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C^*-ALGEBRA TECHNIQUES IN NUMERICAL ANALYSIS

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ABSTRACT. The topic of the present paper is a general approach of studying invertibility problems in algebras the elements of which are sequencesof operators. These sequences can be viewed as approximation sequences for a given operator, and the proposed approach allows to relateproperties of the approximation sequence (stable convergence, limiting sets of spectra, Moore-Penrose invertibility, asymptotic behaviour of the condition numbers) with corresponding properties of a certain function, the symbol of the sequence. This method applies to practically relevantapproximation methods such as the finite section method for Toeplitzoperators and spline projection methods for singular integral equations with piecewise continuous coefficients as well as for Mellin operators.

KEYWORDS: *Approximation methods, C^*-algebra techniques, Toeplitz operators.*
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1. INTRODUCTION

Let H be a Hilbert space and $L(H)$ be the C^*-algebra of all linear and bounded operators on H, and suppose we are given an operator A in $L(H)$ and a sequence (A_n) of operators $A_n \in L(H)$ tending to A in the strong operator topology of H (i.e. $\|A_nx - Ax\| \to 0$ for all $x \in H$). For instance one can have in mind an operator equation

$$Ax = y \quad (x, y \in H) \tag{1.1}$$

which is tried to be solved by a certain "reasonable" approximation method

$$A_nx_n = y \quad (x_n, y \in H). \tag{1.2}$$

In the present paper we are not mainly interested in the applicability of the method (1.2) to the problem (1.1) (i.e. in the convergence of the solutions x_n of (1.2) to a solution x of (1.1), although this aspect is also partially covered) but we would rather like to think of (A_n) as a sequence of approximation operators for the operator A and we ask for relations between the operators A_n with large n and their limitoperator A. Of course, there is no one-to-one correspondence between properties of A and A_n (indeed, these operators are in general of a quite different nature; for example, for solving (1.1) one will certainly choose the operators A_n to be finite matrices, hence compact). Nevertheless, we shall point out that, for large classes of operators A and sequences (A_n), numerous properties of A are reflected by properties of (A_n) at least asymptotically. Note also that the approximation of A by A_n is not a one-way street. Considering for example the Ising modell in statistical physics one is soon led to Toeplitz matrices whose order is proportional to the number of involved particles which can easily exceed 10^{23}. In this situation, any computational effort seems to be non-sense, and one would rather try to replace "giant" matrices by their infinite limit operators and to hope that the limit operators can tell us something about their finite approximations.

In the present paper we shall study the correspondence between (A_n) and A by embedding the sequences we are interested in into a suitably chosen C^*-algebra which owns a special structure. In order to motivate this structure, we start with examining a typical and stimulating example in the following section: the approximation of Toeplitz operators by their finite sections. In Sections 3 and 4 we are going to employ this structure, beginning with general C^*-algebras and proceeding with algebras of sequences of operators, in order to derive relations between the norms, eigenvalues, s-numbers, spectra, pseudospectra etc. of A_n and A, and in the concluding section we shall recall some further concrete algebras of approximation sequences of practical relevance which are just subject to our assumptions.

2. FINITE SECTIONS OF TOEPLITZ OPERATORS

Let us illustrate our programme by an archetypical example: the finite section method for Toeplitz operators with piecewise continuous generating function. Due to its practical relevance (for example, the proof of the Fisher-Hartwig conjecture by A. Böttcher and one of the authors is based essentially on the stable convergence of certain Toeplitz matrices) and to its rich and beauty structure, the finite section method for Toeplitz operators became a standard model in the functional-analytic theory of approximation methods (see, e.g.,[4], [6], [10], [18], [22], [25]).

Let l^2 denote the Hilbert space of all sequences $(x_n)_{n\geqslant 0}$ of complex numbers with inner product$((x_k),(y_k)) = \sum_{k=0}^{\infty} x_k\overline{y_k}$ and corresponding norm $\|(x_k)\| = ((x_k),(x_k))^{1/2}$.

Given a function $a \in L^\infty(\mathbf{T})$, we let

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}z})\,\mathrm{e}^{-\mathrm{i}nz}\,\mathrm{d}z$$

be its nth Fourier coefficient. The *Toeplitz operator* $T(a) : l^2 \to l^2$ generated by a is defined by

$$T(a)(x_k) = (y_k) \quad \text{with} \quad y_k = \sum_{l=0}^{\infty} a_{k-l}x_l.$$

This operator is bounded, and its norm is equal to $\sup_{t\in\mathbf{T}}|a(t)|$. The matrices $T_n(a) = (a_{i-j})_{i,j=0}^{n-1}$ are referred to as *Toeplitz matrices*. Introducing operators $P_n : l^2 \to l^2, (x_k) \mapsto (x_0,\ldots,x_{n-1},0,0,\ldots)$ we can identify the Toeplitz matrix $T_n(a)$ (acting on $\mathbf{C}^n$) with the *finite section* $P_nT(a)P_n$ of the Toeplitz operator $T(a)$ (acting on l^2), and we shall make use of this identification troughout what follows.

