{(1)}\cap\Omega_m^{(2)}$ is a rectangle contained in $G(\mathfrak{X},\leqslant)$ and so the kernel $T(\cdot\,,\cdot)\chi_{\Omega_n^{(1)}\cap\Omega_m^{(2)}}$ is a sum of elementary kernels with supports contained in $\Omega_n^{(1)}\cap\Omega_m^{(2)}$. Thus the corresponding operator belongs to $\mathfrak{R}(\mathcal{L}_1\vee\mathcal{L}_2)$ and so does $T$. ∎

In the rest of this note we specialize in lattices generated by finitely many commuting CD CSLs; for convenience we abbreviate them as FCD CSLs. We prove two theorems on the existence of non-trivial compact operators in algebras of such lattices. The first is an application of Theorem 1 while the second is of independent interest. Both are not valid in the general case of a CSL, thus showing that FCD CSLs are behaving rather nicely.

THEOREM 2. *Let $2 \leqslant p \leqslant \infty$, let $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ be mutually commuting* CD CSLs *and assume that for every $i = 1, 2, \dots, n$ the subspace $C_p \cap \operatorname{Alg} \mathcal{L}_i$ has a complement in $C_p$. Then,* $\operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ *contains a non-zero $C_p$-operator iff it contains a non trivial rank one operator.*

*Proof.* First notice that in the case where $p = 2$, the requirement of $C_p \cap \operatorname{Alg} \mathcal{L}_i$ being complemented, $i = 1, 2, \dots, n$ is always satisfied, since $(C_2, \|\cdot\|_2)$ is a Hilbert space. For the proof, we distinguish two cases:

$p = 2$. Since $\mathcal{L}_i$, $i = 1, 2, \dots, n$ is completely distributive, $\mathfrak{R}(\mathcal{L}_i)$ is $w^*$-dense in $\operatorname{Alg} \mathcal{L}_i$. Theorem 23.18 in [3] implies that $\mathfrak{R}(\mathcal{L}_i)$ is $\|\cdot\|_2$-dense in $C_2 \cap \operatorname{Alg} \mathcal{L}_i$, for all $i$. Theorem 1 shows now that if $\operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ contains a non trivial Hilbert-Schmidt operator, then $\mathfrak{R}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big) \neq \{0\}$.

$p > 2$. Since all subalgebras $C_p \cap \operatorname{Alg} \mathcal{L}_i$ are complemented, Proposition 2.2 in [1] shows that $C_p \cap \operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ is complemented. Let $\pi$ be the idempotent projecting onto $C_p \cap \operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ which satisfies the requirements of Theorem 2.1 in [1]. Then the restriction of $\pi$ on $C_2$ equals $\mathfrak{U}_2$, the orthogonal projection form $C_2$ onto $C_2 \cap \operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ ([1], Proposition 3.1). Since $C_2 \subseteq C_p$ is $\|\cdot\|_p$-dense, we conclude that $C_2 \cap \operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ is $\|\cdot\|_p$-dense in $C_p \cap \operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$. Thus, if $\operatorname{Alg}\Big(\bigvee_{i=1}^{n} \mathcal{L}_i\Big)$ contains a non-zero $C_p$-operator, it contains a non-zero Hilbert-Schmidt operator. The rest of the proof follows from previous considerations. ∎

COROLLARY 3. ([6]) *If $\mathcal{L}$ is a* CSL *of finite width then* $\operatorname{Alg} \mathcal{L}$ *contains a $C_p$-operator iff it contains a non-zero rank one operator.*

It is known that a CSL algebra $\operatorname{Alg} \mathcal{L}_i$ contains a non-zero rank one operator iff there is at least one projection $E$ and $\mathcal{L}$ so that $E_- \triangleq \bigvee\{F \in \mathcal{L};\ F \not\supseteq E\}$ is different from $I$. In what follows, we show that a much simpler lattice theoretic condition is sufficient for the existence of non-trivial rank one operators in algebras of FCD CSLs.

A non-zero element $G$ of a lattice $\mathcal{L}$ is called *joint-irreducible (or prime)* if $E, F \in \mathcal{L}$, $E \vee F = G$ implies $E = G$ or $F = G$. If $I$ is a join-irreducible element then $\mathcal{L}$ is called primary; nest are always primary while Boolean lattices are not. The join-irreducible elements of a lattice $\mathcal{L}$ are of importance in the study of its representations. In [9], J. Orr and S. Power prove that every CD CSL contains join-irreducible elements and, as a result, they obtain new representations for such lattices.

