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LOCAL SPECTRAL PROPERTIES OF CERTAIN MATRIX DIFFERENTIAL OPERATORS IN $L^p(\mathbb{R}^N)^m$

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ABSTRACT. We investigate the local spectral behaviour of (constant coefficient) matrix differential operators, and more general matrix p-multiplier operators, in L^p-spaces (particularly, over $\mathbb{R}^N$). Of particular interest is the decomposability and spectral mapping properties of such operators, together with relevant functional calculi, when they are available.

KEYWORDS: *L^p-spaces, matrix p-multipliers, decomposability, functional calculi.*

AMS SUBJECT CLASSIFICATION: Primary 47A60, 47B40; Secondary 47F05.

0. INTRODUCTION

In this paper we investigate the local spectral behaviour on $L^p(\mathbb{R}^N, \mathbb{R}^m) = L^p(\mathbb{R}^N)^m$, $1 \leqslant p < \infty$, of square systems $Q(D) = \left[Q_{jk}(-\mathrm{i}\frac{\partial}{\partial x_1}, \ldots, -\mathrm{i}\frac{\partial}{\partial x_N})\right]_{j,k=1}^m$ of linear partial differential operators with constant coefficients. For $p \neq 2$ it turns out, as for the case of $m = 1$ considered in [3], that the operator $Q(D)$ is spectral in the sense of N. Dunford ([11]) if and only if all the polynomial entries Q_{jk} are constant. The proof is however more involved since new phenomena occur for $m > 1$.

For $N = 1$ and $m = 1$ the operator $Q(D)$ is necessarily decomposable in the sense of C. Foiaş ([10], [12], [26]; see [3], Corollary 3.7). This is no longer true for $m > 1$. As we shall see, the operator $Q(D)$ with $Q(x) = \begin{bmatrix} x & x^2 \\ 1 & x \end{bmatrix}$, $x \in \mathbf{R}$, is not decomposable on $L^p(\mathbb{R})^2$, even for $p = 2$. There is however an easily verifiable degree condition available which, for the case $N = 1$ and $m \geqslant 2$, is necessary and sufficient for the decomposability of $Q(D)$; see Section 4. For $N \geqslant 2$ there are, already in the case $m = 1$, examples of non-decomposable operators $Q(D)$, ([3], Corollary 3.5). However, in the presence of an ellipticity property for the characteristic polynomial and again a degree condition for the entries of Q, we still have decomposability of $Q(D)$ in $L^p(\mathbf{R}^N)^m$.

As in [3], part of the theory can be done in the more general framework of (not necessarily bounded) multipliers on arbitrary locally compact abelian (l.c.a) groups. So, let G be a l.c.a group with dual group Γ and Haar measures μ, respectively ν, on G, respectively Γ, chosen such that the Plancherel theorem is valid. We write $\mathcal{M}^p(\Gamma)$ for the algebra of all p-multiplier functions on Γ. By $\mathcal{U}^p(\Gamma)$ we will denote the algebra of all local p-multiplier functions on Γ, i.e. the algebra of all (equivalence classes of) ν-measurable functions u on Γ with the property that $\varphi u \in \mathcal{M}^p(\Gamma)$ for all compactly supported functions φ which are the Fourier transform of an $L^1(G)$-function. Notice, for $p = 2$, that $\mathcal{U}^2(\Gamma)$ coincides with the algebra $L^\infty_{\mathrm{loc}}(\Gamma)$ of all locally bounded ν-measurable functions on Γ.

If $\mathcal{A}$ is any algebra, we write $\mathrm{M}_m(\mathcal{A})$ for the algebra of all $(m \times m)$-matrices with entries from $\mathcal{A}$. For $a \in \mathrm{M}_m(\mathcal{M}^p(\Gamma))$ we will write T^p_a for the corresponding multiplier operator on $L^p(G)^m$; see [9]. Suppose $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$. If $f \in L^p(G)^m$ has the property that its Fourier transform $\mathcal{F}f = \widehat{f}$ (computed component-wise) has compact support and $h \in L^1(G)$ is a function such that

(0.1) $\quad$ $\mathrm{supp}(\widehat{h})$ is compact and $\widehat{h} \equiv 1$ in a neighbourhood of $\mathrm{supp}(\widehat{f})$,

then we define $S^p_a f = T^p_{\widehat{h}a} f$. This definition does not depend on the particular choice of the function $h \in L^1(G)$ with property (0.1). It is easy to see that the so defined linear operator S^p_a is closable. Its closure is again denoted by S^p_a. As shown in Section 2, for $1 \leqslant p \leqslant 2$, the domain of S^p_a coincides with $D(S^p_a) = \{f \in L^p(G)^m;\ a\widehat{f} \in \mathcal{F}(L^p(G)^m)\}$ and

$$S^p_a f = \mathcal{F}^{-1}(a\widehat{f}) \quad \text{for } f \in D(S^p_a); \tag{0.2}$$

the product $a\widehat{f}$ makes sense since, by the Hausdorff-Young theorem, $\mathcal{F}(L^p(G)^m) \subset L^q(\Gamma)^m$ where $\frac{1}{p} + \frac{1}{q} = 1$.

In the case when $a = Q$ is a matrix polynomial on $G = \mathbf{R}^N$ it turns out, for $1 \leqslant p < \infty$, that $D(S^p_Q) = \{f \in L^p(\mathbf{R}^N)^m;\ Q(D)f \in L^p(\mathbf{R}^N)^m\}$ and $S^p_Q f = Q(D)f$, for $f \in D(S^p_Q)$, where $Q(D)f$ is formed in the sense of distributions. In the final two sections we shall discuss in detail the decomposability of $Q(D)$, for $p \neq 2$, and the problem of obtaining functional calculi for $Q(D)$.

Because of (0.2), it is necessary to investigate the local spectral properties of matrix multiplication operators on $L^q(\Gamma)^m$. This will be done in the following section.

1. MATRIX MULTIPLICATION OPERATORS IN L^p-SPACES

As usual, a measure space (Ω, Σ, μ) consists of a non-empty set Ω, a σ-algebra Σ of subsets of Ω and a σ-additive measure $\mu : \Sigma \to [0, \infty]$. Recall that a measure space (Ω, Σ, μ) is a *direct sum of finite measure spaces* if there exists a family $\mathcal{F} \subseteq \Sigma$ of pairwise disjoint sets of finite measure such that a subset E of Ω is in Σ if and only if $E \cap F \in \Sigma$ for all $F \in \mathcal{F}$ and $\mu(E) = \sum_{F \in \mathcal{F}} \mu(F \cap E)$.

Every σ-finite measure space is a direct sum of finite measure spaces. If (Ω, Σ, μ) is a measure space and Σ_0 is the conditional σ-ring of all sets in Σ having finite measure, then (Ω, Σ, μ) is a direct sum of finite measure spaces if and only if (Ω, Σ_0, μ) is a direct sum of measure spaces in the sense of [24], Definition 3.1. In that case (Ω, Σ_0, μ) is localizable ([24], Theorem 3.2) and one has the duality $L^1(\mu)^* = L^\infty(\mu)$. Moreover, the algebra of all operators of multiplication by L^∞-functions is a maximal abelian subalgebra of the Banach algebra $\mathcal{L}(L^2(\mu))$ of all bounded linear operators on $L^2(\mu)$; see [24], Theorem 5.1. This fact will also be needed for $L^p(\mu)$. For the sake of completeness we include the proof.

LEMMA 1.1. *Let (Ω, Σ, μ) be a direct sum of finite measure spaces. For every $p \in [1, \infty)$ the algebra $\mathcal{A}^p_\infty = \{M_\varphi;\ \varphi \in L^\infty(\mu)\}$ of all multiplication operators $f \mapsto M_\varphi f = \varphi f$ in $L^p(\mu)$ by L^∞-functions φ is a maximal abelian subalgebra of $\mathcal{L}(L^p(\mu))$.*

Proof. Fix $T \in \mathcal{L}(L^p(\mu))$ with $TM_\varphi = M_\varphi T$ for all $\varphi \in L^\infty(\mu)$. For every $F \in \mathcal{F}$ (with $\mathcal{F}$ as above) we denote by $\mu|F$ the measure μ restricted to the algebra $F \cap \Sigma$ and consider $L^p(\mu|F)$ as a closed subspace of $L^p(\mu)$ (via the embedding $f \to \tilde{f}$ where $\tilde{f} \equiv f$ on F and $\tilde{f} \equiv 0$ on $\Omega \setminus F$). For $f \in L^\infty(\mu|F) \subseteq L^p(\mu|F)$ we have

$$T(f) = T(f\chi_F) = fT(\chi_F) = M_{T(\chi_F)}f,$$

showing that $T(f) \in L^p(\mu|F)$. Since $L^\infty(\mu|F)$ is dense in $L^p(\mu|F)$ it follows that $L^p(\mu|F)$ is invariant for T and the restriction of T to $L^p(\mu|F)$ is $M_{T(\chi_F)}$. This is only possible if $T(\chi_F)$ actually belongs to $L^\infty(\mu|F)$ and $\|T(\chi_F)\|_\infty \leqslant \|T\|$. The function ψ defined to be $T(\chi_F)$ on F, for all $F \in \mathcal{F}$, is then measurable and satisfies $\|\psi\|_\infty = \sup_{F \in \mathcal{F}} \|T(\chi_F)\|_\infty \leqslant \|T\|$ as well as $M_\psi f = T(f)$ for all $f \in L^p(\mu)$. ∎

Since we are mainly interested in spaces $L^p(\mu)$, where μ is the Haar measure on a l.c.a group G we should notice that $(G, \mathcal{B}, \mu)$ is in this case a direct sum of finite measure spaces, where $\mathcal{B}$ denotes the Borel subsets of G; see [24], Theorem 5.2 for a proof. For the rest of this section, (Ω, Σ, μ) will always be a direct sum of finite measure spaces.

If $m \geqslant 1$ is an integer, $a = [a_{jk}]_{j,k=1}^{m}$ is an $(m \times m)$-matrix of measurable functions on Ω and $1 \leqslant p < \infty$, then we define a closed linear matrix multiplication operator M_a^p on $L^p(\mu)^m$ with domain $D(M_a^p) = \{f \in L^p(\mu)^m;\ af \in L^p(\mu)^m\}$ by the formula $M_a^p f = af$, for $f \in D(M_a^p)$. Since m is fixed we do not indicate the dependence of M_a^p on m. We assume that each entry a_{jk} is finite-valued μ-a.e. For $\varphi \in L^\infty(\mu)$ we also consider the operator $M_{\varphi I}^p$ on $L^p(\mu)^m$ where φI is the diagonal $(m \times m)$-matrix with φ down the diagonal. Thus, $M_{\varphi I}^p \in \mathcal{L}(L^p(\mu)^m)$ and $\|M_{\varphi I}^p\| = \|\varphi\|_\infty$, for every $1 \leqslant p < \infty$. If a is as above, we have $M_{\varphi I}^p M_a^p \subseteq M_a^p M_{\varphi I}^p$. Since $D(M_a^p)$ is dense in $L^p(\mu)^m$ we must have

$$(M_a^p)^{-1} M_{\varphi I}^p = M_{\varphi I}^p (M_a^p)^{-1} \tag{1.1}$$

whenever $T = (M_a^p)^{-1}$ exists as a bounded linear operator on $L^p(\mu)^m$. In this case $T = [T_{jk}]_{j,k=1}^{m}$ with $T_{jk} \in \mathcal{L}(L^p(\mu))$ for $j,k = 1, \ldots, m$. Because of (1.1) we have $T_{jk} M_\varphi = M_\varphi T_{jk}$ for all $\varphi \in L^\infty(\mu)$ which, by Lemma 1.1, implies that $T_{jk} = M_{b_{jk}}$ for some $b_{jk} \in L^\infty(\mu)$ and $j,k \in \{1, \ldots, m\}$. Accordingly, if $b = [b_{jk}]_{j,k=1}^{m}$, then $T = (M_a^p)^{-1} = M_b^p$ must be a matrix multiplication operator with μ-essentially bounded entries. Because $M_a^p M_b^p = I$ on $L^p(\mu)^m$ it follows that $b = a^{-1}$, μ-a.e., and $\det(b) = 1/\det(a)$, μ-a.e. Since $\det(b) \in L^\infty(\mu)$, this is only possible if $0 \notin$ ess-range$(\det(a))$. Conversely, if $a^{-1} \in \mathrm{M}_m(L^\infty(\mu))$, then $M_{a^{-1}}^p \in \mathcal{L}(L^p(\mu)^m)$ is a bounded inverse of M_a^p. A similar argument shows that $M_a^p \in \mathcal{L}(L^p(\mu)^m)$ if and only if $a \in \mathrm{M}_m(L^\infty(\mu))$. If we define

$$\Sigma(a) = \{\lambda \in \mathbf{C};\ 0 \notin \text{ess-range}(\det(\lambda - a))\}$$

and $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ we obtain from these considerations the following

LEMMA 1.2. *For every matrix multiplication operator M_a^p on $L^p(\mu)^m$ the set $\Sigma(a)$ is contained in the spectrum $\sigma(M_a^p)$ of M_a^p. Moreover, if $\sigma(M_a^p) \neq \widehat{\mathbf{C}}$, then*

$$\sigma(M_a^p) = \begin{cases} \Sigma(a), & \text{if } a \in \mathrm{M}_m(L^\infty(\mu)) \\ \Sigma(a) \cup \{\infty\}, & \text{if } a \notin \mathrm{M}_m(L^\infty(\mu)). \end{cases}$$

If $m = 1$, then $\Sigma(a) = \text{ess-range}(a)$. Accordingly, for $m \geqslant 1$ we call $\Sigma(a)$ the *essential spectral range* of a. This name is also justified by the easily verified fact that

$$\Sigma(a) = \bigcap_{d \in [a]} \overline{\bigcup_{x \in \Omega} \sigma(d(x))},$$

where $[a]$ denotes the equivalence class of all measurable matrix functions d coinciding μ–a.e. on Ω with a and $\sigma(d(x))$ is the spectrum of the $(m \times m)$-matrix $d(x)$.

In the special case that $(\Omega, \Sigma, \mu) = (\Gamma, \mathcal{B}, \nu)$, where Γ is a l.c.a group with Haar measure ν, and $a = [a_{jk}]_{j,k=1}^{m}$ belongs to $\mathrm{M}_m(C(\Gamma))$, we have $\Sigma(a) = \overline{\bigcup_{\gamma\in\Gamma} \sigma(a(\gamma))}$; here $C(\Gamma)$ is the space of continuous functions on Γ.

Lemma 1.2 (for a σ-finite measure) can be found in [20], a paper which considers the question of when the operators M_a^p are infinitesimal generators of various kinds of C_0-semigroups and integrated semigroups in $L^p(\mu)^m$, $1 \leqslant p < \infty$; see also [22].

Before investigating the local spectral behaviour of matrix multiplication operators on $L^p(\mu)^m$ we recall some definitions. Let X be a Banach space and $\mathcal{C}(X)$ denote the class of all closed linear operators on X. A closed subspace Y of X is said to be *invariant* for $T \in \mathcal{C}(X)$ if $T(D(T) \cap Y) \subset Y$. The restriction operator $T|Y$ with domain $D(T) \cap Y$ is then a closed linear operator. The closed operator T is said to be *decomposable* in the sense of C. Foiaş ([10], [12], [26]) if, for every finite open cover $U_1, \ldots, U_n$ of $\widehat{\mathbb{C}}$, there are closed invariant subspaces $Y_1, \ldots, Y_n$ of T such that $Y_1 + \cdots + Y_n = X$ and $\sigma(T|Y_j) \subset U_j$ for $j = 1, \ldots, n$. We shall frequently use the following known facts.

