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BOUNDARY SETS FOR A CONTRACTION

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ABSTRACT. For any absolutely continuous contraction operator T on Hilbert space we produce a Borel set X_T contained in the unit circle $\mathbb{T}$; X_T localizes a sequence condition which, obtaining on all of $\mathbb{T}$, is equivalent to the membership of T in $\mathbb{A}_{\aleph_0}$ (the most restrictive of the classes of contractions arising from the Scott Brown theory). By consideration of X_T along with other subsets of $\mathbb{T}$ arising naturally from the minimal isometric dilation and minimal coisometric extension of T, we improve known results on the structure of dual operator algebras. Further results include a new characterization of membership in the class $\mathbb{A}_{1,\aleph_0}$ and that $T \in \mathbb{A}_{1,\aleph_0}$ implies $T^n \in \mathbb{A}_{n,\aleph_0}$.

KEYWORDS: *Contraction, Hilbert space, dual operator algebra, minimal coisometric extension, minimal isometric dilation.*

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1. INTRODUCTION

This paper is a continuation of the sequence [11], [7], and [9], and may be described as an attempt to unify and clarify various concepts that appear in these earlier papers by the introduction of a new subset of the unit circle to be associated with any absolutely continuous contraction operator on Hilbert space. This set may be described informally as a way to localize membership in $\mathbb{A}_{\aleph_0}$, the most restrictive of the various classes $\mathbb{A}_{m,n}$ (definitions reviewed below); it is on this subset of the unit circle that the "classical" Scott Brown approximation procedure works for a general absolutely continuous contraction. As a consequence of our new results, we obtain improvements of the major theorems of [9] and [18].

We shall suppose that the reader is familiar with the basics of the theory of dual operator algebras, and the notation and terminology herein coincide with those in [5]. For the reader's convenience, however, we begin by recalling some of the most important notation and terminology that will be needed below. Throughout this paper $\mathbf{D}$ will denote the open unit disc in the complex plane $\mathbf{C}$, $\mathbf{T}$ the boundary of $\mathbf{D}$, $\mathbf{N}$ the set of positive integers, $\mathbf{N}_0$ the set of nonnegative integers, and $\mathbf{Z}$ the set of integers. If $\Lambda \subseteq \mathbf{D}$ we write $\mathrm{NTL}(\Lambda)$ for the subset of $\mathbf{T}$ consisting of all non-tangential limits of sequences from Λ. We will say that Λ is dominating for a subset Σ of $\mathbf{T}$ if $\Sigma \setminus \mathrm{NTL}(\Lambda)$ has Lebesgue measure zero (for the case $\Sigma = \mathbf{T}$, this notion originated in [6]).

For $1 \leqslant p \leqslant +\infty$, the spaces $H^p(\mathbf{T})$ and $L^p(\mathbf{T})$ are the usual Hardy and Lebesgue spaces with respect to normalized Lebesgue measure m on $\mathbf{T}$. Furthermore, $H^1_0(\mathbf{T})$ denotes the subspace of $H^1(\mathbf{T})$ consisting of those functions f whose analytic extension to $\mathbf{D}$ vanishes at 0. If Σ is an arbitrary Borel subset of $\mathbf{T}$, we shall denote by $L^p(\Sigma)$ the (closed) subspace of $L^p(\mathbf{T})$ consisting of all (equivalence classes of) functions f in $L^p(\mathbf{T})$ such that f vanishes almost everywhere (m) on $\mathbf{T} \setminus \Sigma$. The space $H^2(\Sigma)$ is the closure in $L^2(\Sigma)$ of the linear manifold consisting of those functions that agree (a.e.) with some polynomial on Σ, and if $m(\mathbf{T} \setminus \Sigma) \neq 0$, then one knows that $H^2(\Sigma) = L^2(\Sigma)$.

In what follows, $\mathcal{H}$ will denote a separable, infinite dimensional, complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. Moreover $\mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ is the Banach space and ideal of trace-class operators under the trace norm. If $T \in \mathcal{L}(\mathcal{H})$, we shall write $\mathcal{A}_T$ for the dual algebra generated by T and Q_T for its predual $\mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}_T$, so $\mathcal{A}_T = Q_T^*$ under the pairing

$$\langle A, [L] \rangle = \mathrm{trace}(AL), \quad A \in \mathcal{A}_T, L \in \mathcal{C}_1(\mathcal{H}),$$

where $[L]$ (or $[L]_T$) denotes the element of the quotient space Q_T containing the trace-class operator L. In particular, we will deal extensively with cosets of the form $[x \otimes y]$ where $x \otimes y$ is the usual rank-one operator in $\mathcal{L}(\mathcal{H})$ defined by $(x \otimes y)u = (u, y)x$ for u in $\mathcal{H}$. Note that $\|[x \otimes y]\| \leqslant \|x \otimes y\|_{\mathcal{C}_1(\mathcal{H})} = \|x\| \, \|y\|$.

Recall that if T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, then the (Sz.-Nagy–Foiaş) functional calculus $\Phi_T : H^\infty(\mathbf{T}) \to \mathcal{A}_T$ which maps f to $f(T) \triangleq \Phi_T(f)$ is a weak*-continuous algebra homomorphism with range weak*-dense in $\mathcal{A}_T$, and thus is the adjoint of a one-to-one contractive linear transformation $\varphi_T : Q_T \to (L^1/H^1_0)(\mathbf{T}) \triangleq L^1(\mathbf{T})/H^1_0(\mathbf{T})$ (see [5] for details). The class $\mathbb{A} = \mathbb{A}(\mathcal{H})$ is defined to be the set of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which Φ_T is an isometry. It follows easily that in this case Φ_T is a weak*-homeomorphism of $H^\infty(\mathbb{T})$ onto $\mathcal{A}_T$ and φ_T is a surjective isometry. If T is an arbitrary absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and Σ is a Borel subset of $\mathbb{T}$, then we say that Σ is essential for T if $\|f(T)\| \geqslant \|f|\Sigma\|_\infty \stackrel{\Delta}{=} \operatorname{ess\,sup}\|f|\Sigma\|$ for every function f in $H^\infty(\mathbb{T})$. (Thus $T \in \mathbf{A}$ if and only if $\mathbb{T}$ is essential for T.) We shall use some of the results about such sets from [8], where they seem to appear explicitly for the first time.

We recall now the properties $(\mathbf{A}_{m,n})$ and $(\mathbf{A}_{m,n}(r))$, where m and n are any cardinal numbers satisfying $1 \leqslant m, n \leqslant \aleph_0$. A dual subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\mathbf{A}_{m,n})$ if, for every doubly indexed family $\{[L_{ij}]\}_{\substack{0\leqslant i<m\\ 0\leqslant j<n}}$ of elements of $Q_\mathcal{A} = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}$, there exist sequences $\{x_i\}_{0\leqslant i<m}$ and $\{y_j\}_{0\leqslant j<n}$ of vectors from $\mathcal{H}$ such that

$$(1.1) \qquad [L_{ij}] = [x_i \otimes y_j], \quad 0 \leqslant i < m, \quad 0 \leqslant j < n.$$

A dual algebra $\mathcal{A}$ has property $(\mathbf{A}_{m,n}(r))$ for some $r \geqslant 1$ if, for every doubly indexed family $\{[L_{ij}]\}_{\substack{0\leqslant i<m\\ 0\leqslant j<n}}$ of elements of $Q_\mathcal{A}$ such that the rows and columns of the matrix $([L_{ij}])$ are summable, and for every $s > r$, there exist sequences $\{x_i\}_{0\leqslant i<m}$ and $\{y_j\}_{0\leqslant j<n}$ from $\mathcal{H}$ satisfying (1.1) and also the inequalities

$$\|x_i\|^2 \leqslant s \sum_{0\leqslant j<n} \|[L_{ij}]\|, \quad 0 \leqslant i < m,$$
$$\|y_j\|^2 \leqslant s \sum_{0\leqslant i<m} \|[L_{ij}]\|, \quad 0 \leqslant j < n.$$

It is obvious that if m and n are finite cardinals and $\mathcal{A}$ has property $(\mathbf{A}_{m,n}(r))$ for some r, then $\mathcal{A}$ has property $(\mathbf{A}_{m,n})$. Furthermore, an easy scaling argument shows that if $\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0}(r))$ for some r, then $\mathcal{A}$ also has property $(\mathbf{A}_{1,\aleph_0})$. The class $\mathbf{A}_{m,n} = \mathbf{A}_{m,n}(\mathcal{H})$ is defined to be the set of all T in $\mathbf{A}(\mathcal{H})$ such that the dual algebra $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$. The classes $\mathbf{A}_n = \mathbf{A}_{n,n}$, $1 \leqslant n \leqslant \aleph_0$, were introduced in [4] and have played a major role in the theory of dual algebras (see, e.g., [5]). One knows, in particular, that $\mathbf{A} = \mathbf{A}_1(1)$ ([2], [8]). One also knows that $T \in \mathbf{A}_{1,\aleph_0}$ if and only if $T \in \mathbf{A}_{1,\aleph_0}(r)$ for some $r \geqslant 1$ ([9], Theorem 6.2).

2. PRELIMINARIES ON UNITARY DILATIONS

If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, then T has various unitary dilations, and it will be convenient in what follows to employ a certain class of these. We introduce now the notation for such a dilation. Let $\mathcal{D}$ be a complex Hilbert space satisfying $1 \leqslant \dim(\mathcal{D}) \leqslant \aleph_0$, and denote by $L^2(\mathbf{T}, \mathcal{D})$ the Hilbert space of (equivalence classes of) Lebesgue measurable, square integrable functions $x : \mathbf{T} \to \mathcal{D}$ with the inner product defined by

$$(x_1, x_2) = \int_{\mathbf{T}} (x_1(e^{it}), x_2(e^{it}))_{\mathcal{D}} \, dm .$$

One may define an operator U on $L^2(\mathbf{T}, \mathcal{D})$ by setting

$$(Ux)(e^{it}) = e^{it} x(e^{it}), \quad x \in L^2(\mathbf{T}, \mathcal{D}),\ e^{it} \in \mathbf{T}, \tag{2.1}$$

and it is easily seen that U is an absolutely continuous unitary operator in $\mathcal{L}(L^2(\mathbf{T}, \mathcal{D}))$ which is, in fact, a bilateral shift of multiplicity $\dim(\mathcal{D})$. For x and y in $L^2(\mathbf{T}, \mathcal{D})$, we denote by $x \overset{U}{\cdot} y$ the function in $L^1(\mathbf{T})$ defined by

$$(x \overset{U}{\cdot} y)(e^{it}) = (x(e^{it}), y(e^{it}))_{\mathcal{D}}$$

and by $[x \overset{U}{\cdot} y]$ the projection of $x \overset{U}{\cdot} y$ into the quotient space $(L^1/H_0^1)(\mathbf{T})$. Note also that if $x \in L^2(\mathbf{T}, \mathcal{D})$ and Γ is a Borel subset of $\mathbf{T}$, then $\chi_\Gamma x$ (defined pointwise in the obvious way) is another function in $L^2(\mathbf{T}, \mathcal{D})$.

It is well known that if T is any absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, then one may choose $\mathcal{D}$ to be a separable Hilbert space of sufficiently large dimension in order that T will have a unitary dilation of the form (2.1). In other words, up to unitary equivalence, we may regard $\mathcal{H}$ as a subspace of $\mathcal{K} = L^2(\mathbf{T}, \mathcal{D})$ which is semi-invariant under the unitary operator U (defined in (2.1)) and T as the compression of U to the semi-invariant subspace $\mathcal{H}$. (Of course, U may not be the minimal unitary dilation of T, but this will cause no problems.)

2.1. NOTATIONAL CONVENTIONS. Throughout the remainder of this paper, whenever an absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ is under consideration, we shall assume that a space $\mathcal{D}$ and a unitary operator U of the form (2.1) acting on $\mathcal{K} = L^2(\mathbf{T}, \mathcal{D})$ have been fixed such that $\mathcal{H}$ is a subspace of $\mathcal{K}$, semi-invariant under U, and T is the compression of U to $\mathcal{H}$. Moreover, suppose $\{T_j \in \mathcal{L}(\mathcal{H}_j)\}_{j \in J}$ is some finite or countably infinite sequence of absolutely continuous contractions, and the direct sum $\widehat{T} = \bigoplus_{j \in J} T_j$ acting on $\widehat{\mathcal{H}} = \bigoplus_{j \in J} \mathcal{H}_j$ is under consideration. Then, by what has just been said, we may and do suppose that for each $j \in J$, a space $\mathcal{D}_j$ and a unitary dilation $U_j \in \mathcal{L}(L^2(\mathbf{T}, \mathcal{D}_j))$ of T_j of the form (2.1) have been fixed, and furthermore we suppose that the unitary dilation $\widehat{U}$ of $\widehat{T}$ of the form (2.1) that is fixed is $\widehat{U} = \bigoplus_{j \in J} U_j$, which of course may be identified with the operator $x(e^{it}) \mapsto e^{it} x(e^{it})$ on $L^2\Big(\mathbf{T}, \bigoplus_{j \in J} \mathcal{D}_j\Big)$.

