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QUANTUM STOPPING TIMES AND DOOB-MEYER DECOMPOSITIONS

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ABSTRACT. We discuss quantum stopping times, quantum stochastic intervals, stopping quantum L^1-processes by quantum stopping times and the relationship between stopping and the Doob-Meyer decomposition of the squares of quantum martingales.

KEYWORDS: *Quantum stopping time, stochastic interval, quantum martingale, Doob-Meyer decomposition.*

AMS SUBJECT CLASSIFICATION: Primary 81S25; Secondary 46L50.

1. INTRODUCTION

Many of the notions in the classical theory of probability and stochastic processes can be extended or reformulated within a non-commutative context. Indeed, Brownian motion, the superstar of the classical theory, can be considered as a probabilistic realization of abstract (time-zero) boson quantum fields. One can then ask to what extent, if any, can the concepts of classical stochastic analysis be carried over into the realm of boson and even fermion quantum creation and annihilation operators and fields. It turns out that a remarkable amount of the classical theory has a counterpart in the quantum context. For example, one can construct conditional expectations, martingales, stochastic integrals, isometry relations, stochastic differential equations, and martingale representation theorems, all within the non-commutative (i.e., "quantum") domain.

A further construct, very useful in the classical theory, is that of a stopping time. Indeed, most modern approaches to stochastic integration make substantial use of stopping times right from the beginning. A classical stopping time is a random variable, taking values in the "time set" of the theory (usually $\mathbf{N}$, $\mathbf{R}^+$ or $[0,\infty]$), and with certain adaptedness properties. It is these adaptedness properties, suitably reformulated, which allow a definition of a "quantum stopping time". The idea is to replace events by their indicator functions which, in turn, are considered as projections given by multiplication operators.

Having constructed quantum stopping times, one can then discuss the stopping of quantum processes. It turns out that one can indeed set up a satisfactory theory of stopping of various quantum processes including quantum L^2-martingales ([2], [6], [7], [9], [10], [11]). (For convenience, we present a succinct version of some of this analysis below.) In this note, we shall consider the stopping of certain L^1-processes. Specifically, we will show that it is possible to stop the square of an L^2-martingale and also the increasing part of its Doob-Meyer decomposition (and hence also the L^1-martingale part). We examine the relationship between the Doob-Meyer decomposition of the square of a stopped process and the stopped Doob-Meyer decomposition of its square. For boson martingales these are equal, but for fermion martingales equality only holds for even stopping times (the difference being due to an appearance of the fermion parity automorphism).

We are primarily concerned with the (strictly non-Fock) quasifree boson and fermion quantum stochastic theories ([5]) but, to begin with, we consider an abstract set-up comprising a standard filtration ([8]). We suppose that we are given a family $(\mathfrak{A}_t)_{t\in[0,\infty]}$ of von Neumann algebras acting on a Hilbert space $\mathcal{H}$ such that $\mathfrak{A}_s \subseteq \mathfrak{A}_t$ whenever $s \leqslant t$, and where $\mathfrak{A}_\infty$ is generated by the $\mathfrak{A}_t$ with $t < \infty$. We also suppose that there is a cyclic and separating unit vector Ω for $\mathfrak{A}_\infty$ in $\mathcal{H}$, and that there is a family (E_t) of (normal) ω-invariant conditional expectations $E_t : \mathfrak{A}_\infty \to \mathfrak{A}_t$, where ω is the vector state induced by Ω. We will often write $\mathfrak{A}_\infty$ simply as $\mathfrak{A}$. Then, if we denote the closure of $\mathfrak{A}_t\Omega$ in $\mathcal{H}$ by $\mathcal{H}_t$ and the orthogonal projection $\mathcal{H} \to \mathcal{H}_t$ by P_t, we have

$$P_t x\Omega = E_t(x)\Omega$$

for any $x \in \mathfrak{A}$. Furthermore, since $\mathcal{H}_t$ is invariant under $\mathfrak{A}_t$, it follows that $P_t \in \mathfrak{A}_t'$.

This set-up includes the Itô-Clifford (fermion) theory ([3]) and the quasi-free CAR and CCR theories ([5]). In the former case, ω is a tracial state and the von Neumann algebras $\mathfrak{A}_t$ are type II_1 factors.

DEFINITION 1.1. *A quantum stopping time* τ is an increasing family of projections (p_t), say, indexed by $t \in [0, \infty]$ such that $p_t \in \mathfrak{A}_t$ for each t and $p_\infty = \mathbb{1}$.

We use the term increasing synonymously with non-decreasing. Note that since $\mathfrak{A}$ has a separating vector it is σ-finite and so any increasing family of projections such as (p_t) is strongly continuous except possibly for at most countably many values in $[0, \infty]$.

Classically, a stopping time T over a probability space $(\mathcal{X}, \Sigma, \mathbb{P})$ is required to satisfy $\{T \leqslant t\} \in \Sigma_t$, where Σ_t is the σ-algebra supposed to contain the events known up to time t. To each set $\{T \leqslant t\}$ there corresponds its characteristic (indicator) function, which can be thought of as a multiplication projection operator on $L^2(\mathcal{X}, \Sigma, \mathbb{P})$. Furthermore, if $s \leqslant t$ then $\{T \leqslant s\} \subseteq \{T \leqslant t\}$. Accordingly, we think of p_t as corresponding to the "event" $\{\tau \leqslant t\}$, for $0 \leqslant t \leqslant \infty$.

For any $c \geqslant 0$, $\{T + c \leqslant s\} = \{T \leqslant s - c\}$, and so for any quantum stopping time $\tau = (p_s)$ we define $\tau + c$ to be the time (q_s) where $q_s = 0$ for $s \leqslant c$ and otherwise $q_s = p_{s-c}$. Since $p_{s-c} \in \mathfrak{A}_{s-c} \subseteq \mathfrak{A}_s$, we see that $\tau + c$ really is a quantum stopping time. Similarly, for any $\alpha \geqslant 1$, we define $\alpha\tau$ to be the quantum stopping time (q_s) where $q_s = p_{s/\alpha}$. (Note that $\mathfrak{A}_{s/\alpha} \subseteq \mathfrak{A}_s$ if $\alpha \geqslant 1$.)

For classical stopping times S and T, the inequality $S \leqslant T$ holds if and only if $\{T \leqslant t\} \subseteq \{S \leqslant t\}$ for each t. We use this to define an order on quantum stopping times.

DEFINITION 1.2. Let $\sigma = (q_t)$ and $\tau = (p_t)$ be quantum stopping times. We say that $\sigma \leqslant \tau$ if and only if $p_t \leqslant q_t$ for all t.

For quantum stopping times $\sigma = (q_t)$ and $\tau = (p_t)$, we define the quantum stopping times $\sigma \vee \tau = (q_t \wedge p_t)$ and $\sigma \wedge \tau = (q_t \vee p_t)$.

Evidently, $\sigma \vee \tau$ and $\sigma \wedge \tau$ really are quantum stopping times and

$$\sigma \wedge \tau = \tau \wedge \sigma \leqslant \sigma \leqslant \sigma \vee \tau = \tau \vee \sigma.$$

For classical stopping times S and T, and for any $t \geqslant 0$, we have that

$$\{x : S \vee T(x) \leqslant t\} = \{S(x) \leqslant t\} \cap \{T(x) \leqslant t\}$$

and

$$\{x : S \wedge T(x) \leqslant t\} = \{S(x) \leqslant t\} \cup \{T(x) \leqslant t\}.$$

In view of this, and our interpretation of sets in terms of projections, the above definitions of the quantum stopping times $\sigma \vee \tau$ and $\sigma \wedge \tau$ seem to be reasonable.