LEMMA 1.3. *Let T be a closed linear operator on a Banach space X such that $\mathbb{C} \setminus \sigma(T) \neq \emptyset$. Then T is decomposable if and only if $(\lambda - T)^{-1}$ is decomposable for some (all) $\lambda \in \mathbb{C} \setminus \sigma(T)$.*

Proof. Since T is decomposable if and only if $\lambda - T$ is decomposable, the result follows from [4], Lemma 2.4. ∎

PROPOSITION 1.4. *Let X be a Banach space and $\mathcal{A}$ be a regular semisimple Banach algebra with unit. If $\Phi : \mathrm{M}_m(\mathcal{A}) \to \mathcal{L}(X)$ is any unital homomorphism, then every operator in the range $\Phi(\mathrm{M}_m(\mathcal{A}))$, of Φ, is a decomposable operator in X.*

Proof. Let $\mathcal{B} = \mathrm{M}_m(\mathcal{A})$ and write $\Psi : \mathcal{A} \to \mathcal{B}$ for the canonical monomorphism mapping $a \in \mathcal{A}$ to the diagonal matrix with diagonal entries a. Then $\Psi(\mathcal{A})$ is contained in the centre of $\mathcal{B}$. As in the proof of Corollary 3.14 in [2] one sees that the assumptions for b in Theorem 3.7 of [2] are satisfied, for all $b \in \mathcal{B}$, and it follows from this theorem that $\Phi(b)$ is decomposable for all $b \in \mathcal{B}$. ∎

COROLLARY 1.5. (a) *If $a \in \mathrm{M}_m(L^\infty(\mu))$, then M_a^p is decomposable in $L^p(\mu)^m$, for all $p \geqslant 1$.*

(b) *If a is an $\mathrm{M}_m(\mathbb{C})$-valued measurable function (possibly unbounded) such that $\mathbb{C} \setminus \sigma(M_a^p) \neq \emptyset$, then M_a^p is decomposable.*

Proof. (a) is an immediate consequence of Proposition 1.4 since $L^\infty(\mu)$ is a commutative regular unital Banach algebra. Part (b) is a consequence of (a), Lemma 1.3 and the proof of Lemma 1.2. ∎

Since the operators of multiplication with $L^\infty(\mu)$-functions are actually scalar-type spectral operators on $L^p(\mu)$, in the sense of N. Dunford ([11]), one could conjecture that for $a \in \mathrm{M}_m(L^\infty(\mu))$ the operator M_a^p would be at least generalized scalar, that is, admits a $C^\infty(\mathbf{C})$-functional calculus. This is indeed the case for $m = 2$ (cf. [10], Chapter 6, Section 4). However, for $m \geqslant 3$, there are examples ([5], IV, Example 2.3) where the entries of a are even smooth functions on $\mathbf{C}$ but M_a^p is not generalized scalar on $L^p(\mu)^m$ for any $p \geqslant 1$ (with respect to the planar Lebesgue measure). The next result shows that M_a^p is not always decomposable (of course, by Corollary 1.5, a is then necessarily unbounded).

PROPOSITION 1.6. *If $1 \leqslant p < \infty$ and a is a measurable $\mathrm{M}_m(\mathbf{C})$-valued function on Ω such that M_a^p is decomposable in $L^p(\mu)^m$, then $\sigma(M_a^p) \cap \mathbf{C} = \Sigma(a)$.*

Proof. The sets $\Omega_n = \{x \in \Omega;\ |a_{jk}(x)| \leqslant n, \text{ for } j,k = 1,\dots,m\}$ are measurable and $(\Omega_n, \Sigma|\Omega_n, \mu|\Omega_n)$ is still a direct sum of finite measure spaces (where $\Sigma|\Omega_n = \{E \cap \Omega_n;\ E \in \Sigma\}$). Since $b_n = a|\Omega_n$ belongs to $\mathrm{M}_m(L^\infty(\mu|\Omega_n))$, the operator $S_n = M_{b_n}^p$ is bounded and decomposable in $L^p(\mu|\Omega_n)^m$, for each $n \in \mathbf{N}$. If $R_n : L^p(\mu)^m \to L^p(\mu|\Omega_n)^m$ is the canonical restriction mapping we have

$$R_n M_a^p \subseteq S_n R_n, \quad n \in \mathbf{N}. \tag{1.2}$$

Assume now that M_a^p is decomposable and fix an arbitrary $\lambda \in \mathbf{C} \setminus \Sigma(a)$. Since $\Sigma(a)$ is a closed subset in $\mathbf{C}$ there exist a bounded open neighbourhood U of λ and an open set $H \subset \widehat{C}$ such that $U \cup H = \widehat{\mathbf{C}}$, with $\lambda \notin \overline{H}$ and $\overline{U} \cap \Sigma(a) = \emptyset$. Since M_a^p is decomposable, there are closed invariant subspaces Y_1, Y_2 for M_a^p such that $Y_1 + Y_2 = L^p(\mu)^m$, with $\sigma(M_a^p|Y_1) \subset U$, and $\lambda \notin \sigma(M_a^p|Y_2)$. We also have by Lemma 1.2 and the boundedness of S_n that $\sigma(S_n) = \Sigma(b_n) \subseteq \Sigma(a)$. By (1.2) we have $R_n(M_a^p|Y_1) = S_n R_n|Y_1$, and hence, $R_n(Y_1) = \{0\}$ for all $n \in \mathbf{N}$ (by the obvious extension of Corollary 0.13 in [23] to the Banach space setting). Hence, for all $f \in Y_1$, we have $f \equiv 0$, μ-a.e. on $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. This shows that $Y_1 = \{0\}$. That is $L^p(\mu)^m = Y_2$ and $\lambda \notin \sigma(M_a^p)$. ∎

COROLLARY 1.7. *If a is a measurable $\mathrm{M}_m(\mathbf{C})$-valued function such that $\sigma(M_a^p) \cap \mathbf{C} \neq \Sigma(a)$, then M_a^p is not decomposable on $L^p(\mu)^m$ and $\sigma(M_a^p) = \widehat{\mathbf{C}}$.*

EXAMPLE 1.8. Define $a(x) = \begin{bmatrix} x & x^2 \\ 1 & x \end{bmatrix}$ for $x \in \mathbf{R}$. Then $\sigma(M_a^p) = \widehat{\mathbf{C}}$ and M_a^p is not decomposable on $L^p(\mathbf{R})^2$ for all $p \geqslant 1$. Indeed, since a is continuous, we have $\Sigma(a) = \overline{\bigcup_{x \in \mathbf{R}} \sigma(a(x))} = \mathbf{R}$. For $\lambda \in \mathbf{C} \setminus \mathbf{R}$ direct computation shows that

$$(\lambda - a(x))^{-1} = \begin{bmatrix} \frac{\lambda - x}{\lambda(\lambda - 2x)} & \frac{x^2}{\lambda(\lambda - 2x)} \\ \frac{1}{\lambda(\lambda - 2x)} & \frac{\lambda - x}{\lambda(\lambda - 2x)} \end{bmatrix},$$

where not all entries are ess-bounded. So $\sigma(M_a^p) = \widehat{\mathbb{C}}$ properly contains $\Sigma(a) \cup \{\infty\}$. ∎

An operator $T \in \mathcal{C}(X)$ is said to have the *single valued extension property* (SVEP) if, for every X-valued analytic function $f : G \to X$ defined on an arbitrary open set $G \subset \mathbb{C}$ with $f(G) \subset D(T)$, we have $(z - T)f(z) \equiv 0$ on G iff $f \equiv 0$ on G. It is well known that decomposable operators have the SVEP.

Since, in the unbounded case, matrix multiplication operators need not even be decomposable in $L^p(\mu)^m$ for $m \geqslant 2$, the following fact is of some interest.

PROPOSITION 1.9. *For every measurable* $M_m(\mathbb{C})$*-valued function* a *on* Ω *the operator* M_a^p *has the* SVEP.

Proof. We introduce Ω_n, S_n, R_n as in the proof of Proposition 1.6. Let $f : G \to L^p(\mu)^m$ be an analytic $L^p(\mu)^m$-valued function such that $f(G) \subseteq D(M_a^p)$ and $(\lambda - M_a^p)f(\lambda) \equiv 0$ on G. Then $0 \equiv R_n(\lambda - M_a^p)f(z) \equiv (\lambda - S_n)R_n f(\lambda)$. Since $\lambda \to R_n f(\lambda)$ is analytic on G with values in $L^p(\mu|\Omega_n)^m$ and S_n has the SVEP (being decomposable by Corollary 1.5) we obtain $R_n f(\lambda) \equiv 0$ on G. Hence, for every $\lambda \in G$, we have $f(\lambda, x) = 0$ for μ-a.e. $x \in \Omega$, that is, $f(\lambda)$ is a null function for all $\lambda \in G$. ∎

If T is a closed operator on a Banach space X and $F \subseteq \widehat{\mathbb{C}}$ is closed, we write $X_T(F)$ for the linear space of all $x \in X$ such that there exists some analytic X-valued function $f : \widehat{\mathbb{C}} \setminus F \to X$ with $f(\widehat{\mathbb{C}} \setminus F) \subset D(T)$ satisfying $(\lambda - T)f(\lambda) \equiv x$ on $\mathbb{C} \setminus F$. If T is decomposable, then the spaces $X_T(F)$ are closed invariant subspaces for T with $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$; see for example [26]. An operator $T \in \mathcal{C}(X)$ is said to have the *weak property* (δ) if, for every locally finite open cover $\{U_n\}_{n=1}^{\infty}$ of $\mathbb{C}$, the space X coincides with the closed linear span of the union of the submanifolds $X_T(\overline{U}_n)$, $n \in \mathbb{N}$. If, in addition, all the spaces $X_T(F)$ are closed in X, whenever $F \subset \widehat{\mathbb{C}}$ is closed, we say that T has the *Ljubich-Macaev property*. Thus, if T is decomposable and has the weak property (δ), then T also has the Ljubich-Macaev property. The class of unbounded closed linear operators with the Ljubich-Macaev property is not comparable with the class of decomposable operators ([3]).

LEMMA 1.10. *If* $a : \Omega \to M_m(\mathbb{C})$ *is measurable, then* M_a^p *has the weak property* (δ) *in* $L^p(\mu)^m$, *for* $1 \leqslant p < \infty$. *Hence,* M_a^p *has the Ljubich-Macaev property whenever it is decomposable.*

Proof. Let $\{U_k\}_{k=1}^{\infty}$ be any locally finite open cover of $\mathbb{C}$ and fix $f \in X = L^p(\mu)^m$. Let Ω_n, S_n and R_n be as in the proof of Proposition 1.6 and consider the spaces $X^{(n)} = L^p(\mu|\Omega_n)^m$ as closed linear subspaces of X. Then $R_n f \to f$, as

$n \to \infty$, in X. Since S_n is bounded and decomposable on $X^{(n)}$ (by Corollary 1.5), there exists some $k(n) \in \mathbf{N}$ such that $X^{(n)} = X_{S_n}^{(n)}(\overline{U}_1) + \cdots + X_{S_n}^{(n)}(\overline{U_{k(n)}})$. Because of $X_{S_n}^{(n)}(\overline{U}_j) \subseteq X_T(\overline{U}_j)$, with $T = M_a^p$ and $j \in \mathbf{N}$, we see that $R_n f \in \sum_{j=1}^{k(n)} X_T(\overline{U}_j)$. So, $T = M_a^p$ has the weak property (δ). ∎

PROPOSITION 1.11. *Let $1 \leqslant p < \infty$ and $a : \Omega \to \mathrm{M}_m(\mathbf{C})$ be measurable. If the matrix multiplication operator M_a^p has the Ljubich-Macaev property, then $\sigma(M_a^p) \cap \mathbf{C} = \Sigma(a)$.*

Proof. Again let $T = M_a^p$ and Ω_n, S_n, R_n be as in the proof of Proposition 1.6. Fix an arbitrary point $\lambda \in \mathbf{C} \setminus \Sigma(a)$ and let U, H be an open cover of $\widehat{\mathbf{C}}$ as in the proof of Proposition 1.6. By the Ljubich-Macaev property we have $X = L^p(\mu)^m = X_T(\overline{U}) + X_T(\overline{H})$. Fix an arbitrary $f \in X_T(\overline{U})$. By the definition of $X_T(\overline{U})$ there exists an analytic X-valued function $g : \widehat{\mathbf{C}} \setminus \overline{U} \to X$ with $g(\widehat{\mathbf{C}} \setminus \overline{U}) \subset D(T)$ and $(z - T)g(z) \equiv f$ on $\mathbf{C} \setminus \overline{U}$. Then

$$h(z) = \begin{cases} R_n g(z), & z \in \mathbf{C} \setminus \overline{U} \\ (z - S_n)^{-1} R_n f, & z \in \mathbf{C} \setminus \sigma(S_n) \end{cases}$$

is a well defined entire X-valued function vanishing at infinity (notice that $S_n \in \mathcal{L}(L^p(\mu|\Omega_n)^m)$ is decomposable and hence has the SVEP). Thus, by the Liouville theorem, $h \equiv 0$. This shows that for all $z \in \mathbf{C} \setminus \overline{U}$ the function $g(z)$ vanishes μ-a.e. on Ω_n, for all $n \in \mathbf{N}$, that is, $g(z) = 0$ as an element of $X = L^p(\mu)^m$. Hence, $f \equiv (z - T)g(z) \equiv 0$ on $\mathbf{C} \setminus \overline{U}$. This shows that $X_T(\overline{U}) = \{0\}$, that is, $X = X_T(\overline{H})$; notice that $X_T(\overline{H})$ is closed in X. By the definition of $X_T(\overline{H})$ this implies that $(\lambda - T)D(T) = X$. Since T has the SVEP (by Proposition 1.9), the operator $\lambda - T$ must also be injective ([27]). Accordingly, $\lambda \notin \sigma(T)$. ∎

So, a matrix multiplication operator M_a^p with $\sigma(M_a^p) \cap \mathbf{C} \neq \Sigma(a)$ cannot have the Ljubich-Macaev property. This applies, in particular, to Example 1.8.

N. Dunford and J.T. Schwartz characterized all bounded operators $a \in \mathrm{M}_m(L^\infty(Q))$ which are spectral operators in H^m, where $Q : \Sigma \to \mathcal{L}(H)$ is any (selfadjoint) spectral measure in a Hilbert space H; see [11], Section 9, Chapter XV. These results can be adapted to characterize spectrality of operators $a \in \mathrm{M}_m(L^\infty(Q))$ in X^m where $Q : \Sigma \to \mathcal{L}(X)$ is any spectral measure in a Banach space X. However, we will restrict our attention to $X = L^p(\mu)$, $1 \leqslant p < \infty$, and a specific spectral measure Q in $L^p(\mu)$ since this setting suffices for our purposes and because the results are more transparent than for arbitrary Banach spaces X. So, in the remainder of this section we indicate how the Hilbert space results mentioned above can be suitably modified to the setting of matrix multiplication operators in L^p-spaces.

Let (Ω, Σ, ν) be a measure space with $\nu \geqslant 0$ a complete measure which is the direct sum of finite measures. Fix $r \in [1, \infty)$. For each $E \in \Sigma$, let $Q_r(E) \in \mathcal{L}(L^r(\nu))$ be the operator in $L^r(\nu)$ of multiplication by $\chi_E \in L^\infty(\nu)$. Then the mapping $Q_r : \Sigma \to \mathcal{L}(L^r(\nu))$ so defined is a spectral measure. The space $\widehat{V}_r = L^\infty(Q_r)$ of all (equivalence classes of) Σ-measurable functions $\psi : \Omega \to \mathbb{C}$ is a commutative semisimple Banach algebra (with unit and involution) with respect to the norm

$$\|\psi\|_\infty = \inf\{\|\psi\chi_F\|_\infty;\ F \in \Sigma,\ Q_r(F) = I\}, \quad \psi \in L^\infty(Q_r).$$

Since the Q_r-null sets are precisely the ν-null sets we conclude that $L^\infty(Q_r) = L^\infty(\nu)$. Given $\psi \in L^\infty(Q_r)$ the operator $\int \psi \, \mathrm{d}Q_r$, defined via integration with respect to the spectral measure Q_r (see [11]), is the operator in $L^r(\nu)$ of multiplication by ψ and $\|\int \psi \, \mathrm{d}Q_r\| = \|\psi\|_\infty$. Moreover, $V_r = \{\int \psi \, \mathrm{d}Q_r;\ \psi \in L^\infty(Q_r)\}$ is a strong operator closed subalgebra of $\mathcal{L}(L^r(\nu))$ equipped with the involution $(\int \psi \, \mathrm{d}Q_r)^* = \int \overline{\psi} \, \mathrm{d}Q_r$. Accordingly, V_r and $\widehat{V}_r$ are $*$-isomorphic as C^*-algebras via the mapping $\psi \to \int \psi \, \mathrm{d}Q_r$.