Returning now to the situation in which a single absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ is under consideration with unitary dilation U acting on $\mathcal{K} = L^2(\mathbf{T}, \mathcal{D}) \supseteq \mathcal{H}$, we remark that it is well known that the compression of U to the semi-invariant subspace $\mathcal{K}_- = \bigvee_{n=-\infty}^{0} U^n \mathcal{H}$ is the minimal coisometric extension (m.c.e.) of T (cf. [21]), which will be denoted consistently by B_*. Using the Wold decomposition, we may write $\mathcal{K}_- = \mathcal{S} \oplus \mathcal{R}_*$ and $B_* = S^* \oplus R_*$ relative to this decomposition, where S^* is a backward unilateral shift of some multiplicity (not exceeding $\dim(\mathcal{D})$), and R_* is a unitary operator. (Of course, either of the spaces $\mathcal{S}$ or $\mathcal{R}_*$ may be the space (0).) If $\mathcal{R}_* \neq (0)$, then one knows that R_* is an absolutely continuous unitary operator, and thus there exists a Borel subset $\Sigma_* = \Sigma_*(T)$ (unique up to sets of measure zero) such that $m|\Sigma_*$ is a scalar spectral measure for R_*. We shall denote the projections in $\mathcal{L}(\mathcal{K}_-)$ onto $\mathcal{S}$ and $\mathcal{R}_*$ by $P_{\mathcal{S}}$ and $P_{\mathcal{R}_*}$ respectively. Similarly, the restriction of the unitary dilation U to the invariant subspace $\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U^n \mathcal{H}$ is the minimal isometric dilation (m.i.d.) B of T, and (via Wold), we may write $\mathcal{K}_+ = \mathcal{S}_* \oplus \mathcal{R}$ and $B = S_* \oplus R$ relative to this decomposition, where S_* is a (forward) unilateral shift of some multiplicity (not exceeding $\dim(\mathcal{D})$) and R is an absolutely continuous unitary operator. (Again, either summand $\mathcal{S}_*$ or $\mathcal{R}$ may be (0).) We write $\Sigma = \Sigma(T)$ for the Borel subset of $\mathbf{T}$ (again, unique up to sets of measure zero) such that $m|\Sigma$ is a scalar spectral measure for R. We also write $P_{\mathcal{S}_*}$ and $P_{\mathcal{R}}$ for the projections in $\mathcal{L}(\mathcal{K}_+)$ whose ranges are $\mathcal{S}_*$ and $\mathcal{R}$, respectively, and $P_{\mathcal{H}}$ for the projection in $\mathcal{L}(\mathcal{K})$ whose range is $\mathcal{H}$. (Please note that we have thus far introduced projections with three different domains: $\mathcal{K}_-$, $\mathcal{K}_+$, and $\mathcal{K}$.)

NOTE. We alert the reader who has read [11] or [9] that we have made a slight change of notation. In those papers the m.i.d. of an absolutely continuous contraction T did not appear, and what we are denoting as $\mathcal{R}_*$ and Σ_* herein was denoted in those earlier papers by $\mathcal{R}$ and Σ, respectively. We have made this change to conform to the standard notation concerning residual and $*$-residual parts of T appearing in [21].

The following summarizes some known results that we shall need relating the various $[x \overset{U}{\cdot} y]$ and $[x \otimes y]$.

PROPOSITION 2.2. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, U is its unitary dilation in $\mathcal{L}(\mathcal{K}) = \mathcal{L}(L^2(\mathbf{T}, \mathcal{D}))$, B_* is the minimal coisometric extension of T in $\mathcal{L}(\mathcal{K}_-)$, and B is the minimal isometric dilation of T in $\mathcal{L}(\mathcal{K}_+)$.*

(i) *If $x \in \mathcal{K}_-$ and $y \in \mathcal{K}_+$, then $[x \overset{U}{\cdot} y] = [P_{\mathcal{H}}x \overset{U}{\cdot} P_{\mathcal{H}}y]$.*

(ii) *For any f in $L^1(\mathbf{T})$, let $\{c_n(f)\}_{n\in\mathbf{Z}}$ be the sequence of Fourier coefficients of f. Then for $x, y \in \mathcal{H}$,*

$$c_{-n}(x \overset{U}{\cdot} y) = (T^n x, y), \quad c_n(x \overset{U}{\cdot} y) = (T^{*^n} x, y), \quad n \in \mathbf{N}_0.$$

(iii) *For $x, y \in \mathcal{K}$, $\|[x \overset{U}{\cdot} y]\| \leqslant \|x \overset{U}{\cdot} y\|_1 \leqslant \|x\|\,\|y\|$.*

(iv) *The following identities hold:*

$\varphi_T([x\otimes y]_T) = [x \overset{U}{\cdot} y], \quad x, y \in \mathcal{H},$

$\varphi_{B_*}([x\otimes y]_{B_*}) = [x \overset{U}{\cdot} y], \quad x, y \in \mathcal{K}_-,$

$\varphi_B([x\otimes y]_B) = [x \overset{U}{\cdot} y], \quad x, y \in \mathcal{K}_+,$

$\varphi_U([x\otimes y]_U) = [x \overset{U}{\cdot} y], \quad x, y \in \mathcal{K}.$

(v) *If $\{T_j \in \mathcal{L}(\mathcal{H}_j)\}_{j\in J}$ is some finite or countably infinite sequence of absolutely continuous contractions and $\widehat{T} = \bigoplus_{j\in J} T_j$ with unitary dilation $\widehat{U} = \bigoplus_{j\in J} U_j$ as in Subsection 2.1, then for any vectors $\widehat{x} = \bigoplus_{j\in J} x_j$ and $\widehat{y} = \bigoplus_{j\in J} y_j$ in $\widehat{\mathcal{H}}$,*

$$[\widehat{x} \overset{\widehat{U}}{\cdot} \widehat{y}] = \varphi_{\widehat{T}}([\widehat{x}\otimes\widehat{y}]) = \varphi_{\widehat{T}}\Big(\sum_{j\in J}[x_j\otimes y_j]\Big) = \Big[\sum_{j\in J} x_j \overset{U_j}{\cdot} y_j\Big]_{L^1/H_0^1}. \tag{2.2}$$

(vi) *If $T \in \mathbb{A}(\mathcal{H})$ and $x, y \in \mathcal{H}$, then*

$$\|[x\otimes y]_T\| = \|[x\otimes y]_{B_*}\| = \|[x\otimes y]_B\| = \|[x\otimes y]_U\|.$$

Proof. We mention only that (i) is [2], Lemma 3.1, and (ii) and (iii) are essentially contained in [5], Proposition 9.3. The conclusions (iv) and (v) are elementary, and (vi) follows from (iv).

Remark that in this setting (ii) above shows that, for fixed T in $\mathcal{L}(\mathcal{H})$ and x, y in $\mathcal{H}$, the sequence of Fourier coefficients of $x \overset{U}{\cdot} y$ is independent of the particular choice of U of the form (2.1). ∎

The following definition introduces a very useful analog of the sets $\mathcal{X}_\theta(\mathcal{A}_T)$ of [5], Definition 2.7.

DEFINITION 2.3. If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and $0 \leqslant \theta < 1$, we denote by $\widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ the subset of $(L^1/H_0^1)(\mathbf{T})$ consisting of those cosets $[f]$ for which there exist sequences $\{x_n\}$ and $\{y_n\}$ in the (closed) unit ball of $\mathcal{H}$ satisfying

(a) $\limsup \|[f] - \varphi_T([x_n \otimes y_n]_T)\| \leqslant \theta$, and

(b) $\|\varphi_T([x_n \otimes w]_T)\| + \|\varphi_T([w \otimes y_n]_T)\| \to 0$, $w \in \mathcal{H}$.

Observe that since φ_T is contractive, it is sufficient for (b) to have

$$\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\| \to 0, \quad w \in \mathcal{H}.$$

Observe also that by Proposition 2.2, if U is the unitary dilation of T in $\mathcal{L}(L^2(\mathbf{T}, \mathcal{D}))$, then (a) and (b) are equivalent to

(a′) $\limsup \|[f] - [x_n \overset{U}{\cdot} y_n]\| \leqslant \theta$, and

(b′) $\|[x_n \overset{U}{\cdot} w]\| + \|[w \overset{U}{\cdot} y_n]\| \to 0$, $w \in \mathcal{H}$.

PROPOSITION 2.4. *If T is any absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, $0 \leqslant \theta < 1$, then $\varphi_T(\mathcal{X}_\theta(\mathcal{A}_T)) \subseteq \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ and $\widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ is absolutely convex and closed in $(L^1/H_0^1)(\mathbf{T})$. Furthermore the family $\{\widehat{\mathcal{X}}_\theta(\mathcal{A}_T)\}_{\theta \in [0,1)}$ is decreasing (with θ), and if $\{\theta_j\}_{j=1}^\infty$ is any sequence of elements of $(0,1)$ tending to 0, then $\widehat{\mathcal{X}}_0(\mathcal{A}_T) = \bigcap_{j=1}^\infty \widehat{\mathcal{X}}_{\theta_j}(\mathcal{A}_T)$.*

Proof. That $\varphi_T(\mathcal{X}_\theta(\mathcal{A}_T)) \subseteq \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ is immediate from the definitions, and that $\widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ is absolutely convex and closed is proved in the same way as is $\mathcal{X}_\theta(\mathcal{A}_T)$ (see [5], Definition 2.8).

In the last assertion, the containment $\widehat{\mathcal{X}}_0(\mathcal{A}_T) \subseteq \bigcap_{j=1}^\infty \widehat{\mathcal{X}}_{\theta_j}(\mathcal{A}_T)$ is immediate from the definitions. For the reverse containment, let $\{\theta_j\}_{j=1}^\infty$ be such a sequence and let $[f] \in \bigcap_{j=1}^\infty \widehat{\mathcal{X}}_{\theta_j}(\mathcal{A}_T)$. To show $[f] \in \widehat{\mathcal{X}}_0(\mathcal{A}_T)$, it clearly suffices to produce sequences $\{x_n\}$ and $\{y_n\}$ in the closed unit ball of $\mathcal{H}$ satisfying

$$\|[f] - \varphi_T([x_n \otimes y_n])\| < \frac{1}{n}, \quad n \in \mathbf{N}, \tag{2.3}$$

and

$$\|[x_n \otimes w]\| + \|[w \otimes y_n]\| \to 0, \quad w \in \mathcal{H}. \tag{2.4}$$

To achieve (2.4) it suffices to fix a countable dense subset $\{w_k\}$ of $\mathcal{H}$ and arrange the sequences $\{x_n\}$ and $\{y_n\}$ to satisfy

$$\|[x_n \otimes w_k]\| + \|[w_k \otimes y_n]\| < \frac{1}{n}, \quad 1 \leqslant k \leqslant n, \ n \in \mathbf{N}, \tag{2.5}$$

as well as (2.3). We turn to the construction of the requisite sequences.

Let $n \in \mathbf{N}$ be arbitrary, and choose some $j \in \mathbf{N}$ such that $\theta_j < 1/n$. Since $[f] \in \widehat{\mathcal{X}}_{\theta_j}(\mathcal{A}_T)$, there exist by definition sequences $\{x_m^j\}_{m=1}^\infty$ and $\{y_m^j\}_{m=1}^\infty$ in the unit ball of $\mathcal{H}$ satisfying

$$\limsup_m \|[f] - \varphi_T([x_m^j \otimes y_m^j])\| \leqslant \theta_j < \frac{1}{n}, \tag{2.6}$$

and

$$\lim_m (\|[x_m^j \otimes w]\| + \|[w \otimes y_m^j]\|) = 0, \quad w \in \mathcal{H}. \tag{2.7}$$

From (2.6) we conclude that there exists $M_1 \in \mathbf{N}$ such that

$$\|[f] - \varphi_T([x_m^j \otimes y_m^j])\| < \frac{1}{n}, \quad m \geqslant M_1. \tag{2.8}$$

Furthermore, since n is fixed it follows from (2.7) that there exists $M_2 \in \mathbf{N}$ such that

$$\|[x_m^j \otimes w_k]\| + \|[w_k \otimes y_m^j]\| < \frac{1}{n}, \quad 1 \leqslant k \leqslant n, \quad m \geqslant M_2. \tag{2.9}$$

Set $m_0 = \max(M_1, M_2)$ and let $x_n = x_{m_0}^j$ and $y_n = y_{m_0}^j$. It is immediate from (2.8) and (2.9) that x_n and y_n are as needed to satisfy (2.3) and (2.5), and the proof is complete. ∎

3. THE SET X_T

The following proposition introduces the central new construct of this paper. Throughout the paper, expressions such as maximality, uniqueness, and equality of Borel subsets of $\mathbf{T}$ are to be interpreted as pertaining to the equivalence classes arising from the relation "equal a.e. (m)".

PROPOSITION 3.1. *If T is any absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, then there exists a (unique) maximal Borel subset X_T of $\mathbf{T}$ such that*

$$\{[f]_{L^1/H_0^1} : f \in L^1(X_T), \|f\|_1 \leqslant 1\} \subseteq \widehat{\mathcal{X}}_0(\mathcal{A}_T).$$

Proof. Let $\mathcal{F}$ denote the collection of all Borel subsets $F \subseteq \mathbf{T}$ such that

$$\{[f] : f \in L^1(F), \|f\|_1 \leqslant 1\} \subseteq \widehat{\mathcal{X}}_0(\mathcal{A}_T),$$

ordered by inclusion. To obtain a maximal element, we show first that $\mathcal{F}$ is closed under countable disjoint unions. If $F, F' \in \mathcal{F}$, $F \cap F' = \emptyset$, and f belongs to the unit ball of $L^1(F \cup F')$, then it is straightforward to check that the set $\widehat{\mathcal{X}}_0(\mathcal{A}_T)$ is absolutely convex and then, since $[f]$ is an absolute convex combination of the elements $[f|F/\|f|F\|_1]$ and $[f|F'/\|f|F'\|_1]$, that $[f] \in \widehat{\mathcal{X}}_0(\mathcal{A}_T)$. The result for countable disjoint unions $\{F_n\}_{n=1}^{\infty}$ follows in an analogous way from the fact that $\sum\limits_n \int\limits_{F_n} |f| \, dm = \|f\|_1 < +\infty$ for f in $L^1\Big(\bigcup\limits_n F_n\Big)$. Now let $\{F_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ be an increasing sequence such that $m(F_n) \to \sup\limits_{F \in \mathcal{F}} m(F)$. From the above we easily deduce that $\bigcup\limits_n F_n$ is a maximal element in $\mathcal{F}$. ∎

NOTE. We observe for future use that if $T \in \mathbb{A}$, so that φ_T is a surjective isometry, then the set X_T can be identified as the maximal Borel subset Y of $\mathbf{T}$ such that

$$\{\varphi_T^{-1}([f]) : f \in L^1(Y), \|f\|_1 \leqslant 1\} \subseteq \mathcal{X}_0(\mathcal{A}_T).$$

(Condition 2a in the definition of $\mathcal{X}_0(\mathcal{A}_T)$ in [5], Definition 2.7, was subsequently seen to be unnecessarily complicated and replaced by

$$\|[x_i \otimes z]\| + \|[z \otimes y_i]\| \to 0, \quad z \in \mathcal{K}.)$$

We now begin to study various properties of the set X_T introduced above. For this purpose we need some auxiliary operators.

