J. OPERATOR THEORY
35(1996), 117–132

ON COMMUTANT LIFTING WITH FINITE DEFECT

R. AROCENA, T.YA. AZIZOV, A. DIJKSMA and S.A.M. MARCANTOGNINI

Communicated by Florian-Horia Vasilescu

ABSTRACT. In this note we generalize a theorem of J.A. Ball and W.J. Helton on commutant lifting with finite defect to a Kreĭn space setting. The proof of the generalization is based on an invariant subspace theorem.

KEYWORDS: *Krein space, contraction, dilation, fixed point.*

AMS SUBJECT CLASSIFICATION: 46C20, 47A15, 47A20.

1. INTRODUCTION

Let $T \in \mathbf{L}(\mathcal{H})$ be a contraction in the Kreĭn space $\mathcal{H}$. In this paper an operator $W \in \mathbf{L}(\mathcal{G})$ in the Kreĭn space $\mathcal{G}$ will be called *a dilation* of T if $\mathcal{H}$ is a Kreĭn subspace of $\mathcal{G}$ and $PW = TP$, where P is the orthogonal projection in $\mathcal{G}$ onto $\mathcal{H}$, or equivalently, if we write $\mathcal{G} = \mathcal{H} \oplus (\mathcal{G} \ominus \mathcal{H})$, then

$$W = \begin{pmatrix} T & 0 \\ * & * \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{G} \ominus \mathcal{H} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{G} \ominus \mathcal{H} \end{pmatrix}.$$

If the dilation W is minimal, that is,

$$\overline{\operatorname{span}}\{W^n h \mid h \in \mathcal{H}, n \in \mathbb{N} \cup \{0\}\} = \mathcal{G},$$

and isometric then $\mathcal{G} \ominus \mathcal{H}$ is a Hilbert space. A minimal isometric dilation of a contraction on a Kreĭn space $\mathcal{H}$ always exists and is unique up to an isomorphism which coincides with the identity operator on $\mathcal{H}$; see, for example [10]. We recall that the negative index $h_-(H)$ of a selfadjoint operator $H \in \mathbf{L}(\mathcal{H})$ in a Kreĭn space

$(\mathcal{H}, [\cdot, \cdot])$ is defined as the supremum of all $r \in \mathbf{N}$ such that there exists a negative matrix of the form

$$([Hf_k, f_j])_{j,k=1}^{r}, \quad f_1, \ldots, f_r \in \mathcal{H},$$

and $h_-(H) = 0$ if no such r exists. If $\mathcal{H}$ is a Hilbert space then $h_-(H) < \infty$ if and only if the negative spectrum of the operator H consists of a finite number of eigenvalues counted according to multiplicity and this number is $h_-(H)$.

Consider two contractions $T_j \in \mathbf{L}(\mathcal{H}_j)$ and corresponding two minimal isometric dilations $W_j \in \mathbf{L}(\mathcal{G}_j)$, and denote by P_j the orthogonal projection from $\mathcal{G}_j$ onto $\mathcal{H}_j$, $j = 1, 2$. Let $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator such that $AT_1 = T_2 A$. For $\kappa \in \mathbf{N} \cup \{0\}$ we define $\mathbf{LIF}_\kappa(A)$ as the set of all contractions $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2)$ where $\mathcal{E}$ is a closed W_1-invariant subspace of $\mathcal{G}_1$ with $\operatorname{codim} \mathcal{E} = \kappa$ such that

$$\widetilde{A} W_1 \,|\, \mathcal{E} = W_2 \widetilde{A} \quad \text{and} \quad P_2 \widetilde{A} = A P_1 \,|\, \mathcal{E}. \tag{1.1}$$

Here $\mathcal{E}$ is endowed with the inner product of $\mathcal{G}_1$ and may be degenerate; we say that $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2)$ is a contraction if $[\widetilde{A}x, \widetilde{A}x]_2 \leqslant [x, x]_1$, $x \in \mathcal{E}$. We call $\widetilde{A}$ a lifting of A with finite defect κ. $\mathbf{LIF}_\kappa(A)$ depends on the choice of the minimal isometric dilations but is unique up to isomorphism.

In this paper we look for conditions under which $\mathbf{LIF}_\kappa(A)$ is not empty. For example, we prove the following theorem (see Remark 3.4).

THEOREM 1.1. *Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces.*
(i) *If $h_-(I - A^*A) = \kappa$, then $\mathbf{LIF}_\kappa(A) \neq \emptyset$.*
(ii) *If $\mathbf{LIF}_\kappa(A) \neq \emptyset$ then $h_-(I - A^*A) \leqslant \kappa$.*

Theorem 1.1 is a slight generalization of the Ball-Helton lifting theorem in [4]. If $\kappa = 0$, then the theorem reduces to the Sz.-Nagy-Foiaş-Sarason lifting theorem; see [13] and [14]. We prove Theorem 1.1 using an invariant subspace theorem and the lifting theorem of Sz.-Nagy-Foiaş-Sarason. In this way we avoid the approximation argument used by Ball and Helton. We show that Theorem 1.1 remains valid if we assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Kreĭn spaces and that $T_1 \in \mathbf{L}(\mathcal{H}_1)$ is a bicontraction; see Corollary 4.3 and Remark 3.4.

We assume familiarity with operator theory in Pontryagin and Kreĭn spaces and the results in the books [2], [5] and [12] and the paper [10].

2. AN INVARIANT SUBSPACE THEOREM

If $G \in \mathbf{L}(\mathcal{H})$ is a selfadjoint operator in a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ then a subspace $\mathcal{L}$ of $\mathcal{H}$ is called *G-nonnegative* if $(Gx, x) \geqslant 0$, $x \in \mathcal{L}$, and an operator $V \in \mathbf{L}(\mathcal{H})$ is called *G-expansive* if $V^*GV \geqslant G$. Our main result in this section is the following theorem. It is a slight generalization of a theorem of I.S. Iokhvidov; see [2], Theorem 3.9, and the references at the end of [2], Chapter 3. Our proof seems simpler than the proof of a similar result in [15].

THEOREM 2.1. *Let $G \in \mathbf{L}(\mathcal{H})$ be a selfadjoint operator in a Hilbert space $\mathcal{H}$ with $h_-(G) = \kappa < \infty$. Let $V \in \mathbf{L}(\mathcal{H})$ be a G-expansive operator and let $\mathcal{L}$ be a G-nonnegative subspace of $\mathcal{H}$ such that $\overline{V\mathcal{L}} = \mathcal{L}$. Then there exists a V-invariant G-nonnegative subspace $\widetilde{\mathcal{L}}$ of $\mathcal{H}$ with $\mathcal{L} \subseteq \widetilde{\mathcal{L}}$ and* $\operatorname{codim} \widetilde{\mathcal{L}} = \kappa$.

Since $h_-(G) = \kappa < \infty$, a G-nonnegative subspace $\widetilde{\mathcal{L}}$ is maximal G-nonnegative if and only if $\operatorname{codim} \widetilde{\mathcal{L}} = \kappa$.