We say that the *finite section method applies to the operator* $T(a)$ if the equations

$$P_nT(a)P_nx^{(n)} = P_ny$$

are uniquely solvable for all $n \geqslant n_0$ and for all right hand sides $y \in l^2$, and if their solutions $x^{(n)} \in \operatorname{Im} P_n$ converge in the l^2-norm to a solution of the equation

$$T(a)x = y.$$

One of the authors showed in [25] the following result.

THEOREM 2.1. *Let a be a piecewise continuous function (i.e. a function possessing one-sided limits at each point of the unit circle $\mathbf{T}$). Then the finite section method applies to $T(a)$ if and only if both operators $T(a)$ and $T(\tilde{a})$ with $\tilde{a}(t) = a(1/t)$ are invertible.*

One can show that a piecewise continuous function has at most a countable number of discontinuities. In case the number of discontinuities is even finite, Theorem 2.1 goes back to [6]. Moreover one should remark that the invertibility of $T(\tilde{a})$ is already a consequence of that one of $T(a)$, but in the form quoted above the theorem remains valid for matrix valued generating functions as well as for operators acting on l^p-spaces with $1 < p < \infty$, too.

We shall not give a proof of this theorem here, but we want to sketch how C^*-algebra techniques apply to derive this and related results.

First we need an equivalent formulation of applicability of the finite section method. It is not hard to see that the finite section method applies if and only if the sequence $(T_n(a))$ is *stable*, i.e. if and only if the operators $T_n(a) : \operatorname{Im} P_n \to \operatorname{Im} P_n$ are invertible for all *sufficiently large* n,say for $n \geqslant n_0$, and if the norms of their inverses are *uniformly* bounded. The advantage of this reformulation is that stability of a sequence can be translated into an invertibility problem in a suitably chosen C^*-algebra. Indeed, let $\mathcal{F}$ stand for the set of all sequences $(A_n), A_n : \operatorname{Im} P_n \to \operatorname{Im} P_n$. Defining operations by

$$(A_n) + (B_n) = (A_n + B_n) \quad \text{and} \quad (A_n)(B_n) = (A_n B_n),$$

an involution by

$$(A_n)^* = (A_n^*),$$

and a norm by

$$\|(A_n)\| = \sup_n \|A_n\|,$$

one can make $\mathcal{F}$ to become a C^*-algebra. Clearly, the sequence (P_n) is the identity element in this algebra, and it is also easy to check that a sequence $(A_n) \in \mathcal{F}$ is invertible in $\mathcal{F}$ if and only if *all* matrices A_n are invertible and if$\sup_{n\geqslant 0} \|A_n^{-1}\| < \infty$.

Well, this is not yet stability, but there is a simple trick to manage this point. Namely, the set $\mathcal{G}$ of all sequences $(G_n) \in \mathcal{F}$with $\|G_n\| \to 0$ as $n \to \infty$ forms a closed two-sided ideal of the algebra $\mathcal{F}$, and a little thought reveals that the *coset*$(A_n) + \mathcal{G}$ is invertible in the *quotient algebra* $\mathcal{F}/\mathcal{G}$ if and only if the matrices A_n are invertible beginning with a subscript n_0, and if $\sup_{n\geqslant n_0} \|A_n^{-1}\| < \infty$, which exactly means stability.

So we are left with an invertibility problem in the C^*-algebra$\mathcal{F}/\mathcal{G}$. For our purposes it is more convenient to work in a smaller algebra than $\mathcal{F}/\mathcal{G}$. Let $\mathcal{A}$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences $(P_nT(a)P_n)$ with a running through the piecewise continuous functions. One can show (see, e.g., [4], Proposition 7.27) that $\mathcal{G} \subset \mathcal{A}$, hence, one can form the quotient algebra $\mathcal{A}/\mathcal{G}$, and this algebra can be viewed as a $*$-subalgebra of $\mathcal{F}/\mathcal{G}$.

One might ask whether invertibility of a coset $(A_n) + \mathcal{G}$ in$\mathcal{F}/\mathcal{G}$ respective in $\mathcal{A}/\mathcal{G}$ correspond to the same "stabilities". But, fortunately, we are dealing with C^*-algebras which implies that, if for a sequence (A_n) the coset $(A_n) + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$, then it is also invertible in $\mathcal{A}/\mathcal{G}$, thus, no problems arise when working in $\mathcal{A}/\mathcal{G}$ rather than in $\mathcal{F}/\mathcal{G}$.