Fix an integer $m \geqslant 1$. Since $\mathcal{L}(L^r(\nu)^m)$ is isomorphic to $\mathrm{M}_m(\mathcal{L}(L^r(\nu)))$ we can (and do) identify $\mathrm{M}_m(V_r)$ with a subspace of $\mathcal{L}(L^r(\nu)^m)$. Elements $\widehat{a} = [\widehat{a}_{jk}]_{j,k=1}^m$ of $\mathrm{M}_m(\widehat{V}_r)$ are regarded as ν-essentially bounded maps $w \mapsto \widehat{a}(w)$ of Ω into $\mathcal{L}(\mathbb{C}^m) \simeq \mathrm{M}_m(\mathbb{C})$. So, for $w \in \Omega$, we have $\|\widehat{a}(w)\| = \sup\{\|\widehat{a}(w)\xi\|_2;\ \xi \in \mathbb{C}^m, \|\xi\|_2 \leqslant 1\}$, where $\|\cdot\|_2$ denotes the usual Hilbert space norm in $\mathbb{C}^m$. We define $\|\widehat{a}\|_\infty = \inf\{\sup\{\|\widehat{a}(w)\|;\ w \in E\};\ E \in \Sigma \text{ is } \nu\text{-null}\}$. The "same" argument as in [11], p. 1966 shows that $\mathrm{M}_m(\widehat{V}_r)$ is a unital Banach algebra with involution. Moreover, $\mathrm{M}_m(V_r)$ and $\mathrm{M}_m(\widehat{V}_r)$ are $*$-isomorphic as Banach algebras via the correspondence $\widehat{a} = [\widehat{a}_{jk}]_{j,k=1}^m \leftrightarrow a = \left[\int a_{jk} \, \mathrm{d}Q_r\right]_{j,k=1}^m$.

If $\widehat{a} \in \mathrm{M}_m(\widehat{V}_r)$, then $\widehat{a}(w)$ is a linear operator in the finite dimensional space $\mathbb{C}^m$, for $w \in \Omega$. Accordingly, $\widehat{a}(w)$ is a spectral operator of finite type (at most $(m-1)$). Its scalar part will be denoted by $\widehat{A}_{\widehat{a}}(w)$ and its nilpotent part by $\widehat{N}_{\widehat{a}}(w)$. By $F(\,\cdot\,; \widehat{a}(w)) : \mathcal{B} \to \mathcal{L}(\mathbb{C}^m)$ we mean the resolution of the identity of $\widehat{a}(w)$, where $\mathcal{B}$ is the σ-algebra of all Borel subsets of $\mathbb{C}$. The spectral measure $F(\,\cdot\,; \widehat{a}(w))$ is supported by the finite set $\sigma(\widehat{a}(w))$.

Elements $a \in \mathrm{M}_m(V_r)$, when identified as elements of $\mathcal{L}(L^r(\nu)^m)$, are precisely those matrix multiplication operators $M_{\widehat{a}}^r$ in $L^r(\nu)^m$ specified by matrices $\widehat{a}$ whose entries $\widehat{a}_{jk} \in L^\infty(\nu)$. Accordingly, the following two results characterize spectrality of such matrix multiplication operators. Their proofs can essentially be found in pp. 1976-1977 of [11], in the sense that the proofs given there carry over "ad verbatum" to the L^r-setting after noting two points. First, the use of Theorem XV.9.3 in the proof of Theorem 5 on p. 1976 of [11] uses only the $*$-isomorphism between $\mathrm{M}_m(V_r)$ and $\mathrm{M}_m(\widehat{V}_r)$ and not that it is isometric. The second point occurs in the proof of Theorem 6 on p. 1978 of [11] where the isometric isomorphism between $\mathrm{M}_m(V_r)$ and $\mathrm{M}_m(\widehat{V}_r)$ is used in the uniform boundedness argument. An examination of this proof shows the isometric property is not essential; it suffices to have a Banach algebra isomorphism.

LEMMA 1.12. *Let $m \geqslant 1$ be an integer and $r \in [1, \infty)$. Let $a \in \mathrm{M}_m(V_r)$ satisfy*

$$\text{(i)} \qquad \sup\{\|F(\sigma; \widehat{a}(\cdot))\|_\infty;\ \sigma \in \mathcal{B}\} < \infty,$$

and let $\sigma(a)$ be the spectrum of a relative to $\mathcal{L}(L^r(\nu)^m)$. Then, for every bounded Borel function $\varphi : \sigma(a) \to \mathbf{C}$ the $\mathcal{L}(\mathbf{C}^m)$-valued integral

$$\text{(ii)} \qquad w \longmapsto \int_{\sigma(a)} \varphi(\lambda)\, \mathrm{d}F(\lambda; \widehat{a}(w)),\ \ w \in \Omega,$$

is Q_r-essentially bounded. Moreover, the $\mathrm{M}_m(V_r)$-valued integral

$$\text{(iii)} \qquad \sigma \longmapsto \int_\Omega F(\sigma; \widehat{a}(w))\, \mathrm{d}Q_r(w),\ \ \sigma \in \mathcal{B},$$

is a spectral measure in $\mathcal{L}(L^r(\nu)^m)$ and the identity

$$\text{(iv)} \qquad \int_\Omega \Big[\int_{\sigma(a)} \varphi(\lambda)\, \mathrm{d}F(\lambda; \widehat{a}(w)) \Big]\, \mathrm{d}Q_r(w) = \int_{\sigma(a)} \varphi(\lambda) \Big[\int_\Omega F(\mathrm{d}\lambda; \widehat{a}(w))\, \mathrm{d}Q_r(w) \Big]$$

holds in $\mathrm{M}_m(V_r) \subseteq \mathcal{L}(L^r(\nu)^m)$.

Using Lemma 1.12 it can now be argued along the lines of the proof in pp. 1976–1977 of [11] to establish the following result.

PROPOSITION 1.13. *Let $m \geqslant 1$ be an integer and $r \in [1, \infty)$. If $a \in \mathrm{M}_m(V_r)$, then the matrix multiplication operator $M_{\widehat{a}}^r$ is a spectral operator in $\mathcal{L}(L^r(\nu)^m)$ if and only if*

$$\text{(i)} \qquad \sup\{\|F(\sigma; \widehat{a}(\cdot))\|_\infty;\ \sigma \in \mathcal{B}\} < \infty.$$

When this condition is satisfied, $M_{\widehat{a}}^r$ is a spectral operator of type not exceeding $(m-1)$ with the resolution of the identity $F(\,\cdot\,; a) : \mathcal{B} \to \mathcal{L}(L^r(\nu)^m)$ given by

$$\text{(ii)} \qquad F(\sigma; a) = \int_\Omega F(\sigma; \widehat{a}(w))\, \mathrm{d}Q_r(w), \quad \sigma \in \mathcal{B}.$$

Moreover, for ν-a.e. $w \in \Omega$ the scalar part of $\widehat{a}(w)$ is given by $\widehat{a}_{\hat{a}}(w) = \int_{\sigma(a)} \lambda \, \mathrm{d}F(\lambda; \widehat{a}(w))$ with the functions $\widehat{a}_{\hat{a}} : w \mapsto \widehat{A}_{\hat{a}}(w)$ and $\widehat{N}_{\hat{a}} : w \mapsto \widehat{N}_{\hat{a}}(w)$ belonging to $\mathrm{M}_m(\widehat{V}_r)$ and $(\widehat{N}_{\hat{a}})^m = 0$ in $\mathrm{M}_m(\widehat{V}_r)$.

Let $A = \int_\Omega \widehat{A}_{\hat{a}}(w)\, \mathrm{d}Q_r(w)$ and $N = \int_\Omega \widehat{N}_{\hat{a}}(w)\, \mathrm{d}Q_r(w)$. Then A satisfies $A = \int_{\sigma(a)} \lambda \, \mathrm{d}F(\lambda; a)$ with the operators $A, N \in \mathrm{M}_m(V_r)$ commuting, $N^m = 0$ and $A + N = M_{\hat{a}}^r$. In particular, A is the scalar part of $M_{\hat{a}}^r$ and N is the nilpotent part of $M_{\hat{a}}^r$.

REMARK. Let $\widehat{a} \in \mathrm{M}_m(\widehat{V}_r)$. Since each operator $\int \widehat{a}_{jk}\, \mathrm{d}Q_r$ is just the operator of multiplication by $\widehat{a}_{jk}$ in $L^r(\nu)$, for all $j, k \in \{1, \ldots, m\}$, it follows (for each $\sigma \in \mathcal{B}$) that

$$F(\sigma; \widehat{a}(w)) = F(\sigma; a)(w), \quad \text{for } \nu\text{–a.e. } w \in \Omega.$$

2. BASIC FACTS ON MATRIX p-MULTIPLIERS

In this section we derive some basic facts on matrix multiplier operators which will be needed in the sequel and which can be formulated in the general framework of l.c.a groups. In the following, G will be a l.c.a group with dual group Γ and $m \geqslant 1$ is an integer. We will also fix a bounded approximate identity $\{e_\alpha\}_{\alpha \in A}$ of $L^1(G)$ satisfying

$$\widehat{e}_\alpha \geqslant 0 \text{ on } \Gamma \text{ and } e_\alpha \geqslant 0 \text{ on } G, \text{ for all } \alpha \in A, \tag{2.1}$$

$$\|e_\alpha\|_{L^1(G)} = 1, \ \alpha \in A, \tag{2.2}$$

$$\mathrm{supp}(\widehat{e}_\alpha) \text{ is compact, for all } \alpha \in A, \tag{2.3}$$

and

$$\widehat{e}_\alpha \to 1 \text{ uniformly on all compact subsets of } \Gamma; \tag{2.4}$$

see [16], Theorem 33.12.

Since $L^1(G)^m \cap L^p(G)^m$ is dense in $L^p(G)^m$ for $1 \leqslant p < \infty$, we see (using (2.2)) that the operator $T^p_{\hat{e}_\alpha}$ of convolution with e_α satisfies

$$T^p_{\hat{e}_\alpha} f = e_\alpha * f \to f \quad \text{in } L^p(G)^m, \text{ for all } f \in L^p(G)^m, \tag{2.5}$$

where the convolution of e_α with f is defined component-wise. Let $f \in D^{(p)} = \{g \in L^p(G)^m;\ \mathrm{supp}(\widehat{g}) \text{ is compact}\}$. For $f \in D^{(p)}$ and $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$ we define $S^p_a f = T^p_{\varphi a} f$ where $\varphi \in \mathcal{F}(L^1(G))$ has compact support and $\varphi \equiv 1$ in a neighbourhood of $\mathrm{supp}(\widehat{f})$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $D^{(p)}$ converging to zero and $S^p_a f_n \to g$ in $L^p(G)^m$, then

$$e_\alpha * g = \lim_{n\to\infty} e_\alpha * S^p_a f_n = \lim_{n\to\infty} T^p_{\widehat{e}_\alpha a} f_n = 0$$

(because of the continuity of $T^p_{\widehat{e}_\alpha a}$). From (2.5) we conclude that $g = 0$. Hence, S^p_a is closable. The closure will again be denoted by S^p_a. A direct computation shows that $T^p_h S^p_a \subseteq S^p_a T^p_h$, for all $h \in \mathrm{M}_m(\mathcal{M}^p(\Gamma))$ commuting pointwise with a. It follows that for, $1 \leqslant p \leqslant 2$ and all $f \in D(S^p_a)$, we have $e_\alpha * (S^p_a f) = T^p_{\widehat{e}_\alpha a} f$ and hence

$$\mathcal{F}(e_\alpha * (S^p_a f)) = \widehat{e}_\alpha a \widehat{f}, \quad \text{for all } \alpha \in A.$$

Since $\widehat{e}_\alpha \to 1$ uniformly on compact subsets of Γ we conclude that $S^p_a = \mathcal{F}^{-1}(a\widehat{f})$ and thus

$$D(S^p_a) \subset \{f \in L^p(G)^m;\ a\widehat{f} \in \mathcal{F}(L^p(G)^m)\};$$

denote the latter set by $\widetilde{D}(S^p_a)$. For the converse fix $f \in \widetilde{D}(S^p_a)$. Then, for all $\alpha \in A$, we have $e_\alpha * \mathcal{F}^{-1}(a\widehat{f}) = \mathcal{F}^{-1}(\widehat{e}_\alpha a \widehat{f}) = S^p_a(e_\alpha * f)$. Since S^p_a is closed, $f \in D(S^p_a)$ and

$$S^p_a f = \mathcal{F}^{-1}(a\widehat{f}). \tag{2.6}$$

LEMMA 2.1. *Let $1 < p < \infty$. Then, for all $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$, we have $(S^p_a)^* = S^q_{a^t}$, where a^t is the transposed matrix of a and $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Notice that $(S^p_a)^*$ exists since S^p_a is a densely defined, closed linear operator. Obviously $(T^p_u)^* = T^q_{u^t}$ holds for all $u \in \mathrm{M}_m(\mathcal{M}^p(\Gamma))$. Hence, by the definition of S^p_a, we have $\langle S^p_a f, g\rangle = \langle f, S^q_{a^t} g\rangle$, for all $f \in L^p(G)$ and $g \in L^q(G)$ such that the supports of $\widehat{f}$ and $\widehat{g}$ are compact. From this we conclude that $(S^p_a)^* \supseteq S^q_{a^t}$. Conversely, fix $g \in D((S^p_a)^*)$. Then, for all $f \in D(S^p_a)$ and $\alpha \in A$, we have

$$\langle f, S^q_{a^t}(e_\alpha * g)\rangle = \langle e_\alpha * S^p_a f,\ g\rangle = \langle S^p_a(e_\alpha * f), g\rangle = \langle f, e_\alpha * (S^p_a)^* g\rangle$$

and hence $S^q_{a^t}(e_\alpha * g) = e_\alpha * (S^p_a)^* g$. Since $S^q_{a^t}$ is closed and $e_\alpha * g \to g$ we see that $g \in D(S^q_{a^t})$ and obtain $S^q_{a^t} = (S^p_a)^*$. ∎

LEMMA 2.2. *Let* $1 \leqslant p < \infty$ *and* $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$. *Then* $\Sigma(a) \subseteq \sigma(S_a^p)$ *and, for all* $\lambda \in \mathbb{C} \setminus \sigma(S_a^p)$, *we necessarily have* $(\lambda - a)^{-1} \in \mathrm{M}_m(\mathcal{M}^p(\Gamma)) \subseteq \mathrm{M}_m(L^\infty(\Gamma))$.

Proof. If $\lambda \in \mathbb{C} \setminus \sigma(S_a^p)$, then $(\lambda - S_a^p)^{-1} = S_{(\lambda-a)^{-1}}^p \in \mathcal{L}(L^p(G)^m)$; for $1 \leqslant p \leqslant 2$ this equality follows from (2.6), for $p > 2$ by duality using the preceeding lemma. Since $S_{(\lambda-a)^{-1}}^p$ is a bounded translation invariant operator, the entries of $(\lambda - a)^{-1}$ must belong to $\mathcal{M}^p(\Gamma)$ and hence, be essentially bounded. It follows that $\lambda \notin \Sigma(a)$. ∎

PROPOSITION 2.3. *Let* G *be a l.c.a group with dual group* Γ, $1 \leqslant p < \infty$ *and* $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$. *If the operator* S_a^p *is decomposable, then*

(i) $\sigma(S_a^p) \cap \mathbb{C} = \Sigma(a)$.

(ii) $\infty \in \sigma(S_a^p)$ *if and only if* $a \notin \mathrm{M}_m(L^\infty(\Gamma))$.

(iii) ∞ *cannot be an isolated point of* $\sigma(S_a^p)$.

Proof. As $\Sigma(a) = \Sigma(a^t)$, by Lemma 2.1 it suffices to give the proof for $1 \leqslant p \leqslant 2$.

(i) For $n \in \mathbb{N}$, write $\Omega_n = \{x \in \Gamma;\ |a_{jk}(x)| \leqslant n \text{ for } j, k = 1, \ldots, m\}$ and define $R_n : L^p(G)^m \to L^q(\Gamma)^m$, where $\frac{1}{p} + \frac{1}{q} = 1$, by $R_n f = \widehat{f}|\Omega_n$. By the Hausdorff-Young theorem, R_n is a bounded linear operator. Moreover, denoting by S_n the bounded linear operator of multiplication by $a|\Omega_n$ on $L^q(\Omega_n)^m$, we have $R_n S_a^p \subseteq S_n R_n$, for $n \in \mathbb{N}$. Using Lemma 2.2 we obtain, as in the proof of Proposition 1.6, that $\sigma(S_a^p) \cap \mathbb{C} \subseteq \sigma(a)$. Conversely, if $\lambda \in \Sigma(a)$, then $(\lambda - a)^{-1} \notin \mathrm{M}_m(L^\infty(\Gamma))$ and hence $(\lambda - S_a^p)^{-1}$ cannot exist in $\mathcal{L}(L^p(G))$.