Proof of Theorem 2.1. The properties of G imply that $\mathcal{H}$ and G can be decomposed as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and

$$G = \begin{pmatrix} G_+ & 0 \\ 0 & -G_- \end{pmatrix} : \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix},$$

respectively, where $\mathcal{H}_\pm$ is a subspace of $\mathcal{H}$, $\dim \mathcal{H}_- = \kappa$, $G_\pm \in \mathbf{L}(\mathcal{H}_\pm)$ is a nonnegative operator, and G_- is injective. We denote by P_+ the orthogonal projection in $\mathcal{H}$ onto $\mathcal{H}_+$. For each G-nonnegative subspace $\mathcal{L}$ in $\mathcal{H}$ there exists a contraction $K : \mathcal{H}_+ \to \mathcal{H}_-$ with $\operatorname{dom} K = \overline{\operatorname{ran}}\, G_+^{1/2} P_+|\mathcal{L}$ such that

$$\mathcal{L} = \left\{ (I + G_-^{-\frac{1}{2}} K G_+^{\frac{1}{2}})x \mid x \in P_+\mathcal{L} \right\}.$$

The operator K, called the *angular operator*, is uniquely associated with $\mathcal{L}$ and will be denoted by $K_{\mathcal{L}}$. The subspace $\mathcal{L}$ is maximal G-nonnegative if and only if $P_+\mathcal{L} = \mathcal{H}_+$; the proof is the same as the proof of [2], Chapter I, Proposition 8.18, where it is assumed that (in our notation) G_+ is injective and $G_- = I$, but these assumptions are not essential. We denote by $\mathfrak{M}^+$ the collection of all maximal G-nonnegative subspaces of $\mathcal{H}$ and for a G-nonnegative subspace $\mathcal{L}$ of $\mathcal{H}$ we set

$$\mathfrak{M}^+(\mathcal{L}) = \{ \widetilde{\mathcal{L}} \in \mathfrak{M}^+ \mid \mathcal{L} \subseteq \widetilde{\mathcal{L}} \}.$$

Assume $\mathcal{L}$ is a G-nonnegative subspace such that $\overline{V\mathcal{L}} = \mathcal{L}$. Then for every $\mathcal{N} \in \mathfrak{M}^+(\mathcal{L})$, there exists an $\mathcal{M} \in \mathfrak{M}^+(\mathcal{L})$ such that $V\mathcal{N} \subseteq \mathcal{M}$: simply extend the G-nonnegative subspace $V\mathcal{N}$ to a maximal G-nonnegative subspace; $\mathcal{M}$ need not be unique. Using the angular operator notation we have obtained a set valued mapping

$$\Phi : K_{\mathcal{N}} \mapsto \{K_{\mathcal{M}} \mid \mathcal{M} \in \mathfrak{M}^+(\mathcal{L}),\ V\mathcal{N} \subseteq \mathcal{M}\}$$

defined on the set $\operatorname{dom}\Phi = \{K_{\mathcal{N}} \mid \mathcal{N} \in \mathfrak{M}^+(\mathcal{L})\}$. This mapping satisfies the hypotheses of a theorem of Glicksberg ([11]), which in the original formulation reads as follows: *Given a closed point to convex set mapping $\Phi : S \longrightarrow S$ of a convex compact subset S of a convex Hausdorff linear topological space into itself, there exists a fixed point $x \in \Phi(x)$.*

The convex Hausdorff linear topological space we consider here is the space $\mathbf{L}(\mathcal{H}_+, \mathcal{H}_-)$ equipped with the weak topology. Since $\dim \mathcal{H}_- < \infty$, the weak topology coincides with the strong topology. The set

$$S = \operatorname{dom}\Phi = \{K \in \mathbf{L}(\mathcal{H}_+, \mathcal{H}_-) \mid \|K\| \leqslant 1, K \text{ extends } K_{\mathcal{L}}\}$$

is convex and also compact since it is a closed subset of the unit ball in $\mathbf{L}(\mathcal{H}_+, \mathcal{H}_-)$, which is (weakly) compact. Evidently, for each $K \in \operatorname{dom}\Phi$, $\Phi(K)$ is a convex set. It remains to show that $\Phi(K)$ is closed, that is, in terms of nets,

$$K_n \rightarrow K,\ K_n' \in \Phi(K_n),\ K_n' \rightarrow K' \text{ imply } K' \in \Phi(K).$$

We write

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{pmatrix}.$$

If $\mathcal{N}$ and $\mathcal{M}$ are maximal G-nonnegative subspaces then $V\mathcal{N} \subseteq \mathcal{M}$ if and only if

$$V_{21} + V_{22}G_-^{-\frac{1}{2}}K_{\mathcal{N}}G_+^{\frac{1}{2}} = G_-^{-\frac{1}{2}}K_{\mathcal{M}}G_+^{\frac{1}{2}}(V_{11} + V_{12}G_-^{-\frac{1}{2}}K_{\mathcal{N}}G_+^{\frac{1}{2}}). \tag{2.1}$$

By assumption, this equality holds for K_n instead of $K_{\mathcal{N}}$ and K_n' instead of $K_{\mathcal{M}}$. Taking limits we see that (2.1) also holds for K and K' instead of $K_{\mathcal{N}}$ and $K_{\mathcal{M}}$, respectively, and this implies that $K' \in \Phi(K)$.

Thus, by Glicksberg's theorem, there exists an $\widetilde{\mathcal{L}} \in \mathfrak{M}^+(\mathcal{L})$ with $\widetilde{\mathcal{L}} \in \Phi(\widetilde{\mathcal{L}})$, that is, $\widetilde{\mathcal{L}}$ has the desired properties. ∎

REMARK 2.2. Theorem 2.1 remains true if the condition that $h_-(G) = \kappa < \infty$ is replaced by the conditions that, in the notation of the proof, G_- is boundedly invertible and the operator $G_+^{1/2}V_{12}$ is compact. The first condition implies that every maximal G-nonnegative subspace $\mathcal{L}$ in $\mathcal{H}$ has the form

$$\mathcal{L} = \left\{(I + G_-^{-\frac{1}{2}}KG_+^{\frac{1}{2}})x \mid x \in \mathcal{H}_+\right\},$$

and the second condition implies that the set valued mapping Φ is closed, so we can again invoke Glicksberg's theorem.

3. THREE EQUIVALENT THEOREMS

In this section we prove the following three theorems.

THEOREM 3.1. *Let $T_j \in \mathbf{L}(\mathcal{H}_j)$ be a contraction in the Hilbert space $\mathcal{H}_j$ and let $W_j \in \mathbf{L}(\mathcal{G}_j)$ be an isometric dilation of T_j in the Hilbert space $\mathcal{G}_j$, $j = 1, 2$. Let $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that $AT_1 = T_2A$ and $\kappa = h_-(I - A^*A) < \infty$. Then there exist a W_1-invariant subspace $\mathcal{E}$ of $\mathcal{G}_1$ with* $\operatorname{codim}\mathcal{E} = \kappa$ *and an operator $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2)$ such that $\|\widetilde{A}\| \leqslant 1$ and*

$$\widetilde{A}W_1|\mathcal{E} = W_2\widetilde{A}, \quad P_2\widetilde{A} = AP_1|\mathcal{E}.$$

THEOREM 3.2. *Let $T_j \in \mathbf{L}(\mathcal{H}_j)$ be a contraction in a Hilbert space $\mathcal{H}_j$, $j = 1, 2$, and let $W_1 \in \mathbf{L}(\mathcal{G}_1)$ be an isometric dilation of T_1 in a Hilbert space $\mathcal{G}_1$. Let $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that $AT_1 = T_2A$ and $\kappa = h_-(I - A^*A) < \infty$. Then there exists a W_1-invariant subspace $\mathcal{E}$ of $\mathcal{G}_1$ with* $\operatorname{codim}\mathcal{E} = \kappa$ *and $\|AP_1|\mathcal{E}\| \leqslant 1$.*

THEOREM 3.3. *Let $G \in \mathbf{L}(\mathcal{H})$ be a selfadjoint operator in a Hilbert space $\mathcal{H}$ such that $h_-(G) = \kappa < \infty$, and let $V \in \mathbf{L}(\mathcal{H})$ be a G-expansive isometry. Then there exists a V-invariant G-nonnegative subspace $\mathcal{E}$ of $\mathcal{H}$ with* $\operatorname{codim}\mathcal{E} = \kappa$.