In what follows, we need a further family of operators:

$$W_n : l^2 \to l^2, \quad W_n(x_k) = (x_{n-1}, \ldots, x_1, x_0, 0, 0, \ldots).$$

Obviously, $W_n^2 = P_n$ and $W_n P_n = P_n W_n = W_n$. H. Widom established the following fascinating formula relating the finite sections of a Toeplitz operator the generating function of which is a product ab of two functions with the product of the finite sections of the Toeplitz operators with generating functions a and b: If $a, b \in L^\infty(\mathbf{T})$ then

$$T_n(ab) = T_n(a)T_n(b) + P_n H(a)H(\tilde{b})P_n + W_n H(\tilde{a})H(b)W_n$$

where $W(a) : l^2 \to l^2$ refers to the *Hankel operator*

$$H(a)(x_k) = (y_k) \quad \text{with} \quad y_k = \sum_{l=0}^{\infty} a_{k+l+1} x_l.$$

Widom's formula yields that

$$\begin{aligned} T_n(a)T_n(b) - T_n(b)T_n(a) = P_n(H(a)H(\tilde{b}) - H(b)H(\tilde{a}))P_n \\ + W_n(H(\tilde{a})H(b) - H(\tilde{b})H(a))W_n, \end{aligned}$$

and since the operators $H(a)H(\tilde{b}) - H(b)H(\tilde{a})$ and $H(\tilde{a})H(b) - H(\tilde{b})H(a)$ are compact whenever a and b are piecewise continuous (see [4], Theorem 5.34), we see that any two sequences $(T_n(a))$ and $(T_n(b))$ commute modulo sequences of the form $(P_n K_0 P_n + W_n K_1 W_n)$ with compact operators K_0 and K_1. Pushing forward this observation one can even show that the set $\mathcal{J}$ of all sequences (A_n) with

$$A_n = P_n K_0 P_n + W_n K_1 W_n + G_n$$

with K_0, K_1 compact and $(G_n) \in \mathcal{G}$ is contained in $\mathcal{A}$, forms a closed two-sided ideal of $\mathcal{A}$, and that this ideal is nothing else than the commutator ideal of the algebra $\mathcal{A}$.

The commutativity of the quotient algebra $\mathcal{A}/\mathcal{J}$ gives rise to the hope that it could be tackled by means of Gelfand's local spectral theory. Before explaining this point we have to check another problem: suppose we are given a sequence (A_n) for which the coset $(A_n) + \mathcal{J}$ is invertible. In which way is this invertibility related with our original problem, i.e. invertibility of $(A_n) + \mathcal{G}$?

The answer is given by an useful observation by one of the authors (see [25]) which now is usually referred to as a *lifting theorem.*

Namely, one can show that, for each sequence $(A_n) \in \mathcal{A}$, the strong limits

$$\text{s-lim}\, A_n P_n =: W_0(A_n)$$

and

$$\text{s-lim}\, W_n A_n W_n =: W_1(A_n)$$

exist (this is not hard; one only has to bear in mind that $P_n \to I$ strongly and that $W_n T_n(a) W_n = P_n T(\tilde{a}) P_n$) and that a sequence (A_n) is invertible modulo $\mathcal{G}$ if and only if both operators $W_0(A_n)$ and $W_1(A_n)$ are invertible and if the coset $(A_n) + \mathcal{J}$ is invertible in $\mathcal{A}/\mathcal{J}$ (a proof of a more general assertion will be given in the following section).

And what about invertibility in $\mathcal{A}/\mathcal{J}$? Now, it is indeed possible (although not a triviality) to identify the maximal ideal space of this commutative C^*-algebra (it is the cylinder $\mathbf{T} \times [0,1]$ provided with an exotic topology, see [3] and [4], Theorem 7.44). Further, this maximal ideal space coincides exactly with the maximal ideal space of the C^*-algebra generated by all Toeplitz operators with piecewise continuous generating function modulo the compact operators which was characterized by Gohberg and Krupnik (compare [4], Theorem 5.46). This identification can be used to show that if the operators $W_0(A_n)$ and $W_1(A_n)$ are Fredholm then the coset $(A_n) + \mathcal{J}$ is invertible. What results is the following theoremdue to Böttcher and Silbermann ([3] or [4], Sections 7.35–7.45).

THEOREM 2.2. *A sequence $(A_n) \in \mathcal{A}$ is stable if and only if both operators $W_0(A_n)$ and $W_1(A_n)$ are invertible.*

An equivalent formulation of this theorem, given in [27], is

THEOREM 2.3. *The algebra $\mathcal{A}/\mathcal{G}$ is isometrically isomorphic to the smallest closed subalgebra of $L(l^2) \times L(l^2)$ spanned by all pairs $(T(a), T(\tilde{a}))$ with a being piecewise continuous.*

In what follows we are going to show (in an essentially more general situation) that the isomorphism established in Theorem 3 is really a keyobservation. One can express a bulk of relations between an approximation sequence (A_n) and its limit operator A in terms of properties of the pair $(W_0(A_n), W_1(A_n))$.