DEFINITION 3.2. If Γ is any Borel subset of $\mathbf{T}$ (satisfying $0 \leqslant m(\Gamma) \leqslant 1$), we denote by M_Γ the absolutely continuous unitary operator on $L^2(\Gamma)$ defined by

$$(M_\Gamma x)(e^{it}) = e^{it} x(e^{it}), \quad x \in L^2(\Gamma), \quad e^{it} \in \mathbf{T},$$

and by $\tilde{M}_\Gamma$ the direct sum of $\aleph_0$ copies of M_Γ acting on the Hilbert space $\widetilde{L}^2(\Gamma) = \bigoplus\limits_{n \in \mathbf{N}} L^2(\Gamma)$. If $x \in L^2(\Gamma)$ we denote by $\tilde{x}(n)$ the vector in $\widetilde{L}^2(\Gamma)$ given by

$$\tilde{x}(n) = (\underbrace{0, \ldots, 0}_{n-1}, x, 0, \ldots),$$

i.e., the x occurs as the n-th component of $\tilde{x}(n)$ and is the only nonzero component of $\tilde{x}(n)$.

Our first result will be a useful tool in what follows.

PROPOSITION 3.3. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and set $\Gamma = \mathbf{T}\backslash X_T$. Then Γ is the (unique) minimal Borel subset of $\mathbf{T}$ such that $\widehat{T}_\Gamma \triangleq T \oplus \tilde{M}_\Gamma \in \mathbf{A}_{\aleph_0}$.*

Proof. For brevity let us write $\widehat{T} = \widehat{T}_\Gamma$ and $\widehat{\mathcal{H}} = \mathcal{H} \oplus \widetilde{L}^2(\Gamma)$. Suppose first that $f \in L^1(\mathbf{T})$ and $\|f\|_1 \leqslant 1$. We wish to show that $[f]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widehat{T}})$, and to this end we write $f = f|X_T \oplus f|\Gamma$. By Proposition 3.1, $[f|X_T] \in \widehat{\mathcal{X}}_0(\mathcal{A}_T)$, and if we can show that $[f|\Gamma] \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widetilde{M}_\Gamma})$, then a computation like that alluded to in the proof of Proposition 3.1 will show that $[f] \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widehat{T}})$. To see that $[f|\Gamma] \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widetilde{M}_\Gamma})$, note that we can write $f|\Gamma = g\bar{h}$, where $g, h \in L^2(\Gamma)$ and $\|g\|_2^2 = \|h\|_2^2 = \|f\|_1 \leqslant 1$. We define the sequences $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ in the unit ball of $\widetilde{L}^2(\Gamma)$ by $\tilde{x}_n = \tilde{g}(n)$ and $\tilde{y}_n = \tilde{h}(n)$, $n \in \mathbf{N}$. An easy calculation shows that $\tilde{x}_n \overset{U_2}{\cdot} \tilde{y}_n \equiv f|\Gamma$ and also that $\|\tilde{x}_n \overset{U_2}{\cdot} w\|_1 + \|w \overset{U_2}{\cdot} \tilde{y}_n\|_1 \to 0$ for any fixed w in $\widetilde{L}^2(\Gamma)$, where U_2 is the canonical unitary dilation of $\tilde{M}_\Gamma$ given by Subsection 2.1. Thus $[f|\Gamma] \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widetilde{M}_\Gamma})$ and $[f] \in \widehat{\mathcal{X}}_0(\mathcal{A}_{\widehat{T}})$. Therefore, $\varphi_{\widehat{T}}^{-1}(\mathrm{Ball}(L^1/H_0^1)) .= \mathcal{X}_0(\mathcal{A}_{\widehat{T}})$ and, to complete the proof, it suffices to show $\widehat{T} \in \mathbf{A}$. To that end, let $[L]$ be in the unit ball of L^1/H_0^1; then there exist sequences $\{x_n\}$ and $\{y_n\}$ in the unit ball of $\widehat{\mathcal{H}}$ such that $[L] = \lim[x_n \cdot y_n]$. Thus $| < h, [L] > | = \lim|(h(\widehat{T})x_n, y_n)| \leqslant \|h(\widehat{T})\|$. Since this holds for each $[L]$ we have $\|h(\widehat{T})\| \geqslant \|h\|$, so $\Phi_{\widehat{T}}$ is an isometry and $\widehat{T} \in \mathbf{A}$. Hence $\mathcal{X}_0(\mathcal{A}_{\widehat{T}})$ is the unit ball of $Q_{\widehat{T}}$, from which it follows that $\mathcal{A}_{\widehat{T}}$ has property $X_{0,1}$ (cf. [5], Definition 2.8) and thus that, at least, $\widehat{T}_\Gamma \in \mathbf{A}_{\aleph_0}$ ([5], Proposition 6.1).

To see that Γ is the smallest Borel set F such that $\widehat{T}_F \in \mathbf{A}_{\aleph_0}$, we now suppose the contrary, i.e., we suppose that there exists a Borel set $F \subset \Gamma$ such that $m(\Gamma \setminus F) > 0$ and $\widehat{T}_F = T \oplus \tilde{M}_F \in \mathbf{A}_{\aleph_0}(\widehat{\mathcal{H}})$, where $\widehat{\mathcal{H}} \triangleq \mathcal{H} \oplus \widetilde{L}^2(F)$. (We shall show that this leads to a contradiction.) Since $\widehat{T}_F \in \mathbf{A}_{\aleph_0}$, one knows that $\widehat{T}_F \in \bigcap_{n=1}^{\infty} \mathbf{A}_n$ ([5], Theorem 6.3), and thus, as in the proof of [5], Proposition 6.1, for each λ in $\mathbf{D}$, there is a sequence $\{\widehat{x}_{n,\lambda}\}_{n=1}^{\infty}$ of unit vectors in $\widehat{\mathcal{H}}$ such that

$$(3.1) \qquad [C_\lambda]_{\widehat{T}_F} \triangleq \varphi_{\widehat{T}_F}^{-1}([P_\lambda]_{L^1/H_0^1}) = [\widehat{x}_{n,\lambda} \otimes \widehat{x}_{n,\lambda}]_{\widehat{T}_F},$$

and

$$(3.2) \qquad \|[\widehat{x}_{n,\lambda} \otimes \widehat{w}]_{\widehat{T}_F}\| + \|[\widehat{w} \otimes \widehat{x}_{n,\lambda}]_{\widehat{T}_F}\| \to 0, \quad \widehat{w} \in \widehat{\mathcal{H}}.$$

For each $\lambda \in \mathbf{D}$ and $n \in \mathbf{N}$, write $\widehat{x}_{n,\lambda} = x_{n,\lambda} \oplus \tilde{x}_{n,\lambda}$ where $x_{n,\lambda} \in \mathcal{H}$ and $\tilde{x}_{n,\lambda} \in \widetilde{L}^2(F)$. By dropping to subsequences if necessary, we may suppose that, for each $\lambda \in \mathbf{D}$, there exists γ_λ in $[0,1]$ such that $\|\tilde{x}_{n,\lambda}\|^2 \to \gamma_\lambda$. For each θ in $(0,1)$, define

$$\Delta_\theta = \{\lambda \in \mathbf{D} : \gamma_\lambda < \theta\}.$$

It is easy to see that for any fixed λ in Δ_θ, we have $[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$. Indeed, since $\|\tilde{x}_{n,\lambda}\|^2 \to \gamma_\lambda$, we have

$$\limsup \|[(0 \oplus \tilde{x}_{n,\lambda}) \otimes (0 \oplus \tilde{x}_{n,\lambda})]_{\widehat{T}_F}\| \leqslant \gamma_\lambda, \tag{3.3}$$

and since

$$\begin{aligned}[(x_{n,\lambda} \oplus \tilde{x}_{n,\lambda}) &\otimes (x_{n,\lambda} \oplus \tilde{x}_{n,\lambda})]_{\widehat{T}_F} \\ &= [(x_{n,\lambda} \oplus 0) \otimes (x_{n,\lambda} \oplus 0)]_{\widehat{T}_F} + [(0 \oplus \tilde{x}_{n,\lambda}) \otimes (0 \oplus \tilde{x}_{n,\lambda})]_{\widehat{T}_F},\end{aligned} \tag{3.4}$$

it follows from (3.1), (3.3), and the fact that $\varphi_{\widehat{T}_F}$ is an isometry that

$$\limsup \|[P_\lambda]_{L^1/H_0^1} - \varphi_{\widehat{T}_F}([(x_{n,\lambda} \oplus 0) \otimes (x_{n,\lambda} \oplus 0)]\| \leqslant \gamma_\lambda.$$

Furthermore, it follows from Proposition 2.2 (iv) and (v) that

$$\limsup \|[P_\lambda]_{L^1/H_0^1} - \varphi_T([x_{n,\lambda} \otimes x_{n,\lambda}])\| \leqslant \gamma_\lambda < \theta,$$

and from (3.2) we obtain easily by taking $w \in \mathcal{H}$ and $\widehat{w} = w \oplus 0$ that

$$\|[x_{n,\lambda} \otimes w]_T\| + \|[w \otimes x_{n,\lambda}]_T\| \to 0, \quad w \in \mathcal{H}.$$

Thus, by definition, $[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$.

We show next that for each θ in $(0,1)$, the set Δ_θ is dominating for $\Gamma \setminus F$. To see this, suppose that for some $\theta_0 \in (0,1)$, Δ_{θ_0} is not dominating for $\Gamma \setminus F$. Then there exists a Borel subset Γ_0 of $\Gamma \setminus F$ with $m(\Gamma_0) > 0$ such that $\Gamma_0 \cap \mathrm{NTL}(\Delta_{\theta_0}) = \emptyset$. Trivially then, $\Gamma_0 \subseteq \mathrm{NTL}(\mathbb{D} \setminus \Delta_{\theta_0})$. But for λ in $\mathbb{D} \setminus \Delta_{\theta_0}$ we have using $\gamma_\lambda \geqslant \theta_0$, (3.3), and (3.4), that

$$\limsup \|[C_\lambda]_{\widehat{T}_F} - [(0 \oplus \tilde{x}_{n,\lambda}) \otimes (0 \oplus \tilde{x}_{n,\lambda})]_{\widehat{T}_F}\| \leqslant 1 - \gamma_\lambda \leqslant 1 - \theta_0 \tag{3.5}$$

by an argument analogous to the one just used, and since $\varphi_{\widehat{T}_F}$ is an isometry, we have from (3.5) and Proposition 2.2 (iv) that

$$\limsup \|[P_\lambda]_{L^1/H_0^1} - \varphi_{\widetilde{M}_F}([\tilde{x}_{n,\lambda} \otimes \tilde{x}_{n,\lambda}])\| \leqslant 1 - \theta_0, \ \lambda \in \mathbb{D} \setminus \Delta_{\theta_0}.$$

Thus for any such λ, and for any f in $H^\infty(\mathbb{T})$,

$$\begin{aligned}\|f|F\|_\infty \geqslant \|f(\tilde{M}_F)\| &\geqslant |\langle f(\tilde{M}_F), [\tilde{x}_{n,\lambda} \otimes \tilde{x}_{n,\lambda}]_{\widetilde{M}_F}\rangle| \\ &= |\langle f, \varphi_{\widetilde{M}_F}[\tilde{x}_{n,\lambda} \otimes \tilde{x}_{n,\lambda}]\rangle| \\ &= |\langle f, [P_\lambda]\rangle - \langle f, [P_\lambda] - (\varphi_{\widetilde{M}_F}[\tilde{x}_{n,\lambda} \otimes \tilde{x}_{n,\lambda}])\rangle| \\ &\geqslant |f(\lambda)| - (1 - \theta_0)\|f\|_\infty.\end{aligned}$$

Since $\mathrm{NTL}(\mathbf{D} \setminus \Delta_{\theta_0}) \supseteq \Gamma_0$, we obtain from this inequality that

$$(1-\theta_0)\|f\|_\infty + \|f|F\|_\infty \geqslant \|f|\Gamma_0\|_\infty, \quad f \in H^\infty(\mathbf{T}),$$

which is an absurdity since $F \cap \Gamma_0 = \emptyset$ and $\theta_0 > 0$. Thus Δ_θ is dominating for $\Gamma \setminus F$ for each θ in $(0,1)$, and one knows from above that for each λ in Δ_θ, $[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$. It follows immediately from Lemma 4 of [15] and Proposition 2.4, that for each θ in $(0,1)$, $\widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ contains the image of the unit ball of $L^1(\Gamma \setminus F)$ in $(L^1/H_0^1)(\mathbf{T})$. Thus (by Proposition 2.4 again),

$$\widehat{\mathcal{X}}_0(\mathcal{A}_T) = \bigcap_{0<\theta_j<1} \widehat{\mathcal{X}}_{\theta_j}(\mathcal{A}_T) \supseteq \{[f] \in L^1/H_0^1 : f \in L^1(\Gamma \setminus F), \|f\|_1 \leqslant 1\}.$$

Thus $\Gamma \setminus F$ belongs to the set $\mathcal{F}$ of Proposition 3.1, and this contradiction completes the proof. ∎

It will be convenient in what follows to record some consequences of the above proposition and the techniques of its proof.