Theorem 3.3 is a special case of Theorem 2.1. Hence to prove Theorems 3.1–3.3 we only have to show that these theorems are equivalent.

Evidently, *Theorem* 3.1 $\Rightarrow$ *Theorem* 3.2.

Theorem 3.2 $\Rightarrow$ *Theorem* 3.1: The operators in Theorem 3.1 satisfy the hypotheses of Theorem 3.2. Hence there exists a W_1-invariant subspace $\mathcal{E}$ in $\mathcal{G}_1$ with $\operatorname{codim}\mathcal{E} = \kappa$ and $\|AP_1|\mathcal{E}\| \leqslant 1$. Set $\mathcal{H}_1' = \mathcal{E}$, $T_1' = W_1|\mathcal{E}$, $\mathcal{G}_1' = \mathcal{G}_1$, $W_1' = W_1$ and $A' = AP_1|\mathcal{E}$. Then $A'T_1' = T_2A'$ and A' is a contraction. Hence we may apply the Sz.-Nagy-Foiaş-Sarason lifting theorem and we obtain an operator $\widetilde{A}$ which has the properties mentioned in Theorem 3.1.

Theorem 3.2 $\Rightarrow$ *Theorem* 3.3: Consider the operators in the hypothesis of Theorem 3.3. Set $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $T_1 = V$, $\mathcal{G}_1 = \mathcal{H}_1$ and $W_1 = V$. Without loss of generality we may assume that $\|G\| \leqslant 1$. Take $A = (I - G)^{1/2}$, then for all $h \in \mathcal{H}$, $\|AVh\| \leqslant \|Ah\|$ and $T_2Ah = AVh$ defines an operator on $\operatorname{ran} A$ that can be extended to a contraction $T_2 \in \mathbf{L}(\mathcal{H})$ such that $T_2A = AV$. Clearly $h_-(I - A^*A) = h_-(G)$. It follows that there exists a subspace $\mathcal{E}$ in $\mathcal{H}$ as in Theorem 3.2. It is V-invariant and for $h \in \mathcal{H}$, $(Gh, h) = \|h\|^2 - \|Ah\|^2 \geqslant 0$, since $\|A|\mathcal{E}\| \leqslant 1$.

Theorem 3.3 $\Rightarrow$ *Theorem* 3.2: Consider the operators in Theorem 3.2 and set $\mathcal{H} = \mathcal{G}_1$, $A' = AP_1 \in \mathbf{L}(\mathcal{G}_1, \mathcal{H}_2)$, $G = I - A'^*A' = (I - A^*A)P_1 + (I - P_1)$. Since

$\mathcal{G}_1 \ominus \mathcal{H}_1$ is a Hilbert space, $h_-(G) = h_-(I - A^*A) = \kappa$. We set $V = W_1$, then V is G-expansive. Indeed, from

$$A'W_1 = AP_1W_1 = AT_1P_1 = T_2AP_1 = T_2A'$$

it follows that

$$\begin{aligned} V^*GV - G &= W_1^*(I - A'^*A')W_1 - (I - A'^*A') \\ &= A'^*A' - W_1^*A'^*A'W_1 \\ &= A'^*A' - A'^*T_2^*T_2A' \\ &= A'^*(I - T_2^*T_2)A' \geqslant 0. \end{aligned}$$

Let $\mathcal{E}$ be as in Theorem 3.3. It is W_1-invariant, $\operatorname{codim}\mathcal{E} = \kappa$, and for $h \in \mathcal{E}$,

$$\|AP_1h\|^2 = \|A'h\|^2 = ((I - G)h, h) \leqslant \|h\|^2$$

because $\mathcal{E}$ is G-nonnegative.

REMARK 3.4. Theorem 3.1 implies Theorem 1.1 (i) in the Introduction. To see (ii), assume that $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2)$ belongs to $\mathbf{LIF}_\kappa(A)$. Then A is an extension of the contraction $P_2\widetilde{A}P_1 \mid \mathcal{E} \cap \mathcal{H}_1$. The inequalities $\dim(\mathcal{H}_1 \ominus (\mathcal{E} \cap \mathcal{H}_1)) \leqslant \operatorname{codim}\mathcal{E} \leqslant \kappa$ imply $h_-(I - A^*A) \leqslant \kappa$. This argument also shows that Theorem 1.1 (ii) remains true in case $\mathcal{H}_1$ and $\mathcal{H}_2$ are Kreĭn spaces.

4. ALMOST COMMUTANT LIFTING IN KREĬN SPACES

In this section we assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Kreĭn spaces, $T_1 \in \mathbf{L}(\mathcal{H}_1)$ and $T_2 \in \mathbf{L}(\mathcal{H}_2)$ are contractions, $W_1 \in \mathbf{L}(\mathcal{G}_1)$ and $W_2 \in \mathbf{L}(\mathcal{G}_2)$ are minimal isometric dilations of T_1 and T_2, respectively, and $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ satisfies $AT_1 = T_2A$ and $h_-(I - A^*A) = \kappa < \infty$. We set $A' = AP_1 : \mathcal{G}_1 \to \mathcal{H}_2$, where P_1 is the orthogonal projection in $\mathcal{G}_1$ onto $\mathcal{H}_1$. Let J be a fundamental symmetry in the Kreĭn space $(\mathcal{G}_1, [\cdot,\cdot]_1)$ and set $G = J(I - A'^*A')$. Then $G \in \mathbf{L}(\mathcal{G}_1)$ is a selfadjoint operator in the Hilbert space $(\mathcal{G}_1, (\cdot,\cdot)_1)$, where for $x, y \in \mathcal{G}_1$, $(x, y)_1 = [Jx, y]_1$, and

$$(Gx, y)_1 = [(I - A^*A)P_1x, P_1y]_{\mathcal{H}_1} + [(I - P_1)x, (I - P_1)y]_{\mathcal{G}_1 \ominus \mathcal{H}_1}.$$

Since $\mathcal{G}_1 \ominus \mathcal{H}_1$ is a Hilbert space, $h_-(G) = h_-(I - A^*A) = \kappa$. Moreover, W_1 is G-expansive in the Hilbert space $(\mathcal{G}_1, (\cdot,\cdot)_1)$:

$$\begin{aligned} (W_1^*GW_1x, y)_1 &= (GW_1x, W_1y)_1 \\ &= [W_1x, W_1y]_1 - [AP_1W_1x, AP_1W_1y]_2 \\ &= [x, y]_1 - [T_2AP_1x, T_2AP_1y]_2 \\ &\geqslant [x, y]_1 - [A'x, A'y]_2 \\ &= (Gx, y)_1. \end{aligned}$$

On account of Theorem 2.1, there is a W_1-invariant G-nonnegative subspace $\mathcal{E}$ of $\mathcal{G}_1$ with $\operatorname{codim}\mathcal{E} = \kappa$. In this section we fix such an $\mathcal{E}$ and set $\mathcal{E}_0 = \mathcal{E} \cap \mathcal{E}^\perp$, where $\perp$ stands for the orthogonal complement in the Kreĭn space $(\mathcal{G}_1, [\cdot,\cdot]_1)$. Note that $\mathcal{E}_0$ is finite dimensional:

$$\dim \mathcal{E}_0 \leqslant \operatorname{codim} \mathcal{E} = \kappa.$$

We are now able to formulate the main theorem of this paper.