(ii) and (iii). If $\infty \notin \sigma(S_a^p)$, then S_a^p is a translation invariant, bounded linear operator and hence, $a \in \mathrm{M}_m(\mathcal{M}^p(\Gamma)) \subseteq \mathrm{M}_m(L^\infty(\Gamma))$. For the converse direction and for (iii) we observe first, by (i), that the point ∞ can only be an isolated point of $\sigma(S_a^p)$ if $a \in \mathrm{M}_m(L^\infty(\Gamma))$. Hence, assume that $a \in \mathrm{M}_m(L^\infty(\Gamma))$. By (i) there exists some $\lambda \in \mathbb{C} \setminus \sigma(S_a^p)$. Since $(\lambda - S_a^p)^{-1} = S_{(\lambda-a)^{-1}}^p$ is a bounded, translation invariant operator we must have $(\lambda - a)^{-1} \in \mathrm{M}_m(L^\infty(\Gamma))$ and hence, we cannot have $0 \in \Sigma((\lambda - a)^{-1})$. Therefore, $0 \notin \sigma((\lambda - S_a^p)^{-1}) = \Sigma((\lambda - a)^{-1})$ (by (i) since $(\lambda - S_a^p)^{-1}$ is decomposable, cf. [4], Lemma 2.4) and thus $\infty \notin \sigma(S_a^p)$. ∎

Note that the proof shows that if $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$ has essentially bounded entries and S_a^p is decomposable, then $a \in \mathrm{M}_m(\mathcal{M}^p(\Gamma))$.

EXAMPLE. The function $x \mapsto a(x) = e^{i|x|^2}$, $x \in \mathbb{R}^N$, is a local p-multiplier function which is bounded on $\mathbb{R}^N$. Since $a \notin \mathcal{M}^p(\mathbb{R}^N)$ for $p \neq 2$ (cf. [21], Lemma 1.3), the operator S_a^p must be unbounded and cannot be decomposable. ∎

For $p \neq 2$, the semisimple Banach algebra $\mathcal{M}^p(\Gamma)$ is in many cases not regular. However, there always exists a unique maximal, regular, closed subalgebra $\mathcal{R}^p(\Gamma)$ of $\mathcal{M}^p(\Gamma)$ which contains all regular, closed subalgebras of $\mathcal{M}^p(\Gamma)$; see [1], Corollary 2.4. The following fact is a direct consequence of Proposition 1.4.

LEMMA 2.4. *For all $a \in \mathrm{M}_m(\mathcal{R}^p(\Gamma))$, the operator S_a^p is decomposable.*

COROLLARY 2.5. *If $a \in \mathrm{M}_m(\mathcal{U}^p(\Gamma))$ and $(\lambda - a)^{-1} \in \mathrm{M}_m(\mathcal{R}^p(\Gamma))$, for some $\lambda \in \mathbf{C} \setminus \Sigma(a)$, then S_a^p is decomposable.*

Proof. From $(\lambda - a)^{-1} \in \mathrm{M}_m(\mathcal{R}^p(\Gamma))$ we conclude $\lambda \in \mathbf{C} \setminus \sigma(S_a^p)$, that $(\lambda - S_a^p)^{-1} = S^p_{(\lambda - a)^{-1}}$ and that $(\lambda - S_a^p)^{-1}$ is decomposable (by Lemma 2.4). Hence, by [4], Lemma 2.4, S_a^p must be decomposable. ∎

In view of these two facts the following lemma is of some interest.

LEMMA 2.6. *Let $\mathcal{A}$ be a regular Banach algebra and let $\Phi : \mathcal{A} \to \mathcal{M}^p(\Gamma)$ be an algebra homomorphism. Then $\overline{\Phi(\mathcal{A})} \subseteq \mathcal{R}^p(\Gamma)$.*

Proof. It follows from [1], Theorem 1.4 that the operators $L_{\Phi(a)}$ of left multiplication with elements $\Phi(a)$, $a \in \mathcal{A}$, are decomposable as operators in $\mathcal{L}(\mathcal{B})$, where $\mathcal{B} = \overline{\Phi(\mathcal{A})}$. By [6], Theorem 3.6, L_b will be decomposable in $\mathcal{L}(\mathcal{B})$ for every $b \in \mathcal{B}$. Hence, by [13], Theorem 2, $\mathcal{B}$ is a regular Banach algebra and thus contained in $\mathcal{R}^p(\Gamma)$. ∎

Thus, for instance, $BV(\mathbf{R}) \subset \mathcal{R}^p(\mathbf{R})$ and $\mathcal{F}(L^1(G)) \subset \mathcal{R}^p(\Gamma)$ and $\mathcal{N}^k(\mathbf{R}^N) \subset \mathcal{R}^p(\mathbf{R}^N)$ where $\mathcal{N}^k(\mathbf{R}^N)$ is the Mihlin algebra considered in [4], Lemma 1.3.

3. NON-SPECTRALITY OF SYSTEMS OF DIFFERENTIAL OPERATORS

Let $Q = [Q_{jk}]_{j,k=1}^m$ be a matrix whose entries $Q_{jk} : \mathbf{R}^N \to \mathbf{C}$ are polynomials; the algebra of all such polynomials is denoted by $\mathbf{C}[x_1, \ldots, x_N]$. The natural domain of the differential operator $Q(D)$ (where $D = (\frac{1}{i}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_N})$) is

$$\mathcal{D}_Q^p = \{f \in L^p(\mathbf{R}^N)^m; \quad Q(D)f \in L^p(\mathbf{R}^N)^m\},$$

where $Q(D)f$ is formed in the sense of distributions. Write $Q_p(D)$ for the operator with domain $\mathcal{D}_Q^p$ given by $Q_p(D)f = Q(D)f$, for $f \in \mathcal{D}_Q^p$. Then $Q_p(D)$ is a densely defined closed linear operator. A standard argument (using regularization) shows that $Q_p(D)$ coincides with the operator S_Q^p defined in the introduction.

The question of when $Q_p(D)$ is spectral was extensively investigated in [11], Chapter XV, for the situation when $p = 2$ and $m \geqslant 1$. For the case of $m = 1$ and $p \neq 2$ it was shown in [3] that $Q_p(D)$ is spectral if and only if Q is constant. In the case $m \geqslant 2$ a similar statement would be expected. In this section we show that this is indeed the case. The proof is however more involved and we do not give a generalization to more general matrix multiplier operators of the form S_φ^p as was done for the case of $m = 1$ ([3], Proposition 2.4).

We begin with localized versions of Propositions 2.1 and 2.2 in [3]. Let Ω be a bounded open set in $\mathbb{R}^N$ and $p \in [1, 2]$. Denote by $X^p(\Omega)$ the closure, in $L^p(\mathbb{R}^N)$, of $\{f \in L^p(\mathbb{R}^N); \operatorname{supp}(\widehat{f}) \subset \Omega\}$. An essentially bounded function $\varphi : \Omega \to \mathbb{C}$ is said to be a *local p-multiplier* for Ω if φ agrees with the restriction to Ω of some p-multiplier φ_0 on $\mathbb{R}^N$. Since $X^p(\Omega)$ is invariant for $S_{\varphi_0}^p$ we may consider the restriction, R_φ^p, of $S_{\varphi_0}^p$ to $X^p(\Omega)$. This definition of R_φ^p is independent of the p-multiplier φ_0 and satisfies $R_\varphi^p f = \mathcal{F}^{-1}(\varphi \widehat{f})$, for $f \in X^p(\Omega)$. If q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then Young's inequality implies the map $B : X^p(\Omega) \to L^q(\Omega)$ given by $Bf = \widehat{f}|\Omega$, for $f \in X^p(\Omega)$, is bounded and injective. It satisfies

$$BR_\varphi^p = M_\varphi^q B, \tag{3.1}$$

where M_φ^q is the operator in $L^q(\Omega)$ of multiplication by φ. Moreover, M_φ^q is a scalar-type spectral operator with the resolution of the identity given by

$$E_\varphi(\delta) = M_{\chi_\delta \circ \varphi}, \quad \delta \in \mathcal{B}(\mathbb{C}); \tag{3.2}$$

here $\mathcal{B}(\mathbb{C})$ denotes the σ-algebra of all Borel subsets of $\mathbb{C}$. The following statement is an immediate consequence of (3.1) and (3.2) above, of Theorem 1.2 and Corollary 1.5 in [3] and of Theorem XVII.2.10 in [11].

LEMMA 3.1. *Let $1 \leqslant p \leqslant 2$ and Ω be a bounded open set in $\mathbb{R}^N$. Let φ be a local p-multiplier for Ω such that R_φ^p is a spectral operator in $X^p(\Omega)$. Then R_φ^p is of scalar-type and its resolution of the identity F_φ is given by*

$$F_\varphi(\delta)f = \mathcal{F}^{-1}\widehat{f} \cdot (\varphi \circ \chi_\delta), \quad f \in X^p(\Omega), \delta \in \mathcal{B}(\mathbb{C}). \tag{3.3}$$

Moreover, for every bounded Borel function ψ on $\mathbb{C}$, we have

$$\int_{\mathbb{C}} \psi(z)\, \mathrm{d}F_\varphi(z) f = \mathcal{F}^{-1}((\psi \circ \varphi)\widehat{f}) = R_{\psi\circ\varphi}^p f, \tag{3.4}$$

for $f \in X^p(\Omega)$, and there is a constant $K_p > 0$ (independent of ψ) such that

$$\|R_{\psi\circ\varphi}^p\|_{\mathcal{L}(X^p(\Omega))} \leqslant K_p \|\psi\|_{L^\infty(\mathbb{R}^N)}. \tag{3.5}$$

In addition, $\sigma(R_{\psi\circ\varphi}^p) = \text{ess-range}(\psi\circ\varphi)$ with respect to the Lebesgue measure on Ω.

The following result follows from (3.3) and the inclusion $\mathcal{F}(X^1(\Omega)) \subset C_0(\mathbb{R}^N)$.

COROLLARY 3.2. *Let φ be a non-constant local 1-multiplier on a bounded open set $\Omega \subset \mathbf{R}^N$. Then R^1_φ is not a spectral operator.*

PROPOSITION 3.3. *Let Ω be a bounded open set in $\mathbf{R}^N$. Let $1 \leqslant p < 2$ and assume that $\varphi \in C^2(\Omega)$ is a non-constant local p-multiplier on Ω. Then R^p_φ is not a spectral operator on $X^p(\Omega)$.*

Proof. For $p = 1$ this is clear. So, let $1 < p < 2$ and assume that R^p_φ is spectral. Define $\psi_1(z) = \mathrm{Re}(z)$, $\psi_2(z) = \mathrm{Im}(z)$, and $\psi_3(z) = |z|^2$. Then, for each $n \in \mathbf{N}$, the functions $\eta_{j,n} = \exp(in\psi_j)$, for $j = 1, 2, 3$, are local p-multipliers of class $C^2(\Omega)$ satisfying $|\eta_{j,n}| \equiv 1$ on Ω. Moreover, by (3.4) and (3.5), it follows that $\|R^p_{\eta_{j,n}}\|_{\mathcal{L}(X^p(\Omega))} \leqslant K_p$ for all $n \in \mathbf{N}$. Fix any open disc B with $\overline{B} \subset \Omega$ such that φ is non-constant on every (non-empty) open subset of B. By [9], Lemma 5 there exist constants c_j with $|c_j| = 1$ and vectors $x_j \in \mathbf{R}^N$ such that $\eta_{j,n}(\varphi(y)) = \exp(in\psi_j(\varphi(y)) = c_j \exp(i\langle x_j, y\rangle)$, $y \in B$, for each $j = 1, 2, 3$. Hence, on some non-empty open disc $D \subset B$ the functions $\psi_j \circ \varphi$, $1 \leqslant j \leqslant 3$, must be affine linear. This contradicts φ being non-constant on D. ∎

We now come to the main result of the section.

PROPOSITION 3.4. *Let $1 \leqslant p < \infty$ with $p \neq 2$, and let $Q \in \mathrm{M}_m(\mathbf{C}[x_1, \ldots, x_N])$ be a matrix polynomial. Then the operator $Q_p(D)$ is spectral in $L^p(\mathbf{R}^N)^m$ if and only if Q is constant.*

Proof. Since the dual operator to $Q_p(D)$ is the operator $\widetilde{Q}_q(D)$ in $L^q(\mathbf{R}^N)^m$ with $\frac{1}{p} + \frac{1}{q} = 1$, it suffices to prove the statement for $1 \leqslant p < 2$. Assume now that $Q_p(D)$ is spectral in $L^p(\mathbf{R}^N)^m$ $(1 \leqslant p < 2)$ with spectral measure F. Since $Q_p(D)$ is decomposable we must have, by Proposition 2.3, that

$$\sigma(Q_p(D)) = \overline{\bigcup_{x \in \mathbf{R}^N} \sigma(Q(x))}, \tag{3.6}$$

where the closure is taken in $\widehat{\mathbf{C}}$.

As $\mathbf{C}[\lambda, x_1, \ldots, x_N]$ is a unique factorization domain ([28], Section 23) the characteristic polynomial $q(\lambda, x) = \det(\lambda - Q(x))$ of Q has a unique factorization of the form $q(\lambda, x) = \pi_1(\lambda, x)^{n_1} \cdots \pi_r(\lambda, x)^{n_r}$, where $\pi_1, \ldots, \pi_r$ are irreducible and have leading coefficient 1 with respect to λ. Define $q_0(\lambda, x) = \pi_1(\lambda, x) \cdots \pi_r(\lambda, x)$ and denote by $R(x)$ the *resultant* (cf. [28], Section 27) of q_0 and $\frac{\partial q_0}{\partial \lambda}$ at x, where q_0 and $\frac{\partial q_0}{\partial \lambda}$ are considered as polynomials with respect to λ and coefficients coming from $\mathbf{C}[x_1, \ldots, x_N]$. Since q_0 and $\frac{\partial q_0}{\partial \lambda}$ are relatively prime, R is not identically 0 on $\mathbf{R}^N$, cf. [28], Section 27. Here we used the fact that the unique prime factorization of q_0 in $\mathbf{C}[\lambda, x_1, \ldots, x_N]$ and in $\mathbf{K}[\lambda]$ yield the same result, where $\mathbf{K}$ denotes the quotient field of $\mathbb{C}[x_1, \ldots, x_N]$. This follows from [7], p. 377. Fix an arbitrary point $x_0 \in \mathbf{R}^N$ with $R(x_0) \neq 0$. Since R is a polynomial in $x_1, \ldots, x_N$, it follows that $R(x) \neq 0$ for all x in some neighbourhood U of x_0. Again from Section 27 in [28] now applied, for fixed x, to $q_0(\cdot, x)$ and $\frac{\partial q_0}{\partial \lambda}(\cdot, x)$, we see that

$$q_0(\lambda, x) = \prod_{j=1}^{r} \prod_{k=1}^{v_j} (\lambda - \lambda_{kj}(x)), \quad x \in U, \tag{3.7}$$

with $\lambda_{kj}(x)$, for $j = 1, \ldots, r$ and $k = 1, \ldots, v_j$ being pairwise distinct. It follows that

$$\frac{\partial q_0}{\partial \lambda}(\lambda_{kj}(x), x) \neq 0, \text{ for } j \in \{1, \ldots, r\}, \ k \in \{1, \ldots, v_j\}, \ x \in U.$$

By the implicit function theorem there exists a bounded neighbourhood V, of x_0, contained in U such that the functions $\lambda_{kj} : V \to \mathbb{C}$ satisfying (3.7) on V are real analytic (cf. also Supplement S.3.1 in [7]). We may, in addition, assume that the sets $K_{kj} = \overline{\lambda_{kj}(V)}$ are mutually disjoint, compact subsets of $\mathbb{C}$. Let Γ_{kj} be rectifiable systems of curves in $\mathbb{C}$ surrounding K_{kj} and having K_{st} in the exterior of Γ_{kj} for $(s, t) \neq (k, j)$. Then, for $x \in V$, the scalar part of the matrix $Q(x)$ is given by

$$A(x) = \sum_{j=1}^{r} \sum_{k=1}^{v_j} \lambda_{kj}(x) E_{kj}(x),$$

where the idempotents

$$E_{kj}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{kj}} (z - Q(x))^{-1} \, \mathrm{d}z$$

obviously have real analytic entries defined on V.

