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DECOMPOSABLE WEIGHTED ROTATIONS ON THE UNIT CIRCLE

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ABSTRACT. Bounds the rate of uniform convergence of Cesáro averages of rotations of functions by a given angle, for certain functions and angles, give norm bounds on the powers of an associated weighted rotation operator. This implies that these operators are decomposable and hence have many non-trivial invariant subspaces. This paper extends the set of rotations for which the associated weighted rotation is decomposable. The case where the function is a characteristic function of an interval is examined in detail, and stronger results are obtained in this case.

KEYWORDS: *Invariant subspaces, weighted, rotation, operator, decomposable.*

AMS SUBJECT CLASSIFICATION: Primary 47A15; Secondary 28D05.

1. INTRODUCTION

By identifying $\mathbf{T}$, the unit circle in the complex plane, with the interval $[0,1)$, rotation by an angle $2\pi\alpha$ on $\mathbf{T}$ can be identified with translation, modulo one, by α, where $\alpha \in (0,1)$. When α is irrational, rotation by $2\pi\alpha$, or translation by α, is an ergodic transformation. If $\varphi \in L^\infty[0,1)$ then $T = M_\varphi U_\alpha : L^p[0,1) \to L^p[0,1)$ (for $1 \leqslant p \leqslant \infty$) defined by $(Tf)(x) = \varphi(x)f(x+\alpha)$ for $f \in L^p[0,1)$ and $x \in [0,1)$, is a special case of a weighted translation operator (called a Bishop-type operator in [8].) There has been much interest in such operators since they are easily described, with much structure inherited from the function theoretic properties of the weight φ and from the number theoretic properties of the rotation α, and nontrivial invariant subspaces have not been found for all weighted rotation operators. In 1973, A. Davie ([4]) showed that $M_x U_\alpha$ has a nontrivial invariant subspace for almost all α. Most of the results that followed had a similar flavor, with a common approach to finding invariant subspaces for such operators being to obtain bounds on the norms of the powers and then apply a theorem of Wermer ([10]) to obtain invariant subspaces. For a survey of some of the results obtained see [1], [4], [5] and [8]. This theorem of Wermer was generalized by Colojoară and Foiaş as follows.

THEOREM 1.1. ([3]) *For $\mathcal{X}$ a Banach space, and T a bounded invertible operator on $\mathcal{X}$, if*

$$\sum_{n=-\infty}^{\infty} \frac{\log \|T^n\|}{1+n^2} < \infty$$

then T is decomposable, that is, for every finite open covering $\{G_i\}_{1\leqslant i\leqslant n}$ of the spectrum of T there exists a system $\{Y_i\}_{1\leqslant i\leqslant n}$ of spectral maximal spaces of T such that

$$\sigma(T|Y_i) \subset G_i \quad \text{for every } 1 \leqslant i \leqslant n$$

and

$$X = \sum_{i=1}^{n} Y_i.$$

(A spectral maximal subspace Y of an operator $T \in \mathbf{B}(X)$ is a closed invariant subspace such that if Z is another closed invariant subspace of T and $\sigma(T|Z) \subset \sigma(T|Y)$ then $Z \subset Y$.)

Since all invertible weighted rotations have spectrum with circular symmetry, the above theorem will give many invariant subspaces for any weighted rotation with the aforementioned norm bounds.

In this paper we show that for $\log|\varphi|$ of bounded variation, or sufficiently smooth (in terms of its modulus of continuity), and for α having rational approximations $\{p_i/q_i\}_{i=1}^{\infty}$ (these are rational numbers in lowest form such that $|\alpha - p_i/q_i| \leqslant 1/q_i^2$) with the numbers $\{q_i\}_{i=1}^{\infty}$ well distributed in the set of natural numbers (in a sense which will be made precise later) that $M_\varphi U_\alpha$ is decomposable. These results extend results in [1] and [8], mainly by extending the class of α's for which $M_\varphi U_\alpha$ is known to be decomposable. We also consider the special case where $\varphi(x) = r^{\chi_{[0,\beta)}(x)}$. These weighted translation operators were studied by K. Petersen ([9]) when $\beta \in \mathbf{Z}\alpha \pmod 1$. It is shown that in that case $M_r^{\chi_{[0,\beta)}} U_\alpha$ has a spanning set of eigenvectors, and the commutant is described. It is also shown that if $\beta \notin \mathbf{Z}\alpha \pmod 1$ then T has no eigenvectors. We shall show that in this case, for many α and β, $M_r^{\chi_{[0,\beta)}} U_\alpha$ is decomposable.

There are still α for which the above approach cannot be applied, and hence operators $M_\varphi U_\alpha$ which may not be decomposable and for which no non-trivial invariant subspaces are known, even when φ is well-behaved. To obtain invariant subspaces for these remaining exceptional cases, it appears that a new approach will be needed, as is illustrated by an example in Section 4.

Since an invertible weighted translation $M_\varphi U_\alpha$ has spectrum equal to a circle of radius $e^{\int \log|\varphi(y)|\,dy}$ when φ is continuous almost everywhere (see [8]), set $S = e^{-\int \log|\varphi(y)|\,dy} M_\varphi U_\alpha$. It is straightforward to compute the powers of S and obtain that, for all integers n:

$$\log\|S^n\| \leqslant \left\| \sum_{j=0}^{|n|-1} \log|\varphi(x+j\alpha)| - |n| \int_0^1 \log|\varphi(y)|\,dy \right\|_\infty .$$

For a function $f \in L^\infty[0,1)$ and n a positive integer, let

$$C_n(f,\alpha) = \left\| \sum_{j=0}^{n-1} f(x+j\alpha) - n \int_0^1 f(y)dy \right\|_\infty ,$$

then Theorem 1.1 says that $M_\varphi U_\alpha$ is decomposable if

$$\sum_{n=1}^{\infty} \frac{C_n(\log|\varphi|,\alpha)}{n^2} < \infty.$$

(Note the unique ergodicity of translation by α is equivalent to the condition that

$$\frac{C_n(f,\alpha)}{n} \longrightarrow 0 \quad \text{for all } f \in C[0,1)$$

so this is just a slightly stronger condition.)