THEOREM 4.1. *If $W_1\mathcal{E}_0 = \mathcal{E}_0$, then $\mathbf{L}(\mathcal{E},\mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A) \neq \emptyset$.*

If $\kappa = 0$, then $\mathcal{E} = \mathcal{G}_1$, $\mathcal{E}_0 = \{0\}$, and Theorem 4.1 implies that $\mathbf{LIF}_0(A) \neq \emptyset$. This is a result of M.A. Dritschel (see [10], Theorem 3.2.1; for an alternative proof, see [9]) and this result will be used in the proof of Theorem 4.1. Some variants of commutant lifting for bicontractions and contractions in indefinite inner product spaces were proved earlier by T. Constantinescu and A. Gheondea in [6] and [7].

We set $B = A' \,|\, \mathcal{E} = AP_1 \,|\, \mathcal{E}$. Then B is a contraction on $\mathcal{E}$, $BW_1 \,|\, \mathcal{E} = T_2 B$, $B\mathcal{E}_0$ is a nonpositive subspace of $\mathcal{H}_2$, and the equality $W_1\mathcal{E}_0 = \mathcal{E}_0$ implies

$$T_2 B\mathcal{E}_0 = T_2 A P_1 \mathcal{E}_0 = A T_1 P_1 \mathcal{E}_0 = A P_1 W_1 \mathcal{E}_0 = A P_1 \mathcal{E}_0 = B\mathcal{E}_0. \tag{4.1}$$

So the hypothesis of Theorem 4.1 implies an invariant subspace condition on T_2 as well. In the proof of this theorem we use the following lemma.

LEMMA 4.2. *Let $\mathcal{K} = \mathcal{K}_0 \dot{+} \mathcal{K}_1$, direct sum, be a degenerate space (with some Hilbert majorant) in which $\mathcal{K}_0 \subseteq \mathcal{K} \cap \mathcal{K}^\perp$ and $\dim \mathcal{K}_0 < \infty$, let $\mathcal{F}$ be a Kreĭn space and let $B : \mathcal{K} \longrightarrow \mathcal{F}$ be an operator such that $B\mathcal{K}_0$ is a negative subspace of $\mathcal{F}$. Denote by Q the orthogonal projection in $\mathcal{F}$ onto $(B\mathcal{K}_0)^\perp$. Then B is a contraction on $\mathcal{K}$ if and only if $QB \,|\, \mathcal{K}_1$ is a contraction on $\mathcal{K}_1$.*

Proof. Write $\mathcal{F}$ as $\mathcal{F} = B\mathcal{K}_0 + (B\mathcal{K}_0)^\perp$, then B has a matrix representation of the form:

$$B = \begin{pmatrix} B_0 & B_{01} \\ 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} B\mathcal{K}_0 \\ (B\mathcal{K}_0)^\perp \end{pmatrix}.$$

Hilbert space inner products on $\mathcal{K}$ and $\mathcal{F}$ can be obtained from fundamental symmetries of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & G_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -I & 0 \\ 0 & H \end{pmatrix},$$

and then B is a contractive operator if and only if

$$\begin{pmatrix} B_0^* B_0 & B_0^* B_{01} \\ B_{01}^* B_0 & G_1 + B_{01}^* B_{01} - B_1^* H B_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & G_1 \end{pmatrix} - \begin{pmatrix} B_0 & B_{01} \\ 0 & B_1 \end{pmatrix}^* \begin{pmatrix} -I & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} B_0 & B_{01} \\ 0 & B_1 \end{pmatrix} \geqslant 0,$$

where the adjoints are taken in the Hilbert spaces. This inequality holds if and only if

(i) $$X = B_0^* B_0 \geqslant 0,$$

(ii) $$Y = G_1 + B_{01}^* B_{01} - B_1^* H B_1 \geqslant 0,$$

(iii) $$B_0^* B_{01} = X^{\frac{1}{2}} K Y^{\frac{1}{2}}$$

for some contraction $K\colon \overline{\operatorname{ran}}\, Y \to \overline{\operatorname{ran}}\, X$; see [2], Chapter 2, Lemma 3.21. Since B_0^* is injective, $B_0^* = |B_0|U$ for some isometry $U : B\mathcal{K}_0 \to \mathcal{K}_0$. From (iii) we see that $|B_0|UB_{01} = |B_0|KY^{1/2}$, hence

$$UB_{01} = KY^{\frac{1}{2}} \quad \text{and} \quad B_{01}^* B_{01} = B_{01}^* U^* U B_{01} = Y^{\frac{1}{2}} K^* K Y^{\frac{1}{2}} \leqslant Y.$$

It follows from (ii) that $G_1 - B_1^* H B_1 \geqslant 0$, that is, B_1 is a contraction. Conversely, if B_1 is a contraction then (i) and (ii) are valid and $B_{01}^* B_{01} \leqslant Y$. Hence there is a contraction $K_1 : (B\mathcal{K}_0)^\perp \to B\mathcal{K}_0$ such that $B_{01} = K_1 Y^{1/2}$ and $UB_{01} = KY^{1/2}$, $K = UK_1$. It follows that $B_0^* B_{01} = |B_0|UB_{01} = |B_0|KY^{1/2} = X^{1/2}KY^{1/2}$, that is, (iii) is also valid. Hence B is a contraction. ∎

Proof of Theorem 4.1. We consider 4 cases and in each of these we show that the theorem holds.