COROLLARY 3.4. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then*

(i) *$T \in \mathbb{A}_{\aleph_0}$ if and only if $X_T = \mathbf{T}$, and*

(ii) *X_T is essential for T.*

Proof. The first result is immediate from Proposition 3.3, so we concentrate on the second. Let W be an absolutely continuous unitary operator, and recall that there exists a Borel subset Σ of $\mathbf{T}$ (unique up to sets of measure zero) such that $m|\Sigma$ is a scalar spectral measure for W. It is also well known that $W \in \mathbb{A}$ if and only if $\Sigma = \mathbf{T}$, and recall that $W \in \mathbb{A}$ if and only if $\mathbf{T}$ is essential for W. A little work with direct sums of unitary operators then establishes that Σ is the maximal essential set for W. A further computation with the direct sum $T \oplus \tilde{M}_{\mathbf{T} \setminus X_T}$ shows that X_T is essential for T. ∎

We single out as well an argument vital to the proof of Proposition 3.3 which will be used again. Suppose given some direct sum $T \oplus S$ on $\mathcal{H} \oplus \mathcal{H}'$, some $[C_\lambda]_{T\oplus S} = \varphi_{T\oplus S}^{-1}([P_\lambda]_{L^1/H_0^1})$, and some sequence $\{x_n\}_{n=1}^\infty = \{x_n^{\mathcal{H}\oplus\mathcal{H}'}\}_{n=1}^\infty$ such that

$$[C_\lambda]_{T\oplus S} = \left[x_n^{\mathcal{H}\oplus\mathcal{H}'} \otimes x_n^{\mathcal{H}\oplus\mathcal{H}'}\right]_{T\oplus S}, \quad n = 1, 2, \ldots.$$

One may argue, as in that proof, that if for some $\theta > 0$ the squares of the norms of the $\{x_n^{\mathcal{H}}\}_{n=1}^\infty$ are bounded below by θ (equivalently, the squares of the norms of the $\{x_n^{\mathcal{H}'}\}_{n=1}^\infty$ are bounded above by $1-\theta$), then $[P_\lambda] \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$. Henceforth we shall refer to this argument as the *splitting argument*.

The next proposition establishes some elementary properties of the sets X_T.

PROPOSITION 3.5. *Let T and T' be absolutely continuous contractions in $\mathcal{L}(\mathcal{H})$. Then*

(i) $X_{T^*} = X_T^- \triangleq \{\bar{\xi} : \xi \in X_T\}$;

(ii) $X_{T \oplus T'} = X_T \cup X_{T'}$;

(iii) $X_{e^{i\theta}T} = e^{i\theta} X_T, \quad e^{i\theta} \in \mathbb{T}$;

(iv) *if W is any absolutely continuous unitary operator in $\mathcal{L}(\mathcal{H})$ then X_W is the largest Borel subset of $\mathbb{T}$ on which W has infinite (spectral) multiplicity;*

(v) *if T and T' are similar, then $X_T = X_{T'}$;*

(vi) *for any set $\{\lambda_j\}_{j=1}^{\infty} \subseteq \mathbb{D}$, if D is a normal operator with a diagonal matrix* $\mathrm{Diag}(\lambda_j)$ *relative to some orthonormal basis for $\mathcal{H}$, then* $X_D = \mathrm{NTL}(\{\lambda_j\})$;

(vii) *if S is a unilateral shift of multiplicity one, then $X_S = \emptyset$;*

(viii) *if $\widetilde{T}$ is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ having a matrix of the form*

$$\widetilde{T} = \begin{pmatrix} T & * \\ 0 & T' \end{pmatrix}, \tag{3.6}$$

then $X_{\widetilde{T}} \supseteq X_T \cup X_{T'}$.

Proof. Statements (i), (ii), and (iii) can be given routine proofs based on Proposition 3.3 and well known properties of the class $\mathbb{A}_{\aleph_0}$ (cf. [5]). Moreover, again using Proposition 3.3, (iv) and (vi) follow from [14]. Furthermore, (iv) and (viii), applied to a bilateral shift of multiplicity one, together yield (vii), so we content ourselves with the proofs of (v) and (viii).

Regarding (v) it will be useful to note that from (iv) and the definition of $\widetilde{M}_\Gamma$ for any Borel subset Γ of $\mathbf{T}$ we have $X_{\widetilde{M}_\Gamma} = \Gamma$. Also, recall from [5], Remark 2.1 that property ($\mathbb{A}_{\aleph_0}$) is preserved under similarity. Since T is similar to T', $\widetilde{T} = T \oplus \widetilde{M}_{\mathbf{T} \setminus X_T}$ is clearly similar to $\widetilde{T}' = T' \oplus \widetilde{M}_{\mathbf{T} \setminus X_T}$. Furthermore, $\widetilde{T}$ is in $\mathbb{A}_{\aleph_0}$, where we have used Proposition 3.3. Therefore $\mathcal{A}_{\widetilde{T}}$ has property ($\mathbb{A}_{\aleph_0}$), and so does $\mathcal{A}_{\widetilde{T}'}$.

Since $\widetilde{T}$ is in $\mathbb{A}_{\aleph_0}$, we know that for each λ in $\mathbb{D}$, there exist vectors x and y in $\mathcal{H}$ such that $[C_\lambda]_T = [x \otimes y]_T$. Suppose S implements the similarity between T and T', so $STS^{-1} = T'$; it is straightforward to check that $[C_\lambda]_{\widetilde{T}'} = [Sx \otimes (S^{-1^*})y]_{\widetilde{T}'}$, and it follows that $\Phi_{\widetilde{T}}$ is an isometry and $\widetilde{T}' \in \mathbb{A}$. Since $\mathcal{A}_{\widetilde{T}'}$ also has property ($\mathbb{A}_{\aleph_0}$), we have $\widetilde{T}' \in \mathbb{A}_{\aleph_0}$. Using Corollary 3.4 again, we have $X_{\widetilde{T}'} = \mathbf{T}$. Then using (ii) we obtain $X_{T'} \cup X_{\widetilde{M}_{(\mathbf{T} \setminus X_T)}} = X_{T'} \cup (\mathbf{T} \setminus X_T) = X_{\widetilde{T}'} = \mathbf{T}$. By Corollary 3.4 we have $\mathbf{T} \setminus (\mathbf{T} \setminus X_T) \subseteq X_{T'}$, so $X_T \subseteq X_{T'}$. Since the roles of T and T' are symmetric, we have the desired equality.

To prove (viii), recall first that if $T_1 \in \mathbf{A}_{\aleph_0}$ is the compression of an absolutely continuous contraction T_2 to a semi-invariant subspace, then $T_2 \in \mathbf{A}_{\aleph_0}$. Now observe from (3.6) that $\widetilde{T} \oplus \tilde{M}_{(\mathbf{T}\setminus X_T)}$ has a matrix of the form

$$\widetilde{T} \oplus \tilde{M}_{(\mathbf{T}\setminus X_T)} = \begin{pmatrix} \tilde{M}_{\mathbf{T}\setminus X_T} & 0 & 0 \\ 0 & T & * \\ 0 & 0 & T' \end{pmatrix}.$$

Since $\tilde{M}_{(\mathbf{T}\setminus X_T)} \oplus T \in \mathbf{A}_{\aleph_0}$ using (ii), we conclude that $\widetilde{T} \oplus \tilde{M}_{(\mathbf{T}\setminus X_T)} \in \mathbf{A}_{\aleph_0}$ by using the above fact about compressions. From an argument like that in the proof of (v) it follows that $X_{\widetilde{T}} \supseteq \mathbf{T} \setminus (\mathbf{T} \setminus X_T) = X_T$. The containment $X_{T'} \subseteq X_{\widetilde{T}}$ follows similarly. ∎

We remark that, *a priori*, one could consider sets $X_{\theta,T} \subseteq \mathbf{T}$ for $0 \leqslant \theta < 1$ maximal with respect to $\{[f]_{L^1/H^1_0} : f \in (L^1(X_{\theta,T}))_1\} \subseteq \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$. In this notation X_T as defined above is $X_{0,T}$. But clearly $0 \leqslant \theta < \theta' < 1$ implies $X_{\theta,T} \subseteq X_{\theta',T}$ and a moment's calculation with Proposition 3.3 shows that in fact $X_{\theta,T} = X_{0,T} = X_T$ for all $0 \leqslant \theta < 1$.

Recall also that, for some $\Gamma \subseteq \mathbf{T}$, the set

$$\{[f]_{L^1/H^1_0} : f \in L^1(\Gamma), \|f\|_1 \leqslant 1\}$$

is used in the definition of the set X_T. One might consider instead

$$S = \{[f]_{L^1/H^1_0} : f \in L^1(\Gamma), \|[f]\|_{L^1/H^1_0} \leqslant 1\}$$

in some definition analogous to that of X_T. However, as the following result shows, for many Γ this set is surprisingly large (in fact, too large to be useful).

PROPOSITION 3.6. *Let $\Gamma \subseteq \mathbf{T}$ be any measurable set containing a nonempty open set. Then* $\overline{\text{aco}}\{[f]_{L^1/H^1_0} : f \in L^1(\Gamma), \|[f]\| \leqslant 1\} = (L^1/H^1_0)_1$.

Proof. Let Γ be such a set. To prove the claim, it suffices to show that for each $\lambda \in \mathbf{D}$, $[P_\lambda]$ is the limit of elements of $\{[f]_{L^1/H^1_0} : f \in L^1(\Gamma), \|[f]\|_{L^1/H^1_0} \leqslant 1\}$, so let λ be arbitrary. Choose Γ', an open interval contained in Γ, to be such that $0 < \|P_\lambda \chi_{\Gamma'}\|_1 < \varepsilon/2$. Since $\mathbf{T} \setminus \Gamma'$ is closed and does not separate the plane, we may use Runge's Theorem to approximate $P_\lambda \chi_{\mathbf{T}\setminus\Gamma'}$ by some polynomial function p of $z, z^2, \ldots$ so that

$$\|P_\lambda \chi_{\mathbf{T}\setminus\Gamma'} - p\chi_{\mathbf{T}\setminus\Gamma'}\|_1 < \min\left(\frac{\varepsilon}{2}, 1 - \|P_\lambda \chi_{\mathbf{T}\setminus\Gamma'}\|_1\right). \tag{3.7}$$

Then surely

$$\|[P_\lambda \chi_{\mathbf{T}\setminus\Gamma'} - p\chi_{\mathbf{T}\setminus\Gamma'}]\|_{L^1/H^1_0} < \frac{\varepsilon}{2}.$$

Also, since $\|P_\lambda\chi_{\mathbb{T}\setminus\Gamma'} - P_\lambda\| = \|P_\lambda\chi_{\Gamma'}\| < \frac{\varepsilon}{2}$, we have

$$\|[P_\lambda] - [p\chi_{\mathbb{T}\setminus\Gamma'}]\|_{L^1/H_0^1} < \varepsilon.$$

Note also that since $p \in H_0^1$,

$$[p\chi_{\mathbb{T}\setminus\Gamma'}] = [p\chi_{\mathbb{T}\setminus\Gamma'} - p] = -[p\chi_{\Gamma'}],$$

so

$$\|[P_\lambda] - [-p\chi_{\Gamma'}]\|_{L^1/H_0^1} < \varepsilon. \tag{3.8}$$

Also,

$$\begin{aligned}\|[-p\chi_{\Gamma'}]\| &= \|[p\chi_{\mathbb{T}\setminus\Gamma'}]\|_{L^1/H_0^1} \\ &\leqslant \|p\chi_{\mathbb{T}\setminus\Gamma'}\|_1 \\ &\leqslant \|p\chi_{\mathbb{T}\setminus\Gamma'} - P_\lambda\chi_{\mathbb{T}\setminus\Gamma'}\|_1 + \|P_\lambda\chi_{\mathbb{T}\setminus\Gamma'}\|_1 \\ &< 1\end{aligned}$$

using (3.7). Thus since $\Gamma' \subseteq \Gamma$,

$$[-p\chi_{\Gamma'}] \in \{[f]_{L^1/H_0^1} : f \in L^1(\Gamma), \|[f]\|_{L^1/H_0^1} \leqslant 1\},$$

and thus, using (3.8), we have approximated $[P_\lambda]$ as desired. ∎

In the past decade of research on dual algebras, the collections of λ in $\mathbb{D}$ for which $[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)$ for some $0 \leqslant \theta < 1$ and some T in $\mathbb{A}$ have played a large role. We now investigate briefly these sets and their relationship to X_T for an arbitrary absolutely continuous contraction T.

DEFINITION 3.7. If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, we denote by $\Lambda_\theta(\mathcal{A}_T)$ the set

$$\Lambda_\theta(\mathcal{A}_T) \triangleq \{\lambda \in \mathbb{D} : [P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_T)\}, \quad \theta \geqslant 0.$$

PROPOSITION 3.8. *If T is any absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, then $\{\Lambda_\theta(\mathcal{A}_T)\}_{\theta\in[0,1)}$ is a nested family of sets such that*

$$\bigcap_{0<\theta<1} \Lambda_\theta(\mathcal{A}_T) = \Lambda_0(\mathcal{A}_T), \tag{3.9}$$

and for each $0 < \theta < 1$, $X_T = \mathrm{NTL}(\Lambda_\theta(\mathcal{A}_T))$. However, for each Borel set $\Gamma \subset \mathbb{T}$ such that $m(\mathbb{T}\setminus\Gamma) > 0$, there exists $T \in \mathcal{L}(\mathcal{H})$ such that $X_T = \Gamma$ but $\Lambda_0(\mathcal{A}_T) = \emptyset$.

Proof. The result in (3.9) is immediate from the definitions. For the next assertion, pick θ strictly between 0 and 1. That $\mathrm{NTL}(\Lambda_\theta) \subseteq X_T$ follows from a construction almost identical to the last construction in the proof of Proposition 3.3. For the reverse containment, observe that $T \oplus \tilde{M}_{\mathbf{T}\setminus X_T} \in \mathbb{A}_{\aleph_0}$. A repetition of the splitting argument, slightly simplified by Proposition 3.5 (iv), yields the result.