Thus we need to study the growth properties of the sequence $\{C_n(f,\alpha)\}_{n=0}^\infty$ for different functions f and irrational numbers α. Although this sequence is quite irregular, it has two properties which aid in its analysis. First, there are a number of values $\{q_i\}_{i=1}^\infty$ at which $C_{q_i}(f,\alpha)$ is "small" and second, the sequence $\{C_n(f,\alpha)\}_{n=1}^\infty$ is subadditive.

In Section 2 we shall obtain a weighted summability result for subadditive sequences. Then in Section 3, we relate properties of f and α to the existence of values $\{q_i\}_{i=1}^\infty$ at which $C_{q_i}(f,\alpha)$ is small and obtain bounds on these quantities. We then combine these results with Theorem 1.1 to show many weighted rotations are decomposable. Finally, in Section 4, we develop improved bounds on $C_n(f,\alpha)$ in the case where $f(x) = \chi_{[0,\beta)}(x)$ and use these and the results of Section 2 to prove that many operators of the form $M_r^{\chi_{[0,\beta)}} U_\alpha$ are decomposable.

2. WEIGHTED SUMMABILITY OF SUBADDITIVE SEQUENCES

Suppose a positive sequence $\{a_n\}_{n=1}^{\infty}$ is subadditive, that is $a_{n+m} \leqslant a_n + a_m$ for all $n, m \in \mathbf{N}$, and we have bounds on $\{a_n\}_{n=1}^{\infty}$ for certain values of n, then we obtain weighted summability conditions.

NOTATION 2.1. Given $x \in \mathbf{R}$, let $[x]$ denote the *greatest integer* which is less than or equal to x. Let $\{x\} = x - [x]$ denote the *fractional part* of x, and $\langle x \rangle$ denote the *distance* from x *to the nearest integer.*

THEOREM 2.2. *For $\{a_n\}_{n=1}^{\infty}$ a positive subadditive sequence, $\{w_n\}_{n=1}^{\infty}$ a sequence of positive weights, and $\{s_i\}_{i=1}^{\infty}$ an increasing sequence of natural numbers with $s_1 = 1$,*

$$\sum_{n=1}^{\infty} a_n w_n \leqslant \sum_{i=1}^{\infty} a_{s_i} \left(\sum_{n=s_i}^{\infty} b_i(n) w_n \right),$$

where

$$b_i(n) = \left[\frac{s_{i+1}}{s_i} \left\{ \frac{s_{i+2}}{s_{i+1}} \left\{ \cdots \left\{ \frac{s_{m_n}}{s_{m_n-1}} \left\{ \frac{n}{s_{m_n}} \right\} \right\} \cdots \right\} \right\} \right]$$

and $m_n = \sup\{i : s_i \leqslant n\}$.

Proof. The proof is mainly an interchange of the order of summation. Given any natural number n, we can write $n = \sum_{i=1}^{m_n} b_i(n) s_i$ where $b_i(n) \leqslant \left[\frac{s_{i+1}}{s_i}\right]$ and $\{s_i\}_{i=1}^{\infty}$ and m_n are as above. Thus, using the subadditivity of $\{a_n\}_{n=1}^{\infty}$, we have that

$$a_n = a_{\sum_{i=1}^{m_n} b_i(n) s_i} \leqslant \sum_{i=1}^{m_n} a_{b_i(n) s_i} \leqslant \sum_{i=1}^{m_n} b_i(n) a_{s_i},$$

so

$$\sum_{n=1}^{\infty} a_n w_n \leqslant \sum_{n=1}^{\infty} w_n \left(\sum_{i=1}^{m_n} b_i(n) a_{s_i} \right) = \sum_{i=1}^{\infty} a_{s_i} \left(\sum_{n=s_i}^{\infty} b_i(n) w_n \right).$$

We obtain the formula for $b_i(n)$ as follows. Choose $b_{m_n}(n)$ such that $0 \leqslant n - b_{m_n}(n) s_{m_n} < s_{m_n}$, so $b_{m_n}(n) = \left[\frac{n}{s_{m_n}}\right]$, and set $n_{m_n} = n - b_{m_n}(n) s_{m_n}$. Then choose $b_{m_n-1}(n)$ so that $0 \leqslant n_{m_n} - b_{m_n-1}(n) s_{m_n-1} < s_{m_n-1}$, so that

$$b_{m_n-1}(n) = \left[\frac{n_{m_n}}{s_{m_n-1}} \right] = \left[\frac{s_{m_n}}{s_{m_n-1}} \left\{ \frac{n}{s_{m_n}} \right\} \right].$$

Inductively, we define

$$b_k(n) = \left[\frac{n_{k+1}}{s_k}\right] \quad \text{and} \quad n_k = n_{k+1} - b_k(n)s_k,$$

for $k = 1, 2, \ldots, m_n$. Thus, solving for $b_i(n)$ we obtain

$$b_i(n) = \left[\frac{s_{i+1}}{s_i}\left\{\frac{s_{i+2}}{s_{i+1}}\left\{\cdots\left\{\frac{s_{m_n}}{s_{m_n-1}}\left\{\frac{n}{s_{m_n}}\right\}\right\}\cdots\right\}\right\}\right]$$

and the theorem is proven. ∎

Note that for any $k > i$, at each point of the form

$$\left\{\sum_{i<j<k} c_j s_j : c_j \in \mathbb{N},\ c_j \leqslant \left[\frac{s_{j+1}}{s_j}\right]\right\}$$

the function $b_i(n)$ drops to zero, remains zero on the interval of length s_i following the point and increases by a value of one on each interval of length s_i thereafter, until the next point of the above type is reached.