Next we consider stopping a process via a quantum time. An $\mathcal{H}$-valued adapted process is, by definition, a family (ζ_t) satisfying $\zeta_t \in \mathcal{H}_t$ for each $t \in [0, \infty]$.

Similarly, an $\mathfrak{A}$-valued adapted process is a family (a_t) such that $a_t \in \mathfrak{A}_t$ for each t. Thus, if (a_t) is an $\mathfrak{A}$-valued adapted process, the family $(a_t\Omega)$ is an $\mathcal{H}$-valued adapted process. Any quantum stopping time provides an example of an $\mathfrak{A}$-valued adapted process.

If (X_t) is a classical stochastic process and T a stopping time, then (X_t) stopped by T is the random variable $x \mapsto X_{T(x)}(x)$. Suppose that T assumes just the values $s_1 \leqslant \cdots \leqslant s_m$ so that $\mathcal{X} = \{T = s_1\} \cup \cdots \cup \{T = s_m\}$. Then $X_T(x)$ is equal to $X_{s_i}(x)$ on $\{T = s_i\}$ and we have

$$X_T = X_{s_1}\chi_{\{T\leqslant s_1\}} + X_{s_2}(\chi_{\{T\leqslant s_2\}} - \chi_{\{T\leqslant s_1\}}) + \cdots + X_{s_m}(\chi_{\{T\leqslant s_m\}} - \chi_{\{T\leqslant s_{m-1}\}}).$$

This expresses X_T rather like a Stieltjes integral in terms of the events $\{T \leqslant s_i\}$ and motivates our approach to quantum stopping.

Let (ζ_t) be an $\mathcal{H}$-valued process and let $\tau = (p_t)$ be a quantum stopping time. For each finite partition $\theta = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty\}$ of $[0, \infty]$, we set

$$\zeta_\tau^\theta = p_0\zeta_0 + \sum_{i=1}^{n}(p_{t_i} - p_{t_{i-1}})\zeta_{t_i}.$$

The collection Θ of finite partitions of $[0, \infty]$ forms a net when partially ordered by refinement. If the net $(\zeta_\tau^\theta)_{\theta\in\Theta}$ converges in $\mathcal{H}$ we denote its limit by ζ_τ and refer to it as the process (ζ_t) stopped by τ.

If τ is piecewise constant with values $\{p_0 \leqslant p_{s_1} \leqslant \cdots \leqslant p_{s_m} = \mathbb{1}\}$ (and right continuous, so it assumes the value p_{s_i} on the interval $[s_i, s_{i+1})$), then we see that

$$\zeta_\tau^\theta = p_0\zeta_0 + \sum_{i=1}^{m}(p_{s_i} - p_{s_{i-1}})\zeta_{s_i}$$

whenever θ is finer than $\{0 = s_0 < s_1 < \cdots < s_{m-1} < s_m = \infty\}$. Hence $\zeta_\tau = p_0\zeta_0 + \sum_{i=1}^{m}(p_{s_i} - p_{s_{i-1}})\zeta_{s_i}$ in this case.

For given $r \in [0, \infty]$, let $\widehat{r}$ denote the quantum stopping time given by the family (q_s), where

$$q_s = \begin{cases} 0, & s < r \\ \mathbb{1}, & s \geqslant r. \end{cases}$$

$\widehat{r}$ corresponds to the "sure" time r. We see that $\zeta_{\widehat{r}} = \zeta_r$.

If (ζ_t) is constant, $\zeta_t = \zeta$, say, for all t, then

$$\zeta_\tau^\theta = p_0\zeta_0 + \sum_{i=1}^{n}(p_{t_i} - p_{t_{i-1}})\zeta_{t_i} = p_{t_n}\zeta_{t_n} = p_\infty\zeta = \zeta$$

for any θ and any τ. Hence $\zeta_\tau = \zeta$ in this case, as it should.

In preparation for a discussion of stopping of martingales, for any given quantum stopping time $\tau = (p_t)$, we set

$$M_\tau^\theta = p_0 P_0 + (p_{t_1} - p_0)P_{t_1} + \cdots + (p_\infty - p_{t_{n-1}})P_\infty$$

where $\theta = \{0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = \infty\}$ is a given finite partition of $[0, \infty]$. M_τ^θ is an operator on $\mathcal{H}$. Those properties of M_τ^θ we need follow from the following result.

PROPOSITION 1.3. *Let $\mathcal{H}$ be any Hilbert space and let (e_t) and (f_t), $t \in [0, \infty]$, be two families of bounded operators on $\mathcal{H}$ such that*

(i) *$e_t = e_t^*$ and $f_t = f_t^*$ for all t;*

(ii) *$e_s \leqslant e_t$ and $f_s \leqslant f_t$ whenever $s \leqslant t$;*

(iii) *$e_s f_t = f_t e_s$ for all s, t.*

For any given finite partition $\theta = \{0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = \infty\}$ of $[0, \infty]$ let

$$K_\tau^\theta = \sum_{i=0}^{n} \delta e_i f_{t_i}$$

where $\delta e_0 = e_0$ and $\delta e_i = e_{t_i} - e_{t_{i-1}}$ for $i > 0$. (We have suppressed the dependence of δe_i on the partition θ.)

Then the following holds:

(a) *$K_\tau^\theta = K_\tau^{\theta *}$;*

(b) *if θ' is a (finite) refinement of θ, then*

$$K_\tau^{\theta'} \leqslant K_\tau^\theta.$$

Furthermore, the net $(K_\tau^\theta)_{\theta \in \Theta}$ converges strongly to a bounded self adjoint operator K_τ, say. If, in addition, each e_t and each f_t is a projection, then K_τ^θ is a projection for each partition θ and K_τ is also a projection.

Proof. (a) is an immediate consequence of the commutativity of the two families (e_t) and (f_t) of self adjoint operators.

To prove (b), it is enough to suppose that θ' has just one more point than θ. So suppose that $\theta' = \theta \cup \{s\}$, with $t_i < s < t_{i+1}$. Then

$$\begin{aligned}
&(e_s - e_{t_i})f_s + (e_{t_{i+1}} - e_s)f_{t_{i+1}} \\
&\qquad = (e_s - e_{t_i})^{\frac{1}{2}} f_s (e_s - e_{t_i})^{\frac{1}{2}} + (e_{t_{i+1}} - e_s)^{\frac{1}{2}} f_{t_{i+1}} (e_{t_{i+1}} - e_s)^{\frac{1}{2}} \\
&\qquad \leqslant (e_s - e_{t_i})^{\frac{1}{2}} f_{t_{i+1}} (e_s - e_{t_i})^{\frac{1}{2}} + (e_{t_{i+1}} - e_s)^{\frac{1}{2}} f_{t_{i+1}} (e_{t_{i+1}} - e_s)^{\frac{1}{2}} \\
&\qquad = (e_s - e_{t_i})f_{t_{i+1}} + (e_{t_{i+1}} - e_s)f_{t_{i+1}} \\
&\qquad = (e_{t_{i+1}} - e_{t_i})f_{t_{i+1}}.
\end{aligned}$$

Adding the terms $e_0 f_0$ and $\delta e_j f_{t_j}$, $j \neq i+1$, gives

$$K_\tau^{\theta'} \leqslant K_\tau^\theta .$$

Now,

$$\begin{aligned} K_\tau^\theta &= e_0 f_0 + \sum_{i=1}^n \delta e_i \, f_{t_i} = e_0 f_0 + \sum_{i=1}^n \delta e_i{}^{\frac{1}{2}} f_{t_i} \delta e_i{}^{\frac{1}{2}} \\ &\geqslant e_0 f_0 + \sum_{i=1}^n \delta e_i{}^{\frac{1}{2}} f_0 \delta e_i{}^{\frac{1}{2}} \\ &= e_0 f_0 + \sum_{i=1}^n \delta e_i \, f_0 = e_\infty \, f_0 , \end{aligned}$$

and so (K_τ^θ) is a decreasing net which is bounded from below and therefore converges strongly with self adjoint limit.