Case I. *Assume that $\mathcal{D}_0 = B\mathcal{E}_0$ is neutral.* We write $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}_0 \dot{+} \mathcal{E}_1$, direct sum, where $\mathcal{E}_1$ is a Kreĭn subspace, and we write $\mathcal{H}_2 = (\mathcal{D}_0 \oplus \mathcal{H}_2^1) \dot{+} J_2\mathcal{D}_0$, where J_2 is a fundamental symmetry in $\mathcal{H}_2$ and $\mathcal{H}_2^1$ is a Kreĭn subspace. Then $W_1 \,|\, \mathcal{E}$ and B have matrix representations of the form

$$W_1 \,|\, \mathcal{E} = \begin{pmatrix} W_{00} & W_{01} \\ 0 & W_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \\ 0 & B_2 \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ J_2\mathcal{D}_0 \end{pmatrix}.$$

We claim that $B_2 = 0$. To see this we note that the quadratic form

$$[x, x]_1 - [Bx, Bx]_2$$

is nonnegative for $x \in \mathcal{E}$ and zero for $x \in \mathcal{E}_0$. The Cauchy-Schwarz-Bunyakowskii inequality implies that for $x \in \mathcal{E}_0$ and $y \in \mathcal{E}$,

$$[Bx, By]_2 = -([x, y]_1 - [Bx, By]_2) = 0,$$

that is, $B\mathcal{E}_0 \subseteq B\mathcal{E} \cap (B\mathcal{E})^\perp$. This implies that $\operatorname{ran} B \subseteq \mathcal{D}_0 \oplus \mathcal{H}_2^1$ and proves the claim. Since B is a contraction on $\mathcal{E}$ and $\mathcal{D}_0$ is neutral, B_{11} is a contraction on $\mathcal{E}_1$ with values in $\mathcal{H}_2^1$. In the space $\mathcal{H}_2$ the operator T_2 has the matrix representation:

$$T_2 = \begin{pmatrix} T_{00} & T_{01} & T_{02} \\ 0 & T_{11} & T_{12} \\ 0 & 0 & T_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ J_2\mathcal{D}_0 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ J_2\mathcal{D}_0 \end{pmatrix}.$$

The zeros in the first column of T_2 are due to formula (4.1). The zero in the middle column also comes from formula (4.1) as it implies that T_2 acts as a unitary operator on the neutral space $\mathcal{D}_0$ (see [3]): $T_2^*\mathcal{D}_0 = T_2^*T_2\mathcal{D}_0 = \mathcal{D}_0$. This argument also implies that $W_2 \in \mathbf{L}(\mathcal{G}_2)$ has the matrix representation:

$$W_2 = \begin{pmatrix} T_{00} & T_{01} & 0 & T_{02} \\ 0 & T_{11} & 0 & T_{12} \\ 0 & W_{21} & W_{22} & W_{23} \\ 0 & 0 & 0 & T_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \\ J_2\mathcal{D}_0 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \\ J_2\mathcal{D}_0 \end{pmatrix}.$$

Evidently, T_{11} is a contraction in $\mathcal{H}_2^1$ and

$$W_2^1 = \begin{pmatrix} T_{11} & 0 \\ W_{21} & W_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \end{pmatrix}$$

is an isometric dilation of T_{11}. So we have two contractions $W_{11} \in \mathbf{L}(\mathcal{E}_1)$ and $T_{11} \in \mathbf{L}(\mathcal{H}_2^1)$ with minimal isometric dilations $W_{11} \in \mathbf{L}(\mathcal{E}_1)$ and $W_2^1 \in \mathbf{L}(\mathcal{H}_2^1 \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2))$ and a contraction $B_{11} \in \mathbf{L}(\mathcal{E}_1, \mathcal{H}_2^1)$ with the property: $B_{11}W_{11} = T_{11}B_{11}$, which follows from $BW_1 \,|\, \mathcal{E} = T_2 B$. Now [10], Theorem 3.2.1, implies that there exists a contraction $\widetilde{B}$ of the form

$$\widetilde{B} = \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} : \mathcal{E}_1 \longrightarrow \begin{pmatrix} \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \end{pmatrix}$$

such that $\widetilde{B}W_{11} = W_2^1\widetilde{B}$. It is easy to check that if

$$\widetilde{A} = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \\ 0 & B_{21} \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ J_2\mathcal{D}_0 \end{pmatrix}$$

then $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A)$.

Case II. *Assume that $\mathcal{D}_0 = B\mathcal{E}_0$ is a nondegenerate (hence negative) subspace and that $|\sigma(T_2 \,|\, \mathcal{D}_0)| > 1$.* We decompose $\mathcal{E}$ and $W_1 \,|\, \mathcal{E}$ as in Case I. We now have that $\mathcal{H}_2 = \mathcal{D}_0 \oplus \mathcal{H}_2^1$ and we write B, T_2 and W_2 as

$$B = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \end{pmatrix},$$

and

$$(4.2) \qquad W_2 = \begin{pmatrix} T_{00} & T_{01} & 0 \\ 0 & T_{11} & 0 \\ W_{20} & W_{21} & W_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2 \end{pmatrix}.$$

Then $|\sigma(T_{00})| > 1$. We define

$$\mathcal{K}_0 = \overline{\operatorname{span}}\{W_2^n \mathcal{D}_0 \mid n \in \mathbf{N} \cup \{0\}\}.$$

Then $\mathcal{D}_0 \subseteq \mathcal{K}_0 \subseteq \mathcal{D}_0 \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2)$, $\mathcal{K}_0$ is a π_{κ_0}-space with $\kappa_0 = \dim \mathcal{D}_0$, and $W_2^1 = W_2 \,|\, \mathcal{K}_0$ is a dilation of $T_{00} \in \mathbf{L}(\mathcal{D}_0)$. It follows that there is a κ_0- dimensional nonpositive subspace $\mathcal{L}_0$ of $\mathcal{K}_0$ which is W_2^1-invariant. We claim that

$$\sigma(W_2^1 \,|\, \mathcal{L}_0) = \sigma(T_{00}).$$

To prove this we denote by $\mathfrak{R}_\lambda(S)$ the root subspace of the operator S corresponding to the eigenvalue λ of S. Evidently,

$$\mathcal{L}_0 = \operatorname{span}\{\mathfrak{R}_\lambda(W_2^1 \,|\, \mathcal{L}_0) \mid \lambda \in \sigma(W_2^1 \,|\, \mathcal{L}_0)\}$$

and

$$\mathcal{D}_0 = \operatorname{span}\{\mathfrak{R}_\lambda(T_{00}) \mid \lambda \in \sigma(T_{00})\}.$$

We denote by P_0 the orthogonal projection in $\mathcal{K}_0$ onto $\mathcal{D}_0$. Then, as W_2^1 is a dilation of T_{00}, $P_0 \mathfrak{R}_\lambda(W_2^1 \,|\, \mathcal{L}_0) \subseteq \mathfrak{R}_\lambda(T_{00})$. Since $\mathcal{L}_0$ is a nonpositive subspace and $\mathcal{K}_0 \ominus \mathcal{D}_0$ is a Hilbert space, the restriction $P_0 \,|\, \mathcal{L}_0$ is an injection on $\mathcal{L}_0$. From $\dim \mathcal{D}_0 = \dim \mathcal{L}_0$ it follows that $P_0 \mathfrak{R}_\lambda(W_2^1 \,|\, \mathcal{L}_0) = \mathfrak{R}_\lambda(T_{00})$, and hence the claim is true. Consequently,

$$|\sigma(W_2^1 \,|\, \mathcal{L}_0)| > 1.$$

W_2^1 is isometric, and therefore $\mathcal{L}_0$ is a neutral subspace of $\mathcal{K}_0$. We denote the angular operator of $\mathcal{L}_0$ by U:

$$(4.3) \qquad \mathcal{L}_0 = \{x + Ux \mid x \in \mathcal{D}_0\}.$$

The operator U is an isometry from $|\mathcal{D}_0|$ to $\mathcal{K}_0 \ominus \mathcal{D}_0$, where the first space stands for the anti space of $\mathcal{D}_0$ which here means that $|\mathcal{D}_0|$ is a Hilbert space. We may write $\mathcal{G}_2$ as $\mathcal{G}_2 = \mathcal{D}_0 \oplus \mathcal{H}_2^1 \oplus U|\mathcal{D}_0| \oplus \mathcal{G}_2^1$, where $\mathcal{G}_2^1$ is a subspace of $\mathcal{K}_0 \ominus \mathcal{D}_0 \subseteq \mathcal{G}_2 \ominus \mathcal{H}_2$. We set $\widehat{\mathcal{G}}_2 = \mathcal{D}_0 \oplus \mathcal{H}_2^1 \oplus |\mathcal{D}_0| \oplus \mathcal{G}_2^1$ and define the operator $\widehat{U} : \widehat{\mathcal{G}}_2 \to \mathcal{G}_2$ by $\widehat{U} = \operatorname{diagonal}[I, I, U, I]$. Then $\widehat{U}$ is an isomorphism which coincides with the identity on $\mathcal{H}_2$ and so $\widehat{W}_2 = \widehat{U}^* W_2 \widehat{U}$ is a minimal isometric dilation of T_2, isomorphic to W_2. Let

$$\widehat{W}_2 = \begin{pmatrix} T_{00} & T_{01} & 0 & 0 \\ 0 & T_{11} & 0 & 0 \\ \widehat{W}_{20} & \widehat{W}_{21} & \widehat{W}_{22} & \widehat{W}_{23} \\ \widehat{W}_{30} & \widehat{W}_{31} & \widehat{W}_{32} & \widehat{W}_{33} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ |\mathcal{D}_0| \\ \mathcal{G}_2^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ |\mathcal{D}_0| \\ \mathcal{G}_2^1 \end{pmatrix}$$

be the matrix representation of $\widehat{W}_2$. We show that

$$\widehat{W}_{22} = T_{00} - \widehat{W}_{20}, \quad \widehat{W}_{21} = T_{01}, \quad \widehat{W}_{23} = 0, \quad \widehat{W}_{32} = -\widehat{W}_{30}.$$

Let $\widehat{\mathcal{L}} = \{(x, 0, x, 0) \in \mathcal{D}_0 \oplus \mathcal{H}_2^1 \oplus |\mathcal{D}_0| \oplus \mathcal{G}_2^1 \mid x \in \mathcal{D}_0\}$. Then $\widehat{\mathcal{L}} = \widehat{U}^* \mathcal{L}_0$, and so $\widehat{\mathcal{L}}$ is $\widehat{W}_2$-invariant. This readily implies the first and fourth equalities. The second equality follows from the first one, the surjectivity of T_{00} and the fact that $\widehat{\mathcal{L}}_2 = \widehat{W}_2 \widehat{\mathcal{L}}_2 \perp \widehat{W}_2 \mathcal{H}_2^1$. The remaining one follows in a similar way from the orthogonality of $\widehat{\mathcal{L}}_2$ and $\mathcal{G}_2^1$. We now have a situation as in Case I: From $BW_1 \,|\, \mathcal{E} = T_2 B$ we have $B' W_1 \,|\, \mathcal{E} = T_2' B'$ where

$$B' = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \\ B_{00} & B_{01} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ |\mathcal{D}_0| \end{pmatrix}$$

is a contraction and

$$T_2' = \begin{pmatrix} T_{00} & T_{01} & 0 \\ 0 & T_{11} & 0 \\ \widehat{W}_{20} & T_{01} & T_{00} - \widehat{W}_{20} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ |\mathcal{D}_0| \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ |\mathcal{D}_0| \end{pmatrix}$$

is a contractive dilation of $T_2 : T_2' = \widehat{W}_2 \mid \mathcal{D}_0 \oplus \mathcal{H}_2^1 \oplus |\mathcal{D}_0|$ and $B' \mathcal{E}_0 = \widehat{\mathcal{L}}_0$ is neutral. The minimal dilations of $W_1 | \mathcal{E}$ and T_2' are W_1 and $\widehat{W}_2$, respectively. From Case I we conclude that $\mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A') \neq \emptyset$ and hence $\mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A) \neq \emptyset$.

Case III. *Assume that $\mathcal{D}_0 = B\mathcal{E}_0$ is a nondegenerate (hence negative) subspace.* We decompose $\mathcal{D}_0$ as $\mathcal{D}_0 = \mathcal{D}_0^0 \dot{+} \mathcal{D}_0^1$, direct sum, such that

$$T_2 \mathcal{D}_0^i = \mathcal{D}_0^i, \ i = 0, 1, \quad |\sigma(T_2 \,|\, \mathcal{D}_0^0)| = 1, \quad |\sigma(T_2 \,|\, \mathcal{D}_0^1)| > 1.$$

Here

$$\mathcal{D}_0^0 = \operatorname{span}\{\mathfrak{R}_\lambda(T_2 \,|\, \mathcal{D}_0) \mid \lambda \in \sigma(T_2 \,|\, \mathcal{D}_0), |\lambda| = 1\},$$

and it can be shown that $T_2 \,|\, \mathcal{D}_0^0$ acts on $\mathcal{D}_0^0$ as a unitary operator and $\mathfrak{R}_\lambda(T_2 \,|\, \mathcal{D}_0) = \ker(T_2 \,|\, \mathcal{D}_0 - \lambda)$. We write $\mathcal{H}_2 = \mathcal{D}_0^0 \oplus \mathcal{H}_2^1$, where $\mathcal{H}_2^1 = \mathcal{H}_2 \ominus \mathcal{D}_0^0$, and

$$T_2 = \begin{pmatrix} T_{00} & 0 \\ 0 & T_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{D}_0^0 \\ \mathcal{H}_2^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0^0 \\ \mathcal{H}_2^1 \end{pmatrix}.$$

We claim that $\mathcal{E}_0$ can be decomposed as $\mathcal{E}_0 = \mathcal{E}_0^0 + \mathcal{E}_0^1$, direct sum, such that

$$B\mathcal{E}_0^i = \mathcal{E}_0^i, \; i = 0, 1, \quad \text{and} \quad W_1\mathcal{E}_0^0 = \mathcal{E}_0^0.$$

To prove the claim we choose subspaces $\mathcal{F}_0^i \subseteq \mathcal{E}_0$ such that $B\mathcal{F}_0^i = \mathcal{D}_0^i$ and $\ker B \cap \mathcal{E}_0 \subseteq \mathcal{F}_0^i$, $i = 0, 1$. We define

$$\mathcal{E}_0^0 = \operatorname{span}\{W_1^n \mathcal{F}_0^0 \,|\, n \in \mathbf{N} \cup \{0\}\}.$$