Using this description of $b_i(n)$, we can get a good bound on the quantity on the right in the inequality in Theorem 2.2. In particular, to obtain our invariant subspace results, when $w_n = 1/n^2$ we obtain the following.

COROLLARY 2.3. *If $\{a_n\}_{n=1}^{\infty}$ is subadditive and $\{s_i\}_{i=1}^{\infty}$ is an increasing sequence of natural numbers with $s_1 = 1$ then,*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \leqslant \sum_{i=1}^{\infty} \frac{a_{s_i}}{s_i}\left(\frac{2}{s_i} + \log\left(\frac{s_{i+1}}{s_i}\right)\right).$$

Proof. Using the bounds $b_i(n) = 0$ for $n \in [1, s_i)$ and $n \in [s_{i+1}, s_{i+1} + s_i)$, $b_i(n) \leqslant n/s_i$ for $n \in [s_i, s_{i+1})$, and $b_i(n) \leqslant s_{i+1}/s_i$ for $n \in [s_{i+1} + s_i, \infty)$, and Theorem 2.2, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a_n}{n^2} &\leqslant \sum_{i=1}^{\infty} \frac{a_{s_i}}{s_i}\left(\sum_{n=s_i}^{s_{i+1}} \frac{1}{n} + s_{i+1}\sum_{n=s_i+s_{i+1}}^{\infty} \frac{1}{n^2}\right)\\ &\leqslant \sum_{i=1}^{\infty} \frac{a_{s_i}}{s_i}\left(\frac{1}{s_i} + \log\left(\frac{s_{i+1}}{s_i}\right) + \frac{1}{s_{i+1}}\right)\\ &\leqslant \sum_{i=1}^{\infty} \frac{a_{s_i}}{s_i}\left(\frac{2}{s_i} + \log\left(\frac{s_{i+1}}{s_i}\right)\right)\end{aligned}$$

by the integral comparison test. ∎

3. ERGODIC AVERAGES AND INVARIANT SUBSPACES

As mentioned in the introduction, we shall apply the results of the previous to $\{C_n(f,\alpha)\}_{n=1}^{\infty}$. To do so, we need bounds on $C_n(f,\alpha)$ for certain values of n. These values will depend on α and can be given most conveniently in terms of continued fractions.

Given α, there exists unique $\{a_n\}_{n=1}^{\infty}$, positive natural numbers, called the partial quotients of α, such that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

which we will denote $\alpha = [a_0, a_1, a_2, a_3, \ldots]$. (See [6] for an introduction to continued fractions. All the properties we shall need are developed there.) The convergents of α are

$$\frac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i]$$

and have many well known properties. We shall use the following properties:

(3.1) The convergents satisfy the recurrence relations

$$q_{i+1} = a_i q_i + q_{i-1} \ \ (q_{-1} = 1,\ q_0 = 0), \quad p_{i+1} = a_i p_i + p_{i-1} \ \ (p_{-1} = 0,\ p_0 = 1).$$

(3.2) The convergents approximate α in the sense that

$$\frac{1}{q_i(q_{i+1} + q_i)} < \left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{q_i q_{i+1}} < \frac{1}{q_i^2}.$$

(3.3) The convergents are in lowest form, that is $\gcd(p_i, q_i) = 1$.

(3.4) The convergent denominators grow at least exponentially fast. In fact, $q_i \geqslant \tau^{i-1}$, where τ is the golden ratio, so in particular for all $\varepsilon > 0$, $\sum_{i=1}^{\infty} q_i^{-\varepsilon} < \infty$.

We shall see that bounds can be obtained for $C_n(f,\alpha)$, when n is a convergent denominator of α, for many functions f. There are two main classes of functions for which we shall obtain bounds; functions of bounded variation and smooth functions. To obtain the bounds on $C_{q_i}(f,\alpha)$ we refer the reader to the monograph of M.R. Herman on diffeomorphisms of the circle ([7]). Although we only consider the above two classes of functions in this paper, the results that follow can be extended to any function f for which bounds can be obtained on $C_{q_i}(f,\alpha)$. For example, results can be obtained for functions f whose Fourier coefficients satisfy certain properties, but these would be no better than known results (see [1] and [5]).

DEFINITION 3.1. A function f on $[0,1)$ is of *bounded variation* if

$$\|f\|_{\mathrm{BV}} = \sup\left\{|f(x_0) - f(x_n)| + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : \right.$$

$$\left. 0 \leqslant x_0 < x_1 < \cdots < x_{n-1} < x_n < 1\right\} < \infty.$$

PROPOSITION 3.2. (Denjoy-Koksma Inequality, ([7])) *For $\alpha \in [0,1)$ irrational, p_i/q_i a convergent of α, and f of bounded variation:*

$$C_{q_i}(f,\alpha) \leqslant \|f\|_{\mathrm{BV}}.$$

DEFINITION 3.3. The *modulus of continuity* of a function f on $[0,1)$ is defined as

$$\omega_f(\delta) = \sup\{|f(x) - f(y)| : |x-y| < \delta\},$$

for all $\delta > 0$. Then, for $s > 0$, let Λ_s be the Holder class of functions f on $[0,1)$ such that $\omega_f(\delta) \leqslant K\delta^s$ for some constant $K > 0$ and for all $\delta > 0$.