If the e_t and f_t are orthogonal projections, then $\delta e_i \delta e_j = \delta_{ij} \delta e_i$, for any partition θ. Hence

$$\delta e_i f_{t_i} \delta e_j f_{t_j} = f_{t_i} \delta e_i \delta e_j f_{t_j} = f_{t_i} \delta_{ij} \delta e_i f_{t_j} = \delta_{ij} \delta e_i f_{t_i} .$$

Summing over i and j (from 0 to n) gives

$$K_\tau^\theta K_\tau^\theta = K_\tau^\theta$$

and we conclude that K_τ^θ is a projection. The same is then also true of the strong limit K_τ. ∎

As an immediate corollary, we have the following.

COROLLARY 1.4. *For any quantum stopping time $\tau = (p_t)$, the net $(M_\tau^\theta)_{\theta \in \Theta}$ converges strongly to a projection on $\mathcal{H}$ — denoted by M_τ.*

Proof. This follows immediately from the proposition by setting $e_t = p_t$ and $f_t = P_t$. ∎

Another interesting application of the proposition is when $e_t = p_t$, where $(p_t) = \tau$ is a quantum stopping time, and $f_t \in \mathfrak{A}'$ for all t. In particular, if ω is a central state (i.e., is tracial), then $f_t \Omega = f_t^* \Omega = J f_t \Omega = J f_t J \Omega$, where, as usual, J denotes the modular conjugation operator, so that

$$K_\tau^\theta \Omega = \sum_{i=0}^n \delta p_i f_{t_i} \Omega = \sum_{i=0}^n \delta p_i J f_{t_i} J \Omega$$

converges in $\mathcal{H}$. This means that it is possible to stop any $L^2(\mathfrak{A})$ process (ζ_t) of the form $(\zeta_t) = (f_t\Omega)$, where $f_t = f_t^*$ is any increasing family of elements of $\mathfrak{A}$, *without* the assumption that (ζ_t) be adapted.

Let $\sigma = (q_s)$ and $\tau = (p_s)$ be quantum stopping times with $\sigma \leqslant \tau$, so that $p_s \leqslant q_s$ for all s. Then

$$M_\tau^\theta M_\sigma^\theta = \sum_{i,j=0}^{n} \delta p_i P_{t_i} \delta q_j P_{t_j} = \sum_{i,j=0}^{n} P_{t_i} \delta p_i \delta q_j P_{t_j}.$$

If $j \geqslant i+1$, then $p_i q_{j-1} = p_i$, since $\sigma \leqslant \tau$, and so $\delta p_i \delta q_j = 0$ whenever $j \geqslant i+1$. Hence

$$\begin{aligned}
M_\tau^\theta\, M_\sigma^\theta &= \sum_{j \leqslant i}^{n} P_{t_i} \delta p_i \delta q_j P_{t_j} = \sum_{j \leqslant i}^{n} \delta p_i P_{t_j} \delta q_j = \sum_j \Big(\sum_{i \geqslant j} \delta p_i \Big) P_{t_j} \delta q_j \\
&= \sum_j (p_\infty - p_{j-1}) P_{t_j} \delta q_j = p_\infty \sum_j P_{t_j} \delta q_j - \sum_j P_{t_j} p_{j-1} \delta q_j \\
&= p_\infty M_\sigma^\theta \qquad (\text{since } p_{j-1}\delta q_j = p_{j-1}(q_j - q_{j-1}) = p_{j-1} - p_{j-1} = 0) \\
&= M_\sigma^\theta .
\end{aligned}$$

Notice that we have only used that $p_\infty = \mathbb{1}$ in the very last step of the computation. In fact, the conclusion remains valid provided that $p_\infty \geqslant q_s$ for all s.

This leads to the following.

PROPOSITION 1.5. *If σ and τ are quantum stopping times such that $\sigma \leqslant \tau$, then $M_\sigma \leqslant M_\tau$.*

Proof. From the above argument, we see that $M_\sigma^\theta \leqslant M_\tau^\theta$ for any θ. The result now follows by taking the limit as θ refines. ∎

Setting $\tau = \widehat{s}$, we see that $M_\tau = M_{\widehat{s}} = P_s$, so that M_τ is an extension of the notion of conditional expectation.

Suppose now that (ζ_t) is a bounded $\mathcal{H}$-valued martingale. Then (ζ_t) is closed by some ζ, say, in $\mathcal{H}$; that is, ζ_t can be written as $\zeta_t = P_t\zeta$, for all t (see e.g., [8]). If $\tau = (p_s)$ is a quantum stopping time, we see that

$$\zeta_\tau^\theta = \sum \delta p_i \zeta_{t_i} = \sum \delta p_i P_{t_i} \zeta = M_\tau^\theta \zeta.$$

It follows that $\lim_\theta \zeta_\tau^\theta$ exists and, in fact, is equal to $M_\tau \zeta$. In other words, we can stop any bounded $\mathcal{H}$-valued martingale.

If $\sigma \leqslant \tau$, then for any $\zeta \in \mathcal{H}$, we have that

$$M_\sigma M_\tau \zeta = M_\sigma \zeta.$$

This can be interpreted as saying that if we stop the $\mathcal{H}$-valued martingale closed by ζ by τ and then stop the $\mathcal{H}$-valued martingale closed by this stopped "random variable" by σ, then we obtain the same result by stopping the martingale closed by ζ directly by σ. This is an analogue of Doob's optional stopping theorem.

Let $\tau = (p_s)$ be a quantum stopping time, and for any $t \in [0,\infty]$ let $\widehat{t} = (q_s)$ be the quantum stopping time corresponding to the sure time t. Let $\sigma = \tau \wedge \widehat{t} = (f_s)$. Then

$$f_s = p_s \vee q_s = \begin{cases} p_s, & s < t \\ \mathbb{1}, & s \geqslant t. \end{cases}$$