Let T be the restriction of $Q_p(D)$ to $X^p(V)^m$. Since $Q_p(D)$ intertwines with translations the same is true for the values of the resolution of the identity F of $Q_p(D)$; see [11], Corollary XVIII.2.4. Thus the values of F are matrix-p-multiplier operators and $X^p(V)^m$ is invariant for F. We conclude that T is spectral and the restriction F_0 of F to $X^p(V)^m$ is the spectral measure for T. Notice that, for every $\delta \in B(\mathbb{C})$, the entries of the matrix symbols $\widehat{F}(\delta)$ corresponding to $F(\delta)$ are essentially bounded. It follows that the operators M_Q^q and $M_{\widehat{F}(\delta)}^q$ of multiplication with Q and $\widehat{F}(\delta)$, respectively, define bounded operators on $L^q(V)^m$. Thus, M_Q^q is spectral on $L^q(V)^m$ with spectral measure $M_{\widehat{F}(\cdot)}^q$. By Proposition 1.13, the scalar part A_q of M_Q^q is the operator M_A^q of multiplication with the scalar part $A(x)$ of

$Q(x)$ on $L^q(V)^m$. Accordingly, by [3], Theorem 1.2, the scalar part S of T is given by $Sf = \mathcal{F}^{-1}(M_A^q \widehat{f})$, for $f \in X^p(V)^m$. Now $A(x)$ is similar to the diagonal matrix

$$D(x) = \operatorname{diag}\{\underbrace{\lambda_{11}(x), \ldots, \lambda_{11}(x)}_{n_1}, \ldots, \underbrace{\lambda_{v_1 1}, \ldots, \lambda_{v_1 1}}_{n_1}, \ldots$$

$$\ldots, \underbrace{\lambda_{1r}, \ldots, \lambda_{1r}}_{n_r}, \ldots, \underbrace{\lambda_{v_r r}, \ldots, \lambda_{v_r r}}_{n_r}\}$$

and so $A(x) = C(x)^{-1} D(x) C(x)$, where $C(x)$ can be chosen to have real analytic entries which are bounded in a neighbourhood of $\overline{V}$ (using the fact that the eigenprojections E_{kj} depend analytically on x and, if necessary, shrinking V). Since the operator $f \mapsto Sf = \mathcal{F}^{-1}(M_A^q \widehat{f})$ is a scalar-type spectral operator on $X^p(V)^m$ it follows that $f \mapsto \mathcal{F}^{-1}(D(\cdot)\widehat{f})$ must have the same property. From this we conclude that the operator $f \mapsto \mathcal{F}^{-1}(\lambda_{kj}(\cdot)\widehat{f})$ is a scalar-type spectral operator on $X^p(V)$. Being real analytic in a neighbourhood of $\overline{V}$, the functions $\lambda_{kj}(\cdot)$ are local p-multipliers on V. By Proposition 3.3, we see that $\lambda_{kj}(\cdot)$ is constant on V for all k, j. This shows that $q(\lambda, x) = \det(\lambda - Q(x)) = q(\lambda)$ is independent of x. Hence, by (3.6), $\sigma(Q_p(D)) = \{\lambda \in \mathbf{C};\ q(\lambda) = 0\}$ is a finite subset of $\mathbf{C}$. In particular, $Q_p(D)$ must be bounded. This is only possible if all the entries of Q are bounded on $\mathbf{R}^N$ which implies that Q is constant. ∎

4. A DECOMPOSABILITY CRITERION FOR CERTAIN SYSTEMS OF DIFFERENTIAL OPERATORS

Before giving the criterion we first recall some facts from the theory of decomposable operators. Let X be a Banach space and $T \in \mathcal{C}(X)$. If Ω is an open subset of $\widehat{\mathbf{C}}$ we write $\mathcal{O}(\Omega; X)$ for the Fréchet space of all analytic X-valued functions on Ω. The operator T is said to have *Bishop's property* (β) if, for every sequence $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{O}(\Omega, X)$ with $f_n(\Omega) \subset D(T)$ and $z \mapsto Tf_n(z) \in \mathcal{O}(\Omega, X)$, for all $n \in \mathbf{N}$, and satisfying $(z - T)f_n(z) \to 0$ uniformly on compact subsets of Ω as $n \to \infty$, also $f_n(z)$ tends to zero uniformly on compact subsets of Ω as $n \to \infty$, [8]. If T is decomposable, then T necessarily has property (β); see [26], Lemma IV.4.16. If T has property (β), then for all closed sets $F \subset \widehat{\mathbf{C}}$, the linear subspace $X_T(F)$ is closed and invariant for T (cf. the proof of [26], Corollary IV.4.18) and satisfies $\sigma(T|X_T(F)) \subset F$ (by [26], Proposition IV.3.8). Hence, T is decomposable if and only if T has property (β) and the following *property* (δ): for each (finite) open cover $\{U_1, \ldots, U_n\}$ of $\widehat{\mathbf{C}}$ there exist closed subsets $F_j \subset U_j$ $(1 \leqslant j \leqslant n)$ such that $X = X_T(F_1) + \cdots + X_T(F_n)$.

PROPOSITION 4.1. *Let $m \geqslant 1$ be an integer. Let $Q = [Q_{jk}]_{j,k=1}^m$ be a matrix of polynomials in N real variables such that, for some $\lambda \in \mathbb{C}$,*

(i) *the characteristic polynomial $x \mapsto q(\lambda, x) = \det(\lambda - Q(x))$ of Q is elliptic in $\mathbb{R}^N$ with respect to x, and*

(ii) *the degree (with respect to x) of each minor of $\lambda - Q(x)$ does not exceed $\deg(q(\lambda, \cdot))$.*

Then the differential operator $Q_p(D)$ is decomposable on $L^p(\mathbb{R}^N)^m$, for all $p \in (1, \infty)$.

Proof. Fix $\lambda \in \mathbb{C}$ satisfying (i) and (ii). Because of the ellipticity condition (i), the set $N(q, \lambda) = \{x \in \mathbb{R}^N;\ q(\lambda, x) = 0\}$ must be compact. Hence, for some $r > 0$, $N(q, \lambda) \subset U_r(0) = \{x \in \mathbb{R}^N;\ |x| < r\}$. Fix $\varphi \in C^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \subset U_{r+1}(0)$ and $\overline{U_r(0)} \cap \operatorname{supp}(1 - \varphi) = \emptyset$. If F is any closed subset of $\mathbb{R}^N$, we write $\mathcal{E}^p(F)$ for the closed subspace $\{f \in L^p(\mathbb{R}^N)^m;\ \operatorname{supp}(\widehat{f}) \subseteq F\}$ of $L^p(\mathbb{R}^N)^m$. Obviously, $\mathcal{E}^p(F)$ is invariant for $Q_p(D)$. Notice that $\operatorname{ran}(S_\varphi^p) \subset \mathcal{E}^p(\operatorname{supp}(\varphi)) \subset D(Q_p(D))$ and $\operatorname{ran}(S_{1-\varphi}^p) \subset \mathcal{E}^p(\mathbb{R}^N \setminus U_r(0))$. Hence, $L^p(\mathbb{R}^N)^m = \mathcal{E}^p(\operatorname{supp}(\varphi)) + \mathcal{E}(\mathbb{R}^N \setminus U_r(0))$. In the first two steps we prove that the restricted operators $Q_p(D)|\mathcal{E}(\operatorname{supp}(\varphi))$ and $Q_p(D)|\mathcal{E}(\mathbb{R}^N \setminus U_r(0))$ are decomposable. In a third step we will then derive the decomposability of $Q_p(D)$ from these facts.

(a) To verify decomposability of $Q_p(D)|\mathcal{E}^p(\operatorname{supp}(\varphi))$, fix $k > N/2$. The unital Banach algebra $C^k(\overline{U_{r+1}(0)})$ consisting of all k-times continuously differentiable functions on $U_{r+1}(0)$ whose partial derivatives of order not exceeding k extend continuously to $\overline{U_{r+1}(0)}$, is obviously semisimple and regular. For $h \in \mathrm{M}_m(C^k(\overline{U_{r+1}(0)}))$ we define $\Psi(h) \in \mathcal{L}(\mathcal{E}^p(\operatorname{supp}(\varphi)))$ as follows. Fix $\psi \in C^k(\overline{U_{r+1}(0)})$ with $\operatorname{supp}(\psi) \subset U_{r+1}(0)$ and $\operatorname{supp}(\varphi) \cap \operatorname{supp}(1 - \psi) = \emptyset$. The operator $\Psi(h) = S_{\psi h}^p|\mathcal{E}^p(\operatorname{supp}(\varphi))$ does not depend on the special choice of ψ with these properties. Continuity of $\Psi(h)$ is easily obtained from the Mihlin multiplier theorem. The so defined $\Psi : C^k(\overline{U_{r+1}(0)}) \to \mathcal{E}^p(\operatorname{supp}(\varphi))$ is a continuous unital homomorphism. By Proposition 1.4, every operator in $\operatorname{ran}(\Psi)$ and hence, in particular $\Psi(Q) = Q_p(D)|\mathcal{E}^p(\operatorname{supp}(\varphi))$, is decomposable.

(b) To establish decomposability of $Q_p(D)|\mathcal{E}^p(\mathbb{R}^N \setminus U_r(0))$, note first that the function $x \mapsto q(\lambda, x)$ has no zeros in $\mathbb{R}^N \setminus U_r(0)$ and hence, $(\lambda - Q(x))^{-1}$ exists in $\mathbb{R}^N \setminus N(q, \lambda)$ which is an open neighbourhood of $\mathbb{R}^N \setminus U_r(0)$. By Cramer's rule and (ii) the entries of this matrix function are of the form $\frac{h}{q(\lambda, \cdot)}$ where h is a polynomial of degree at most $\deg(q(\lambda, \cdot))$. Direct computation shows that, for any function $\rho \in C^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(1 - \rho) \subset U_r(0)$ and $\operatorname{supp}(\rho) \cap N(q, \lambda) = \emptyset$, the function $\frac{\rho h}{q(\lambda, \cdot)}$ belongs to the Mihlin algebra $\mathcal{N}^k(\mathbb{R}^N)$. Hence, by the Mihlin multiplier theorem, $A = S_{\rho(\lambda - Q)^{-1}}^p$ belongs to $\mathcal{L}(L^p(\mathbb{R}^N)^m)$. Moreover, $\mathcal{E}^p(\mathbb{R}^N \setminus U_r(0))$ is invariant for A and we have the identities $(\lambda - Q(D))A|\mathcal{E}^p(\mathbb{R}^N \setminus U_r(0)) = 1|\mathcal{E}^p(\mathbb{R}^N \setminus U_r(0))$, and

$A(\lambda - Q_p(D))|D(Q_p(D)) \cap \mathcal{E}^p(\mathbf{R}^N \setminus U_r(0)) = 1|D(Q_p(D)) \cap \mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$. This shows that

$$(4.1) \qquad A|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0)) = (\lambda - Q_p(D))|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0)).$$

The mapping $a \mapsto S_a^p|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$ defines a homomorphism Θ from $\mathrm{M}_m(\mathcal{N}^k(\mathbf{R}^N))$ to $\mathcal{L}(\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0)))$. Since $A|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0)) \in \mathrm{ran}(\Theta)$ we see as in (a) that $A|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$ is decomposable. By (4.1) and [4], Lemma 2.4, the decomposability of $Q_p(D)|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$ follows.

(c) To verify the decomposability of $Q_p(D)$ it suffices, as mentioned before, to establish properties (β) and (δ) for $Q_p(D)$. To prove (β) let $\Omega \subset \widehat{\mathbf{C}}$ be open and $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathcal{O}(\Omega, L^p(\mathbf{R}^N)^m)$ such that $z \mapsto Q_p(D)f_n(z) \in \mathcal{O}(\Omega, L^p(\mathbf{R}^N)^m)$ and $f_n(\Omega) \subset D(Q_p(D))$, for all $n \in \mathbf{N}$, and $(z - Q_p(D))f_n(z) \to 0$ uniformly on compact subsets of Ω. By continuity of S_φ^p and $S_{1-\varphi}^p$ it follows that the two sequences of functions

$$z \mapsto S_\varphi^p(z - Q_p(D))f_n(z) = (z - Q_p(D))S_\varphi^p f_n(z)$$

and

$$z \mapsto S_{1-\varphi}^p(z - Q_p(D))f_n(z) = (z - Q_p(D))S_{1-\varphi}^p f_n(z),$$

tend uniformly to 0 on compact subsets of Ω, as $n \to \infty$. Since $Q_p(D)|\mathcal{E}^p(\mathrm{supp}(\varphi))$ and $Q_p(D)|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$ are decomposable they have property (β). Accordingly, $S_\varphi^p f_n(z) \to 0$ and $S_{1-\varphi}^p f_n(z) \to 0$ and hence, also $f_n(z) \to 0$ uniformly on compact subsets of Ω, as $n \to \infty$. This shows that $Q_p(D)$ has property (β). Since both $Q_p(D)|\mathcal{E}^p(\mathrm{supp}(\varphi))$ and $Q_p(D)|\mathcal{E}^p(\mathbf{R}^N \setminus U_r(0))$ have property (δ) we conclude that $Q_p(D)$ must have property (δ) and the proof is complete. ∎

REMARK. In the special case that there exists some $\lambda \in \mathbf{C} \setminus \Sigma(Q)$ satisfying (i) and (ii), the proof is somewhat easier. Indeed, direct computation shows, with $k > N/2$, that $(\lambda - Q)^{-1} \in \mathrm{M}_m(\mathcal{N}^k(\mathbf{R}^N))$. Hence, $(\lambda - Q_p(D))^{-1} = S_{(\lambda - Q)^{-1}}^p$ exists and is decomposable by the remarks at the end of Section 2. However, for the general case the partition of unity argument in the preceding proof is needed, since it may happen that $\Sigma(Q) = \mathbf{C}$. For example, with $Q(x,y) = \begin{bmatrix} 0 & x + iy \\ 1 & 0 \end{bmatrix}$ we have $q(\lambda, (x,y)) = \lambda^2 - (x + iy)$ which shows that $\Sigma(Q) = \mathbf{C}$. Nevertheless, conditions (i) and (ii) are fulfilled for $\lambda = 0$ and $Q_p(D)$ is therefore decomposable by Proposition 4.1.

The next fact shows that the degree condition in Proposition 4.1 is quite natural.

LEMMA 4.2. *Let $Q \in M_m(\mathbb{C}[x_1, \ldots, x_N])$, $1 \leqslant p < \infty$ and assume there exists $\lambda \in \mathbb{C} \setminus \sigma(Q_p(D))$. Then the degree condition of Proposition* 4.1 (ii) *must be satisfied at* λ.

Proof. This follows by means of Cramer's rule after noting that the entries of $(\lambda - Q(x))^{-1}$ are bounded (by Lemma 2.2). ∎

EXAMPLES 4.3. We consider the following examples, to be found in [14] and [17], [18], [19], where they are studied from the point of view of semigroup theory:

$$A(x_1, x_2, x_3) = \mathrm{i} \begin{bmatrix} 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_3 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 & x_1 & 0 \\ 0 & x_3 & -x_2 & 0 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 & 0 & 0 \\ x_2 & -x_1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B(x_1, x_2) = \mathrm{i} \begin{bmatrix} 0 & 0 & 0 & (a+b)^{\frac{1}{2}} x_1 & (a+b)^{\frac{1}{2}} x_2 \\ 0 & 0 & 0 & a^{\frac{1}{2}} x_1 & -a^{\frac{1}{2}} x_2 \\ 0 & 0 & 0 & a^{\frac{1}{2}} x_2 & a^{\frac{1}{2}} x_1 \\ (a+b)^{\frac{1}{2}} x_1 & a^{\frac{1}{2}} x_1 & a^{\frac{1}{2}} x_2 & 0 & 0 \\ (a+b)^{\frac{1}{2}} x_2 & -a^{\frac{1}{2}} x_2 & a^{\frac{1}{2}} x_1 & 0 & 0 \end{bmatrix}$$

$$C(x_1, x_2, x_3) = -\mathrm{i} \begin{bmatrix} x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & -x_3 \end{bmatrix}$$

$$E(x_1) = -c\mathrm{i}x_1^3 + \mathrm{i}dx_1.$$

The operator $A(D)$ is related to Maxwell's equation, for $a, b > 0$ the operator $B(D)$ is related to elastic waves in a 2-dimensional homogeneous medium, $C(D)$ is related to the Neutrino equation, and $E(D)$ is related to the linearized Korteweg-de Vries equation, where $c, d \in \mathbb{R} \setminus \{0\}$. The characteristic polynomials are given by

$$\begin{aligned} q_A(\lambda, x) &= \lambda^2(\lambda^2 + |x|^2)^2, \\ q_B(\lambda, x) &= \lambda(\lambda^2 + a|x|^2)(\lambda^2 + (2a+b)|x|^2), \\ q_C(\lambda, x) &= \lambda^2 + |x|^2, \text{ and} \\ q_E(\lambda, x) &= \lambda + \mathrm{i}(cx^3 - dx), \end{aligned}$$

and hence, are elliptic in x for $\lambda \neq 0$. Direct computation shows that the degree condition is satisfied for $\lambda \neq 0$. Hence, Proposition 4.1 tells us that $A_p(D)$, $B_p(D)$, $C_p(D)$ and $E_p(D)$ are decomposable in $L^p(\mathbb{R}^3)^6$, resp. $L^p(\mathbb{R}^2)^5$, resp. $L^p(\mathbb{R}^3)^2$, resp. $L^p(\mathbb{R})$, for every $p \in (1, \infty)$. The case of $E_p(D)$ also follows from [3]. ∎

In the case $N = 1$ we can give a complete description of all decomposable matrix differential operators in $L^p(\mathbf{R})^m$. For this we shall need the following fact.