PROPOSITION 3.4. ([7]) *For $\alpha \in [0,1)$ is irrational, and p_i/q_i a convergent of α, and f with modulus of continuity $\omega_f(\delta)$*

$$C_{q_i}(f,\alpha) \leqslant q_i \omega_f\left(\frac{1}{q_i}\right).$$

Combining the above propositions with Corollary 2.3 and Theorem 1.1, we have the following theorems.

THEOREM 3.5. *If $\varphi \in L^\infty[0,1)$ is such that $\log|\varphi| \in \mathrm{BV}[0,1]$ and α is irrational with convergent denominators $\{q_i\}_{i=1}^{\infty}$ satisfying*

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

then the weighted rotation operator $M_\varphi U_\alpha$ is decomposable.

The condition

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

which occurs in Theorem 3.5 is equivalent to

$$\sum_{i=1}^{\infty} \frac{\log(q_{i+1})}{q_i} < \infty$$

which is known in dynamical systems as the *Brjuno condition* and is a natural condition for several "small-divisors problems" (see page 210 of [2]). It arises in that context as a sufficient condition for convergence of a certain series which defines an analytic transformation which transforms the dynamical system $x' = e^{2\pi i\alpha}x + xf(x)$ into $y' = e^{2\pi i\lambda}y$. However, there seems to be no direct connection between the "small-divisor problems" and Theorem 3.5.

THEOREM 3.6. *If $\varphi \in L^\infty[0,1)$ is such that $\log|\varphi|$ has modulus of continuity $\omega_{\log|\varphi|}$ and α is irrational with convergent denominators $\{q_i\}_{i=1}^\infty$ such that*

$$\sum_{i=1}^{\infty} \omega_{\log|\varphi|}\left(\frac{1}{q_i}\right)\log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

then the weighted rotation operator $M_\varphi U_\alpha$ is decomposable.

Proof of Theorems 3.5 *and* 3.6. Use the bounds obtained in Propositions 3.2 and 3.4, in Corollary 2.3 to obtain that, under the conditions stated in Theorems 3.5 and 3.6,

$$\sum_{n=1}^{\infty} \frac{C_n(\log|\varphi|,\alpha)}{n^2} < \infty.$$

(Here we use property 3.4 of convergents mentioned above). The results now follow from Theorem 1.1. ∎

COROLLARY 3.7. *If $\varphi \in L^\infty[0,1)$ is such that $\log|\varphi|$ is in the Holder class Λ_s and α is irrational with convergent denominators $\{q_i\}_{i=1}^\infty$ satisfying*

$$\sum_{i=1}^{\infty} \frac{1}{q_i^s}\log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

then the weighted rotation operator $M_\varphi U_\alpha$ is decomposable.

Theorems 3.5 generalizes results of [8], while Theorem 3.6 and Corollary 3.7 generalize results of [1], mainly by increasing the set of allowable α's.

In most previously obtained results, a necessary condition was that α not be a Liouville number. That is, there exists some constant C_α and some number n such that $|\alpha - p/q| \geqslant C_\alpha/q^n$ for all $p, q \in \mathbf{N}$ relatively prime. The set of all such α has full measure, however it is meager. An equivalent formulation is that there exists a number $n \in \mathbf{N}$ and a constant $k > 0$ such that $q_{i+1} \leqslant k q_i^n$ for all convergent denominators $\{q_i\}_{i=1}^\infty$. However, the condition that α have convergent denomimators satisfying

$$\sum_{i=1}^{\infty} \frac{1}{q_i}\log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

is satisfied when $q_{i+1} \leqslant K e^{q_i^\rho}$ for some $\rho < 1$. It is relatively straightforward to construct non-Liouville numbers α satisfying the above inequality via continued fractions. If $\{a_n\}_{n=1}^\infty$ are the partial quotients of α and $\{q_i\}_{i=1}^\infty$ are the convergent denominators then

$$\sum_{i=1}^{\infty} \frac{1}{q_i}\log\left(\frac{q_{i+1}}{q_i}\right) \leqslant \sum_{n=1}^{\infty} \frac{1}{\prod_{0<j<n} a_j}\log a_n$$

so our condition on the convergent denominators will be satisfied if we have a bound on the rate of growth of the partial quotients of α, however this bound is very lax. For example, if α has partial fraction expansion $\alpha = [a_0, a_1, \ldots]$ where the partial quotients are defined recursively by $a_0 = 0$, $a_1 = 1$ and $a_n = [\exp(a_1 a_2 \cdots a_{n-1}/n^2)]$ then the convergent denominators of α satisfy

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \log\left(\frac{q_{i+1}}{q_i}\right) < \infty$$

but α is a Liouville number.

4. INVARIANT SUBSPACES FOR $M_{r\chi_{[0,\beta)}} U_\alpha$

By Theorem 3.5, $M_{r\chi_{[0,\beta)}} U_\alpha$ is decomposable for all β when α is irrational with convergent denominators $\{q_i\}_{i=1}^{\infty}$ satisfying

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \log\left(\frac{q_{i+1}}{q_i}\right) < \infty.$$

To obtain invariant subspaces for $M_{r\chi_{[0,\beta)}} U_\alpha$ for a larger set of α, we must get bounds on $C_n(\chi_{[0,\beta)}, \alpha)$ for values of n other than convergent denominators. The following lemmas are the first steps in obtaining those bounds.