Since W_1 is an isometry, $W_1\mathcal{E}_0 = \mathcal{E}_0$ and $W_1\mathcal{E}_0^0 \subseteq \mathcal{E}_0^0$, the subspace $\mathcal{E}_0^0$ is finite dimensional and $W_1\mathcal{E}_0^0 = \mathcal{E}_0^0$. Hence

$$\mathcal{D}_0^0 = T_{00}\mathcal{D}_0^0 = T_2 B\mathcal{F}_0^0 = BW_1\mathcal{F}_0^0 = T_{00}BW_1\mathcal{F}_0^0 = BW_1^2\mathcal{F}_0^0 = \cdots = BW_1^n\mathcal{F}_0^0,$$

and this implies that $\mathcal{D}_0^0 = B\mathcal{E}_0^0$. Evidently, $\mathcal{E}_0^0 \cap \mathcal{F}_0^1 = \ker B \cap \mathcal{E}_0$, and therefore the claim is true with $\mathcal{E}_0^0$ and $\mathcal{E}_0^1 = \mathcal{F}_0^1 \ominus (\ker B \cap \mathcal{E}_0)$. We decompose $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}_0^0 \dot{+} \mathcal{E}^1$, direct sum, such that $\mathcal{E}_0^1 = \mathcal{E}^1 \cap (\mathcal{E}^1)^\perp$. We write

$$W_1 \,|\, \mathcal{E} = \begin{pmatrix} W_{00} & W_{01} \\ 0 & W_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0^0 \\ \mathcal{E}^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{E}_0^0 \\ \mathcal{E}^1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0^0 \\ \mathcal{E}^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0^0 \\ \mathcal{H}_2^1 \end{pmatrix},$$

and we note that $BW_1 \,|\, \mathcal{E} = T_2 B$ implies $B_{11}W_{11} = B_{11}T_{11}$. Since B is a contraction, on account of Lemma 4.2, B_{11} also is a contraction. Moreover, $B_{11}\mathcal{E}_0^1 = \mathcal{D}_0^1$, $\mathcal{D}_0^1$ is a nondegenerate subspace, and $|\sigma(T_{11} \,|\, \mathcal{D}_0^1)| > 1$. Thus we have a situation as in Case II. If $W_2^1 \in \mathbf{L}(\mathcal{G}_2^1)$ is the minimal isometric dilation of T_{11}, then there exists a contraction of the form

$$A_1 = \begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix} : \mathcal{E}^1 \longrightarrow \begin{pmatrix} \mathcal{H}_2^1 \\ \mathcal{G}_2^1 \ominus \mathcal{H}_2^1 \end{pmatrix}$$

such that $A_1 W_{11} \,|\, \mathcal{E}^1 = W_2^1 A_1$. The operator

$$\widetilde{A} = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \\ 0 & B_{21} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0^0 \\ \mathcal{E}^1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0^0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2^1 \ominus \mathcal{H}_2^1 \end{pmatrix}$$

belongs to $\mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A)$.

Case IV. *General case.* Using arguments as in the beginning of Case III we find that $\mathcal{E}_0$ and $\mathcal{D}_0$ can be written as the direct sums

$$\mathcal{E}_0 = \mathcal{E}_0^0 \dot{+} \mathcal{E}_0^1, \quad \mathcal{D}_0 = \mathcal{D}_0^0 \dot{+} \mathcal{D}_0^1$$

such that

$$\mathcal{D}_0^0 = \mathcal{D}_0 \cap \mathcal{D}_0^\perp, \quad B\mathcal{E}_0^i = \mathcal{D}_0^i, i = 0, 1, \quad W_1 \mathcal{E}_0^0 = \mathcal{E}_0^0.$$

We consider direct sum decompositions of the form

$$\mathcal{E} = \mathcal{E}_0^0 \dot{+} \mathcal{E}_1, \quad \mathcal{E}_1 = \mathcal{E}_0^1 \dot{+} \mathcal{E}^1, \quad \mathcal{H}_2 = \mathcal{D}_0^0 \dot{+} \mathcal{H}_2^1 \dot{+} J_2 \mathcal{D}_0^0,$$

and observe that we can use the same notation as in Case I with $\mathcal{D}_0$ replaced by $\mathcal{D}_0^0$. There are two differences with Case I. In Case I we have that $\mathcal{D}_0 = \mathcal{D}_0^0$ and B_{11} is an operator from the Kreĭn space $\mathcal{E}_1$ to the Kreĭn space $\mathcal{H}_2^1$, and we used [10], Theorem 3.2.1, to obtain a dilation of B_{11}. In the general case $\mathcal{D}_0 \neq \mathcal{D}_0^0$ and B_{11} is an operator from the degenerate space $\mathcal{E}_1$ to the Kreĭn space $\mathcal{H}_2^1$. Now we invoke, instead of [10], Theorem 3.2.1, Case III and obtain a dilation of B_{11} such that the operator $\widetilde{A}$, defined at the end of the proof of Case I, belongs to $\mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A)$. ∎

REMARK. Returning to Case II in the proof above, from the formula for B' at the end of Case II we see that $\widetilde{A} \in \mathbf{L}(\mathcal{E}, \mathcal{G}_2) \cap \mathbf{LIF}_\kappa(A)$ has the following matrix representation:

$$\widetilde{A} = \begin{pmatrix} B_{00} & B_{01} \\ 0 & B_{11} \\ UB_{00} & B_{21} \end{pmatrix} : \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{H}_2^1 \\ \mathcal{G}_2 \ominus \mathcal{H}_2, \end{pmatrix}$$

for some operator B_{21}. Using formula (4.2) for the dilation W_2 and (1.1) we obtain the equation for UB_{00}:

$$UB_{00}W_{00} - W_{22}UB_{00} = W_{20}B_{00}. \tag{4.4}$$

Since $|\sigma(W_{00})| > 1$ and W_{22} is an isometry and therefore $|\sigma(W_{22})| \leqslant 1$, there are positive numbers ε_0, ρ_0 and ρ_2 such that

$$\sigma(W_{00}) \subset \{\lambda \in \mathbf{C} \mid \rho_2 + \varepsilon_0 < |\lambda| < \rho_0\} \quad \text{and} \quad \sigma(W_{22}) \subset \{\mu \in \mathbf{C} \mid |\mu| < \rho_2\},$$

and $\Gamma_0 \cap \Gamma_2 = \emptyset$, where $\Gamma_0 = \{\lambda = \rho_0 e^{i\varphi} \mid 0 \leqslant \varphi < 2\pi\} \cup \{\lambda = (\rho_2 + \varepsilon_0)e^{-i\varphi} \mid 0 \leqslant \varphi < 2\pi\}$ and $\Gamma_2 = \{\mu = \rho_2 e^{i\varphi} \mid 0 \leqslant \varphi < 2\pi\}$. By, for example, [8], Chapter I, Formula (3.10), these conditions imply the existence of a unique solution UB_{00} of the equation (4.4):

$$UB_{00} = -\frac{1}{4\pi^2} \int_{\Gamma_0}\int_{\Gamma_2} \frac{(W_{22} - \mu)^{-1} W_{20} B_{00} (W_{00} - \lambda)^{-1}}{\lambda - \mu} \, d\mu \, d\lambda,$$

where $\lambda \in \Gamma_0$ and $\mu \in \Gamma_2$. Using this formula for U, which is defined on ran B_{00}, we can calculate the W_2^1-invariant subspace $\mathcal{L}_0$ (see (4.3)).

COROLLARY 4.3. *If* $T_1 \in \mathbf{L}(\mathcal{H}_1)$ *is a bicontraction, then* $\mathbf{LIF}_\kappa(A) \neq \emptyset$.