To establish the last claim, let Γ be a Borel subset of $\mathbf{T}$ such that $m(\Gamma) < m(\mathbf{T})$. Let $T = \tilde{M}_\Gamma \oplus M_{\mathbf{T}\setminus\Gamma}$ acting on $\mathcal{K} = \mathcal{H}^1 \oplus \mathcal{H}^2$ and write vectors with respect to the obvious decomposition. Observe first that $T \in \mathbb{A}$ and $X_T = \Gamma$ by Proposition 3.5 (iv). Since $T \in \mathbb{A}$ via the Note after Proposition 3.1 we may, and do, place our arguments in Q_T instead of L^1/H_0^1.

Pick $\lambda \in \mathbf{D}$, and suppose that $\lambda \in \Lambda_0(\mathcal{A}_T)$. That is, there exist sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in the unit ball of $\mathcal{K}$ satisfying

$$\|[C_\lambda]_T - [x_n \otimes y_n]_T\| \to 0,$$

and

$$\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\| \to 0, \quad w \in \mathcal{K}.$$

A brief calculation using $\langle I_\mathcal{K}, [C_\lambda]\rangle = 1$ and the $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in the unit ball of $\mathcal{K}$ shows that in fact

$$\|[C_\lambda]_T - [x_n \otimes x_n]_T\| \to 0,$$

and

$$\|[x_n \otimes w]_T\| + \|[w \otimes x_n]_T\| \to 0, \quad w \in \mathcal{K}.$$

We first claim that with $x_n = x_n^1 \oplus x_n^2$ we have $\lim \|x_n^2\| = 0$. If not, by dropping to a subsequence $\{x_{n_k}\}$ we may assume that $\lim \|x_{n_k}^2\|^2 \geqslant \delta > 0$. A repetition of the splitting argument then shows that $\lambda \in \Lambda_\delta(\mathcal{A}_{M_{\mathbf{T}\setminus\Gamma}})$. It follows easily that $\lambda \in \Lambda_\delta(\mathcal{A}_B)$, where B is the bilateral shift of multiplicity one. But if ϕ_μ is any Möbius transform it is well known that $\phi_\mu(B)$ is unitarily equivalent to B, and an argument as in [5], Chapter VI, shows that $\mathbf{D} \subseteq \Lambda_\delta(\mathcal{A}_B)$. But then $B \in \mathbb{A}_{\aleph_0}$, which is absurd. Thus $\lim \|x_n^2\| = 0$, or equivalently, $\lim \|x_n^1\| = 1$.

Consider next $\tilde{B} = T \oplus \tilde{M}_{\mathbf{T}\setminus\Gamma} = \tilde{M}_\Gamma \oplus M_{\mathbf{T}\setminus\Gamma} \oplus \tilde{M}_{\mathbf{T}\setminus\Gamma}$ acting on $\mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3$; note first that $\tilde{B} \in \mathbb{A}_{\aleph_0}$. It follows easily from the above that

$$\|[C_\lambda]_{\tilde{B}} - [(x_n^1 \oplus 0 \oplus 0) \otimes (x_n^1 \oplus 0 \oplus 0)]_{\tilde{B}}\| \to 0.$$

Since $\tilde{B} \in \mathbb{A}_{\aleph_0}$, we may solve equations in $Q_{\tilde{B}}$ from initial data and with a good control on the norms of the resulting vectors (cf. [5], Chapter III). It follows that we may find sequences $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ in $\mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3$ such that

$$[C_\lambda]_{\tilde{B}} = [v_n \otimes w_n]_{\tilde{B}}, \quad n \in \mathbf{N},$$

and

$$(3.10)\qquad \begin{aligned} \|v_n - (x_n^1 \oplus \mathbf{0} \oplus \mathbf{0})\| &\to 0 \\ \|w_n - (x_n^1 \oplus \mathbf{0} \oplus \mathbf{0})\| &\to 0. \end{aligned}$$

In fact, by a standard device used in the proof of [5], Theorem 4.12, we may actually find a sequence $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{H}^1 \oplus \mathcal{H}^2 \oplus \mathcal{H}^3$ such that

$$(3.11)\qquad [C_\lambda]_{\tilde{B}} = [u_n \otimes u_n]_{\tilde{B}}, \quad n \in \mathbb{N},$$

and

$$(3.12)\qquad \|u_n - (x_n^1 \oplus \mathbf{0} \oplus \mathbf{0})\| \to 0.$$

From (3.11) it follows that

$$(3.13)\qquad P_\lambda = u_n \cdot u_n, \quad n \in \mathbb{N},$$

by a calculation of Fourier coefficients. (A general definition of "$\cdot$" is found in Section 1, but in this case u_n may clearly be viewed as a "tuple" of functions in $L^2(\mathbf{T})$. In this case "$\cdot$" corresponds to the sum of coordinatewise function multiplications.)

From (3.12) and (3.13) we obtain a contradiction, since $\{x_n^1 \oplus \mathbf{0} \oplus \mathbf{0}\}$ is a sequence of functions each with support on Γ, and the support of P_λ is not contained in Γ. Thus $\lambda \notin \Lambda_0(\mathcal{A}_T)$, and we have $\Lambda_0(\mathcal{A}_T)$ is empty, as desired. ∎

We may now obtain the beginnings of some mapping theorems for X_T.

PROPOSITION 3.9. *Let T be an absolutely continuous contraction. Then*
(i) $(X_T)^m \subseteq X_{T^m}$, $m \in \mathbb{N}$, *and*
(ii) *if ϕ_μ is any Möbius transformation, then $\phi_\mu(X_T) = X_{\phi_\mu(T)}$.*

Proof. We only prove (i) since the proof of (ii) is similar. Let $0 < \theta < 1$ be arbitrary, and note that by Proposition 3.8 it suffices to show that $(\Lambda_\theta(\mathcal{A}_T))^m \subseteq \Lambda_\theta(\mathcal{A}_{T^m})$. Suppose first that $T \in \mathbb{A}(\mathcal{H})$. If $\lambda \in \Lambda_\theta(\mathcal{A}_T)$, let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in the unit ball of $\mathcal{H}$ satisfying

$$\limsup_n \|[C_\lambda]_T - [x_n \otimes y_n]_T\| \leqslant \theta,$$

and

$$\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\| \to 0, \quad w \in \mathcal{H}.$$

A routine computation using

$$\|[x \otimes y]_T\| = \sup_{f \in H^\infty, \|f\| \leqslant 1} |\langle f(T), [x \otimes y]_T \rangle|$$

and the analogous fact in Q_{T^m} shows that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are satisfactory sequences to put λ^m in $\Lambda_\theta(\mathcal{A}_{T^m})$. Thus the result holds for $T \in \mathbf{A}$.

For T merely an absolutely continuous contraction an analogous argument, now of necessity in L^1/H_0^1 instead of Q_T, works equally well. Alternatively, one can use Proposition 3.3 and for $\widetilde{T} = T \oplus \widetilde{M}_\Gamma$ acting on $\mathcal{H} \oplus \mathcal{H}'$ define some auxiliary sets $\Lambda_\theta^{\mathcal{H}}(\mathcal{A}_{\widetilde{T}})$ to consist of those λ for which there exist sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in the unit ball of $\mathcal{H}$ so that $\limsup_n \|[C_\lambda]_{\widetilde{T}} - [(x_n \oplus 0) \otimes (y_n \oplus 0)]_{\widetilde{T}}\| \leqslant \theta$ and $\|[(x_n \oplus 0) \otimes (w \oplus 0)]_{\widetilde{T}}\| + \|[(w \oplus 0) \otimes (y_n \oplus 0)]_{\widetilde{T}}\| \to 0$, for all $w \in \mathcal{H}$. ∎

The matter of the reverse containment to that of (i) will be discussed at length in Section 5.

4. OTHER BOUNDARY SETS FOR A CONTRACTION

We turn next to a consideration of some other subsets of $\mathbf{T}$ associated with an absolutely continuous contraction T. Recently, various authors have used the sets $\Sigma_*(T)$ and $\Sigma(T)$ (defined in Section 1) to settle some fundamental questions in the theory of singly generated dual operator algebras (see, for example, [2], [8], and [9]). We use these sets along with X_T to give, among other results, a new characterization of the class $\mathbf{A}_{1,\aleph_0}$.

The papers [1], [3], and [2], as well as the parallel work in [8] (each of which culminated in the proof that $\mathbf{A} = \mathbf{A}_1$) contain results fundamental for work "on $\mathbf{T}$" for operators in the class $\mathbf{A}$. We assemble some of these results here, preparatory to their use in conjunction with the set X_T. The following is from [3]. The reader is referred to Section 2 for our notation and conventions concerning unitary dilations.

THEOREM 4.1. *Suppose $T \in \mathbf{A}(\mathcal{H})$ has a unitary dilation U acting on $L^2(\mathbf{T}, \mathcal{D})$. Given $f \in L^1(\mathbf{T})$, $\varepsilon > 0$, and vectors $w_1, w_2, \ldots, w_n$ in $L^2(\mathbf{T}, \mathcal{D})$, there exist vectors x and y in $\mathcal{H}$ such that*

(i) $\|x\| \leqslant \|f\|^{1/2}$, $\|y\| \leqslant \|f\|^{1/2}$,

(ii) $(x, w_j)_{L^2(\mathbf{T},\mathcal{D})} = (y, w_j)_{L^2(\mathbf{T},\mathcal{D})} = 0$, $1 \leqslant j \leqslant n$, *and*

(iii) $\|f - x \overset{U}{\cdot} y\|_1 < \varepsilon$.

The following is the vanishing lemma in [2], Lemma 3.3 (see also [11], Lemma 3.8).

LEMMA 4.2. *Let T, U, and $L^2(\mathbb{T},\mathcal{D})$ be as in the previous theorem.*

(i) *If $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{S}$ is a sequence weakly convergent to zero and $w \in \mathcal{K}_-$, then $\|[x_n \overset{U}{\cdot} w]\| \to 0$.*

(ii) *If $\{y_n\}_{n=1}^{\infty} \subseteq \mathcal{S}_*$ is a sequence weakly convergent to zero and $w \in \mathcal{K}_+$, then $\|[w \overset{U}{\cdot} y_n]\| \to 0$.*

We now give the characterization of the absolutely continuous contractions in the class $\mathbb{A}$ which is implicit in [2] and [8], Lemma 5.1.

THEOREM 4.3. *Let T be an absolutely continuous contraction. Then $T \in \mathbb{A}$ if and only if $\mathbb{T} = X_T \cup \Sigma_*(T) \cup \Sigma(T)$ (where this equality is interpreted as up to a set of measure zero).*

Proof. It is straightforward to show that if $\mathbb{T} = X_T \cup \Sigma_*(T) \cup \Sigma(T)$ then $\mathbb{T}$ is essential for T using the techniques in [11] and [2], and thus $T \in \mathbb{A}$.

For the reverse direction, let $\Gamma = \mathbb{T} \setminus (\Sigma_*(T) \cup \Sigma(T))$. If $m(\Gamma) = 0$ there is nothing to prove, so suppose not and let f in $L^1(\Gamma)$ be of norm one. Using Theorem 4.1 it is easy to construct sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$ each weakly convergent to zero and satisfying

$$\|f - x_n \overset{U}{\cdot} y_n\|_1 \to 0. \tag{4.1}$$

Since $f = f|\Gamma$ has norm one and $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are in the unit ball it is clear that

$$\|\chi_{\mathbb{T}\setminus\Gamma} x_n\| \to 0, \quad \text{and} \quad \|\chi_{\mathbb{T}\setminus\Gamma} y_n\| \to 0.$$

So clearly

$$\|\chi_{\Sigma_*(T)} x_n\| \to 0, \quad \text{and} \quad \|\chi_{\Sigma(T)} y_n\| \to 0. \tag{4.2}$$

Observe that for all x in $\mathcal{H}$

$$\chi_{\Sigma_*(T)} P_{\mathcal{R}_*} x = P_{\mathcal{R}_*} x, \quad \text{and} \quad \chi_{\Sigma(T)} P_{\mathcal{R}} x = P_{\mathcal{R}} x. \tag{4.3}$$

From (4.2) and (4.3) we have

$$\|P_{\mathcal{R}_*} x_n\| \to 0, \quad \text{and} \quad \|P_{\mathcal{R}} y_n\| \to 0. \tag{4.4}$$

Since $x_n = P_{\mathcal{S}} x_n \oplus P_{\mathcal{R}_*} x_n$ and $y_n = P_{\mathcal{S}_*} y_n \oplus P_{\mathcal{R}} y_n$, and using (4.1) and (4.4), it follows that

$$\|f - P_{\mathcal{S}} x_n \overset{U}{\cdot} P_{\mathcal{S}_*} y_n\| \to 0. \tag{4.5}$$

Certainly $\{P_S x_n\}_{n=1}^{\infty}$ and $\{P_{S_*} y_n\}_{n=1}^{\infty}$ are weakly convergent to zero. Thus since $\mathcal{H} \subseteq \mathcal{K}_+ \cap \mathcal{K}_-$ we may cite Lemma 4.2 to conclude

$$\|[P_S x_n \overset{U}{\cdot} w]\| + \|[w \overset{U}{\cdot} P_{S_*} y_n]\| \to 0, \quad w \in \mathcal{H}. \tag{4.6}$$

Finally, transferring (4.5) into L^1/H_0^1 and using Proposition 2.2 (i), we obtain

$$\|[f] - [P_{\mathcal{H}} P_S x_n \overset{U}{\cdot} P_{\mathcal{H}} P_{S_*} y_n]\| \to 0,$$

and

$$\|[P_{\mathcal{H}} P_S x_n \overset{U}{\cdot} w]\| + \|[w \overset{U}{\cdot} P_{\mathcal{H}} P_{S_*} y_n]\| \to 0, \quad w \in \mathcal{H}.$$

By the remark following the definition of $\widehat{\mathcal{X}}_0(\mathcal{A}_T)$ we have $[f] \in \widehat{\mathcal{X}}_0(\mathcal{A}_T)$ and hence $\Gamma = \mathbf{T} \setminus (\Sigma_*(T) \cup \Sigma(T)) \subseteq X_T$ as desired. ∎

The following corollary extends this result to T assumed only to be an absolutely continuous contraction, and the next proposition to operators in the classes $C_{\cdot 0}$, $C_{\cdot 1}$, and so on. Recall that for such a T, $\operatorname{ess}(T)$ denotes the maximal subset of $\mathbf{T}$ essential for T.