NOTATION 4.1. Recall that for $x \in \mathbf{R}$, $[x]$ denotes the greatest integer which is less than or equal to x, $\{x\} = x - [x]$ denotes the fractional part of x, and $\langle x \rangle$ denotes the distance from x to the nearest integer. Also, for q a natural number, let $\tau_{1/q}$ denote translation by $1/q$ modulo 1 on the interval $[0,1)$.

LEMMA 4.2. *For $\alpha \in [0,1)$ irrational with p, q natural numbers such that $|\alpha - p/q| < q^{-2}$, $q \geqslant 2$ and $\gcd(p,q) = 1$, and for $\beta \in [0,1)$, set*

$$I = \bigcup_{k=0}^{q-1} \tau_{\frac{1}{q}}^{-k} \left[\langle q\alpha \rangle, \frac{1}{q}\{q\beta\} - \langle q\alpha \rangle\right), \tag{4.1}$$

$$J = \bigcup_{k=0}^{q-1} \tau_{\frac{1}{q}}^{-k} \left[\frac{1}{q}\{q\beta\} + \langle q\alpha \rangle, \frac{1}{q} - \langle q\alpha \rangle\right), \tag{4.2}$$

$$N = (I \cup J)^c. \tag{4.3}$$

Then the following are true:

(i) *If* $x \in I$, $[q\beta]+1$ *of* $\{x+j\alpha\}_{j=0}^{q-1}$ *are in* $[0,\beta)$.

(ii) *If* $x \in J$, $[q\beta]$ *of* $\{x+j\alpha\}_{j=0}^{q-1}$ *are in* $[0,\beta)$.

(iii) *If* $x \in N$, *at most* $[q\beta]+2$, *and at least* $[q\beta]-1$ *of* $\{x+j\alpha\}_{j=0}^{q-1}$ *are in* $[0,\beta)$.

Proof. Suppose $x \in I$, then $\{x+j(p/q)\}_{j=0}^{q-1}$ are also in I, and are evenly distributed at a distance of $1/q$ apart (identifying 0 with 1). Now

$$\left|\left(x+j\frac{p}{q}\right)-(x+j\alpha)\right| < j\left|\alpha-\frac{p}{q}\right| < q\left|\alpha-\frac{p}{q}\right| < |q\alpha-p| \leqslant \langle q\alpha\rangle$$

so each $\{x+j\alpha\}_{j=0}^{q-1}$ is at most at distance $\langle q\alpha\rangle$ from I. So

$$\{x+j\alpha\}_{j=0}^{q-1} \in \bigcup_{k=0}^{q-1} \tau_{\frac{1}{q}}^{-k}\left[0,\frac{1}{q}\{q\beta\}\right)$$

and only one point is in each interval $\tau_{\frac{1}{q}}^{-k}[0,\frac{1}{q}\{q\beta\})$. Now $\beta \in [[q\beta]/q,([q\beta]+1)/q)$, so we have $[q\beta]$ of the points $\{x+j\alpha\}_{j=0}^{q-1}$ in $[0,\beta)$ corresponding to the intervals $\tau_{\frac{1}{q}}^{-k}[0,\{q\beta\}/q)$ with $k<[q\beta]$ and we have one more point when $k=[q\beta]$ since

$$\tau_{\frac{1}{q}}^{-[q\beta]}\left[0,\frac{1}{q}\{q\beta\}\right) = \left[\frac{[q\beta]}{q},\frac{[q\beta]+\{q\beta\}}{q}\right) = \left[\frac{[q\beta]}{q},\beta\right).$$

Thus, if $x \in I$, $[q\beta]+1$ of $\{x+j\alpha\}_{j=0}^{q-1}$ are in $[0,\beta)$.

When $x \in J$, this last point (corresponding to the interval where $k=[q\beta]$) misses $[0,\beta)$, and hence only $[q\beta]$ of $\{x+j\alpha\}_{j=0}^{q-1}$ are in $[0,\beta)$.

Finally, when $x \in N$, the location of two of the points $\{x+j\alpha\}_{j=0}^{q-1}$ cannot be established, so we obtain the above bounds. ∎

Using this lemma, we obtain the following bounds.

LEMMA 4.3. *For* α,β *as in Lemma* 4.2, *and* $\{q_i\}_{i=1}^{\infty}$ *the convergent denominators of* α;

(i) *For* $k=1,2,\ldots,\left[\frac{q_{i+1}\{q_i\beta\}}{q_i}\right]$,

$$k(1-\{q_i\beta\})-6 \leqslant \sup_{x\in[0,1)} \sum_{j=0}^{kq_i-1} \chi_{[0,\beta)}\{x+j\alpha\}-kq_i\beta \leqslant k(1-\{q_i\beta\})+6.$$

(ii) *For* $k=\left[\frac{q_{i+1}\{q_i\beta\}}{q_i}\right],\ldots,\left[\frac{q_{i+1}}{q_i}\right]$,

$$\frac{\{q_i\beta\}}{q_i}(q_{i+1}-kq_i)-6 \leqslant \sup_{x\in[0,1)} \sum_{j=0}^{kq_i-1} \chi_{[0,\beta)}\{x+j\alpha\}-kq_i\beta \leqslant \frac{\{q_i\beta\}}{q_i}(q_{i+1}-kq_i)+6.$$

Proof. Define

$$c_n(x) = \sum_{j=0}^{n-1} \chi_{[0,\beta)}\{x + j\alpha\} - n\beta.$$

Then $c_n(x)$ has the following properties:

(4.4) For $k = 1, \dots, \left[\frac{q_{i+1}}{q_i}\right]$

$$c_{kq_i}(x) = \sum_{j=0}^{k-1} c_{q_i}\left(\{x + j\{q_i\alpha\}\}\right).$$

(4.5) By Lemma 4.2,

$$c_{q_i}(x) = \begin{cases} 1 - \{q_i\beta\}, & \text{if } x \in I; \\ -\{q_i\beta\}, & \text{if } x \in J; \\ n - \{q_i\beta\}, & \text{if } x \in N; \end{cases}$$

where $n = -1, 0, 1$ or 2.