THEOREM 4. *Let $\{\mathcal{L}_i\}_{i=1}^n$ be any finite family of commuting completely distributive* CSLs. *If $\mathcal{L} = \bigvee_{i=1}^n \mathcal{L}_i$ contains a join-irreducible element, then* $\operatorname{Alg}\mathcal{L}$ *contains a non-trivial rank one operator.*

*Proof.* We first prove the result in the case where $I$ is a join-irreducible element. The proof follows by induction on $n$. By the way of contradiction assume that $\operatorname{Alg}\left(\bigvee_{i=1}^{n-1} \mathcal{L}_i\right)$ contains non-zero rank one operators but $\operatorname{Alg}\mathcal{L}$ does not; let $E$ be an element of $\left(\bigvee_{n=1}^{n-1} \mathcal{L}_i\right)$ such that $E_- \neq I$ ($E_-$ is computed in $\bigvee_{i=1}^{n-1} \mathcal{L}_i$).

Observe now that if $F$ is an element of $\mathcal{L}_n$ and $R$ is an arbitrary rank one operator, then $EFR(E_-)^\perp(F_-)^\perp = 0$, since $\operatorname{Alg}\mathcal{L}$ does not contain rank-one operators. This mean that for any $F$ in $\mathcal{L}_n$ such that $EF \neq 0$, we have $E_- \vee F_- = I$ ($F_-$ is computed in $\mathcal{L}_n$).

We now define $\pi(E) = \bigvee\{P \in \mathcal{L}_n \mid P \text{ is orthogonal to } E\}$; clearly $\pi(E) \in \mathcal{L}_n$ and $\pi(E) \neq I$. Moreover, since $\mathcal{L}_n$ is completely distributive, $\pi(E) = \bigwedge\{F_- \mid F \in \mathcal{L}_n, F \not\subseteq \pi(E)\}$. (Theorem 5.2 in [8].) Hence,

$$\begin{aligned} E_- \vee \pi(E) &= \bigwedge\{E_- \vee F_- \mid F \in \mathcal{L}_n,\ F \not\subseteq \pi(E)\} \\ &\supseteq \bigwedge\{E_- \vee F_- \mid F \in \mathcal{L}_n,\ EF \neq 0\} \\ &\supseteq \bigwedge\{E_- \vee F_- \mid F \in \mathcal{L}_n,\ E_- \vee F_- = I\} \\ &= I. \end{aligned}$$

Since $\pi(E)$, $E_-$ are both different from the identity we conclude that $\mathcal{L}$ is not primary, a contradiction. For the general case, if $E \in \mathcal{L}$ is join-irreducible, consider the lattice $E\mathcal{L} = \{EF \mid F \in \mathcal{L}\}$ acting on the Hilbert space $E(\mathfrak{H})$. It is easily seen that $E\mathcal{L}$ is FCD CSL and $I$ is a join-irreducible element of $E\mathcal{L}$. Thus $\operatorname{Alg} E\mathcal{L} = E(\operatorname{Alg}\mathcal{L})$ contains a non-trivial rank one operator and the conclusion follows. ∎

The present work was stimulated by the fact that it seemed to be unknown whether or not every CSL can be generated by finitely many CD CSLs. Using Theorems 2 and 4, we are in position to show that this question has a negative answer.

COROLLARY 5. *Let $\mathfrak{X} = [0,1] \times [0,1]$, $\mu = \lambda \times \lambda$ where $\lambda$ is the Lebesgue measure, and let $\leqslant$ be the partial order with graph*

$$G = \{(x,x) \mid 0 \leqslant x \leqslant 1\} \cup \{(x,y) \mid 0 \leqslant x \leqslant \frac{1}{2} \leqslant y \leqslant 1 \text{ and } y - x \in K\}$$

*where $K$ is a closed nonwhere dense subset of* [0,1] *with positive measure. Then* $\mathcal{L}(\mathfrak{X}, \leqslant, \mu)$ *is not generated by finitely many* CD CSL's.

*Proof.* T. Trent has shown that the algebra of a lattice of the above form contains Hilbert-Schmidt but no rank one operators. The conclusion follows now from Theorem 2. ∎

For our next example, let $2^\infty$ be the Cantor space of all sequences $(x_i)$ of zeros and ones, and define $(x_i) \leqslant (y_i)$ to mean $x_i \leqslant y_i$, for every $i = 1, 2, \ldots$. For each real number $p$, $0 < p < 1$, let $m_p$ be the infinite product measure $m_0 \times m_0 \times \ldots$, where $m_0$ assigns mass $p$ to $\{1\}$ and mass $1 - p$ to $\{0\}$.