We recall that a contraction in a Pontryagin space is a bicontraction. Hence $\mathbf{LIF}_\kappa(A) \neq \emptyset$ if $\mathcal{H}_1$ is a Pontryagin space.

Proof of Corollary 4.3. A contraction T on a Kreĭn space $\mathcal{H}$ is a bicontraction if and only if $0 \in \rho(P_- T P_-)$, where P_- stands for the orthogonal projection in $\mathcal{H}$ onto the negative subspace of a fundamental decomposition of $\mathcal{H}$. The hypothesis of the corollary implies that W_1 is a bicontraction. Also a contraction is a bicontraction if and only if it maps every maximal nonpositive subspace onto a maximal nonpositive subspace. So W_1 has this property, and we claim that if $\mathcal{L}$ is a maximal nonpositive subspace of $\mathcal{E}$, then $W_1\mathcal{L}$ also is a maximal nonpositive subspace of $\mathcal{E}$. To see this we write $\mathcal{G}_1$ as the direct sum

$$\mathcal{G}_1 = (\mathcal{E}_0 \dot{+} \mathcal{E}_1 \dot{+} J_1\mathcal{E}_0) \oplus \mathcal{F}^+ \oplus \mathcal{F}^-,$$

where J_1 is a fundamental symmetry in $\mathcal{G}_1$, $\mathcal{E}_1$ is a Kreĭn space and $\mathcal{F}^+ \oplus \mathcal{F}^-$ is a fundamental decomposition of $\mathcal{G}_1 \ominus (\mathcal{E}_0 \dot{+} \mathcal{E}_1 \dot{+} J_1\mathcal{E}_0)$. A nonpositive subspace $\mathcal{L}$ of $\mathcal{E}$ is maximal nonpositive in $\mathcal{E}$ if and only if $\mathcal{L} \oplus \mathcal{F}^-$ is maximal nonpositive in $\mathcal{G}_1$. For the proof of the claim, assume that $\mathcal{L}$ is a maximal nonpositive subspace of $\mathcal{E}$. Then $W_1(\mathcal{L} \oplus \mathcal{F}^-) = W_1\mathcal{L} \dot{+} W_1\mathcal{F}^-$ is maximal nonpositive in $\mathcal{G}_1$. Since $W_1\mathcal{L} \subseteq \mathcal{E}$ and $\dim W_1\mathcal{F}^- = \dim \mathcal{F}^-$, we find that $W_1\mathcal{F} \oplus \mathcal{F}^-$ is maximal nonpositive in $\mathcal{G}_1$; see, for example, [1], Lemma 2.11. Hence $W_1\mathcal{L}$ is maximal nonpositive in $\mathcal{E}$, and this prove the claim.

Note that if $\mathcal{L}$ is a maximal nonpositive subspace of $\mathcal{E}$ then necessarily $\mathcal{E}_0 \subset \mathcal{L}$. Consider the subspace $\mathcal{L} = \mathcal{E}_0 \dot{+} \mathcal{E}_1^-$ where $\mathcal{E}_1^-$ is the negative subspace of a fundamental decomposition of $\mathcal{E}_1$. Then $\mathcal{L}$ is maximal nonpositive in $\mathcal{E}$, hence $W\mathcal{L}$ is also maximal nonpositive in $\mathcal{E}$ and $\mathcal{E}_0 \subset W_1\mathcal{L} \cap (W_1\mathcal{L})^\perp$. From $\mathcal{L} \cap \mathcal{L}^\perp = \mathcal{E}_0$ it follows that $W_1\mathcal{L} \cap W_1\mathcal{L}^\perp = W_1\mathcal{E}_0$, and hence $\mathcal{E}_0 \subseteq W_1\mathcal{E}_0$. Since both spaces must have the same finite dimension, equality prevails. The corollary now follows from Theorem 4.1. ∎

The research of T.Ya. Azizov was supported by the Netherlands Organization for Scientific Research, NWO, and by the International Science Foundation (grant NZP300*).*

REFERENCES

1. T.Ya. Azizov, A. Dijksma, Contractive linear relations in Pontryagin spaces, Report W-9503, Department of Mathematics, University of Groningen.
2. T.Ya. Azizov, I.S. Iokhvidov, *Foundation of the theory of linear operators in spaces with an indefinite metric*, [Russian], Nauka, Moscow, 1986; English transl., *Linear operators in spaces with an indefinite metric*, Wiley, New York, 1989.
3. T.Ya. Azizov, H. Langer, Some spectral properties of contractive and expansive operators in indefinite inner product spaces, *Math. Nachr.* **162**(1993), 247–259.
4. J.A. Ball, W.J. Helton, A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation theory, *J. Operator Theory* **9**(1983), 107–142.
5. J. Bognar, *Indefinite Inner Product Spaces*, Springer Verlag Berlin, 1974.
6. T. Constantinescu, A. Gheondea, On unitary dilation and characteristic functions in indefinite inner product spaces, in *Oper. Theory Adv. Appl.*, vol. 24, Birkhäuser Verlag, Basel, 1987, pp. 87–102.
7. T. Constantinescu, A. Gheondea, Minimal signatures in lifting operators, *J. Operator Theory* **22**(1989), 345–367.
8. Yu.L. Daleckiĭ, M.G. Kreĭn, *Stability of solutions of differential equations in Banach space*, Nauka, Moscow, 1970; English transl., Amer. Math. Soc., Rhode Island, 1974.
9. A. Dijksma, M.A. Dritschel, S.A.M. Marcantognini, H.S.V. de Snoo, The commutant lifting theorem for contractions on Kreĭn spaces, in *Oper. Theory Adv. Appl.*, vol. 61, Birkhäuser Verlag, Basel, 1993, pp. 65–83.
10. M.A. Dritschel, J. Rovnyak, Contraction operators in Kreĭn spaces, in *Oper. Theory Adv. Appl.*, vol. 47, Birkhäuser Verlag, Basel, 1990, pp. 221–305.
11. I.L. Glicksberg, A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points, *Proc. Amer. Math. Soc.* **3**(1952), 170–174.
12. I.S. Iokhvidov, M.G. Kreĭn, H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie Verlag, Berlin, 1982.
13. D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127**(1967), 179–203.

14. B. SZ.-NAGY, C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.
15. S. TREIL, A. VOLBERG, A fixed point approach to Nehari's problem and its applications, in *Oper. Theory Adv. Appl.*, vol. 71, Birkhäuser Verlag, Basel, 1994, pp. 165–186.

RODRIGO AROCENA
Facultad de Ciencias
Universidad de la Republica
José M. Montero 3006, ap. 503
Montevideo
URUGUAY
e-mail: rarocena@cmat.edu.uy

TOMAS YA. AZIZOV
Department of Mathematics
Voronezh State University
Universitetskaja pl., 1
394693 Voronezh
RUSSIA
e-mail: azizov@mathd.vucnit.voronezh.su

AAD DIJKSMA
Department of Mathematics
University of Groningen
P.O. Box 800
9700 AV Groningen
THE NETHERLANDS
e-mail: a.dijksma@math.rug.nl

STEFANIA A.M. MARCANTOGNINI
Department of Mathematics
Simón Bolívar University
Apart. Postal 89000
Caracas 1086-A
VENEZUELA
e-mail: stefania@usb.ve

Received March 24, 1995; revised June 30, 1995.