Fix $x \in [0, 1)$ and consider the points $\{x + j\{q_i\alpha\}\}_{j=0}^{k-1}$ associated with points on the unit circle. These are k points, each at a distance of exactly $\langle q_i\alpha\rangle$ from the next. When

$$k < \left[\frac{1}{q_i\langle q_i\alpha\rangle}\right],$$

these points are all within $1/q_i$ of each other.

Let I, J and N denote the sets in Lemma 4.2 corresponding to the convergent p_i/q_i. The width of an interval of N is $2\langle q_i\alpha\rangle$, and at most two such intervals of N meet any interval of width $1/q_i$. Similarly, the width of an interval of I (resp. J) is $(1/q_i)\{q_i\beta\} - 2\langle q_i\alpha\rangle$ (resp. $(1/q_i)(1 - \{q_i\beta\}) - 2\langle q_i\alpha\rangle$) and only one interval of I (resp. J) meets any interval of width $1/q_i$. Therefore, given the k points $\{x + j\{q_i\alpha\}\}_{j=0}^{k-1}$, $c_{kq_i}(x)$ will be the largest when the supremum of the largest number of the $c_{q_i}(\{x + j\{q_i\alpha\}\})$ is attained. This will happen when 4 of these points are in N, $\frac{1}{q_i}\frac{\{q_i\beta\}}{\langle q_i\alpha\rangle} - 2$ are in I (or $k - 4$ if there isn't that many points) and the rest are in J. Thus $\sup_{x\in[0,1)} c_{kq_i}(x)$ increases at a rate of $1 - \{q_i\beta\}$ until $k\langle q_i\alpha\rangle > \frac{\{q_i\beta\}}{q_i}$ and then decreases at a rate of $-\{q_i\beta\}$. We have an error of ± 4 corresponding to the four points in N.

Convergent denominators satisfy

$$\frac{1}{q_i(q_{i+1} + q_i)} < \langle q_i\alpha\rangle < \frac{1}{q_{i+1}} \quad \text{so} \quad \left[\frac{q_{i+1}}{q_i}\right] < \left[\frac{1}{q_i\langle q_i\alpha\rangle}\right] < \left[\frac{q_{i+1}}{q_i}\right] + 1$$

so replacing $\left[\frac{1}{q_i\langle q_i\alpha\rangle}\right]$ by $\left[\frac{q_{i+1}}{q_i}\right]$ above will only increase the bound by ± 2, to give the estimate in the statement of the lemma to within ± 6. ∎

COROLLARY 4.4. *For α, β as in Lemma 4.2, and $\{q_i\}_{i=1}^{\infty}$ the convergent denominators of α;*

(i) *For $k = 1, 2, \ldots, \left[\frac{q_{i+1}}{q_i}\right]$,*

$$C_{kq_i}\left(\chi_{[0,\beta)}, \alpha\right) \leqslant \begin{cases} k(1 - \langle q_i\beta\rangle) + 6, & \text{if } 1 \leqslant k \leqslant \left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]; \\ \frac{\langle q_i\beta\rangle q_{i+1}}{q_i} + 6, & \text{if } \left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right] < k \leqslant \left[\frac{q_{i+1}}{q_i}\right]. \end{cases}$$

(ii) *For $k = 1, 2, \ldots, \left[\frac{q_{i+1}}{2q_i}\right]$,*

$$C_{kq_i}\left(\chi_{[0,\beta)}, \alpha\right) \geqslant k\langle q_i\beta\rangle - 6.$$

Proof. If we set

$$c_n^{\beta}(x) = \sum_{j=0}^{n-1} \chi_{[0,\beta)}\{x + j\alpha\} - n\beta,$$

then

$$C_n\left(\chi_{[0,\beta)}, \alpha\right) = \sup_{x\in[0,1)} \left\{c_n^{\beta}(x), -c_n^{\beta}(x)\right\}.$$

Lemma 4.3 supplies the upper bound on $c_n^{\beta}(x)$, and will also supply the lower bound by using the identity $c_n^{\beta}(x) = -c_n^{(1-\beta)}(x)$. Using these bounds, and the fact that $1 - \langle q_i\beta\rangle = \max(\{q_i\beta\}, \{q_i(1-\beta)\})$ the result follows. ∎

With the above bounds, we can bound the rate of growth of $\left\{C_n\left(\chi_{[0,\beta)}, \alpha\right)\right\}_{n=1}^{\infty}$ using Corollary 2.3 with $\{s_i\}_{i=1}^{\infty} = \{kq_i : k = 1, \ldots, [q_{i+1}/q_i]\}_{i=1}^{\infty}$.