COROLLARY 6. *The lattices* $\mathcal{L}(2^\infty, \leqslant, m_p)$, $0 < p < 1$, *are not generated by any finite family of* CD CSLs.

*Proof.* W. Arveson has shown in [2], p. 519, that the lattices $\mathcal{L}(2^\infty, \leqslant, m_p)$ are primary and $J$. Froelich ([4]) proved that their algebras consist of non-compact operators. The conclusion follows now from Theorem 4. ∎

REMARKS. (i) Theorem 4 leads to the following interesting characterization of complete distributivity in the class of FCD CSLs: *A* FCD CSL $\mathcal{L}$ *is completely distributive iff for every semi-invariant projection $P$ the lattice $P\mathcal{L}$ contains a join-irreducible element.*

(ii) We would like to show that if the assumption of mutual commutativity is dropped, Theorem 1 is no longer valid; *there is a lattice $\mathcal{L}$ generated by four (non-commuting) nests whose algebra contains trace class but no rank one operators.*

Indeed, let $\mathfrak{H} = L^2([0,1], \lambda)$, where $\lambda$ is the Lebesgue measure and let

$$\begin{aligned} M_1 &= \mathfrak{H} \oplus 0, \\ M_2 &= 0 \oplus \mathfrak{H}, \\ M_3 &= \{(x, x) \mid x \in \mathfrak{H}\}, \\ M_4 &= \{(x, Vx) \mid x \in \mathfrak{H}\}, \end{aligned}$$

where $V$ is the Volterra integral operator. Define $\mathcal{L}_i = \{0, M_i, I\}$, $i = 1, 2, 3, 4$, and let $\mathcal{L} = \bigvee_{i=1}^{4} \mathcal{L}_i$. Easy calculations show that $\operatorname{Alg}\mathcal{L} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in B(\mathfrak{H}), AV = VA \right\}$. Since the invariant subspace lattice of $V$ consists of infinite dimensional subspaces, there are not finite rank operators commuting with $V$ and thus $\operatorname{Alg}\mathcal{L}$ contains no finite rank operators. On the other hand, $\begin{pmatrix} V^2 & 0 \\ 0 & V^2 \end{pmatrix}$ belongs to $\operatorname{Alg}\mathcal{L}$ and so $\operatorname{Alg}\mathcal{L}$ contains non-trivial trace class operators.

## REFERENCES

1. M. ANOUSSIS, E.G. KATSOULIS, Complemented subspaces of $C_p$ spaces and CSL algebras, *J. London Math. Soc. (2)* **45**(1992), 301–313.
2. W.B. ARVESON, Operator algebras and invariant subspaces, *Ann. of Math. (2)* **100** (1974), 433–532.
3. K. DAVIDSON, *Nest algebras*, Pitman Res. Notes Math. Ser., vol. 191, New York 1988.
4. J. FROELICH, Compact operators, invariant subspaces and spectral synthesis, *J. Funct. Anal.* **81**(1988), 1–37.
5. A. HOPENWASSER, R.L. MOORE, Finite rank operators in reflexive operator algebras, *J. London Math. Soc. (2)* **27**(1993), 331–338.
6. E.G. KATSOULIS, R.L. MOORE, On compact operators in certain reflexive operator algebras, *J. Operator Theory* **25**(1991), 177–182.
7. E.G. KATSOULIS, S.C. POWER, Compact perturbations of certain CSL algebras, *Proc. Amer. Math. Soc.* **114**(1992), 1041–1044.
8. W.E. LONGSTAFF, Strongly reflexive lattices, *J. London Math. Soc. (2)* **11**(1975), 491–498.
9. J.L. ORR, S.C. POWER, Representation and refinement for reflexive operator algebras with completely distributive lattice, *Indiana Univ. Math. J.* **40**(1991), 617–638.
10. S. ROSENOER, Trace-class operators in CSL algebras, *Canad. Math. Bull.* **35**(1992), 416–422.

*N. ANOUSSIS*
*Department of Mathematics*
*University of the Aegean*
*83200 Karlovasi*
*GREECE*

*E.G. KATSOULIS*
*Department of Mathematics*
*East Carolina University*
*Greenville, NC 27858*
*U.S.A.*

Received October 21, 1994; revised April 26, 1995.