COROLLARY 4.4. *Let T be an absolutely continuous contraction. Then* $\operatorname{ess}(T) = X_T \cup \Sigma_*(T) \cup \Sigma(T)$.

Proof. It suffices to consider $T \oplus M_{\mathbf{T}\setminus \operatorname{ess}(T)}$. ∎

PROPOSITION 4.5. *Let T be an absolutely continuous contraction. Then*
(i) $T \in C_{\cdot 0}$ *implies* $\operatorname{ess}(T) = X_T \cup \Sigma_*(T)$;
(ii) $T \in C_{0\cdot}$ *implies* $\operatorname{ess}(T) = X_T \cup \Sigma(T)$;
(iii) $T \in C_{00}$ *implies* $\operatorname{ess}(T) = X_T$;
(iv) $T \in C_{1\cdot}$ *implies* $\operatorname{ess}(T) = X_T \cup \Sigma_*(T)$; *and*
(v) $T \in C_{\cdot 1}$ *implies* $\operatorname{ess}(T) = X_T \cup \Sigma(T)$.

Proof. We prove only (i) and (iv), since the others follow easily. For (i) we begin with the case $T \in \mathbf{A} \cap C_{\cdot 0}$. From [10], Lemma 2.7, we have that if $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is any (bounded) sequence converging weakly to zero then

$$\|[x_n \otimes w]_T\| \to 0, \quad w \in \mathcal{H}. \tag{4.7}$$

For some $f \in L^1(\mathbf{T})$ of unit norm supported on $\mathbf{T} \setminus \Sigma_*(T)$ a construction as in the proof of Theorem 4.3 yields a sequence of vectors $\{x_n\}_{n=1}^{\infty}$ and $\{P_{\mathcal{H}} P_{S_*} y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$ such that

$$\|f - x_n \overset{U}{\cdot} P_{\mathcal{H}} P_{S_*} y_n\|_1 \to 0.$$

Now use (4.7) and Lemma 4.2.

For an absolutely continuous contraction T in $C_{\cdot 0}$ but not in $\mathbf{A}$, choose $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbf{D}$ such that $\mathrm{NTL}(\{\lambda_n\}_{n=1}^{\infty}) = \mathbf{T} \setminus \mathrm{ess}(T)$. Form $T \oplus \mathrm{Diag}(\lambda_n)$ and combine the result just proved with Proposition 3.5 (vi).

The result (iv) follows from the proof of [9], Theorem 7.2 or from [8], Corollary 7.4. ∎

Recall that with the class $\mathbf{A}_{1,\aleph_0}$ there are associated sets $\mathcal{E}_\theta^r$ (see [11] and [9]): for $T \in \mathbf{A}(\mathcal{H})$ and $0 \leqslant \theta < 1$ the set $\mathcal{E}_\theta^r$ is defined to be the set of all $[L] \in Q_T$ for which there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$ such that

$$\limsup_{n\to\infty} \|[L] - [x_n \otimes y_n])\| \leqslant \theta,$$

and

$$\|[x_n \otimes w]_T\| \to 0, \quad w \in \mathcal{H}.$$

Similarly, associated with $\mathbf{A}_{\aleph_0,1}$ there is a set $\mathcal{E}_\theta^\ell$ defined to be the set of all $[L] \in Q_T$ for which there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$ such that

$$\limsup_{n\to\infty} \|[L] - [x_n \otimes y_n])\| \leqslant \theta,$$

and

$$\|[w \otimes y_n]_T\| \to 0, \quad w \in \mathcal{H}.$$

(These sets were originally defined with further condition, later seen to be superfluous, namely that $\{y_n\}_{n=1}^{\infty}$ [respectively, $\{x_n\}_{n=1}^{\infty}$] tends weakly to zero.) The reader may be impelled to embark upon the definition of yet another subset of $\mathbf{T}$ as follows: define, for T an absolutely continuous contraction, the set $\widehat{\mathcal{E}}_\theta^r(\mathcal{A}_T)$ to be the set of $[f]$ in L^1/H_0^1 for which there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$ satisfying

$$\limsup_{n\to\infty} \|[f] - \varphi_T([x_n \otimes y_n])\| \leqslant \theta,$$

and

$$\|\varphi_T([x_n \otimes w]_T)\| \to 0, \quad w \in \mathcal{H}.$$

Then find a set F_T maximal in $\mathbf{T}$ for which $\{[f]_{L^1/H_0^1} : f \in L^1(F_T), \|f\|_1 \leqslant 1\} \subseteq \widehat{\mathcal{E}}_\theta^r(\mathcal{A}_T)$.

An impediment is that $\widehat{\mathcal{E}}_\theta^r(\mathcal{A}_T)$ is not known to be convex, so the "well-definition" of F_T is unclear (indeed, even in the case $T \in \mathbf{A}$ it is not known in general whether $\mathcal{E}_\theta^r(\mathcal{A}_T) \subseteq Q_T$ is convex). The following shows this scheme to associate a boundary set with $\widehat{\mathcal{E}}_\theta^r(\mathcal{A}_T)$ is in fact unnecessary.

THEOREM 4.6. *Let T be an absolutely continuous contraction. Then $T \in \mathbb{A}_{1,\aleph_0}$ [respectively, $T \in \mathbb{A}_{\aleph_0,1}$] if and only if $\mathbf{T} = X_T \cup \Sigma_*(T)$ [respectively, $\mathbf{T} = X_T \cup \Sigma(T)$].*

Proof. It follows easily from the techniques of [9] that if $\mathbf{T} = X_T \cup \Sigma_*(T)$ then $T \in \mathbb{A}$ and in fact $T \in \mathbb{A}_{1,\aleph_0}$, so we concentrate on the reverse implication. From [9], Theorem 5.4 we know that T has a "full analytic invariant subspace" $\mathcal{M}$. It suffices for our purposes to note that $\mathcal{M} = \bigvee_{\lambda \in \mathbf{D}} \ker(T|\mathcal{M} - \lambda)^*$ so $T|\mathcal{M} \in C_{\cdot 0}$ by [10] and that $\sigma(T|\mathcal{M}) = \overline{\mathbf{D}}$ so $T|\mathcal{M} \in \mathbb{A}$. It then follows from [9] that $T|\mathcal{M} \in \mathbb{A}_{1,\aleph_0}$. By Proposition 4.5 we have $X_{T|\mathcal{M}} \cup \Sigma_*(T|\mathcal{M}) = \mathbf{T}$. Since any coisometric extension of T is a coisometric extension of $T|\mathcal{M}$, we have $\Sigma_*(T|\mathcal{M}) \subseteq \Sigma_*(T)$, and $X_{T|\mathcal{M}} \subseteq X_T$ from Proposition 3.5, and the result follows. The other statement follows from this one upon taking adjoints and using $\Sigma_*(T^*) = \overline{\Sigma(T)}$. ∎

Before considering the extension of the last result to absolutely continuous contractions we need the following.

PROPOSITION 4.7. *Let T be an absolutely continuous contraction. Then for each $0 < \theta < 1$,*

$$\{[f] : f \in (L^1(\Sigma_*(T)))_1\} \subseteq \widehat{\mathcal{E}}_\theta^r(\mathcal{A}_T).$$

Proof. Consider first the case in which $T \in \mathbb{A}$. A modification of the argument in [11], Theorem 3.11, allows one to construct the requisite sequences. The key is to use the fact that if B is the bilateral shift on $\mathcal{K}$ and x, y are in $\mathcal{K}$ then $\|[B^n x \otimes y]_B\| \to 0$. The case T merely an absolutely continuous contraction follows from a consideration of $T \oplus M_{\mathbf{T} \setminus \Sigma_*(T)}$ and the usual splitting argument. ∎

We remark that for the case $T \in \mathbb{A}$ we may actually get

$$\{[f] : f \in (L^1(\Sigma_*(T)))_1\} \subseteq \widehat{\mathcal{E}}_0^r(\mathcal{A}_T)$$

and this provides a direct proof that the properties $E_{0,1}^r$ and $F_{0,1}^r$ of [9] are equivalent.

The following proposition now gives us a boundary set associated with $\widehat{\mathcal{E}}_0^r(\mathcal{A}_T)$.

PROPOSITION 4.8. *Let T be an absolutely continuous contraction. Set $E_T^r = X_T \cup \Sigma_*(T)$ and $E_T^\ell = X_T \cup \Sigma(T)$. Then*

$$\{[f] : f \in (L^1(E_T^r))_1\} \subseteq \widehat{\mathcal{E}}_0^r(\mathcal{A}_T)$$

and E_T^r is a maximal subset of $\mathbf{T}$ with respect to this property. Similarly,

$$\{[f] : f \in (L^1(E_T^\ell))_1\} \subseteq \widehat{\mathcal{E}}_0^\ell(\mathcal{A}_T)$$

and E_T^ℓ is maximal with respect to this property.

Proof. By duality it suffices to prove the first claim. To show the containment, write f in $(L^1(E_T^r))_1$ as $f|_{\Sigma_*(T)} + f|_{X_T \setminus \Sigma_*(T)}$. Then use Proposition 4.7, the sequences of vectors available for each summand, and the two-sided vanishing conditions arising from $\widehat{\mathcal{X}}_0(\mathcal{A}_T)$ on the sequence for $f|_{X_T\setminus\Sigma_*(T)}$ to get a sequence for $[f]$.

For the maximality, suppose that Γ is a Borel subset of $\mathbb{T}$ such that $E_T^r \subseteq \Gamma$, $m(\Gamma \setminus E_T^r) > 0$, and

$$\{[f] : f \in (L^1(\Gamma))_1\} \subseteq \widehat{\mathcal{E}}_0^r(\mathcal{A}_T).$$

It follows easily that $\widetilde{T} = T \oplus M_{\mathbb{T}\setminus\Gamma} \subseteq \mathbb{A}_{1,\aleph_0}$ and thus $\mathbb{T} = X_{\widetilde{T}} \cup \Sigma_*(\widetilde{T})$. But this requires $\mathbb{T} = X_T \cup \Sigma_*(T) \cup (\mathbb{T} \setminus \Gamma)$, which contradicts the choice of Γ. ∎

We observe in passing that the following, first noticed in [19], is immediate from Proposition 4.8 and Theorem 4.6.

COROLLARY 4.9. *Let T be an element of $\mathbb{A}_{1,\aleph_0}$. Then $\mathcal{E}_0^r(\mathcal{A}_T) = (Q_T)_1$, and is, in particular, convex.*

The techniques of [9] and the proof of Proposition 4.8 yield as well the following, where $\sigma_r(T)$ denotes the right spectrum of T.

COROLLARY 4.10. *Let T be an absolutely continuous contraction. Then* $\mathrm{NTL}(\sigma_r(T) \cap \mathbb{D}) \subseteq E_T^r = X_T \cup \Sigma_*(T)$.

The result of Proposition 4.8 shows that E_T^r is the appropriate subset of $\mathbb{T}$ to associate with $\widehat{\mathcal{E}}_0^r(\mathcal{A}_T)$. We leave to the reader the results for this set analogous to those for X_T contained in Proposition 3.5.

The following may be deduced from Theorem 4.3 and elementary considerations. For ease of notation, we leave to the interested reader some generalizations from T in $\mathbb{A}$ to T merely assumed an absolutely continuous contraction.

COROLLARY 4.11. *If $T \in \mathbb{A}$ and $\Sigma_*(T) \cap \Sigma_*(T^*) \subseteq X_T \cup X_{T^*}$ then $T \oplus T^* \in \mathbb{A}_{1,\aleph_0} \cap \mathbb{A}_{\aleph_0,1}$.*

COROLLARY 4.12. *For $n \in \mathbb{N}$, denote by S_n the unilateral shift of multiplicity n. If T is an absolutely continuous contraction then $T \oplus S_n^* \in \mathbb{A}_{1,\aleph_0}$ implies $T \in \mathbb{A}_{1,\aleph_0}$.*

COROLLARY 4.13. *Let $T \in \mathbb{A} \cap C_{\cdot 0}$. For any subspace $\mathcal{M}$ semi-invariant for T,* $\mathrm{ess}(T_{\mathcal{M}}) \setminus X_T \subseteq \Sigma_*(T_{\mathcal{M}})$.

We close this section by sharpening somewhat some results of Saina and Ouannasser. They consider in [20] and [18] respectively the multiplicity on $\Sigma_*(T)$

and $\Sigma(T)$ of the operators R_* and R; in particular, Ouannasser uses this to answer the long standing question "If $T \in \mathbb{A}$, is $T^{(n)} \in \mathbb{A}_n$?" in the affirmative. The machinery of X sets developed above allows one to see that the crucial sets for consideration of multiplicity are in fact $\Sigma_*(T) \setminus X_T$ and $\Sigma(T) \setminus X_T$. This approach yields the following modest generalizations of the main results of [20] and [18] respectively.

THEOREM 4.14. *Let $T \in \mathbb{A}_{1,\aleph_0}$ and let $B_* = S^* \oplus R_*$ be its minimal coisometric extension. Suppose that on $\Sigma_*(T) \setminus X_T$ the operator R_* has uniform multiplicity at least n. Then $T \in \mathbb{A}_{n,\aleph_0}$.*

THEOREM 4.15. *Let $T \in \mathbb{A}(\mathcal{H})$ and let*

$$T \cong \begin{pmatrix} T_0 & * \\ 0 & T_1 \end{pmatrix}$$

be its triangularization with respect to $\mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0 = \{x \in \mathcal{H} : \|T^n x\| \to 0\}$. Denote the minimal isometric dilation of T_0 by $S_0 \oplus R_0$ and let Σ^0 denote the Borel set such that $m|\Sigma^0$ is a scalar spectral measure for R_0. Denote the minimal coisometric extension of T_1 by $S_1^* \oplus R_1$ and let Σ_*^1 denote the Borel set such that $m|\Sigma_*^1$ is a scalar spectral measure for R_1. If R_0 [respectively, R_1] has uniform multiplicity at least n on $\Sigma^0 \setminus X_T$ [respectively, $\Sigma_*^1 \setminus X_T$] then $T \in \mathbb{A}_n$.