Let $s < t$ be given and let $\theta = \{0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1} \leqslant t_n = \infty\}$ be any finite partition of $[0,\infty]$ containing both the points s and t. Then

$$M^\theta_\sigma = \sum_i \delta f_i \, P_{t_i} = \sum_{\{i\,:\,t_i \leqslant t\}} \delta f_i \, P_{t_i}$$

since $\delta f_i = 0$ whenever t_i is such that $t_i > t$. Suppose that $s = t_k$ and that $t = t_m$, with $k < m$. Then

$$\begin{aligned}
P_s M^\theta_{\tau\wedge\hat{t}} &= P_s \sum_{i \leqslant m} \delta f_i \, P_{t_i} \\
&= P_s\big(f_0 P_0 + \delta f_1 P_{t_1} + \cdots + \delta f_k P_s + \cdots + \delta f_m P_t\big) \\
&= P_s\big(P_0 f_0 + P_{t_1}\delta f_1 + \cdots + P_s \delta f_k + \cdots + P_t \delta f_m\big) \\
&= P_0 f_0 + P_{t_1}\delta f_1 + \cdots + P_s\delta f_k + P_s\big(P_{t_{k+1}}\delta f_{k+1} + \cdots + P_t \delta f_m\big) \\
&= P_0 f_0 + P_{t_1}\delta f_1 + \cdots + P_s\delta f_k + P_s\big(\underbrace{\delta f_{k+1} + \cdots + \delta f_m}_{(f_{t_m} - f_{t_k}) = (\mathbb{1} - f_{t_k})}\big) \\
&= P_0 f_0 + P_{t_1}\delta f_1 + \cdots + P_s\delta f_k + P_s(\mathbb{1} - f_{t_k}) \\
&= P_0 f_0 + P_{t_1}(f_{t_1} - f_0) + \cdots + P_s(\mathbb{1} - f_{t_{k-1}}) \\
&= M^\theta_{\tau\wedge\hat{s}}.
\end{aligned}$$

Taking the limit along θ, gives

$$P_s \, M_{\tau\wedge\hat{t}} = M_{\tau\wedge\hat{s}} \,.$$

In particular, this implies the following.

PROPOSITION 1.6. *For any $\zeta \in \mathcal{H}$ and $t \in [0,\infty]$, set $\zeta_t = M_{\tau\wedge\hat{t}}\,\zeta$. Then (ζ_t) is an $\mathcal{H}$-valued martingale.*

2. QUANTUM STOCHASTIC INTERVALS

Suppose that S and T are classical stopping times with $S \leqslant T$. The stochastic interval $(S, T]$ is defined as

$$\begin{aligned}(S,T] &= \{(t,x) : S(x) < t \leqslant T(x)\} \\ &= \{S < t\} \cap \{T < t\}^c.\end{aligned}$$

If we think of t as a parameter, $(S, T]$ can be thought of as a process rather than as a set in a product space. As usual, we identify sets with their characteristic functions which are, in turn, thought of as projections. In this way, we can think of stochastic intervals in terms of the families of projections associated with the stopping times. Now, $\{S < t\} = \bigcup_{s<t} \{S \leqslant s\}$, so we make the following definition.

Definition 2.1. Let $\tau = (p_s)$ be a quantum stopping time. For each $s \in (0, \infty)$ let $p_{s-} = \sup_{r<s} p_r$, and let τ_- be the quantum stopping time (p'_s), where $p'_s = p_{s-}$ for $s \in (0, \infty)$, $p'_0 = p_0$ and $p'_\infty = \mathbb{1}$.

Note that τ_- is a quantum stopping time and since $p_{s-} \leqslant p_s$ for each s, we see that $\tau_- \geqslant \tau$. Let $\sigma = (q_s)$ be a quantum stopping time with $\sigma \leqslant \tau$. Then $q_s \geqslant p_s$ for any s so that

$$p_r \leqslant q_r \leqslant q_s \leqslant q_{t-}$$

for any $r \leqslant s < t$. Hence $p_{t-} \leqslant q_{t-}$.

Now, in the classical context, for each t, we have

$$\chi_{\{S<t\}\cap\{T<t\}^c} = \chi_{\{S<t\}} - \chi_{\{S<t\}}\chi_{\{T<t\}},$$

which motivates the following definition.

Definition 2.2. For quantum stopping times σ and τ with $\sigma \leqslant \tau$, the *quantum stochastic interval* $(\sigma, \tau]$ is the process

$$\begin{aligned}(\sigma,\tau] &= (q_{t-} - q_{t-} \wedge p_{t-}) \\ &= (q_{t-} - p_{t-}).\end{aligned}$$

Similarly,

$$\begin{aligned}[S,T] &= \{(t,x) : S(x) \leqslant t \leqslant T(x)\} = \{S \leqslant t\} \cap \{T < t\}^c, \\ [S,T) &= \{(t,x) : S(x) \leqslant t < T(x)\} = \{S \leqslant t\} \cap \{T \leqslant t\}^c, \\ (S,T) &= \{(t,x) : S(x) < t < T(x)\} = \{S < t\} \cap \{T < t\}^c,\end{aligned}$$

and so we define the other quantum stochastic intervals as follows:

$$[\sigma,\tau] = (q_t - q_t \wedge p_{t-}) = (q_t - p_{t-}) \qquad (\text{since } q_t \geqslant p_t \geqslant p_{t-}),$$
$$[\sigma,\tau) = (q_t - p_t) \qquad (\text{since } q_t \geqslant p_t),$$
$$(\sigma,\tau) = q_{t-} - q_{t-} \wedge p_t.$$

Note that $q_t \geqslant p_t$ and $q_{t-} \geqslant p_{t-}$ but it is not necessarily true that $q_{t-} \geqslant p_t$.

Suppose that X_t is a classical martingale, closed by X. Then

$$X_t = X_0 + \int_0^t \mathrm{d}X_s$$

where the integral is a stochastic integral. For any stopping time T, we have

$$X_T = X_0 + \int_0^T \mathrm{d}X_s.$$

We can think of this last integral as $\int_0^\infty [0,T)\,\mathrm{d}X_s$.

If we translate this into the quantum case, we obtain the formula

$$\zeta_\tau = \zeta_0 + \int_0^\infty [0,\tau)\,\mathrm{d}\zeta_s = \zeta_0 + \int_0^\infty (\mathbb{1} - p_s)\,\mathrm{d}\zeta_s,$$

where $(\zeta_t) = (P_t\zeta)$ is the $\mathcal{H}$-valued martingale closed by ζ, and τ is any quantum stopping time. We have interpreted $[0,\tau)$ as the stochastic interval $[\widehat{0},\tau)$, where $\widehat{0} = (q_s)$ with $q_s = \mathbb{1}$ for all s. This is a formal argument in as much as we have not actually defined what we mean by the stochastic integral on the right hand side. In fact, we can use this formula to construct such a stochastic integral — that is, to show that the usual Itô-type construction works in this case. Note that the stochastic integral will be defined in terms of stopping and not by some form of isometry relation, as is usually done. Of course, in the first instance, the integrands are restricted to those formed from increasing, adapted families of projections — but see the next remark below.

Let $\zeta \in \mathcal{H}$, and set $\zeta_t = P_t\zeta$. We wish to investigate the meaning of the stochastic integral

$$\int_0^\tau \mathrm{d}\zeta_s \equiv \int_0^\infty [0,\tau)\,\mathrm{d}\zeta_s = \int_0^\infty (\mathbb{1} - p_s)\,\mathrm{d}\zeta_s.$$