3. IDEAL-LIFTING HOMOMORPHISMS

3.1. LIFTING IDEALS BY HOMOMORPHISMS. Let $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras with unit elements, let $\mathcal{I}$ be a closed two-sided $*$-ideal of $\mathcal{B}$, and let $W : \mathcal{B} \to \mathcal{C}$ be a $*$-homomorphism which sends the unit element of $\mathcal{B}$ into that of $\mathcal{C}$. Recall that a $*$-homomorphism is automatically continuous, and that its norm is less than or equal to 1 (compare [14], Theorem 4.1.8 (a)). We say that the homomorphism *W lifts the ideal $\mathcal{I}$* (or that it is *$\mathcal{I}$-lifting*) if the image of the ideal $\mathcal{I}$ under this homomorphism is a closed two-sided $*$-ideal of $\mathcal{C}$, and if the restriction of W onto $\mathcal{I}$ yields an *isomorphism* between these two ideals. For example, every homomorphism lifts the zero-ideal, and the identical homomorphism of $\mathcal{B}$ lifts every ideal.

PROPOSITION 3.1. *Let $\mathcal{B}, \mathcal{C}, \mathcal{I}$ be as above and let $W : \mathcal{B} \to \mathcal{C}$ be an $\mathcal{I}$-lifting homomorphism. Then an element b of $\mathcal{B}$ is invertible in $\mathcal{B}$ if and only if its coset $b + \mathcal{I}$ is invertible in the quotient algebra $\mathcal{B}/\mathcal{I}$ and if its image $W(b)$ is invertible in $\mathcal{C}$.*

Proof. The invertibility of b evidently implies that one of $b + \mathcal{I}$ and of $W(b)$. So let, conversely, $b + \mathcal{I}$ and $W(b)$ be invertible, and denote the identity element of $\mathcal{B}$ by e. Invertibility of $b + \mathcal{I}$ involves the existence of elements $a \in \mathcal{B}$ and $j \in \mathcal{I}$ such that $ab = e + j$. Since $W(\mathcal{I})$ is an ideal, we conclude from $W(j) \in W(\mathcal{I})$ that $W(j)W(b)^{-1}$ is in $W(\mathcal{I})$, too. Further, the isomorphism of $\mathcal{I}$ and $W(\mathcal{I})$ entails that there is a uniquely determined element k in $\mathcal{I}$ such that $W(k) = W(j)W(b)^{-1}$. Set $\widehat{a} = a - k$. Then $a + \mathcal{I} = \widehat{a} + \mathcal{I}$, and

$$\widehat{a}b = (a - k)b = e + (j - kb).$$

Since, by definition, $j - kb \in \mathcal{I}$ and $W(j - kb) = 0$ we conclude (again employing the isomorphism between $\mathcal{I}$ and $W(\mathcal{I})$) that $j - kb = 0$ and, thus, $\widehat{a}b = e$. The invertibility of b from the right hand side can be shown analogously. ∎

The next result states that one can even "glue" liftable ideals.

LIFTING THEOREM, PART 1. *Let $\mathcal{B}$ be a unital C^*-algebra and let T be an index set. Suppose that, for each $t \in T$, there are given a unital C^*-algebra $\mathcal{C}_t$, a closed two-sided $*$-ideal $\mathcal{I}_t$ of $\mathcal{B}$, and an $\mathcal{I}_t$-lifting homomorphism $W_t : \mathcal{B} \to \mathcal{C}_t$.Let $\mathcal{I}$ denote the smallest closed two-sided ideal of $\mathcal{B}$which contains all ideals $\mathcal{I}_t$. Then an element b of $\mathcal{B}$ is invertible (in $\mathcal{B}$) if and only if all elements $W_t(b)$ are invertible (in $\mathcal{C}_t$) and if the coset $b + \mathcal{I}$ is invertible (in $\mathcal{B}/\mathcal{I}$).*

Proof. Clearly, invertibility of b involves that one of $b+\mathcal{T}$ and of $W_t(b)$ for all t. For the reverse implication, let $b+\mathcal{T}$ and $W_t(b)$ be invertible for all t. Then there are elements $a \in \mathcal{B}$ and $j \in \mathcal{T}$ such that $ab = e + j$ where e again denotes the unit element of $\mathcal{B}$. By the definition of the ideal $\mathcal{T}$, one can find elements $j_{t_i} \in \mathcal{I}_{t_i}$ with $i \in \{1, 2, \ldots, n\}$ such that

$$\|j - j_{t_1} - j_{t_2} - \cdots - j_{t_n}\| < \frac{1}{2}.$$

As in the proof of the preceding proposition, there are elements $k_{t_i} \in \mathcal{I}_{t_i}$ such that $W_{t_i}(k_{t_i}) = W_{t_i}(j_{t_i})W_{t_i}(b)^{-1}$. Set $\widehat{a} = a - k_{t_1} - k_{t_2} - \cdots - k_{t_n}$ and $\widehat{j} = j_{t_1} + j_{t_2} + \cdots + j_{t_n}$. Then $a + \mathcal{I} = \widehat{a} + \mathcal{I}$ and

$$\widehat{a}b = e + j - \widehat{j} + (j_{t_1} - k_{t_1}b) + \cdots + (j_{t_n} - k_{t_n}b).$$

Again as in Proposition 1, we get that $j_{t_i} - k_{t_i}b = 0$ for all i, hence

$$\widehat{a}b = e + j - \widehat{j}.$$

Because of $\|j - \widehat{j}\| < 1/2$, this equality gives the left invertibility of b, and its right invertibility can be shown analogously. ∎

A particular case of this result (with $T = \{0, 1\}$) goes back to one of the authors ([25]). A general version first appeared in [22]; for a slight generalization (also including the Banach algebra case) and some comments we refer to [10].