THEOREM 4.5. *Given $\beta \in [0, 1)$, and $\alpha \in [0, 1)$ irrational with convergent denominators $\{q_i\}_{i=1}^{\infty}$*

$$\text{if} \quad \sum_{i=1}^{\infty} \frac{1}{q_i} \ln\left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right) < \infty \quad \text{then} \quad \sum_{n=1}^{\infty} \frac{C_n(\chi_{[0,\beta)}, \alpha)}{n^2} < \infty.$$

Proof. Consider the sequence

$$\left\{kq_i : k = 1, 2, \ldots, \left[\frac{q_{i+1}}{q_i}\right]\right\} \quad \text{for} \quad i = 1, 2, \ldots.$$

Since this sequence is well distributed in the set of natural numbers, $b_{kq_i}(n)$ will be zero for most values of n, and will never be greater than one. It will be nonzero only on

$$[kq_i, (k+1)q_i),\ [q_{i+1} + kq_i, q_{i+1} + (k+1)q_i),\ [2q_{i+1} + kq_i, 2q_{i+1} + (k+1)q_i) \quad \text{etc.}$$

So $b_{kq_i}(n)$ will be zero k times as often as it is one, and using the bound

$$\int_{a+kq_i}^{a+(k+1)q_i} \frac{1}{x^2}\,\mathrm{d}x \leqslant \frac{1}{k} \int_{a}^{a+(k+1)q_i} \frac{1}{x^2}\,\mathrm{d}x$$

and the integral comparison test, we obtain that

$$\begin{aligned}\sum_{n=kq_i}^{\infty} \frac{b_{kq_i}(n)}{n^2} &\leqslant \sum_{n=kq_i}^{(k+1)q_i} \frac{1}{n^2} + \frac{1}{k}\int_{q_{i+1}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \\ &\leqslant \frac{1}{k^2 q_i} - \frac{1}{k}\frac{1}{x}\Big|_{q_{i+1}}^{\infty} \\ &\leqslant \frac{1}{k^2 q_i} + \frac{1}{k q_{i+1}}.\end{aligned}$$

Since $q_{i+1} \geqslant kq_i$ the above implies that

$$\sum_{n=kq_i}^{\infty} \frac{b_{kq_i}(n)}{n^2} \leqslant \frac{2}{k^2 q_i}.$$

Applying Corollary 2.3, with the sequence above, and the bounds from Corollary 4.4, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{C_n(f,\alpha)}{n^2} &\leqslant 2\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\left[\frac{q_{i+1}}{q_i}\right]} \frac{C_{kq_i}\left(\chi_{[0,\beta)},\alpha\right)}{k^2 q_i}\right) \\ &\leqslant 2\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]} \frac{k(1-\langle q_i\beta\rangle)}{k^2 q_i} + \sum_{k=\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]+1}^{\left[\frac{q_{i+1}}{q_i}\right]-1} \frac{\langle q_i\beta\rangle q_{i+1}}{q_i}\frac{1}{k^2 q_i}\right. \\ &\qquad \left. + \sum_{k=1}^{\left[\frac{q_{i+1}}{q_i}\right]} \frac{6}{k^2 q_i} + \left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i} + 6\right)\frac{1}{\left[\frac{q_{i+1}}{q_i}\right]^2 q_i}\right).\end{aligned}$$

Consider each summand separately. The last term is the "rollover term", that is the term associated with the gap between $\left[\frac{q_{i+1}}{q_i}\right] q_i$ and the next point in the sequence, q_{i+1}. Since this gap is small we can bound the sum over $i = 1, 2, \ldots$ of the last terms easily. It is clearly bounded by $7\sum_{i=1}^{\infty} \frac{1}{q_{i+1}}$ which is finite since convergent denominators grow at least exponentially fast.

The sum over $i = 1, 2 \ldots$ of the third terms is

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\left[\frac{q_{i+1}}{q_i}\right]} \frac{6}{k^2 q_i} \leqslant \sum_{i=1}^{\infty} \frac{6}{q_i} \sum_{k=1}^{\left[\frac{q_{i+1}}{q_i}\right]} \frac{1}{k^2} \leqslant \sum_{i=1}^{\infty} \frac{12}{q_i}$$

which is also finite.

The sum over $i = 1, 2, \ldots$ of the second terms is

$$\sum_{i=1}^{\infty} \sum_{k=\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]+1}^{\left[\frac{q_{i+1}}{q_i}\right]-1} \frac{\langle q_i\beta\rangle q_{i+1}}{k^2 q_i^2} \leqslant \sum_{i=1}^{\infty} \frac{\langle q_i\beta\rangle q_{i+1}}{q_i^2} \int_{\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]}^{\left[\frac{q_{i+1}}{q_i}\right]} \frac{1}{x^2}\,\mathrm{d}x$$

$$\leqslant \sum_{i=1}^{\infty} \frac{\langle q_i\beta\rangle q_{i+1}}{q_i^2} \left(\frac{q_i}{q_{i+1}\langle q_i\beta\rangle} - \frac{q_i}{q_{i+1}} \right)$$

$$\leqslant \sum_{i=1}^{\infty} \langle q_i\beta\rangle \left(\frac{1}{q_i\langle q_i\beta\rangle} - \frac{1}{q_i} \right) \leqslant \sum_{i=1}^{\infty} \frac{1}{q_i}$$

which is also finite.

The sum over $i = 1, 2, \ldots$ of the first terms is clearly finite if

$$\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right] = 1$$

while if this quantity is greater than 1 we have that

$$\sum_{i=1}^{\infty} \frac{(1 - \langle q_i\beta\rangle)}{q_i} \sum_{k=1}^{\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]} \frac{1}{k} \leqslant \sum_{i=1}^{\infty} \frac{1}{q_i} \left(1 + \int_{1}^{\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right]} \frac{1}{x}\,\mathrm{d}x \right)$$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{q_i} \left(1 + \ln\left(\left[\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right] \right) \right)$$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{q_i} + \sum_{i=1}^{\infty} \frac{1}{q_i} \ln\left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i} \right)$$

which is finite if and only if

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \ln\left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i} \right) < \infty$$

so the theorem is proven. ∎

COROLLARY 4.6. *If α is an irrational number with convergent denominators $\{q_i\}_{i=1}^{\infty}$ and $\beta \in (0,1)$ is such that*

$$\sum_{i=1}^{\infty} \frac{1}{q_i} \ln\left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right) < \infty$$

then the weighted translation operator $M_{r^{\chi_{[0,\beta)}}} U_\alpha$ is decomposable.