Let $\theta = \{0 = t_0 < t_1 < \cdots < t_n = \infty\}$ be a finite partition of $[0, \infty]$, as usual. Then we have

$$\begin{aligned}
&\sum_{i=0}^{n-1} (\mathbb{1} - p_{t_i})(\zeta_{t_{i+1}} - \zeta_{t_i}) \\
&\qquad = (\zeta_\infty - \zeta_0) - p_0(\zeta_{t_1} - \zeta_0) - p_{t_1}(\zeta_{t_2} - \zeta_{t_1}) - \cdots - p_{t_{n-1}}(\zeta_{t_n} - \zeta_{t_{n-1}}) \\
&\qquad = \zeta - \zeta_0 - \big(p_0(P_{t_1} - P_0) + p_{t_1}(P_{t_2} - P_{t_1}) + \cdots + p_{t_{n-1}}(P_{t_n} - P_{t_{n-1}})\big)\zeta \\
&\qquad = \zeta - \zeta_0 + \big(p_0 P_0 + (p_{t_1} - p_0)P_{t_1} + (p_{t_2} - p_{t_1})P_{t_2} + \cdots \\
&\qquad\qquad\qquad \cdots + (p_{t_n} - p_{t_{n-1}})P_{t_n}\big)\zeta - p_\infty P_\infty \zeta \\
&\qquad = -\zeta_0 + \sum_{i=0}^{n} \delta p_i P_{t_i} \zeta \qquad (\text{since } p_\infty P_\infty \zeta = \zeta) \\
&\qquad = -\zeta_0 + M_\tau^\theta \zeta.
\end{aligned}$$

The left hand side is an Itô-type approximation to the quantum stochastic integral $\int_0^\infty (\mathbb{1} - p_s)\, d\zeta_s$. It is not a priori clear whether this has a limit or not. However, we do know that the right hand side does have a limit as θ refines and so, therefore, must the left hand side. We have therefore proved the following.

PROPOSITION 2.3. *For any $\zeta \in \mathcal{H}$ and any quantum stopping time, $\tau = (p_s)$, the quantum stochastic integral, $\int_0^\infty p_s\, d\zeta_s$, with $(\zeta_s) = (P_s \zeta)$, exists as a strong limit in $\mathcal{H}$ of the usual Itô approximations. Moreover, we have the formulae*

$$M_\tau \zeta = \zeta_\tau = \zeta_0 + \int_0^\tau d\zeta_s = \zeta_0 + \int [0, \tau)\, d\zeta_s = \zeta_0 + \int_0^\infty p_s^\perp\, d\zeta_s = \zeta - \int_0^\infty p_s\, d\zeta_s.$$

This last equality can be rewritten as

$$M_\tau^\perp \zeta = \int_0^\infty p_s\, d\zeta_s.$$

We have used the equality $\int_0^\infty d\zeta_s = \zeta - \zeta_0$.

For quantum stopping times $\sigma = (q_s)$ and $\tau = (p_s)$ with $\sigma \leqslant \tau$, we define the quantum stochastic integral

$$\int_\sigma^\tau d\zeta \equiv \int_0^\infty [\sigma, \tau)\, d\zeta = \int_0^\infty (q_s - p_s)\, d\zeta_s.$$

Then

$$\int_\sigma^\tau d\zeta = \int_0^\infty (q_s - p_s)\, d\zeta_s = \int_0^\infty (p_s^\perp - q_s^\perp)\, d\zeta = \int_0^\infty p_s^\perp\, d\zeta - \int_0^\infty q_s^\perp\, d\zeta = M_\tau\zeta - M_\sigma\zeta.$$

REMARK 2.4. To allow more general integrands, we proceed as follows. Suppose that f is a finite linear combination of increasing, adapted, projection-valued maps and denote $\sup_s \|f(s)\|$ by κ. For any partition θ, we have

$$\begin{aligned}
\Big\|\sum_\theta f(s_i)(\zeta_{s_{i+1}} - \zeta_{s_i})\Big\|^2 &= \Big\|\sum_\theta f(s_i)\Delta P_{s_i}\zeta\Big\|^2 \qquad (\text{where } \Delta P_{s_i} = P_{s_{i+1}} - P_{s_i})\\
&= \Big\|\sum_\theta \Delta P_{s_i} f(s_i)\zeta\Big\|^2\\
&= \Big(\sum_\theta \Delta P_{s_i} f(s_i)\zeta, \sum_\theta \Delta P_{s_j} f(s_j)\zeta\Big)\\
&= \sum_\theta (\Delta P_{s_i} f(s_i)\zeta, \Delta P_{s_i} f(s_i)\zeta) \qquad (\text{by orthogonality})\\
&= \sum_\theta \|f(s_i)\Delta P_{s_i}\zeta\|^2\\
&\leqslant \kappa^2 \sum_\theta \|\Delta P_{s_i}\zeta\|^2\\
&= \kappa^2 \Big(\sum_\theta \Delta P_{s_i}\zeta, \sum_\theta \Delta P_{s_j}\zeta\Big)\\
&= \kappa^2 \sum_\theta (\Delta P_{s_i}\zeta, \zeta) \qquad (\text{by orthogonality})\\
&= \kappa^2 \|(P_\infty - P_0)\zeta\|^2.
\end{aligned}$$

Thus we may define $\int_0^\infty f(s)\, d\zeta_s$ by taking the limit as θ refines, and we have that

$$\Big\|\int_0^\infty f(s)\, d\zeta_s\Big\| \leqslant \sup_s \|f(s)\|\, \|\zeta\|.$$

This inequality allows us to define (by continuity) $\int_0^\infty g(s)\, d\zeta_s$ for any map g obtained as a uniform norm limit of a sequence (f_n) of maps of the above form. Indeed, $(\int_0^\infty f_n(s)\, d\zeta_s)$ is a Cauchy sequence in $\mathcal{H}$, and so we define $\int_0^\infty g(s)\, d\zeta_s$ as $\lim_n \int_0^\infty f_n(s)\, d\zeta_s$. (One readily checks that this does not depend on the particular sequence (f_n) converging to g.)

3. STOPPING AND DOOB-MEYER DECOMPOSITIONS

We wish now to discuss the relationship between stopped martingale processes and associated Doob-Meyer decompositions. This will necessitate a discussion of stopping L^1-processes. Let $\tau = (q_s)$ be a quantum stopping time and (ζ_t) a bounded $\mathcal{H}$-valued martingale. We have seen that (ζ_t) stopped by τ is the element

$$\zeta_\tau = \lim_\theta M_\tau^\theta \zeta_\infty = \lim \sum_{i=0}^{n} \delta q_i P_{t_i} \zeta_\infty$$

in $\mathcal{H}$, where θ is the finite partition of $[0,\infty]$ given by $\theta = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty\}$. Now, we embed ζ_τ into $L^1(\mathfrak{A}) = \mathfrak{A}_*$ according to [8]

$$\zeta_\tau \mapsto \omega_{\zeta_\tau}(\,\cdot\,) = (\,\cdot\,\Omega, J\zeta_\tau),$$

where the conjugate linear isometry J is the modular conjugation operator. Let $\widetilde{q}_s = Jq_sJ \in \mathfrak{A}'$. Then, for any $x \in \mathfrak{A}$,

$$\begin{aligned}
\omega_{\zeta_\tau}(x) &= \lim_\theta \Big(x\Omega, J\sum_{i=0}^{n} \delta q_i P_{t_i}\zeta_\infty\Big) = \lim_\theta \Big(x\Omega, J\sum_{i=0}^{n} \delta q_i JJP_{t_i}\zeta_\infty\Big) \\
&= \lim_\theta \Big(x\Omega, \sum_{i=0}^{n} \delta\widetilde{q}_i JP_{t_i}\zeta_\infty\Big) = \lim_\theta \sum_{i=0}^{n} (x\delta\widetilde{q}_i\Omega, JP_{t_i}\zeta_\infty) \\
&= \lim_\theta \sum_{i=0}^{n} \omega_{\zeta_{t_i}}(x\delta\widetilde{q}_i)\,,
\end{aligned}$$

where $\omega_{\zeta_s}(\,\cdot\,) = (\,\cdot\,\Omega, J\zeta_s)$ is the embedding of ζ_s into L^1.