The lifting theorem states that, sometimes and under *well-defined* additional conditions (invertibility of all $W_t(b)$), invertibility of an element modulo an ideal can imply invertibility of the element itself. This observation often simplifies invertibility problems drastically, since, in many applications, one can choose the ideal-lifting homomorphisms in such a way that the quotient algebra $\mathcal{B}/\mathcal{I}$ possesses a much nicer structure that the algebra $\mathcal{B}$ itself (e.g. $\mathcal{B}/\mathcal{I}$ can have a large center or can even be commutative whereas $\mathcal{B}$ itself is far away from commutativity).

Let us, for example, show how the lifting theorem cited in Section 2 is covered by the general one: The algebra $\mathcal{B}$ corresponds to $\mathcal{A}/\mathcal{G}$, and the index set contains two elements only, $T = \{0, 1\}$. The algebras $\mathcal{C}_0$ and $\mathcal{C}_1$ are both equal to $L(l^2)$, and the ideals $\mathcal{I}_0$ and $\mathcal{I}_1$ can be identified with

$$\{(P_n K_0 P_n) + \mathcal{G} \quad \text{with } K_0 \text{ being compact}\}$$

and

$$\{(W_n K_1 W_n) + \mathcal{G} \quad \text{with } K_1 \text{ being compact}\},$$

respectively. Thus, $\mathcal{I} \cong \mathcal{J}$. Only with the homomorphisms W_0 and W_1 introduced in Section 2 some care is needed, since both homomorphisms act on $\mathcal{A}$, not on the quotient $\mathcal{A}/\mathcal{G}$. But, evidently, the ideal $\mathcal{G}$ belongs to the kernels of both W_0 and W_1, thus, the quotient homomorphisms

$$\widehat{W_i} : \mathcal{A}/\mathcal{G} \to L(l^2), (A_n) + \mathcal{G} \mapsto W_i(A_n) \quad \text{for} \quad i = 0, 1$$

are correctly defined, and these quotient homomorphisms are exactly the homomorphisms figuring in the general lifting theorem. Indeed, $\widehat{W_i}$ is $\mathcal{I}_i$-lifting, and the image of this ideal is just the ideal of all compact operators on l^2.

3.2. THE SEPARATION PROPERTY. In the above example, one can moreover show that

$$W_i(\mathcal{I}_j) = 0 \quad \text{whenever} \quad i, j \in \{0, 1\} \quad \text{and} \quad i \neq j.$$

Under this additional assumption, we can complete the general lifting theorem as follows.

LIFTING THEOREM, PART 2. *Let the conditions of the general lifting theorem be satisfied and suppose moreover that the homomorphisms W_t separate the ideals $\mathcal{I}_t$, i.e.*

$$W_t(\mathcal{I}_s) = 0 \quad \textit{whenever} \quad s \neq t.$$

Then

(i) $\mathcal{I}_s \cap \mathcal{I}_t = \{0\}$;

(ii) *invertibility of the coset $b + \mathcal{I}$ implies invertibility of the cosets $W_t(b) + W_t(\mathcal{I}_t)$ for all $t \in T$;*

(iii) *invertibility of the coset $b + \mathcal{I}$ implies invertibility of $W_t(b)$ for almost all $t \in T$ with only finitely many exceptions.*

Proof. (i) Let $j \in \mathcal{I}_s \cap \mathcal{I}_t$ and $s \neq t$. Then $W_r(j) = 0$ for all $r \neq t$ since $j \in \mathcal{I}_t$, and $W_t(j) = 0$ since $j \in \mathcal{I}_s$ and $s \neq t$. Moreover, $j \in \mathcal{T}$, and this shows that for each invertible element b of $\mathcal{B}$, the element $b + j$ is invertible again. Hence, j belongs to the radical of $\mathcal{B}$ which is known to consist of the zero element only.

(ii) If $j \in \mathcal{T}$ then $W_t(j) \in W_t(\mathcal{I}_t)$. Indeed, this is evident in case j is a finite sum of elements $j_{t_i} \in \mathcal{I}_{t_i}$, and it is a simple consequence of continuity of W_t in the general case. Thus, if $b + \mathcal{T}$ is invertible in $\mathcal{B}/\mathcal{I}$ then there are elements $a \in \mathcal{B}$ and $j \in \mathcal{T}$ such that $ab = e + j$, and applying the homomorphism W_t to both sides of this identity we get the left invertibility of $W_t(b)$ modulo elements in the ideal $W_t(\mathcal{I}_t)$. Invertibility from the right can be shown analogously.