Proof. Follows from Theorem 1.1 and Theorem 4.5. ∎

For all α, if $\beta = \{m\alpha\}$ for some integer m then clearly $M_{r^{\chi_{[0,\beta)}}} U_\alpha$ has a nontrivial invariant subspace. Of course, this follows from a result of Petersen ([9]). However, Corollary 4.6 covers many new operators. Here we give just one way of constructing examples.

COROLLARY 4.7. *If α has partial fraction expansion $[a_0, a_1, \ldots]$ with $a_0 = 0$ and all other partial quotients a_i odd, with*

$$\sum_{i \text{ even}} \frac{1}{\prod_{0<j<i} a_j} \ln a_i < \infty$$

then $M_{r^{\chi_{[0,\frac{1}{2})}}} U_\alpha$ has a nontrivial invariant subspace.

Proof. As usual, let p_i/q_i denote the convergents of α. All a_i odd implies that q_i is odd if and only if i is even, and p_i is odd if and only if i is odd. Now, $\beta = 1/2$, so $\langle q_i\beta\rangle = 0$ if q_i is even, and $\langle q_i\beta\rangle = 1/2$ if q_i is odd. Thus

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{q_i} \ln\left(\frac{q_{i+1}\langle q_i\beta\rangle}{q_i}\right) &\leqslant \sum_{i \text{ even}} \frac{1}{q_i} \ln\left(\frac{q_{i+1}}{q_i}\right) \\ &\leqslant \sum_{i \text{ even}} \frac{1}{\prod_{0<j<i} a_j} \ln a_i.\end{aligned}$$

(The last step follows from the fact that $q_{i+1} = a_i q_i + q_{i-1}$.) The result follows from Corollary 4.6. ∎

Note that we can extend these results to $\beta = r/s$ or $\beta = (r/s)\alpha$. Also, the condition that all the a_i be odd is not necessary in general, but simplifies the statement of the corollary. Note that Corollary 4.7 states that we can get invariant subspaces for $M_{r^{\chi_{[0,\frac{1}{2})}}} U_\alpha$ while having only some of the partial quotients constrained, while the results of Section 3 require growth bounds on all the partial quotients.

The following result shows that the growth rate of $\left\{C_n(\chi_{[0,\beta)}, \alpha)\right\}_{n=1}^{\infty}$ does indeed depend on α and β, and that the only growth bound that we know is universally true for all α and β is that $\frac{C_n(\chi_{[0,\beta)},\alpha)}{n} \to 0$. (This follows from the unique ergodicity of translation by α.) This example also shows that the method of using bounds on the norms of the powers of an weighted translation operator to obtain invariant subspaces is not generalizable to all cases, even when the weight is $r^{\chi_{[0,\frac{1}{2})}}$.

COROLLARY 4.8. *There exist $\alpha \in [0,1)$ irrational such that*

$$\sum_{n=1}^{\infty} \frac{C_n(\chi_{[0,\frac{1}{2})},\alpha)}{n^2} = \infty.$$

Proof. Let $\alpha = [a_0, a_1, \ldots]$ with $a_0 = 0$, all the partial quotients even, and

$$\ln\left(\frac{a_i}{2}\right) \geqslant \prod_{0<j<i} a_j^2.$$

Since all the partial quotients are even, q_i is odd for all $i > 1$, so $\langle q_i \frac{1}{2} \rangle = 1/2$ for all $i > 1$. Thus

$$\sum_{n=1}^{\infty} \frac{C_n(\chi_{[0,\beta)},\alpha)}{n^2} \geqslant \sum_{i=1}^{\infty} \sum_{k=1}^{\left[\frac{q_{i+1}}{2q_i}\right]} \frac{C_{kq_i}(\chi_{[0,\beta)},\alpha)}{k^2 q_i^2}$$
$$\geqslant \sum_{i=1}^{\infty} \sum_{k=1}^{\left[\frac{q_{i+1}}{2q_i}\right]} \frac{k}{2k^2 q_i^2} - \sum_{k=1}^{\left[\frac{q_{i+1}}{2q_i}\right]} \frac{6}{k^2 q_i^2}$$

by Corollary 4.4 (ii). The second term is finite since

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\left[\frac{q_{i+1}}{2q_i}\right]} \frac{6}{k^2 q_i^2} \leqslant \sum_{i=1}^{\infty} \frac{6}{q_i^2}\left(1 + \int_1^{\frac{q_{i+1}}{q_i}} \frac{1}{x^2}\,dx\right)$$
$$\leqslant \sum_{i=1}^{\infty} \frac{6}{q_i^2}\left(2 - \frac{q_i}{q_{i+1}}\right) < \sum_{i=1}^{\infty} \frac{6}{q_i^2} < \infty.$$

We can bound the sum of the first terms by

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\left[\frac{q_{i+1}}{2q_i}\right]} \frac{k}{2k^2 q_i^2} \geqslant K_1 \sum_{i=1}^{\infty} \frac{1}{q_i^2} \int_1^{\frac{q_{i+1}}{2q_i}} \frac{1}{x}\,dx$$
$$\geqslant K_1 \sum_{i=1}^{\infty} \frac{1}{q_i^2} \ln\left(\frac{q_{i+1}}{2q_i}\right)$$
$$\geqslant K_2 \sum_{i=1}^{\infty} \frac{1}{\prod_{j<i} a_j^2} \ln\left(\frac{a_i}{2}\right)$$

for some constants K_1, K_2. This is infinite because of our conditions on the partial quotients of α. ∎

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