For elements η, ζ in $\mathcal{H}$, their product $(\eta\zeta)$ is the element $(\,\cdot\, J\eta, J\zeta)$ in L^1 ([8]). We see that the embedding $\zeta \mapsto \omega_\zeta$ above, corresponds to the product $(\Omega\zeta)$. Guided by this discussion, we make the following definition.

DEFINITION 3.1. Let (η_t) and (ζ_t) be $\mathcal{H}$-valued processes and let $\tau = (q_s)$ be a quantum stopping time. For any finite partition $\theta = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty\}$ of $[0,\infty]$, we define $(\eta\zeta)_\tau^\theta$ to be the L^1-process given by

$$(\eta\zeta)_\tau^\theta = \sum_{i=0}^{n} \delta\widetilde{q}_i\,(\eta_{t_i}\zeta_{t_i}),$$

where $\delta\widetilde{q}_i\,(\eta_{t_i}\zeta_{t_i})$ is the element of $L^1(\mathfrak{A}) = \mathfrak{A}_*$ given by $x \mapsto (x\delta\widetilde{q}_i J\eta_{t_i}, J\zeta_{t_i})$ for $x \in \mathfrak{A}$.

If this net of maps converges in L^1 (with respect to the weak topology), we define the limit to be the stopped L^1-process $(\eta\zeta)_\tau$.

The next proposition says that if two $\mathcal{H}$-valued processes can be stopped by τ, then so can their product. Moreover, the stopped product is equal to the product of the stopped processes.

PROPOSITION 3.2. *Suppose that (η_t) and (ζ_t) are $\mathcal{H}$-valued processes such that both $\sum_{i=0}^{n} \delta q_i \eta_{t_i}$ and $\sum_{i=0}^{n} \delta q_i \zeta_{t_i}$ converge in $\mathcal{H}$, to η_τ and ζ_τ, say, as θ refines. Then $(\eta\zeta)_\tau^\theta$ converges weakly in L^1 to $(\eta_\tau\zeta_\tau)$.*

Proof. For any $x \in \mathfrak{A}$, we have

$$\begin{aligned}(\eta\zeta)_\tau^\theta(x) = \sum_{i=0}^{n}(x\delta\widetilde{q}_i J\eta_{t_i}, J\zeta_{t_i}) &= \sum_{i=0}^{n}(x\delta\widetilde{q}_i J\eta_{t_i}, \delta\widetilde{q}_i J\zeta_{t_i}) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n}(x\delta\widetilde{q}_i J\eta_{t_i}, \delta\widetilde{q}_j J\zeta_{t_j})\end{aligned}$$

since the increments $\delta\widetilde{q}_i$ are mutually orthogonal projections

$$\begin{aligned}&= \Big(x\sum_{i=0}^{n}\delta\widetilde{q}_i J\eta_{t_i}, \sum_{j=0}^{n}\delta\widetilde{q}_j J\zeta_{t_j}\Big) \\ &= \Big(xJ\sum_{i=0}^{n}\delta q_i\eta_{t_i}, J\sum_{j=0}^{n}\delta q_j\zeta_{t_j}\Big) \\ &\to (xJ\eta_\tau, J\zeta_\tau) \\ &= (\eta_\tau\zeta_\tau)(x),\end{aligned}$$

as required. ∎

COROLLARY 3.3. *For any bounded $\mathcal{H}$-valued martingales (η_t), (ζ_t), the net $(\eta\zeta)_\tau^\theta$ converges weakly in L^1 to $(M_\tau\eta_\infty M_\tau\zeta_\infty)$.*

Proof. Since $\eta_t = P_t\eta_\infty$ and $\zeta_t = P_t\zeta_\infty$, we have

$$\begin{aligned}\sum \delta q_i\eta_{t_i} &= \sum \delta q_i P_{t_i}\eta_\infty \\ &= M_\tau^\theta\eta_\infty \\ &\to M_\tau\eta_\infty.\end{aligned}$$

Similarly, $\sum \delta q_i\zeta_{t_i} \to M_\tau\zeta_\infty$ and the result now follows from the Proposition 3.2. ∎

We recall that for any $\zeta \in \mathcal{H}$, $|\zeta|^2$ is defined to be the product $(\zeta\zeta) \in L^1$.

COROLLARY 3.4. *Suppose that (ζ_s) is an $\mathcal{H}$-valued martingale closed by ζ and let τ be a quantum stopping time. For each t, let $N_t = |\zeta_t|^2$. Then $N_{\tau\wedge\hat{t}}$ exists, and*

$$N_{\tau\wedge\hat{t}} = |\zeta_{\tau\wedge\hat{t}}|^2.$$

Proof. For fixed t, let $\sigma = \tau \wedge \hat{t} = (f_s)$. Then, by the preceding corollary, we have

$$\begin{aligned} N^{\theta}_{\sigma} &= (\zeta\zeta)^{\theta}_{\sigma} \\ &= \sum_{i=0}^{n} (\,\cdot\, \delta \tilde{f}_i J\zeta_{t_i}, J\zeta_{t_i}) \\ &\to (\,\cdot\, J\zeta_{\sigma}, J\zeta_{\sigma}) \\ &= |\zeta_{\sigma}|^2. \quad ∎ \end{aligned}$$

Let $(\zeta_t) = (P_t\zeta)$ be the $\mathcal{H}$-valued martingale closed by ζ and let τ be a quantum stopping time. We wish to consider the Doob-Meyer decomposition of the square of the $\mathcal{H}$-valued martingale $(\zeta_{\tau\wedge\hat{t}})$. To do this, we shall consider the boson (CCR) and fermion (CAR) theories separately following the notation of [8] and [9].

4. CCR THEORY

We consider the quasifree quantum stochastic calculus corresponding to the gauge invariant quasifree state ω on the CCR over $L^2(\mathbb{R}^+)$ with two-point function

$$\omega(a^*(f)a(g)) = \int_0^\infty f(s)\overline{g(s)}\,\gamma(s)\,\mathrm{d}s$$

where $\gamma \in L^\infty_{\mathrm{loc}}$, $\gamma > 0$ almost everywhere (no Fock part) an f and g belong to the domain of $\gamma^{1/2}$ as a multiplication operator on $L^2(\mathbb{R}^+)$. The creation and annihilation operators a^* and a are realized as (unbounded) operators on the tensor product of two copies of the symmetric Fock space over $L^2(\mathbb{R}^+)$. The von Neumann algebra $\mathfrak{A}_t$ is generated by the Weyl operators with test function with support in the interval $[0,t]$, and the vector $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector, is cyclic and separating for $\mathfrak{A}$.

The martingale representation theorem ([12], [15]) says that for any $\mathcal{H}$-valued martingale (ζ_t), there exist unique $\alpha \in \mathbb{C}$ and adapted processes $\xi \in L^2_{\mathrm{loc}}(\mathbb{R}^+, (1+\gamma(s))\,\mathrm{d}s, \mathcal{H})$ and $\eta \in L^2_{\mathrm{loc}}(\mathbb{R}^+, \gamma(s)\,\mathrm{d}s, \mathcal{H})$ such that

$$\zeta_t = \alpha\Omega + \int_0^t \mathrm{d}a^*_s\,\xi(s) + \int_0^t \mathrm{d}a_s\,\eta(s)$$

for all $t \in [0,\infty)$. We write $(\zeta_t) \leftrightarrow (\alpha, \xi, \eta)$ for notational convenience. If (ζ_t) is bounded, then ξ and η are square-integrable and the formula is valid for $t = \infty$.