(iii) Let again $ab = e + j$ with elements $a \in \mathcal{B}$ and $j \in \mathcal{T}$, and choose an element $\hat{j} = j_{t_1} + j_{t_2} + \cdots + j_{t_n}$ with $j_{t_i} \in \mathcal{I}_{t_i}$ and $\|j - \hat{j}\| < 1/2$. Applying W_t to both sides of the equality

$$ab = e + j_{t_1} + j_{t_2} + \cdots + j_{t_n} + j - \hat{j}$$

we obtain

$$W_t(a)W_t(b) = e_t + W_t(j - \hat{j}) \quad \text{for all} \quad t \in T \setminus \{t_1, t_2, \ldots, t_n\}$$

where $e_t = W_t(e)$ denotes the unit element in $\mathcal{C}_t$. Because of $\|W_t(j - \hat{j})\| < 1/2$, this yields left invertibility of $W_t(b)$ for all t with only finitely many exceptions. Invertibility from the right can be shown analogously. ∎

Let us remark that this result remains valid (with the zero ideal replaced by an ideal in the radical) in the Banach algebra case, too.

3.3. SYMBOLS. For our next step into the world of lifting theorems we remember once more our intimate guide: the finite section method of Toeplitz operators. We have seen in Section 2 that already Fredholmness of $W_0(A_n)$ and $W_1(A_n)$ implies invertibility of $(A_n) + \mathcal{J}$. Well, Fredholmness of $W_t(A_n)$ corresponds to invertibility of $W_t(b)$ modulo elements in the ideal $W_t(\mathcal{I}_t)$, and this suggests the following completion of the general lifting theorem.

LIFTING THEOREM, PART 3. *Let all hypotheses of the second part be satisfied and suppose, moreover, that the converse of assertion* (ii) *is true, i.e. that, if all cosets* $W_t(b) + W_t(\mathcal{I}_t)$ *are invertible, then the coset* $b + \mathcal{T}$ *is invertible. Then* (iv) *an element* $b \in \mathcal{B}$ *is invertible if and only if all elements* $W_t(b)$ *are invertible;* (v) *for each* $b \in \mathcal{B}$,

$$\|b\| = \sup_{t \in T} \|W_t(b)\|,$$

and the function $T \to \mathbf{R}$, $t \mapsto \|W_t(b)\|$ *attains its supremum.*

In general, the additional assumption in the third part of the lifting theorem proves to be the strongest and hardest of our hypotheses. In applications it is usually verified by means of so-called *local principles* which can be viewed as abstract versions of the well-known method of "freezing coefficients" in the theory of partial and pseudo differential operators. In case $\mathcal{B}/\mathcal{I}$ is commutative, the classical spectral theory of Gelfand can serve as a local principle, whereas non-commutativity requires one of the various generalizations of Gelfand's theory due to Simonenko, Allan/Douglas, Gohberg/Krupnik, Krupnik. (See [4], Chapter 1, for a brief overview, and compare [15]. See also [10], Chapter 1.)

On the other hand, the forcing of this additional assumption is, in a sense, a main goal of the application of the whole lifting story. Namely, it tells us that we have found out *enough* lifting homomorphisms in order to examine invertibility.

Proof. Assertion (iv) is immediate from the first part of the lifting theorem in combination with our new hypothesis.

(v) Let $\mathcal{S}$ stand for the set of all bounded functions on Ttaking at $t \in T$ a value in $\mathcal{C}_t$. Provided with pointwise operations

$$(f+g)(t) = f(t) + g(t), \quad (fg)(t) = f(t)g(t),$$

the pointwise involution

$$(f^*)(t) = f(t)^*,$$

and the supremum norm

$$\|f\| = \sup_{t \in T} \|f(t)\|,$$

this set becomes a C^*-algebra, and the mapping smb : $\mathcal{B} \to \mathcal{S}$ associating with $b \in \mathcal{B}$ the function$f : t \mapsto W_t(b)$ is a $*$-homomorphism. Since $*$-homomorphisms cannot increase the norm, we have

$$\|\text{smb}(b)\|_{\mathcal{S}} \leqslant \|b\|_{\mathcal{B}}. \tag{3.1}$$

We claim that the kernel of the mapping smb is trivial. Indeed, if $j \in \ker \text{smb}$ then $b+j$ is invertible whenever b is invertible. Thus, j is in the radical of $\mathcal{B}$ which consists of zero element only. Hence, smb is actually a $*$-isomorphism between the C^*–algebras $\mathcal{B}$ and smb$(\mathcal{B})$ which involves that equality holds in (3.1).