In general, the question of when membership in some class $\mathbb{A}_{m,n}$ yields information about multiplicity is open; see [14], [12], and [13] for positive results for normal operators and $C_{\cdot 0}$ or C_{11} operators with finite defect indices.

5. MAPPING THEOREMS FOR X_T

In this section we return to the question raised at the end of Section 3 regarding the relationship between $(X_T)^n$ and X_{T^n}. While the containment $(X_T)^n \subseteq X_{T^n}$ is straightforward, the reverse containment is not so easy but potentially more important. Suppose one has an operator T whose norm equals its spectral radius, and one is looking for invariant subspaces; there is clearly no harm in assuming T is a contraction. The techniques of [16] and [17] allow one to exchange a contraction T satisfying merely $\sigma(T) \cap \mathbf{T} \neq \emptyset$ for another contraction $h(T)$ with $\sigma(h(T)) \cap \mathbf{D}$ dominating on a large subset of $\mathbf{T}$ and also satisfying $\operatorname{Lat}(h(T)) = \operatorname{Lat}(T)$. If one knows how to deduce from $S^2 \in \mathbb{A}_{\aleph_0}$ the existence of a non-trivial invariant subspace for S, an application of this to $h(T)$ (plus elementary arguments) produces a non-trivial invariant subspace for T. (See [17] for a full discussion of this and related approaches.)

The passage from information about T^2 to information about T appears to be hard. Therefore the containment $X_{T^2} \subset (X_T)^2$ and its analogies for E_T^r and E_T^ℓ are a start, allowing one to deduce something about T from information about T^2. We begin with some preliminary lemmas.

LEMMA 5.1. *Let N be a positive integer. Any h in H^∞ can be written*

$$h(z) = h_0(z^N) + zh_1(z^N) + \cdots + z^{N-1}h_{N-1}(z^N), \tag{5.1}$$

where $h_j \in H^\infty$, $0 \leqslant j \leqslant N-1$; moreover, the $\{h_j\}$ may be taken to satisfy as well

$$\|h_j\|_\infty \leqslant \|h\|_\infty, \quad 0 \leqslant j \leqslant N-1.$$

Proof. Given N and h as above, it is clearly possible to write h as in (5.1) with analytic h_j, so it merely remains to check the norm conditions. Let ρ denote some N-th root of unity so that $1, \rho, \ldots, \rho^{N-1}$ are the N-th roots of 1. Assuming h is written as in (5.1) we have

$$h(\rho^k z) = \sum_{j=0}^{N-1} \rho^{kj} z^j h_j(z^N), \quad 0 \leqslant k \leqslant N-1. \tag{5.2}$$

Observe that

$$\sum_{k=0}^{N-1} \rho^{-k\ell}\rho^{km} = 0, \quad 0 \leqslant m, \ell \leqslant N-1, \quad m \neq \ell.$$

Using this, and adding the N equalities in (5.2), we obtain

$$Nh_0(z^N) = \sum_{k=0}^{N-1} h(\rho^k z),$$

from which the norm condition on h_0 follows easily. Similarly,

$$\sum_{k=0}^{N-1} \rho^{-k\ell} h(\rho^k z) = Nz^\ell h_\ell(z^N), \quad 0 < \ell \leqslant N-1,$$

from which the condition follows for the other h_ℓ. ∎

The next two lemmas are preparatory to mapping theorems for X_T under powers, and we omit their computational proofs.

LEMMA 5.2. *Let Γ be a Borel subset of $\mathbf{T}$, N a positive integer, and ρ an N-th root of unity as in the proof of Lemma 5.1. There exists a Borel subset γ of $\mathbf{T}$ such that $\gamma^N = \Gamma$ and $\rho^k\gamma \cap \rho^j\gamma = \emptyset$, $0 \leqslant k, j < N-1$, $k \neq j$. Further, let $\tilde{\gamma} = \bigcup_{0 \leqslant k < N-1} \rho^k \gamma$; then for any h in H^∞, $\int_{\tilde{\gamma}} h(z)\,\mathrm{d}m(z) = \int_{\Gamma} h_0(w)\,\mathrm{d}m(w)$, where h_0 is defined as in Lemma 5.1 and m is Lebesgue measure on $\mathbf{T}$. Note also that $m(\tilde{\gamma}) = m(\Gamma)$, and that $\tilde{\gamma} = \{s \in \mathbf{T} : s^N \in \Gamma\}$.*

LEMMA 5.3. *Let T in $\mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction, N a positive integer, and ρ an N-th root of unity as in the proof of Lemma 5.1. Let $\widetilde{T} = T \oplus \rho T \oplus \cdots \oplus \rho^{N-1}T$, and for u in $\mathcal{H}$ let $\tilde{u} = \frac{1}{\sqrt{N}}(u \oplus u \oplus \cdots \oplus u)$ in $\mathcal{H}^{(N)}$. Then (with h_0 defined for h in H^∞ as in Lemma 5.1),*

$$(h(\widetilde{T})\tilde{u}, \tilde{v})_{\mathcal{H}^{(N)}} = (h_0(T^N)u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{H}, h \in H^\infty.$$

We may now give a theorem which includes a mapping theorem for X_T.

THEOREM 5.4. *Suppose T is an absolutely continuous contraction and N a positive integer such that $T^N \in \mathbb{A}_{\aleph_0}$. Then $\widetilde{T} = T \oplus \rho T \oplus \cdots \oplus \rho^{N-1}T$ (where $1, \rho, \ldots, \rho^{N-1}$ are the N-th roots of unity) belongs to $\mathbb{A}_{\aleph_0}$. It follows that for any T and N, $X_{T^N} = (X_T)^N$.*

Proof. For any Borel subset F of $\mathbf{T}$, denote by $\tilde{\chi}_F$ the normalized characteristic function of F. Now let Γ be a subset of X_{T^N}. There exist sequences of vectors $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in the unit ball of $\mathcal{H}$ satisfying

$$\begin{aligned} &\|[\tilde{\chi}_\Gamma] - \varphi_{T^N}([x_n \otimes y_n])\|_{L^1/H_0^1} \to 0, \quad \text{and} \\ &\|[x_n \otimes w]\|_{Q_{T^N}} + \|[w \otimes y_n]\|_{Q_{T^N}} \to 0, \quad w \in \mathcal{H}. \end{aligned} \tag{5.3}$$

Equivalently,

$$\sup_{\|h\| \leqslant 1} \left| \frac{1}{m(\Gamma)} \int_{\Gamma} h(s)\,\mathrm{d}m(s) - (h(T^N)x_n, y_n) \right| \to 0,$$

and

$$\|[x_n \otimes w]\|_{Q_{T^N}} + \|[w \otimes y_n]\|_{Q_{T^N}} \to 0, \quad w \in \mathcal{H}.$$

Let $\tilde{\gamma}$ be associated with Γ as in Lemma 5.2. We will show that $[\tilde{\chi}_{\tilde{\gamma}}]$ is in $\mathcal{X}_0(\mathcal{A}_{\widetilde{T}})$.

First,

$$\begin{aligned}\|[\tilde{\chi}_{\tilde{\gamma}}] - \varphi_{\widetilde{T}}([\tilde{x}_n \otimes \tilde{y}_n])\|_{L^1/H_0^1} &= \sup_{\|g\|\leqslant 1} \left| \frac{1}{m(\tilde{\gamma})} \int_{\tilde{\gamma}} g(s)\, dm(s) - (g(\widetilde{T})\tilde{x}_n, \tilde{y}_n) \right| \\ &= \sup_{\|g\|\leqslant 1} \left| \frac{1}{m(\Gamma)} \int_{\Gamma} g_0(s)\, dm(s) - (g_0(T^N)x_n, y_n) \right| \\ &\leqslant \|[\tilde{\chi}_\Gamma] - \varphi_{T^N}([x_n \otimes y_n])\|_{L^1/H_0^1},\end{aligned}$$

where we have used the previous lemmas in the calculations. Hence $\|[\tilde{\chi}_{\tilde{\Gamma}}] - \varphi_{\widetilde{T}}([\tilde{x}_n \otimes \tilde{y}_n])\|_{L^1/H_0^1} \to 0$.

To complete the proof that $[\tilde{\chi}_{\tilde{\gamma}}]$ is in $\widehat{\mathcal{X}}_0(\mathcal{A}_{\widetilde{T}})$, we need to check the vanishing conditions. So let $\overline{w} = (w_1 \oplus w_2 \oplus \cdots \oplus w_N)$ be arbitrary in $\mathcal{H}^{(N)}$. Then

$$\begin{aligned}\|\varphi_{\widetilde{T}}([\tilde{x}_n \otimes \overline{w}])\| &= \sup_{\|g\|\leqslant 1} |(g(\widetilde{T})\tilde{x}_n, \overline{w})| \\ &\leqslant \sup_{\|g\|\leqslant 1} \sum_{j=0}^{N-1} \frac{1}{\sqrt{N}} \left|(g_j(T^N)x_n, (T^*)^j w_{j+1})\right|,\end{aligned}$$

where the $\{g_j\}$ are associated with g as in Lemma 5.1. But since the $\{x_n\}_{n=1}^{\infty}$ satisfy the vanishing condition (5.3) in Q_{T^N}, it is easy to show that this latter sum goes to zero with n. The other part of the vanishing condition is proved similarly.

The final claim follows easily from Proposition 3.9 and the proof of the first claim, and we are done. ∎

We may obtain next, after a proposition about minimal coisometric extensions and powers, a result about the powers of an operator in $\mathbb{A}_{1,\aleph_0}$. To ease the notation in what follows we will denote by $B_T = S_T^* \oplus R_T$ acting on $\mathcal{K} = \mathcal{S} \oplus \mathcal{R}$ the minimal co-isometric extension of T.

PROPOSITION 5.5. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and let n be a positive integer. Then $R_{(T^n)}$ is unitarily equivalent to $(R_T)^n$.*

Proof. We give only the proof for the case $n = 2$. Obviously $B_T^2 = (S_T^*)^2 \oplus (R_T)^2$ acting on $\mathcal{K}$ is *some* co-isometric extension of T^2, so we may assume that there exists $\mathcal{K}_1 \subseteq \mathcal{K}$, reducing for B_T^2, such that $B_T^2|\mathcal{K}_1$ is the minimal co-isometric extension of T^2 (see [11]). It is enough to show that $\mathcal{R} \subseteq \mathcal{K}_1$ or equivalently that $\mathcal{K}_1^\perp \subseteq \mathcal{S}$. Equivalently, since B_T is a co-isometry, we may show

$$\mathcal{K}_1^\perp \subseteq \bigcup_{n=1}^{\infty} \operatorname{Ker}(B_T^{2n}). \tag{5.4}$$

To prove (5.4) we use some facts about the minimal coisometric extension from [21]. Since B_T is minimal, we have that $\bigvee_{n\geqslant 0}(B_T^*)^n\mathcal{H} = \mathcal{K}$. Also, there is a subspace $\mathcal{L}$ of $\mathcal{K}$ defined by

$$\mathcal{L} = \overline{\{(B_T^* - T^*)h : h \in \mathcal{H}\}} \tag{5.5}$$

so that

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{L} \oplus B_T^*\mathcal{L} \oplus \cdots \oplus (B_T^*)^n\mathcal{L} \oplus \cdots . \tag{5.6}$$

Because $\mathcal{K}_1$ is reducing for B_T^2, we have

$$(B_{T^2}|\mathcal{K}_1)^* = ((B_T^2)|\mathcal{K}_1)^* = ((B_T^2)^*|\mathcal{K}_1) = (B_T^*)^2|\mathcal{K}_1. \tag{5.7}$$

Let $\mathcal{L}_2$ be the space for B_{T^2} analogous to $\mathcal{L}$, defined by

$$\mathcal{L}_2 = \overline{\{(B_{T^2}^* - (T^2)^*)h : h \in \mathcal{H}\}}. \tag{5.8}$$

Clearly from (5.7)

$$\mathcal{L}_2 = \overline{\{((B_T^*)^2 - T^{*2})h : h \in \mathcal{H}\}}. \tag{5.9}$$

Using (5.7) again, we have the decomposition

$$\mathcal{K}_1 = \mathcal{H} \oplus \mathcal{L}_2 \oplus (B_T^*)^2\mathcal{L}_2 \oplus \cdots \oplus (B_T^*)^{2n}\mathcal{L}_2 \oplus \cdots . \tag{5.10}$$

It is easy to show that $\mathcal{L}_2 \subseteq \mathcal{L} \oplus B_T^*\mathcal{L}$ and that for any u and v in $\mathcal{L}$ such that $B_T^*u \oplus v \perp \mathcal{L}_2$, we have $B_T^*u \oplus v \in \operatorname{Ker}(B_T^2)$. Similarly one may compute $((B_T^*)^2\mathcal{L} \oplus (B_T^*)^3\mathcal{L}) \ominus (B_T^*)^2\mathcal{L}_2 \subseteq \operatorname{Ker}(B_T^4)$, and so on. Then (5.4) follows from the decompositions given by (5.6) and (5.10). ∎

THEOREM 5.6. *Suppose T is in $\mathbf{A}_{1,\aleph_0}$. Then for any positive integer N, $T^N \in \mathbf{A}_{N,\aleph_0}$.*

Proof. It suffices to use Theorem 5.4 and Theorem 4.14, counting multiplicities on the complement of X_{T^N} using Proposition 5.5. ∎

A combination of Theorem 5.4 and Proposition 5.5 shows that the set $X_T \cup \Sigma_*(T)$ also maps perfectly under powers, with additional multiplicity information on $X_{T^n} \setminus \Sigma_*(T^n)$. Also, if T is an absolutely continuous contraction, then $\operatorname{ess}(T) = \operatorname{ess}(T^{(\aleph_0)}) = X_{T^{(\aleph_0)}}$. It is then straightforward to combine this with Theorem 5.4 to deduce a mapping theorem about essential sets. We leave these and allied results to the interested reader.