The Doob-Meyer decomposition for the "square" of a martingale was given in [8]. The result is that for any $\mathcal{H}$-valued martingale (ζ_t), there exists an L^1-martingale (Z_t) and a unique (natural) increasing L^1-process (A_t), null at $t = 0$, such that

$$|\zeta_t|^2 = Z_t + A_t$$

for any $t \geqslant 0$. The L^1-process (A_t) is given explicitly by

$$A_t(\,\cdot\,) = \int_0^t (\,\cdot\, J\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s + \int_0^t (\,\cdot\, J\eta(s), J\eta(s))\gamma(s)\,\mathrm{d}s,$$

where $(\zeta_t) \leftrightarrow (\alpha, \xi, \eta)$ as above.

Suppose that $(\zeta_t) = (P_t\zeta)$ is an $\mathcal{H}$-valued martingale, closed by ζ. For any finite partition θ of $[0,\infty]$ and quantum stopping time $\tau = (q_s)$, we define A_τ^θ by

$$A_\tau^\theta = \sum_{i=0}^{n} \delta\widetilde{q}_i\, A_{t_i}$$

where $\delta\widetilde{q}_i\, A_{t_i}$ denotes the functional $x \mapsto A_{t_i}(x\delta\widetilde{q}_i)$ for $x \in \mathfrak{A}$.

PROPOSITION 4.1. *The limit $A_\tau \equiv \lim_\theta A_\tau^\theta$ exists in L^1, with respect to the weak topology, and*

$$A_\tau(\,\cdot\,) = \int_0^\infty (\,\cdot\, Jq_s^\perp\xi(s), Jq_s^\perp\xi(s))(1+\gamma(s))\,\mathrm{d}s + \int_0^\infty (\,\cdot\, Jq_s^\perp\eta(s), Jq_s^\perp\eta(s))\gamma(s)\,\mathrm{d}s.$$

Proof. Since $\mathfrak{A}_0 = \mathbb{C}\mathbb{1}$, we may assume that $q_0 = 0$. Let $x \in \mathfrak{A}$ be given. By resumming, we see that

$$\begin{aligned}
A_\tau^\theta(x) &= \sum_{i=0}^{n} A_{t_i}(x\delta\widetilde{q}_i) \\
&= A_0(x\widetilde{q}_0) + A_{t_1}(x(\widetilde{q}_{t_1} - \widetilde{q}_0)) + A_{t_2}(x(\widetilde{q}_{t_2} - \widetilde{q}_{t_1})) + \cdots \\
&\qquad\qquad \cdots + A_{t_n}(x(\widetilde{q}_{t_n} - \widetilde{q}_{t_{n-1}})) \\
&= A_{t_n}(x\widetilde{q}_{t_n}) - (A_{t_1} - A_0)(x\widetilde{q}_0) - \cdots - (A_{t_n} - A_{t_{n-1}})(x\widetilde{q}_{t_{n-1}}) \\
&= A_\infty(x) - \sum_{i=0}^{n-1} (A_{t_{i+1}} - A_{t_i})(x\widetilde{q}_{t_i}).
\end{aligned}$$

The sum for $A_\tau^\theta(x)$ consists of two separate parts, one involving ξ and the other involving η. The former is equal to

$$\int_0^\infty (xJ\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s - \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (x\widetilde{q}_{t_i}J\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s$$

$$= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} (x\widetilde{q}_{t_i}^{\perp}J\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s$$

$$= \int_0^\infty (x\widetilde{q}_\theta^{\perp}(s)J\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s$$

$$= \Lambda_\theta(x),$$

say, where $\widetilde{q}_\theta$ is the piecewise constant projection-valued map

$$\widetilde{q}_\theta(s) = \sum_{i=0}^{n-1} \widetilde{q}_{t_i}\chi_{[t_i,t_{i+1})}.$$

We claim that, as θ refines, $\Lambda_\theta(x)$ converges to

$$\int_0^\infty (x\widetilde{q}_s{}^{\perp}J\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s.$$

To see this, we may suppose that $x \geqslant 0$. Then $x\widetilde{q}_\theta^{\perp}(s) = y\widetilde{q}_\theta^{\perp}(s)y \geqslant 0$, where y is the positive square root of x. But the net $y\widetilde{q}_\theta^{\perp}(s)y$ decreases as θ refines and converges strongly almost everywhere to $y\widetilde{q}_s^{\perp}y = x\widetilde{q}_s^{\perp}$. A dominated convergence argument gives the required result.

The contribution involving η is dealt with in a similar way, and the proof is complete. ∎

COROLLARY 4.2. *Let (ζ_t) be a bounded $\mathcal{H}$-valued martingale closed by ζ, and let τ be a quantum stopping time and let $|\zeta_t|^2 = Z_t + A_t$ be the Doob-Meyer decomposition of $|\zeta_t|^2$. Then both Z_τ and A_τ exist and we have*

$$(|\zeta_t|^2)_\tau = Z_\tau + A_\tau.$$

Proof. We have seen that the L^1-process $(|\zeta_t|^2) = ((\zeta_t\zeta_t))$ can be stopped by τ, the result being $|\zeta_\tau|^2$. Also, we have just seen that (A_t) can be stopped by τ and so, therefore, can (Z_t), by linearity. The formula $(|\zeta_t|^2)_\tau = Z_\tau + A_\tau$ follows, again by linearity. ∎

COROLLARY 4.3. *For any bounded $\mathcal{H}$-valued martingale (ζ_t) closed by ζ, say, and any quantum stopping time τ, the Doob-Meyer decomposition of the "square" of the stopped martingale $(\zeta_{\tau\wedge\hat{t}})$ is equal to the stopped Doob-Meyer decomposition of the "square" of the martingale (ζ_t); that is,*

$$|\zeta_{\tau\wedge\hat{t}}|^2 = Z_{\tau\wedge\hat{t}} + A_{\tau\wedge\hat{t}}$$

is the Doob-Meyer decomposition of $|\zeta_{\tau\wedge\hat{t}}|^2$, where $|\zeta_t|^2 = Z_t + A_t$ is the decomposition of $|\zeta_t|^2$.

Proof. Fix t and let $\sigma = \tau \wedge \hat{t}$. We have

$$|\zeta_\sigma|^2 = (|\zeta_s|^2)_\sigma = Z_\sigma + A_\sigma.$$

We shall show that (A_σ) is the (natural) increasing part of the Doob-Meyer decomposition of $t \mapsto |\zeta_\sigma|^2 = |\zeta_{\tau\wedge\hat{t}}|^2$ as given in [8].