For the second assertion of (v) we assume there is an $b \in \mathcal{B}$ suchthat

$$\|W_t(b)\| < \sup_{t \in T} \|W_t(b)\| \quad \text{for all } t \in T. \tag{3.2}$$

Since

$$\begin{aligned}
\|W_t(b)\|^2 &= \|W_t(b)^* W_t(b)\| \\
&= \|(W_t(b)^* W_t(b))^{\frac{1}{2}} (W_t(b)^* W_t(b))^{\frac{1}{2}}\| \\
&= \|(W_t(b)^* W_t(b))^{\frac{1}{2}}\|^2 \\
&= \|W_t((b^*b)^{\frac{1}{2}})\|^2
\end{aligned}$$

we can suppose without loss the element b in (3.2) to be self-adjoint.For self-adjoint elements b, inequality (3.2) entails that

$$\rho(W_t(b)) < \sup_{t \in T} \rho(W_t(b)) =: M \tag{3.3}$$

where $\rho(\cdot)$ denotes the spectral radius. Set $c = b - Me$. The elements $W_t(c) = W_t(b) - Me_t$ are invertible for all $t \in T$ since $\rho(W_t(b)) < M$, and the first part of the lifting theorem gives invertibility of $c = b - Me$. Then, clearly, $b - me$ is invertible for all m belonging to some neighborhood U of M. On the other hand, since $\sup_{t \in T} \rho(W_t(b)) = M$, for each neighborhood U of M there is a $t_U \in T$ and an $m_U \in U$ such that $W_{t_U}(b) - m_U e_{t_U}$ is not invertible and, hence, $b - m_U e$ is not invertible. This contradiction proves the assertion. ∎

Part (iv) of the previous theorem states that the mapping smb, introduced in the course of proving assertion (v) and assigning with each element of $\mathcal{B}$ an algebra-valued function on the index set T, is a symbol mapping in the following sense: a C^*-homomorphism s of a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is called a *symbol map* (and $s(a) \in \mathcal{B}$ is referred to as the *symbol* of $a \in \mathcal{A}$) if, for each $a \in \mathcal{A}$, invertibility of $s(a)$ in $\mathcal{B}$ implies invertibility of a in $\mathcal{A}$. Since the converse implication holds true for every unital homomorphism we conclude that

$$\text{spectrum of } a \text{ in } \mathcal{A} = \text{spectrum of } s(a) \text{ in } \mathcal{B}$$

whenever s is a symbol mapping.

3.4. MOORE-PENROSE INVERTIBILITY. Our final goal is Moore-Penrose invertibility. Recall that an element b of a C^*-algebra $\mathcal{B}$ is *Moore-Penrose invertible* if there is an element a in $\mathcal{B}$ such that

$$bab = b, \quad aba = a, \quad ab = (ab)^*, \quad ba = (ba)^*. \tag{3.4}$$

If the Moore-Penrose inverse of b exists then it is uniquely determined, and its standard notation is $b^\dagger$.

In case $\mathcal{B}$ is the algebra $L(H)$ of all bounded linear operators on a Hilbert space H, the Moore-Penrose inverse of an operator A exists if and only if this operator is normally solvable (i.e. the image of H is closed), and in this case one has

$$A^\dagger = (A^*A + P)^{-1}A^*$$

where P is the orthogonal projection onto the kernel of A. Indeed, here is a proof for completeness: If the operator $A^*A + P$ is invertible then it is easy to see that$A^\dagger$ satisfies the axioms (3.4). So we are left with verifying invertibility of $A^*A + P$. Let x belong to the kernel of this operator. Then, since $AP = 0$ and $PA^* = 0$, one has$(I - P)A^*A(I - P)x + Px = 0$ which immediately gives $Px = 0$ and$(I-P)A^*A(I-P)x = 0$. The latter equality implies $((I-P)A^*A(I-P)x, x) = (A(I-P)x, A(I-P)x) = 0$, i.e. $A(I-P)x = 0$. But P is the orthogonal projection onto the kernel of$A$, hence, $(I - P)x = 0$ and, consequently, $x = 0$. Further, since $A^*A + P$ is self-adjoint and, hence,

$$\ker(A^*A + P) + \overline{\operatorname{Im}(A^*A + P)} = H,$$

it remains to show that the range of $A^*A + P$ is closed. Using once more that $(I - P)A^*A = 0$, and recalling that the range of a projection is always closed, one easily gets that $A^*A + P$ is normally solvable if and only if A^*A is so. For the normal solvability of A^*A one can argue as follows: Let $z = \lim A^*Ax_n$. Since A^* is normally solvable whenever A is so we conclude that z belongs to the range of A^*, that is, $z = A^*y$ with some $y \in H$. Let further Q denote the orthogonal projection onto the (closed) range of A. Then $QA = A$ and $A^*Q = A^*$, and one can suppose without loss that $y = Qy$. Thus, for all $w \in H$,

$$(A^*Ax_n - z, w) = (A^*Ax_n - A^*y, w) = (QAx_n - Qy, Aw) \to 0$$

as $n \to \infty$. Since the range of A is closed and coincides with the range of Q, we conclude that

$$(QAx_n - Qy, Qv) = (QAx_n - Qy, v) \to 0$$

for all $v \in H$ and, thus, $Ax_n \to y$. The normal solvability of A yields $y \in \mathrm{Im}(A)$ and, hence, $z \in \mathrm{Im}(A^*A)$, and we are done.