LEMMA 4.4. *Let $q \in \mathbf{C}[\lambda, x]$ be a polynomial in two variables of the form*

$$q(\lambda, x) = \lambda^d + \sum_{j=0}^{d-1} p_j(x)\lambda^j,$$

where $p_0, \ldots, p_{d-1} \in \mathbf{C}[x]$. Then $N(q) = \{\lambda \in \mathbf{C};\ q(\lambda, x) = 0$ for some $x \in \mathbf{R}\}$ has planar Lebesgue measure zero.

Proof. Note that q has a representation of the form $q = q_1^{k_1} \cdots q_r^{k_r}$ with irreducible polynomials $q_1, \ldots, q_r$ which are again monic in λ. Obviously $N(q) = \bigcup_{j=1}^{r} N(q_j)$. Hence, without loss of generality, we may assume that q is irreducible. Let $R(x)$ be the resultant of q and $\frac{\partial q}{\partial \lambda}$ and let $x_1, \ldots, x_s$ be the finitely many zeros of R (cf. the proof of Theorem 3.5). If A is any subset of $\mathbf{R}$ we define $\Lambda(A) = \{\lambda \in \mathbf{C};\ q(\lambda, x) = 0$ for some $x \in A\}$. With this notation we have $N(q) = \Lambda(\mathbf{R})$. Obviously, $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$ holds for all subsets A and B of $\mathbf{R}$. Since $\Lambda(\{x_1, \ldots, x_s\})$ is a finite set and $\mathbf{R} \setminus \{x_1, \ldots, x_s\}$ can be written as a countable union of compact sets it suffices to show, for each compact set $K \subset \mathbf{R} \setminus \{x_1, \ldots, x_s\}$, that the set $\Lambda(K)$ has planar Lebesgue measure zero. So, fix an arbitrary compact set $K \subset \mathbf{R} \setminus \{x_1, \ldots, x_s\}$. If $u \in K$, then as in the proof of Theorem 3.1 we can find an open neighbourhood $V_{(u)}$ of u and finitely many real analytic functions $\lambda_1(\cdot), \ldots, \lambda_t(\cdot)$ on $V(u)$ such that $q(\lambda, x) = \prod_{j=1}^{t} (\lambda - \lambda_j(x))$, for all $x \in V(u)$. It follows that $\lambda \in \Lambda(V(u))$ if and only if $\lambda = \lambda_j(x)$ for some $x \in V(u)$ and some $j \in \{1, \ldots, t\}$. Hence, $\Lambda(V(u)) = \bigcup_{j=1}^{t} \lambda_j(V(u))$ has planar Lebesgue measure zero by the "Mini-Sard" theorem; see [15], Appendix 1. Since K is compact there are finitely many $u_1, \ldots, u_n \in K$ such that $K \subset \bigcup_{j=1}^{n} V(u_j)$. It follows that $\Lambda(K) \subset \bigcup_{j=1}^{n} \Lambda(V(u_j))$ must have planar Lebesgue measure zero and the proof is complete. ∎

PROPOSITION 4.5. *Let $1 \leqslant p < \infty$. For a matrix polynomial $Q \in \mathrm{M}_m(\mathbf{C}[x])$ in one variable the following statements are equivalent:*

(i) *$Q_p(D)$ is decomposable in $L^p(\mathbf{R})^m$.*

(ii) $\sigma(Q_p(D)) \cap \mathbf{C} = \Sigma(Q)$.

(iii) $\sigma(Q_p(D)) \neq \widehat{\mathbf{C}}$.

(iv) *For all $\lambda \in \widehat{\mathbf{C}} \setminus \Sigma(Q)$ the degree (with respect to x) of each minor of $(\lambda - Q(x))$ does not exceed $\deg(q_Q(\lambda, \cdot))$, where $q_Q(\lambda, x) = \det(\lambda - Q(x))$.*

(v) *There exists some* $\lambda \in \widehat{\mathbb{C}} \setminus \Sigma(Q)$ *satisfying the degree condition in* (iv).

(vi) *For each minor* $A_{jk}(\lambda, x)$ *of* $(\lambda - Q(x))$, *the maximal degree* $d_x(A_{jk})$ *does not exceed* $d_x(q_Q)$, *where* $d_x(A_{jk}) = \max\{\deg(A_{jk}(\lambda, \cdot));\ \lambda \in \mathbb{C}\}$.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 2.3 and (ii) $\Rightarrow$ (iii) is a consequence of the preceding lemma (which shows that $\Sigma(Q)$ must have planar Lebesgue measure zero and hence cannot equal $\mathbb{C}$). The implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) follow from Lemma 4.2 (together with Lemma 2.2 in the second case). To prove (iv) $\Rightarrow$ (vi) notice that $\Sigma(Q) \neq \mathbb{C}$, by Lemma 4.4, and that the sets $U_{jk} = \{\lambda \in \mathbb{C};\ \deg(A_{jk}(\lambda, \cdot)) = d_x(A_{jk})\}$, for $j, k = 1, \ldots, m$, and $U = \{\lambda \in \mathbb{C};\ \deg(q_Q(\lambda, \cdot)) = d_x(q_Q)\}$ are dense in $\mathbb{C}$. Hence, there exists some $\lambda \in (\mathbb{C} \setminus \Sigma(Q)) \cap U \cap \bigcap_{j,k=1}^{m} U_{jk}$. Since the degree condition is satisfied at λ, we obtain (vi). If (vi) holds, then for every $\lambda \in (\mathbb{C} \setminus \Sigma(Q)) \cap U \cap \bigcap_{j,k=1}^{m} U_{jk}$, the degree condition is satisfied and we obtain (v). Finally, (v) $\Rightarrow$ (i) is true by Proposition 4.1 (after noting that the ellipticity condition in Proposition 4.1 (i) is always fulfilled for $N = 1$). ∎

EXAMPLE 4.6. Consider the matrix polynomial Q given by

$$Q(x) = \begin{bmatrix} -ib_1x^2 - 2b_2x - ib_1 & -ib_2x^2 + 2b_1x - ib_2 \\ -2b_4x & -ib_4x^2 - ib_4 \end{bmatrix},$$

where $b_1, b_2, b_4 \in \mathbb{R} \setminus \{0\}$ with $b_1 \neq b_4$. The characteristic polynomial of Q has degree 4 in x, whereas the minors of $\lambda - Q$ have degree $\leqslant 2$ in x. Hence, by Proposition 4.5 the operator $Q_p(D)$ is decomposable for each $p \in [1, \infty)$. It has been shown in [17], pp. 12–13 that $Q_p(D)$ does not generate a strongly continuous or even an (exponentially bounded) integrated semigroup. We shall see in the following section that $Q_p(D)$ admits a rich functional calculus.

For $N = 1$ Proposition 4.5 shows that the equality $\sigma(Q_p(D)) \cap \mathbb{C} = \Sigma(Q)$ is necessary and sufficient for $Q_p(D)$ to be decomposable. In the case of several variables ($N \geqslant 2$) this is no longer true. As an example, consider the 2-variable matrix polynomial Q given by $Q(x, y) = \begin{bmatrix} Q_1(x, y) & 0 \\ 0 & Q_2(x, y) \end{bmatrix}$, where $Q_1(x, y) = x + iy$ and $Q_2(x, y) = x - y^2$. Suppose $p \neq 2$. Obviously $\Sigma(Q) = \mathbb{C}$ and hence $\sigma(Q_p(D)) \cap \mathbb{C} = \Sigma(Q)$. Since $Q_p(D) = Q_{1,p}(D) \oplus Q_{2,p}(D)$ and $Q_{2,p}(D)$ is not decomposable ([3], Corollary 3.5), it follows that $Q_p(D)$ cannot be decomposable.

5. A FUNCTIONAL CALCULUS FOR CERTAIN SYSTEMS OF MATRIX O.D.E.'S

In this section we construct a functional calculus for ordinary matrix differential operators (with constant coefficients) in $L^p(\mathbf{R})^m$. To a large extent we follow the ideas of C. Apostol ([5]), where a functional calculus was constructed for operator matrices of the form $[f_{jk}(T)]_{j,k=1}^m$ with T being a bounded generalized scalar operator and the functions f_{jk} are analytic in a neighbourhood of $\sigma(T)$.

Every square matrix $A \in \mathrm{M}_m(\mathbf{C})$ has a unique representation of the form $A = D+N$, where D is similar to a normal matrix and $N^m = 0$. So, we have a natural $C^{(m-1)}(\mathbf{C})$-functional calculus $\varphi \mapsto \varphi(A)$, extending the analytic functional calculus, given by

$$\varphi \mapsto \varphi(A) = \sum_{j=1}^{m-1} \frac{\partial^j \varphi}{\partial \overline{z}^j}(D)\,\frac{1}{j!}N^j. \tag{5.1}$$

Hence, given a matrix polynomial $Q \in \mathrm{M}_m(\mathbf{C}[x])$ it is natural to attempt to formulate the functional calculus

$$\varphi \mapsto \Phi(\varphi) = S^p_{\varphi(Q)}, \tag{5.2}$$

where $\varphi(Q)(x) = \varphi(Q(x))$, for $x \in \mathbf{R}$, is defined via (5.1). The class $\mathcal{A}_Q$ of admissible functions φ in (5.2) has to be prescribed in such a way that the matrix function $x \mapsto \varphi(Q(x))$ is an element of $\mathrm{M}_m(\mathcal{M}^p(\mathbf{R}))$. In order to do this it turns out that the critical points of Q cause some difficulties; in particular, $C^\infty(\widehat{\mathbf{C}})$ may not be admissible for (5.2). For example, if

$$Q(x) = \begin{bmatrix} x & 1 & 2x \\ 1 & ix & 1 \\ 0 & -1 & -x \end{bmatrix}, \tag{5.3}$$

and $\varphi \in C^\infty(\widehat{\mathbf{C}})$ is any function satisfying $\varphi(z) = \overline{z}$ in a neighbourhood of 0, then it is shown in [5] that the matrix function $x \mapsto \varphi(Q(x))$ is not bounded near $x = 0$ and hence, does not even define a local matrix p-multiplier function. For this example,

$$q_Q(\lambda, x) = \det(\lambda - Q(x)) = (\lambda^2 - x^2)(\lambda - ix)$$

and we see that Q satisfies condition (vi) in Proposition 4.5. Therefore, $Q_p(D)$ is decomposable for all $p \in [1,\infty)$ and $\sigma(Q_p(D)) \cap \mathbf{C} = \Sigma(Q) = \mathbf{R} \cup i\mathbf{R}$. The difficulties illustrated by this example can be overcome if we require (complex) analyticity for φ near the eigenvalues corresponding to the critical points of Q.

For a given matrix polynomial $Q \in M_m(\mathbb{C}[x])$ we write $q_0 = q_1 \cdots q_r$ where $q_1, \ldots, q_r$ are irreducible and $q_Q(\lambda, x) = \det(\lambda - Q(x)) = \prod_{j=1}^{r} q_j(\lambda, x)^{k_j}$, with $k_1, \ldots, k_r \in \mathbb{N}$. The set Δ_Q of *critical points* of Q is then the set of zeros of the resultant R of q_0 and $\frac{\partial q_0}{\partial \lambda}$. We also have to take care of the behaviour of Q near ∞. With the notation as in the proof of Lemma 4.4 we define $\Lambda_Q = \Lambda(\Delta_Q) \cup \Lambda(\{\infty\}) \cup \{\infty\}$, where $\Lambda(\{\infty\})$ is the set of zeros of the polynomial $q_\infty(\lambda) = \lim_{x \to \infty} x^{-\alpha} q_Q(\lambda, x)$, $\lambda \in \mathbb{C}$, and $\alpha = d_x(q_Q)$ is the maximal degree with respect to x of q_Q. So, $\Lambda(\{\infty\})$ is a finite set, possibly empty. Accordingly, Λ_Q is a finite subset of $\widehat{\mathbb{C}}$. We now define the algebra

$$\mathcal{A}_Q = \{\varphi \in C^m(\widehat{\mathbb{C}});\ \varphi \text{ analytic in a neighbourhood } U_\varphi \text{ of } \Lambda_Q\};$$

see also [5], p. 1513. Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact subsets in $\widehat{\mathbb{C}}$ satisfying

$$\operatorname{int}(K_n) \supset K_{n+1} \supset \bigcap_{j=1}^{\infty} K_j = \Lambda_Q,$$

and define, for $n \in \mathbb{N}$, the set $\mathcal{A}_Q(K_n) = \{\varphi \in C^m(\widehat{\mathbb{C}});\ f|\operatorname{int}(K_n) \text{ is analytic}\}$. Then $\mathcal{A}_Q(K_n)$ is a closed subspace of the Banach space $C^m(\widehat{\mathbb{C}})$ and hence, is itself a Banach space. Since $\mathcal{A}_Q = \bigcup_{n=1}^{\infty} \mathcal{A}_Q(K_n)$ we may endow $\mathcal{A}_Q$ with the corresponding natural inductive limit topology $\mathcal{A}_Q = \operatorname{ind}_{n \to \infty} \mathcal{A}_Q(K_n)$, and note that the algebra $\mathcal{A}_Q$ is *quasi-admissible* in the sense of [26], Definition IV.9.2.

For the construction of the functional calculus we shall need the following:

LEMMA 5.1. *Let $Q \in M_m(\mathbb{C}[x])$ be a non-constant matrix polynomial and let U be an open neighbourhood in $\widehat{\mathbb{C}}$ of the set $\Lambda(\{\infty\}) \cup \{\infty\}$. Then there exists $c > 0$ such that, for all $x \in \mathbb{R} \setminus [-c, c]$, we have $\sigma(Q(x)) \subset U$.*

Proof. If this where not the case, we could find a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, with $|x_n| \to \infty$, such that for each $n \in \mathbb{N}$ the matrix $Q(x_n)$ has an eigenvalue $\lambda_n \in \widehat{\mathbb{C}} \setminus U$. Since $\widehat{\mathbb{C}} \setminus U$ is a compact subset of $\mathbb{C}$, by passing to a subsequence, if necessary, we may assume that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is convergent to some point $\mu \in \widehat{\mathbb{C}} \setminus U$. Because of $0 = q_Q(\lambda_n, x_n) x_n^{-d_x(q_Q)} \to q_\infty(\mu)$ we see that $\mu \in \Lambda(\{\infty\})$, contradicting $\mu \in \widehat{\mathbb{C}} \setminus U$. ∎

PROPOSITION 5.2. *Let $1 < p < \infty$ and $Q \in M_m(\mathbb{C}[x])$ be a non-constant matrix polynomial in one variable satisfying the degree condition* (vi) *in Proposition* 4.5. *Then the mapping $\Phi : \mathcal{A}_Q \to \mathcal{L}(L^p(\mathbb{R})^m)$ defined by* (5.2) *is a continuous, unital, algebra homomorphism with the following properties:*

(i) Φ *extends the analytic functional calculus, in the sense that if* $\varphi \in \mathcal{A}_Q$ *coincides with an analytic function* h *in some neighbourhood of* $\sigma(Q_p(D)) = \Sigma(Q) \cup \{\infty\}$, *then* $\Phi(\varphi) = h(Q_p(D))$, *where the operator* $h(Q_p(D))$ *is formed by means of the analytic functional calculus.*

(ii) $\operatorname{supp}(\Phi) = \sigma(Q_p(D))$.