Suppose that $(\zeta_t) \leftrightarrow (\alpha, \xi, \eta)$, as above, and suppose that $\sigma = \tau \wedge \hat{t} = (f_s)$; so that $f_s = q_s$ for $s < t$, but otherwise $f_s = \mathbb{1}$. For $x \in \mathfrak{A}$,

$$\begin{aligned}
A_{\tau\wedge\hat{t}}(x) &= A_\sigma(x) \\
&= \int_0^\infty (xJf_s^\perp\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s + \int_0^\infty (xJf_s^\perp\eta(s), J\eta(s))\gamma(s)\,\mathrm{d}s \\
&= \int_0^t (xJq_s^\perp\xi(s), J\xi(s))(1+\gamma(s))\,\mathrm{d}s + \int_0^t (xJq_s^\perp\eta(s), J\eta(s))\gamma(s)\,\mathrm{d}s \\
&= \int_0^t (xJq_s^\perp\xi(s), Jq_s^\perp\xi(s))(1+\gamma(s))\,\mathrm{d}s + \int_0^t (xJq_s^\perp\eta(s), Jq_s^\perp\eta(s))\gamma(s)\,\mathrm{d}s.
\end{aligned}$$

The result now follows because $\zeta_{\tau\wedge\hat{t}} \leftrightarrow (\alpha, q^\perp\xi, q^\perp\eta)$ ([9]) and so the (natural) increasing part of the Doob-Meyer decomposition of $|\zeta_{\tau\wedge\hat{t}}|^2$ is precisely $A_{\tau\wedge\hat{t}}$, as calculated above. ∎

5. CAR THEORY

The discussion for the quasifree stochastic theory of the CAR parallels that for the CCR, except that the parity automorphism now comes into play as it does, for example, in connection with Markov solutions to quantum stochastic differential equations ([1], [4]).

We shall consider the quasifree quantum stochastic calculus corresponding to the gauge invariant quasifree state ω on the CAR over $L^2(\mathbb{R}^+)$ with two-point function

$$\omega(b^*(f)b(g)) = \int_0^\infty f(s)\overline{g(s)}\,\rho(s)\,\mathrm{d}s$$

where $0 < \rho < 1$ almost everywhere (there is no Fock part) and $f, g \in L^2(\mathbb{R}^+)$. The creation and annihilation operators b^* and b are realized as operators on the tensor product of two copies of the antisymmetric Fock space over $L^2(\mathbb{R}^+)$. The von Neumann algebra $\mathfrak{A}_t$ is generated by the operators $\{b(f)\}$ where f runs over those elements of $L^2(\mathbb{R}^+)$ with support in the interval $[0, t]$, and the vector $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector, is cyclic and separating for $\mathfrak{A}$.

For fermions, the martingale representation theorem ([13], [14]) states that for any $\mathcal{H}$-valued martingale (ζ_t), there exist unique $\alpha \in \mathbb{C}$ and adapted processes $\xi \in L^2_{\mathrm{loc}}(\mathbb{R}^+, (1-\rho(s))\,\mathrm{d}s, \mathcal{H})$ and $\eta \in L^2_{\mathrm{loc}}(\mathbb{R}^+, \rho(s)\,\mathrm{d}s, \mathcal{H})$ such that

$$\zeta_t = \alpha\Omega + \int_0^t \mathrm{d}a_s^*\,\xi(s) + \int_0^t \mathrm{d}a_s\,\eta(s)$$

for all $t \in [0, \infty)$. As before, we write $\zeta_t \leftrightarrow (\alpha, \xi, \eta)$. If (ζ_t) is bounded, then ξ and η are square-integrable and the formula is valid for $t = \infty$.

The Doob-Meyer decomposition for the "square" of a fermion martingale is as follows [8]. For any $\mathcal{H}$-valued martingale (ζ_t), the "square", $|\zeta_t|^2$, can be uniquely written as

$$|\zeta_t|^2 = Z_t + A_t\,,$$

for $t \geqslant 0$, where (Z_t) is an L^1-martingale and (A_t) an increasing L^1-process, null at $t = 0$. The process (A_t) is given explicitly by

$$A_t(\,\cdot\,) = \int_0^t (\,\cdot\,J\xi(s), J\xi(s))(1-\rho(s))\,\mathrm{d}s + \int_0^t (\,\cdot\,J\eta(s), J\eta(s))\rho(s)\,\mathrm{d}s,$$

where $(\zeta_t) \leftrightarrow (\alpha, \xi, \eta)$ as above.

Let $(\zeta_t) = (P_t\zeta)$ be an $\mathcal{H}$-valued martingale closed by ζ and let $\tau = (q_s)$ be a quantum stopping time. We will compute the increasing part of the Doob-Meyer decomposition of $|\zeta_{\tau\wedge\hat{t}}|^2$ and compare it with the increasing part of the Doob-Meyer decomposition of $|\zeta_t|^2$ stopped by $\tau \wedge \hat{t}$.

Suppose that, according to the martingale representation theorem, (ζ_t) is given by

$$\zeta_t = \alpha\Omega + \int_0^t db_s^*\, \xi(s) + \int_0^t db_s\, \eta(s)$$

for $\alpha \in \mathbb{C}$ and processes $\xi \in L^2(\mathbb{R}^+, (1-\rho(s))\, ds, \mathcal{H})$ and $\eta \in L^2(\mathbb{R}^+, \rho(s)\, ds, \mathcal{H})$. Then we know ([9]) that $(\zeta_{\tau\wedge\hat{t}})$ is given by

$$\zeta_{\tau\wedge\hat{t}} = \alpha\Omega + \int_0^t db_s^*\, \beta(q_s^\perp)\xi(s) + \int_0^t db_s\, \beta(q_s^\perp)\eta(s)$$

where β denotes the parity automorphism on the CAR algebra $\mathfrak{A}$. Hence, according to [8], the increasing part of the Doob-Meyer decomposition of $|\zeta_{\tau\wedge\hat{t}}|^2$, (A_t') say, is given by

$$A_t'(\,\cdot\,) = \int_0^t (\,\cdot\, J\beta(q_s^\perp)\xi(s),\, J\beta(q_s^\perp)\xi(s))\,(1-\rho(s))\, ds$$
$$+ \int_0^t (\,\cdot\, J\beta(q_s^\perp)\eta(s),\, J\beta(q_s^\perp)\eta(s))\,\rho(s)\, ds.$$

On the other hand, we can compute the increasing part of $(|\zeta_t|^2)$ stopped by $\tau \wedge \hat{t}$. Indeed, the increasing part of $(|\zeta_t|^2)$, (A_t) say, is given by

$$A_t(\,\cdot\,) = \int_0^t (\,\cdot\, J\xi(s), J\xi(s))\,(1-\rho(s))\, ds + \int_0^t (\,\cdot\, J\eta(s), J\eta(s))\,\rho(s)\, ds.$$

Then (A_t), stopped by $\tau \wedge \hat{t}$, is given by

$$A_{\tau\wedge\hat{t}}(\,\cdot\,) = \int_0^t (\,\cdot\, \widetilde{q}_s^\perp J\xi(s), J\xi(s))\,(1-\rho(s))\, ds + \int_0^t (\,\cdot\, \widetilde{q}_s^\perp J\eta(s), J\eta(s))\,\rho(s)\, ds$$
$$= \int_0^t (\,\cdot\, \widetilde{q}_s^\perp J\xi(s), \widetilde{q}_s^\perp J\xi(s))\,(1-\rho(s))\, ds + \int_0^t (\,\cdot\, \widetilde{q}_s^\perp J\eta(s), \widetilde{q}_s^\perp J\eta(s))\,\rho(s)\, ds$$
$$= \int_0^t (\,\cdot\, Jq_s^\perp\xi(s), Jq_s^\perp\xi(s))\,(1-\rho(s))\, ds + \int_0^t (\,\cdot\, Jq_s^\perp\eta(s), Jq_s^\perp\eta(s))\,\rho(s)\, ds.$$

We see that if τ is an even quantum stopping time, that is, if $\beta(q_s) = q_s$ for (almost all) $s \in [0, \infty)$, then

$$A_{\tau \wedge \hat{t}} = A'_{\tau \wedge \hat{t}}$$

for all $t \geqslant 0$.

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