6. TRIANGULAR FORMS

In this section we consider the containment reverse to that of Proposition 3.5 (viii), namely, how is the X set of an operator in upper triangular form related to the X sets of the operators on its diagonal? We begin with an easy observation whose proof is omitted.

LEMMA 6.1. *Let $T \in \mathbb{A}$, $\mathcal{M}$ a subspace semi-invariant for T, $0 \leqslant \theta < 1$, and $[L]$ in Q_T be given. Suppose there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{M}$ such that*

$$\limsup_{n\to\infty} \|[L]_T - [x_n \otimes y_n]_T\| \leqslant \theta,$$

and

$$\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\| \to 0, \quad w \in \mathcal{M}.$$

Then $\varphi_T([L]) \in \widehat{\mathcal{X}}_\theta(\mathcal{A}_{T_{\mathcal{M}}})$.

We may now give a theorem on triangular forms which provides a partial answer to the general question.

THEOREM 6.2. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ having a matrix of the form*

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}. \tag{6.1}$$

Suppose $T \in \mathbb{A}_{\aleph_0}$ and either $T_1 \in C_{0\cdot}$ or $T_2 \in C_{0\cdot}$. Then $\mathbf{T} = X_T = X_{T_1} \cup X_{T_2}$.

Proof. It clearly suffices to prove the containment $\mathbf{T} \subseteq X_{T_1} \cup X_{T_2}$ in light of Proposition 3.5 (viii) and $T \in \mathbb{A}_{\aleph_0}$. For each $\lambda \in \mathbb{D}$, there exists a sequence $\{x_n\}_{n=1}^{\infty} = \{x_n^\lambda\}_{n=1}^{\infty}$ of unit vectors in $\mathcal{H}$ such that

$$[C_\lambda]_T = [x_n \otimes x_n]_T$$

and

$$\|[x_n \otimes w]_T\| + \|[w \otimes x_n]_T\| \to 0, \quad w \in \mathcal{H}.$$

Write each $x_n = u_n \oplus v_n$ relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. By dropping to a subsequence if necessary, we may assume that, for some γ_λ with $0 \leqslant \gamma_\lambda \leqslant 1$,

$$\|u_n^\lambda\|^2 \to \gamma_\lambda.$$

It follows that

$$\|v_n^\lambda\|^2 \to 1 - \gamma_\lambda.$$

From these we get easily, as in the proof of Proposition 3.3, that

$$\limsup_{n\to\infty} \|[P_\lambda] - \varphi_{T_2}([v_n \otimes v_n])\|_{L^1/H_0^1} \leqslant \gamma_\lambda^{\frac{1}{2}}, \tag{6.2}$$

and

$$\limsup_{n\to\infty} \|[P_\lambda] - \varphi_{T_1}([u_n \otimes u_n])\|_{L^1/H_0^1} \leqslant (1-\gamma_\lambda)^{\frac{1}{2}}.$$

Moreover, we have

$$\begin{aligned} &\|[v_n \otimes z]_{T_2}\| = \|[x_n \otimes z]_T\| \to 0, \quad z \in \mathcal{H}_2, \quad \text{and} \\ &\|[w \otimes u_n]_{T_1}\| = \|[w \otimes x_n]_T\| \to 0, \quad w \in \mathcal{H}_1. \end{aligned} \tag{6.3}$$

From standard facts, if $T_2 \in C_{0\cdot}$, we have also

$$[z \otimes v_n]_{T_2} \to 0, \quad z \in \mathcal{H}_2.$$

In the case $T_1 \in C_{0\cdot}$, we may reach the same conclusion with a little more work: we may write $\|[z\otimes v_n]_{T_2}\| = \|[z\otimes v_n]_T\| = \|[z\otimes x_n]_T - [z\otimes u_n]_T\| \leqslant \|[z\otimes x_n]_T\| + \|[z\otimes u_n]_T\|$, and, since $[z\otimes x_n]_T \to 0$, it is enough to show that $[z \otimes u_n]_T \to 0$. But observe that the sequence $\{u_n\}_{n=1}^\infty \subseteq \mathcal{H}_1$ must be contained in the backward shift space $\mathcal{S}$ of the minimal coisometric extension B_* of T, because $\|T^m u_n\| = \|T_1^m u_n\| \to 0$ as $m \to 0$, since $T_1 \in C_{0\cdot}$. It is then a standard computation to show that $\|[z\otimes u_n]_T\| = \|[P_{\mathcal{S}} z \otimes u_n]_{B_*}\| \to 0$, since $S^* \in C_{0\cdot}$. Thus, under the hypotheses of the theorem in either case, we have

$$[z \otimes v_n]_{T_2} \to 0, \quad z \in \mathcal{H}_2. \tag{6.4}$$

By combining (6.2), (6.3), and (6.4), we have that

$$[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_{\gamma_\lambda^{\frac{1}{2}}}(\mathcal{A}_{T_2}), \quad \lambda \in \mathbf{D}.$$

For each $0 \leqslant \alpha \leqslant 1$, let $\mathbf{D}_\alpha = \{\lambda \in \mathbf{D} : \gamma_\lambda \leqslant \alpha\}$, and $\mathbf{D}'_\alpha = \{\lambda \in \mathbf{D} : \gamma_\lambda > \alpha\}$. It is clear from the definition that the sets $\widehat{\mathcal{X}}_\beta(\mathcal{A}_{T_2})$ increase with β. It then follows from the above computations and Lemma 6.1 that, under the hypotheses of the theorem,

$$[P_\lambda]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_{\alpha^{\frac{1}{2}}}(\mathcal{A}_{T_2}), \quad \lambda \in \mathbf{D}_\alpha.$$

From Proposition 3.8 we have

$$\mathrm{NTL}(\mathbf{D}_\alpha) \subseteq X_{T_2}, \quad 0 \leqslant \alpha < 1.$$

Let $F = \mathbf{T} \setminus X_{T_2}$, and Λ some subset of $\mathbf{D}$ such that $\mathrm{NTL}(\Lambda) = F$. Then for each α, $0 \leqslant \alpha < 1$, $\mathrm{NTL}(\Lambda \cap \mathbf{D}_\alpha) = \emptyset$ and consequently

$$\mathrm{NTL}(\Lambda \cap \mathbf{D}'_\alpha) = F, \quad 0 \leqslant \alpha < 1,$$

where as usual we interpret these equalities as up to sets of Lebesgue measure zero.

We now turn to a fundamental lemma.

LEMMA 6.3. *Assume the hypotheses as in the theorem, and let* $f \in (L^1(F))_1$, $\{w_1, \ldots, w_p\} \subseteq \mathcal{H}_1$, *and* $\varepsilon > 0$ *be given. Then there exist* s *and* t *in the unit ball of* $\mathcal{H}_1$ *such that*

$$\|\varphi_T^{-1}([f]) - [s \otimes t]\|_T < \varepsilon,$$

and

$$\|[s \otimes w_k]\|, \|[w_k \otimes t]\| < \varepsilon, \quad 1 \leqslant k \leqslant p.$$

Before proving this lemma, note that by a standard procedure (give yourself a dense sequence $\{w_k\}_{k=1}^{\infty}$ in $\mathcal{H}_1$ and a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ decreasing to 0) we obtain by repeated applications of it the following result: for any $f \in (L^1(F))_1$ there exist sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}_1$ such that

$$\|[f] - \varphi_T([s_n \otimes t_n])\|_{L^1/H_0^1} \to 0,$$

and

$$\|[s_n \otimes w]\|, \|[w \otimes t_n]\| \to 0, \quad w \in \mathcal{H}_1.$$

This result, by virtue of Lemma 6.1, shows that

$$[f]_{L^1/H_0^1} \in \widehat{\mathcal{X}}_0(\mathcal{A}_{T_1}), \quad f \in (L^1(F))_1.$$

Consequently $F \subseteq X_{T_1}$, so $X_{T_2} \cup X_{T_1} = \mathbb{T}$, and, modulo the proof of Lemma 6.3, the theorem is proved.

Proof of Lemma 6.3. Select $\alpha < 1$ but sufficiently close to 1 so that

$$(1-\alpha)^{\frac{1}{2}} < \frac{\varepsilon}{3}, \tag{6.5}$$

and

$$(1-\alpha)^{\frac{1}{2}} \max\{\|w_k\| : k = 1, \ldots, p\} < \frac{\varepsilon}{3}.$$

Since $\mathrm{NTL}(\Lambda \cap \mathbb{D}'_\alpha) = F$ we can find sequences $\{\beta_i\}_{i=1}^{N} \subseteq \mathbb{C}$ and $\{\lambda_i\}_{i=1}^{N} \subseteq \Lambda \cap \mathbb{D}'_\alpha$ such that

$$\left\|\varphi_T^{-1}([f]) - \sum_{i=1}^{N} \beta_i [C_{\lambda_i}]\right\| < \frac{\varepsilon}{3}, \tag{6.6}$$

and

$$\sum_{i=1}^{N} |\beta_i| < 1.$$

For each of these λ_i there exists a sequence $\{x_{i,n}\}_{n=1}^{\infty} = \{u_{i,n} \oplus v_{i,n}\}_{n=1}^{\infty}$ of unit vectors in $\mathcal{H}$ satisfying

$$[C_{\lambda_i}] = [x_{i,n} \otimes x_{i,n}], \quad n \in \mathbf{N},$$
$$\|[x_{i,n} \otimes w]_T\| + \|[w \otimes x_{i,n}]_T\| \to 0, \quad w \in \mathcal{H},$$
$$\|u_{i,n}^{\lambda}\|^2 \to \gamma_{\lambda_i} \geqslant \alpha,$$

and

$$\|v_{i,n}^{\lambda}\|^2 \to 1 - \gamma_{\lambda_i} \leqslant 1 - \alpha.$$

For any N-tuple $\nu = (n_1, \dots, n_N)$ set

$$s_\nu = \sum_{i=1}^{N} \beta_i^{\frac{1}{2}} x_{i,n_i},$$
$$t_\nu = \sum_{i=1}^{N} \overline{\beta}_i^{\frac{1}{2}} x_{i,n_i},$$

and write $s_\nu = s_\nu^1 \oplus s_\nu^2$ and $t_\nu = t_\nu^1 \oplus t_\nu^2$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Following the "classical" procedure (see the proof of [5], Lemma 2.9) it is easy to select ν so as to satisfy

$$\Big\|[s_\nu \otimes t_\nu] - \sum_{i=1}^{N} \beta_i [C_{\lambda_i}]\Big\|_{Q_T} < \frac{\varepsilon}{3}, \quad \|s_\nu\|, \|t_\nu\| \leqslant 1,$$

$$\|[s_\nu \otimes w_k]\|, \|[w_k \otimes t_\nu]\| < \frac{\varepsilon}{2}, \quad k = 1, \dots, p,$$

(6.7)
$$\|s_\nu^2\|^2 \Big(\approx \sum_{i=1}^{N} |\beta_i| \|v_{i,n_i}\|^2\Big) \leqslant (1-\alpha), \quad \text{and}$$

$$\|t_\nu^2\|^2 \Big(\approx \sum_{i=1}^{N} |\beta_i| \|v_{i,n_i}\|^2\Big) \leqslant (1-\alpha).$$

From the inequalities (6.6), (6.7), (6.5), and

$$\|[s_\nu \otimes t_\nu] - [s_\nu^1 \otimes t_\nu^1]\| = \|[s_\nu^2 \otimes t_\nu]\| \leqslant \|s_\nu^2\| \leqslant (1-\alpha)^{\frac{1}{2}},$$

we obtain

$$\|\varphi_T^{-1}([f]) - [s_\nu^1 \otimes t_\nu^1]\| < \varepsilon.$$

We have also, for $k = 1, \dots, p$,

$$\begin{aligned} \|[s_\nu^1 \otimes w_k]\| &= \|[s_\nu \otimes w_k] - [s_\nu^2 \otimes w_k]\| \\ &\leqslant \|[s_\nu \otimes w_k]\| + \|[s_\nu^2 \otimes w_k]\| \\ &\leqslant \frac{\varepsilon}{2} + (1-\alpha)^{\frac{1}{2}} \|w_k\| \\ &< \varepsilon. \end{aligned}$$

Similar computations yield

$$\|[w_k \otimes t_\nu^1]\| < \varepsilon, \quad k = 1, \ldots, p.$$

This completes the construction of the vectors required for Lemma 6.3 and thus its proof, which in turn completes the proof of Theorem 6.2. ∎

We have the following generalization, to T assumed merely an absolutely continuous contraction, as a corollary.

COROLLARY 6.4. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ having a matrix of the form*

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}. \tag{6.8}$$

Suppose either $T_1 \in C_{0\cdot}$ or $T_2 \in C_{0\cdot}$. Then $X_T = X_{T_1} \cup X_{T_2}$.

Proof. If $T \in \mathbb{A}_{\aleph_0}$ the conclusion follows from the theorem. If not, let $F = \mathbf{T} \setminus X_T$ and let D be some strictly contractive diagonal normal operator such that $F = \mathrm{NTL}(\sigma(D) \cap \mathbf{D})$. It is routine that $D \in C_{00}$, and $F = X_D$ from Proposition 3.5 (vi). The result follows from the theorem upon consideration of $\widetilde{T} = D \oplus T$ and a little work with Proposition 3.5 (ii). ∎

We leave the obvious corollary for an upper triangular form with some diagonal entry in $C_{\cdot 0}$ to the reader, and note also that we know of no counterexample to the theorem with these various $C_{\alpha\beta}$ hypotheses deleted.

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