(iii) *For all* $\varphi \in \mathcal{A}_Q$ *we have* $\Phi(\varphi)Q_p(D) \subseteq Q_p(D)\Phi(\varphi)$.

(iv) *If* $\varphi \in \mathcal{A}_Q$ *has compact support contained in* $\mathbf{C}$, *then* $Q_p(D)\Phi(\varphi) = \Phi(id_{\mathbf{C}} \cdot \varphi)$.

Proof. Fix $n \in \mathbf{N}$ and $\varphi \in \mathcal{A}_Q(K_n)$. If u is any point in $\mathbf{R} \setminus \Delta_Q$ then, as in the proof of Proposition 3.4, there exists a bounded open neighbourhood V_u of u such that $x \mapsto D(x)$ is real analytic in V_u. Here $Q(x) = D(x) + N(x)$ is the canonical Jordan decomposition for $Q(x)$ and $D(x) = C(x)^{-1}\operatorname{diag}(x)C(x)$ where $\operatorname{diag}(x)$ is a diagonal matrix. Moreover, $\operatorname{diag}(x)$ and $C(x)$ depend real analytically on x. So, if $\psi \in C^\infty(\mathbf{R})$ satisfies $\operatorname{supp}(\psi) \subset V_u$, then we obtain (by (5.1) and direct computation) for the entries $b_{jk}(x)$ of $\psi(x)\varphi(Q(x))$ that $\operatorname{supp}(b_{jk}) \subseteq V_u$ and

$$(5.4) \qquad \|b_{jk}(x)\|_{C^1(\widehat{\mathbf{R}})} \leqslant C_{\psi,Q}\|\varphi\|_{C^m(\widehat{\mathbf{C}})} = C_{\psi,Q}\|\varphi\|_{\mathcal{A}_Q(K_n)}.$$

Let now $u \in \Delta_Q$ be a singular point. Since $x \mapsto \sigma(Q(x))$ is upper semicontinuous there exists a bounded open neighbourhood V_u of u such that $\sigma(Q(x)) \subset W$ for all $x \in V_u$, where W is a bounded neighbourhood of $\Lambda(\Delta_Q)$ with $\overline{W} \subset \operatorname{int}(K_n)$. If ψ is any function in $C^\infty(\mathbf{R})$ with $\operatorname{supp}(\psi) \subset V_u$ we see that $(\lambda - Q(x))^{-1}$ exists for all $x \in V_u$ and all $\lambda \in \operatorname{int}(K_n \setminus W)$. Hence, if Γ is a finite system of closed rectifiable curves in $\operatorname{int}(K_n)$ surrounding W, we see that

$$\psi(x)\varphi(Q(x)) = \frac{\psi(x)}{2\pi i}\int_\Gamma \varphi(\lambda)(\lambda - Q(x))^{-1}\,d\lambda.$$

We note that φ is analytic in $\operatorname{int}(K_n)$, which contains $\sigma(Q(x))$ for $x \in V_u$, and $\varphi \mapsto \varphi(Q(x))$ is an extension of the analytic functional calculus. Since $(\lambda, x) \mapsto (\lambda - Q(x))^{-1}$ is analytic with respect to λ and real analytic with respect to x in $(\mathbf{C} \setminus W) \times V_u$ (by Cramer's rule) we see that the entries of $\psi(x)\varphi(Q(x))$ are in $C^1(\mathbf{R})$ with compact support contained in V_u and satisfy an estimate of the type (5.4).

Finally, for $u = \infty$, by the preceding lemma we can find a neighbourhood V_∞ of ∞ such that $\sigma(Q(x)) \subset \operatorname{int}(K_{n+1})$ for all $x \in V_\infty$. Using the analytic functional calculus for unbounded domains containing $\sigma(Q(x))$ we see that, for all $\psi \in C^\infty(\widehat{\mathbf{R}})$

satisfying $\operatorname{supp}(\psi) \subset V_\infty$ and $\psi(x) = 1$ for all $|x| \geqslant R$ (where R is a constant with $\{x \in \mathbb{R};\ |x| \geqslant R\} \subset V_\infty$) we have

$$\psi(x)\varphi(Q(x)) = \psi(x)\Big[\varphi(\infty)1 - \frac{1}{2\pi i}\int_\Gamma \varphi(\lambda)(\lambda - Q(x))^{-1}\,d\lambda\Big]. \tag{5.5}$$

Here Γ is a finite system of closed rectifiable curves in $\operatorname{int}(K_n)$ having K_{n+1} in its exterior. Since Q satisfies condition (vi) in Proposition 4.5, the set of all $\lambda \in \mathbb{C}$ for which the degree, with respect to x, of one of the minors exceeds $d_x(q_Q)$ is at most finite. So, we may assume that Γ avoids these "bad points" and hence that for all $\lambda \in \Gamma$ the entries of $(\lambda - Q(x))^{-1}$ are rational functions in x with the property that the degree of the denominator is greater or equal to the degree of the numerator. It follows from (5.5) that the entries $b_{jk}(x)$ of $\psi(x)\varphi(Q(x))$ satisfy an estimate of the form

$$\|b_{jk}\|_{\mathcal{N}^1(\mathbb{R})} = \sup_{x\in\mathbb{R}} |b_{jk}(x)| + \sup_{x\in\mathbb{R}} |x b'_{jk}(x)| \leqslant C_{\psi,Q} \sup_{\lambda\in\Gamma} |\varphi(\lambda)| \leqslant C_{\psi,Q}\|\varphi\|_{\mathcal{A}_Q(K_n)},$$

where $\mathcal{N}^1(\mathbb{R})$ is the Mihlin algebra; see Section 2.

Using the compactness of the 1-point compactification $\widehat{\mathbb{R}}$ and a partition of unity argument we see that the mapping $\varphi \longmapsto \varphi(Q)$ is a continuous, unital homomorphism from $\mathcal{A}_Q(K_n)$ to $\mathrm{M}_m(\mathcal{N}^1(\mathbb{R}))$, for all $n \in \mathbb{N}$. Since $h \mapsto S^p_h$ is a continuous, unital homomorphism from $\mathcal{N}^1(\mathbb{R})$ to $\mathcal{L}(L^p(\mathbb{R}))$, by the Mihlin multiplier theorem, we conclude that Φ is indeed a continuous, unital, algebra homomorphism from $\mathcal{A}_Q$ into $\mathcal{L}(L^p(\mathbb{R})^m)$, for all $p \in (1,\infty)$.

(i) For every function $h \in \mathcal{A}_Q$ which coincides with a rational function r in a neighbourhood of $\sigma(Q_p(D)) = \Sigma(Q) \cup \{\infty\}$ we must have $\Phi(h) = S^p_{h\circ Q} = S^p_{r\circ Q} = r(Q_p(D))$. By the continuity properties of the analytic functional calculus and the mapping Φ this implies that Φ extends the analytic functional calculus.

(ii) If $\varphi \in \mathcal{A}_Q$ vanishes in a neighbourhood of $\sigma(Q_p(D))$ then, by the proof of (i), $\Phi(\varphi) = 0$. Hence, Φ vanishes on $\widehat{\mathbb{C}} \setminus \sigma(Q_p(D))$. Conversely, assume that $K \subset \widehat{\mathbb{C}}$ is a closed set such that Φ vanishes on $\widehat{\mathbb{C}} \setminus K$. Fix $\lambda \in \widehat{\mathbb{C}} \setminus U$. Then there exists $\varphi \in \mathcal{A}_Q$ such that $\lambda \notin \operatorname{supp}(\varphi)$ and $\operatorname{supp}(1-\varphi) \cap K = \emptyset$. It follows that $h : \widehat{\mathbb{C}} \to \mathbb{C}$ defined by

$$h(z) = \begin{cases} \frac{\varphi(z)}{\lambda - z}, & \text{for } z \in \mathbb{C} \cap \operatorname{supp}(\varphi) \\ 0, & \text{for } z \in (\mathbb{C} \setminus \operatorname{supp}(\varphi)) \cup \{\infty\}, \end{cases}$$

belongs to $\mathcal{A}_Q$ and satisfies

$$\Phi(h)(\lambda - Q_p(D))f = S^p_{h(Q)} S^p_{\lambda - Q} f = S^p_{h(Q)(\lambda - Q)} f = S^p_\varphi f = S^p_1 f = f,$$

for all $f \in D(Q_p(D))$, and

$$(\lambda - Q_P(D))\Phi(h) = S^p_{(\lambda-Q)}S^p_{h(Q)} = S^p_{(\lambda-Q)h(Q)} = S^p_\varphi = \mathbb{1}.$$

Hence, $\lambda \in \widehat{\mathbf{C}}\setminus\sigma(Q_p(D))$ showing that $\sigma(Q_p(D)) \subseteq K$. This implies that $\sigma(Q_p(D))$ is contained in the support of Φ (i.e. the smallest closed subset of $\widehat{\mathbf{C}}$ such that Φ vanishes on its complement).

(iii) follows from the fact that $\Phi(\varphi) = S^p_{\varphi(Q)}$, that $\varphi(Q) \in \mathrm{M}_m(\mathcal{M}^p(\mathbf{R}))$, and that $\varphi(Q)$ commutes pointwise with Q, for all $\varphi \in \mathcal{A}_Q$.

(iv) is straightforward. ∎

For $m = 2$ it was noted in [10], Chapter 6, Section 4, that the functional calculus $\varphi \mapsto \varphi(Q)$ for matrix functions is particularly transparent. Fix $Q \in \mathrm{M}_2(\mathbf{C}[x])$. Then

$$q_Q(\lambda, x) = \lambda^2 - \text{ trace}(Q(x))\lambda + \det Q(x)$$

and, for $x \in \mathbf{R}$, the eigenvalues of $Q(x)$ are given by

$$\lambda_1(x) = a_Q(x) - (b_Q(x))^{\frac{1}{2}} \quad \text{and} \quad \lambda_2(x) = a_Q(x) + (b_Q(x))^{\frac{1}{2}}, \tag{5.6}$$

where $a_Q(x) = \frac{1}{2}\text{trace}(Q(x))$ and $b_Q(x) = a_Q(x)^2 - \det Q(x)$. With this notation we have, for $\varphi \in C^1(\Omega_Q)$, where Ω_Q is an open set in $\mathbf{C}$ containing $\Sigma(Q)$, that

$$\varphi(Q(x)) = \begin{cases} \varphi(\lambda_1(x))\mathbb{1} + \frac{\varphi(\lambda_2(x))-\varphi(\lambda_1(x))}{\lambda_2(x)-\lambda_1(x)}(Q(x) - \lambda_1(x)\mathbb{1}), & \text{if } \lambda_1(x) \neq \lambda_2(x) \\ \varphi(\lambda_1(x))\mathbb{1} + \frac{\partial\varphi}{\partial\lambda}(\lambda_1(x))(Q(x) - \lambda_1(x)), & \text{if } \lambda_1(x) = \lambda_2(x), \end{cases}$$

where $\frac{\partial\varphi}{\partial\lambda} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial\mathrm{Re}(\lambda)} - \mathrm{i}\frac{\partial\varphi}{\partial\mathrm{Im}(\lambda)}\right)$.

We now restrict ourselves to the setting where $b_Q(x) \not\equiv 0$. Then b_Q has only finitely many zeros in $\mathbf{R}$, say $x_1, \dots, x_d$, and we have for all $x \in \mathbf{R} \setminus \{x_1, \dots, x_d\}$ that

$$\varphi(Q(x)) = \varphi(\lambda_1(x))\mathbb{1} + \frac{\varphi(\lambda_2(x)) - \varphi(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}(Q(x) - \lambda_1(x)\mathbb{1}). \tag{5.7}$$

To obtain a functional calculus for $Q_p(D)$ the algebra of admissible functions φ needs to be chosen in such a way that the entries of (5.7) are in $\mathcal{M}^p(\mathbf{R})$.

We will require the following version of Mihlin's multiplier theorem; it follows from the standard version ([25], Chapter IV, Section 3) by translation and a partition of unity argument.

LEMMA 5.3. *Let F be a finite subset of $\mathbb{R}$. If $h \in C^1(\mathbb{R} \setminus F)$ is a bounded function such that $|h'(x)| \leqslant B \sum_{u \in F} |x-u|^{-1}$, for $x \notin F$, and some $B > 0$, then $h \in \mathcal{M}^p(\mathbb{R})$ for every $1 < p < \infty$.*

As a consequence we have the following application.

LEMMA 5.4. *Let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \in \mathrm{M}_2(\mathbb{C}[x])$ satisfy $b_Q(x) \not\equiv 0$. Let $\Omega_Q \subset \mathbb{C}$ be open with $\Sigma(Q) \subset \Omega_Q$. Then, for every compact interval J such that the finite set $F = b_Q^{-1}(\{0\})$ is contained in $\mathrm{int}(J)$, there exists a constant $B > 0$ such that, for all $\varphi \in C_b^2(\Omega_Q)$ and $x \in J \setminus F$, we have*

(i) $|(\varphi \circ \lambda_k)'(x)| \leqslant B\Big(\sum_{u \in F} |x-u|^{-1}\Big)\|\varphi\|_{C_b^2(\Omega_Q)}$ *for $k = 1, 2$,*

(ii) $\left|\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{(\varphi \circ \lambda_2 - \varphi \circ \lambda_1)}{\lambda_2 - \lambda_1} q_{jk}\right](x)\right| \leqslant B\Big(\sum_{u \in F} |x-u|^{-1}\Big)\|\varphi\|_{C_b^2(\Omega_Q)}$ *for all $j, k \in \{1, 2\}$,*

(iii) $\left|\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{(\varphi \circ \lambda_2 - \varphi \circ \lambda_1)}{\lambda_2 - \lambda_1} \cdot (q_{jj} - \lambda_1)\right](x)\right| \leqslant B\Big(\sum_{u \in F} |x-u|^{-1}\Big)\|\varphi\|_{C_b^2(\Omega_Q)}$ *for $j = 1, 2$.*

In particular, $\varphi(Q) \in \mathrm{M}_m(\mathcal{U}^p(\mathbb{R}))$, for every $1 < p < \infty$.

Proof. Fix $u \in F$. It follows from (5.6) that

$$\lambda_1'(x) = a_Q'(x) - \frac{b_Q'(x)}{2b_Q(x)}(b_Q(x))^{\frac{1}{2}}. \tag{5.8}$$

If u is a zero of order k for b_Q, then it is of order $(k-1)$ for b_Q' and it follows that $x \mapsto \frac{(x-u)b_Q'(x)}{2b_Q(x)}$ and hence also $(x-u)\lambda_1'(x)$ is bounded near u. Using the chain rule we see that $(x-u)(\varphi \circ \lambda_1)'(x)$ must be bounded near u. A similar argument applies to λ_2 in place of λ_1 and (i) follows.

To establish (ii) and (iii) note first that $\lambda_2(x) - \lambda_1(x) \equiv b_Q(x)^{1/2}$ and that the function $x \mapsto h(x) = \frac{\varphi(\lambda_2(x)) - \varphi(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}$ is bounded in modulus by $\|\varphi\|_{C_b^2(\Omega_Q)}$. Fix again an arbitrary point $u \in F$. Then, for $x \in \mathbb{R} \setminus F$, we have

$$(x-u)\frac{\mathrm{d}}{\mathrm{d}x}(h \cdot q_{jk})(x) = (x-u)h(x)q_{jk}'(x) + (x-u)h'(x)q_{jk}(x)$$

and

$$(x-u)\frac{\mathrm{d}}{\mathrm{d}x}(h \cdot \lambda_1)(x) = (x-u)h(x)\lambda_1'(x) + (x-u)h'(x)\lambda_2(x).$$

Since the terms $(x-u)q'_{jk}(x)$ and $q_{jk}(x)$ and $\lambda_1(x)$ and $(x-u)\lambda'_1(x)$ are all bounded near u (the last expression by the proof of (i)), it suffices to estimate the term

$$(5.9)\quad (x-u)h'(x) = \frac{(x-u)}{b_Q(x)^{\frac{1}{2}}}\frac{\mathrm{d}}{\mathrm{d}x}(\varphi\circ\lambda_2-\varphi\circ\lambda_1)(x) - \frac{(x-u)b'_Q(x)}{2b_Q(x)}b_Q(x)^{\frac{1}{2}}h(x)$$

near u. As seen before, $|h(x)| \leqslant \|\varphi\|_{C_b^2(\Omega_Q)}$ and $\frac{(x-u)b'_Q(x)}{2b_Q(x)}$ is bounded near u. Hence, we have only to consider the first expression on the right-hand-side of (5.9) which (via the chain rule and (5.8)) is the sum of

$$s(x) = \frac{(x-u)}{b_Q(x)^{\frac{1}{2}}}\left[\left(\frac{\partial\varphi}{\partial z}(\lambda_2(x)) - \frac{\partial\varphi}{\partial z}(\lambda_1(x))\right)a'_Q(x) + \left(\frac{\partial\varphi}{\partial\bar z}(\lambda_2(x)) - \frac{\partial\varphi}{\partial\bar z}(\lambda_1(x))\right)\overline{a'_Q(x)}\right]$$

and

$$t(x) = \frac{(x-u)}{b_Q(x)^{\frac{1}{2}}}\left[\left(\frac{\partial\varphi}{\partial z}(\lambda_2(x)) + \frac{\partial\varphi}{\partial z}(\lambda_1(x))\right)\frac{\mathrm{d}b_Q^{\frac{1}{2}}(x)}{\mathrm{d}x} + \left(\frac{\partial\varphi}{\partial\bar z}(\lambda_2(x)) + \frac{\partial\varphi}{\partial\bar z}(\lambda_1(x))\right)\overline{\frac{\mathrm{d}b_Q^{\frac{1}{2}}(x)}{\mathrm{d}x}}\right].$$

Applying a mean value argument to s gives

$$|s(x)| \leqslant C_{J,Q}|x-u|\cdot\|\varphi\|_{C_b^2(\Omega_Q)}, \quad \text{for } x\in\mathbf{R}\setminus F.$$

For t we have

$$|t(x)| \leqslant 4\cdot\|\varphi\|_{C_b^2(\Omega_Q)}\cdot\left|\frac{(x-u)}{b_Q(x)^{\frac{1}{2}}}\cdot\frac{\mathrm{d}b_Q^{\frac{1}{2}}(x)}{\mathrm{d}x}\right|,$$

where the factor

$$\left|\frac{(x-u)}{b_Q(x)^{\frac{1}{2}}}\cdot\frac{\mathrm{d}b_Q^{\frac{1}{2}}(x)}{\mathrm{d}x}\right| = \left|\frac{(x-u)b'_Q(x)}{2b_Q(x)}\right|$$

is bounded near u by the proof of (i). ∎

Lemma 5.4 shows that to determine, for a given function $\varphi\in C_b^2(\Omega_Q)$, whether the matrix function $x\mapsto\varphi(Q(x))$ belongs to $\mathrm{M}_2(\mathcal{M}(\mathbf{R}))$, $1<p<\infty$, reduces via Lemma 5.3 to checking that estimates of the kind (i)–(iii) hold near infinity. The following result shows that such a condition on φ (less restrictive than analyticity near $\Lambda(\{\infty\}) \cup \{\infty\})$ can be easily formulated if the matrix Q satisfies a certain degree condition.

Given an open set $\Omega \subseteq \mathbb{C}$ we define

$$\mathcal{A}_1^2(\Omega) = \Big\{\varphi \in C_b^2(\Omega);\ \|\varphi\|_{2,1} = \|\varphi\|_{C_b^2(\Omega)} + \sup_{z\in\Omega} |z| \Big(\Big|\frac{\partial\varphi}{\partial z}(z)\Big| + \Big|\frac{\partial\varphi}{\partial \bar z}(z)\Big|\Big) < \infty\Big\}.$$

This is a Banach algebra with respect to $\|\cdot\|_{2,1}$ and is quasi-admissible in the sense of [26], Definition IV.9.2.

PROPOSITION 5.5. *Let* $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \in \mathrm{M}_2(\mathbb{C}[x])$ *be a matrix polynomial such that* $b_Q \not\equiv 0$. *Let* $\Omega_Q \subset \mathbb{C}$ *be open with* $\Sigma(Q) \subseteq \Omega_Q$. *Suppose that*

$$\max\{\deg(q_{11} - q_{22}),\ \deg(q_{12}),\ \deg(q_{21})\} \leqslant \frac{1}{2}\deg(b_Q). \tag{5.10}$$

Then $\Phi : \mathcal{A}_1^2(\Omega_Q) \to \mathcal{L}(L^p(\mathbb{R})^2)$ *defined by* $\Phi(\varphi) = S^p_{\varphi(Q)}$, *for* $\varphi \in \mathcal{A}_1^2(\Omega_Q)$, *is a continuous, unital homomorphism having the properties* (i)–(iv) *in Proposition* 5.2 (*with* $\mathcal{A}_Q$ *replaced by* $\mathcal{A}_1^2(\Omega_Q)$).

Proof. It is clear that Φ will be a unital homomorphism if $\Phi(\mathcal{A}_1^2(\Omega_Q)) \subset \mathrm{M}_2(\mathcal{M}^p(\mathbb{R}))$. To prove this inclusion and the continuity of Φ it suffices to obtain estimates of the type (i)–(iii) near ∞. That is, we have to show that there exists $R > 0$ such that, for all $x \in \mathbb{R} \setminus (-R, R)$, we have (with the notation as in the proof of Lemma 5.4), that

(i)$_\infty$ $|x(\varphi\circ\lambda_j)'(x)| \leqslant C_{R,Q}\|\varphi\|_{2,1}$, for $j = 1, 2$,

(ii)$_\infty$ $\Big|x\frac{\mathrm{d}}{\mathrm{d}x}(q_{jk}h)(x)\Big| \leqslant C_{R,Q}\|\varphi\|_{2,1}$, for distinct $j, k \in \{1, 2\}$, and

(iii)$_\infty$ $\Big|x\frac{\mathrm{d}}{\mathrm{d}x}(h(q_{jj} - \lambda_1))(x)\Big| \leqslant C_{R,Q}\|\varphi\|_{2,1}$, for $j = 1, 2$,

for some constant $C_{R,Q}$ independent of φ and x. For (i)$_\infty$ we note, via the chain rule, that

$$|x(\varphi\circ\lambda_j)'(x)| \leqslant \Big|x\frac{\lambda_j'(x)}{\lambda_j(x)}\Big| |\lambda_j(x)| \Big(\Big|\frac{\partial\varphi}{\partial z}(\lambda_j(x))\Big| + \Big|\frac{\partial\varphi}{\partial\bar z}(\lambda_j(x))\Big|\Big) \leqslant \Big|x\frac{\lambda_j'(x)}{\lambda_j(x)}\Big| \|\varphi\|_{2,1}.$$

Thus we have only to prove that $\Big|x\frac{\lambda_j'(x)}{\lambda_j(x)}\Big|$ is bounded near infinity. If $\det Q \equiv 0$ the proof of (i)$_\infty$ is trivial. Direct computation (using $b_Q = a_Q^2 - \det Q$) shows, for $\det Q \not\equiv 0$ and $|x| > \max\{u;\ \det Q(u) = 0\}$, that

$$\begin{aligned} &\frac{x a_Q'(x)}{a_Q(x) - b_Q(x)^{\frac12}} - \frac{x b_Q'(x)}{2 b_Q(x)^{\frac12}(a_Q(x) - b_Q(x)^{\frac12})} \\ &\qquad = \frac{x\lambda_1'(x)}{\lambda_1(x)} = \frac{x\det(Q)'(x)(a_Q(x) + b_Q(x)^{\frac12})}{2\det(Q(x)) b_Q(x)^{\frac12}} - \frac{x a_Q'(x)}{b_Q(x)^{\frac12}}. \end{aligned} \tag{5.11}$$

If $\deg(b_Q) < 2\deg(a_Q)$ it is clear from the left-hand-side of (5.10) that $\left|\frac{x\lambda_1'(x)}{\lambda_1(x)}\right|$ is bounded near ∞. For the case $\deg(b_Q) \geqslant 2\deg(a_Q)$ the expression on the right-hand-side of (5.11) shows that $\left|\frac{x\lambda_1'(x)}{\lambda_1(x)}\right|$ is bounded at ∞. A similar argument applies to $\left|\frac{x\lambda_2'(x)}{\lambda_2(x)}\right|$. This completes the proof for $(\mathrm{i})_\infty$.

Suppose w is a polynomial with $\deg(w) \leqslant \frac{1}{2}\deg(b_Q)$. Direct computation shows that

$$(5.12)\qquad \begin{aligned} x\frac{\mathrm{d}}{\mathrm{d}x}(w\cdot h)(x) = \frac{w(x)}{b_Q(x)^{\frac{1}{2}}}\cdot x\frac{\mathrm{d}}{\mathrm{d}x}(\varphi\circ\lambda_2 - \varphi\circ\lambda_1) \\ + (\varphi\circ\lambda_2 - \varphi\circ\lambda_1)\cdot x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{b_Q^{\frac{1}{2}}}\right)(x). \end{aligned}$$

Since $\frac{w(x)}{b_Q(x)^{1/2}}$ and hence, also $x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{b_Q^{1/2}}(x)\right)$, are bounded near ∞ (by the degree condition) we see by $(\mathrm{i})_\infty$ that, for some sufficiently large $R > 0$, we must have

$$(5.13)\qquad \left|x\frac{\mathrm{d}}{\mathrm{d}x}(w.h)(x)\right| \leqslant C_{w,R}\|\varphi\|_{2,1}, \quad \text{for } |x| > R,$$

where the constant $C_{w,R}$ does not depend on φ. From (5.13) and (5.10) we now obtain $(\mathrm{ii})_\infty$ with $w = q_{jk}$ (where $j,k \in \{1,2\}$ and $j \neq k$). Also $(\mathrm{iii})_\infty$ follows in a similar way by replacing w by $\lambda_j(x) - q_{jj}(x)$ in (5.12) and noting that $\frac{\lambda_j(x)-q_{jj}(x)}{b_Q(x)^{1/2}} = \mp\left(\frac{q_{11}(x)-q_{22}(x)}{2b_Q(x)^{1/2}} + 1\right)$ is bounded near ∞ by (5.10) and hence, also $x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\lambda_j - q_{jj}}{b_Q^{1/2}}\right)(x) = \mathrm{O}(1)$, for $|x| \to \infty$. ∎

Proposition 5.5 is applicable to the class of matrix differential operators listed in Example 4.6. Hence, these operators all have an $\mathcal{A}_1^2(\Omega)$-functional calculus, for $1 < p < \infty$ and every open set $\Omega \supset \Sigma(Q)$.

The following example shows that condition (5.10) cannot be weakened to the degree condition (vi) in Proposition 4.5. Let $Q(x) = \begin{bmatrix} 0 & x^k \\ 1 & 0 \end{bmatrix}$, where $k \geqslant 1$ is some fixed integer. Then $q_Q(\lambda, x) = \lambda^2 + x^k$ and condition (vi) in Proposition 4.5 is fulfilled. Let $\Omega_Q = \{z \in \mathbf{C};\ |\mathrm{Re}(z)| < 1\}$. If $\varphi_0 \in C_b^2(\Omega_Q)$ is a function such that $\varphi_0(z) = \sin(\frac{1}{\sqrt{z}})$ for $|\mathrm{Im}(z)| > 1$, then $\varphi_0 \in \mathcal{A}_{2,1}(\Omega_Q)$. However, the $(1,2)$ entry of $\varphi_0(Q(x))$ is $(1 - \mathrm{i}x^{k/2})\sin((\mathrm{i}x^{k/2})^{-1/2})$ which is not even bounded near ∞. It may be of interest to note that even in this case we can still obtain a rich functional calculus based on the algebra

$$\{\varphi \in C_b^2(\Omega_Q);\ z(\varphi(z) - \varphi(-z)),\ z(\varphi'(z) - \varphi'(-z)),\ \text{and} \\ z^2(\varphi'(z) + \varphi'(-z)) \text{ are } \mathrm{O}(1) \text{ at } \infty\},$$

instead of $\mathcal{A}_1^2(\Omega_Q)$.

So far we have restricted ourselves in this section to the case of $N = 1$. In the case of several variables the construction of sufficiently rich functional calculi is in general not so straightforward. A general result like Proposition 5.2 is not possible and the set of critical points will no longer be discrete. We conclude with a class of examples where a particularly nice functional calculus is possible.

If $\varphi \in \mathcal{M}^p(\mathbb{R}^N)$ and $\mathcal{A} = [a_{jk}]_{j,k=1}^m \in \mathrm{M}_m(\mathbb{C})$, then we define $S_\varphi^p A = [a_{jk} S_\varphi^p]_{j,k=1}^m \in \mathcal{L}(L^p(\mathbb{R}^N)^m)$; we identify $\mathcal{L}(L^p(\mathbb{R}^N)^m)$ with $\mathrm{M}_m(\mathcal{L}(L^p(\mathbb{R}^N))$ in the usual natural way.

EXAMPLE 5.6. Let $A_1, \ldots, A_N$ be N commuting diagonizable matrices in $\mathrm{M}_m(\mathbb{C})$. The joint spectrum $\sigma(A_1, \ldots, A_N)$ consists of r ($\leqslant m$) joint eigenvalues $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jN})$, for $j = 1, \ldots, r$. Moreover, $Q(x) = \sum_{j=1}^N x_j A_j$ coincides with $\sum_{j=1}^r \left(\sum_{k=1}^N \lambda_{jk} x_k \right) P_j$, $x \in \mathbb{R}^N$, where P_j is the joint eigenprojection for λ_j. We note that $\sum_{j=1}^r P_j = \mathbb{1}_m$ is the unit matrix in $\mathrm{M}_m(\mathbb{C})$ and $P_j P_k = 0$ for $j \neq k$ in $\{1, 2, \ldots, m\}$. The functionals $x \mapsto L_j(x) = \sum_{k=1}^N \lambda_{jk} x_k$ are real linear. Theorem 2.2 of [4] shows that each operator $(L_j)_p(D)$, $1 \leqslant j \leqslant r$, has a nice functional calculus. As in [4], for any integer $k \geqslant 1$ denote by $\mathcal{H}^k$ the regular, semisimple Banach algebra of all those $h \in C(\mathbb{C}) \cap C^{2k}(\mathbb{C} \setminus \{0\})$ satisfying

$$\|h\|_{\mathcal{H}^k} = \sum_{\beta \leqslant (k,k)} \beta!^{-1} \sup_{\xi \in \mathbb{R}^2} \left| \xi^\beta \frac{\partial^\beta h}{\partial \xi^\beta}(\xi) \right| < \infty.$$

Now, for $h \in \mathcal{H}^N$, define $\Phi(h) = \sum_{j=1}^r S_{(h \circ L_j)}^p P_j$. It follows from [4], Theorem 2.2 that Φ is a continuous homomorphism with the following properties:

(i) For all $h \in \mathcal{H}^N$ with compact support $\Phi(h) Q_p(D) \subseteq Q_p(D) \Phi(h) = \Phi(h \cdot \mathrm{id}_{\mathbb{C}})$.

(ii) There exists a sequence $\{\rho_n\}_{n=1}^\infty$ in $C_c^\infty(\mathbb{C}) \subset \mathcal{H}^N$ such that $\Phi(\rho_n) \to I$, as $n \to \infty$, in the strong operator topology. Moreover,

$$D(Q_p(D)) = \{ f \in L^p(\mathbb{R}^N)^m ;\ \lim_{n \to \infty} \Phi(\rho_n \cdot \mathrm{id}_{\mathbb{C}}) f \text{ exists in } L^p(\mathbb{R}^N)^m \}$$

and $Q_p(D) f = \lim_{n \to \infty} \Phi(\rho_n \cdot \mathrm{id}_{\mathbb{C}}) f$, for all $f \in D(Q_p(D))$.

(iii) $Q_p(D)$ is decomposable, has the Ljubich-Macaev property and satisfies $\sigma(Q_p(D)) = \overline{Q(\mathbb{R}^N)} = \mathrm{supp}(\Phi)$, where the closure is taken in $\widehat{\mathbb{C}}$.

(iv) Every operator in the range of Φ is generalized scalar. In particular, $(\lambda - Q_p(D))^{-1} = \Phi(\frac{1}{\lambda - \mathrm{id}_{\mathbb{C}}})$ is generalized scalar whenever $\lambda \in \mathbb{C} \setminus \sigma(Q_p(D))$. ∎

Matrix differential operators of this type occur in [9].

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