There is still another equivalent characterization of Moore-Penrose invertibility which, in contrast to the condition of normal solvability, also works for elements in arbitrary C^*-algebras with identityelement:

PROPOSITION 3.2. *An element a of a C^*-algebra $\mathcal{B}$ with identity is Moore-Penrose invertible if and only if the element a^*a is invertible, or if 0 is an isolated point of the spectrum of a^*a.*

This result is certainly also well-known; only for completeness we present its proof here. The arguments used in its second part are due to T. Ehrhardt.

Proof. **Sufficiency part.** If a^*a is invertible then the element $b := (a^*a)^{-1}a^*$ (belonging also to $\mathcal{B}$) evidently satisfies the axioms (3.4) and is, hence, the Moore-Penrose inverse of a. So suppose 0 to be an isolated point of the spectrum of a^*a. The C^*-subalgebra $\mathcal{C}$ of $\mathcal{B}$ which is generated by the identity element e and by a^*a is, by the spectral theorem, isometrically isomorphic to the C^*-algebra of all continuous complex-valued functions on the spectrum of a^*a. Let $p \in \mathcal{C}$ be the (uniquely determined) element corresponding to the (continuous!) function which takes the value 1 at 0 and which vanishes at all points in $\sigma(a^*a) \setminus \{0\}$. The element p is a projection in $\mathcal{C}$ which commutes with a^*a and for which $a^*ap = 0$ and a^*a+p is invertible. Having in mind that then $ap = 0$ (since$\|a^*ap\| = \|pa^*ap\| = \|ap\|^2$) one can straightforwardly check that the element $b = (a^*a + p)^{-1}a^* \in \mathcal{C}$ is the Moore-Penrose inverse of a.

Necessity part. Let a be Moore-Penrose invertible and $a^\dagger = b$. If $\lambda \neq 0$ is an arbitrary complex number fulfilling the inequality $|\lambda| < \|bb^*\|^{-1}$ then the element $e - \lambda bb^*$ is invertible in $\mathcal{B}$, and a straightforward calculation shows that moreover

$$(e - \lambda bb^*)^{-1}bb^* - \frac{1}{\lambda}(e - ba)$$

is the inverse of $a^*a - \lambda$. ∎

A C^*-subalgebra of a C^*-algebra with identity is called unital if it contains the identity element. It is a remarkable property of unital C^*-subalgebras of C^*-algebras that they are inverse closed with respect to usual invertibility, that is, if an element a of a C^*-subalgebra $\mathcal{C}$ of a C^*-algebra $\mathcal{B}$ has an inverse in $\mathcal{B}$ then this inverse necessarily belongs to $\mathcal{C}$. Thus, the spectrum of a considered as an element of $\mathcal{B}$ coincides with the spectrum of a viewed as an element of $\mathcal{C}$. Together with the previous proposition this yields the following.

COROLLARY 3.3. *Unital C^*-subalgebras of C^*-algebras are inverse closed with respect to Moore-Penrose invertibility, that is, if an element of a C^*-subalgebra $\mathcal{C}$ of a C^*-algebra $\mathcal{B}$ has a Moore-Penrose inverse in $\mathcal{B}$ then this Moore-Penrose inverse necessarily belongs to $\mathcal{C}$.*

We agree upon calling an ideal $\mathcal{T}$ of a C^*-algebra $\mathcal{B}$a *Moore-Penrose ideal* if the invertibility of an element $b \in \mathcal{B}$ modulo $\mathcal{T}$ implies its Moore-Penrose invertibility. Obviously, the zero ideal is always a Moore-Penrose ideal, and the ideal $K(H)$ of all compact linear operators on the Hilbert space H is a Moore-Penrose ideal in the algebra $L(H)$ (the latter follows simply from the fact that an operator is invertible modulo compact operators if and only if it is Fredholm, and that Fredholm operators have a finite-dimensional cokernel $H/\mathrm{Im}\,A$ by definition, which on its hand implies that their image $\mathrm{Im}\,A$ is closed).

PROPOSITION 3.4. *Let $\mathcal{B}$ be a C^*-algebra with identity, $\mathcal{T}$ be a Moore-Penrose ideal in $\mathcal{B}$, and let $b \in \mathcal{B}$ be invertible modulo $\mathcal{T}$. Then there is a uniquely determined projection k in $\mathcal{T}$ (i.e. an element satisfying $k^* = k$ and $k^2 = k$) such that*

$$b^\dagger = (b^*b + k)^{-1}b^*. \tag{3.5}$$

Proof. Due to the definition of Moore-Penrose invertibility, the element $b^\dagger b$ is a projection, hence $k := e - b^\dagger b$ is a projection, too. We represent the C^*-algebra, generated by e, b and $b^\dagger$, isometrically isomorphic as an algebra of operators on a Hilbert space, and we denote the operator corresponding to an element a by $\widehat{a}$. Clearly, the operator $\widehat{b}$ is Moore-Penrose invertible, and $\widehat{b^\dagger} = \widehat{b}^\dagger$. Further, the operator $\widehat{k} = \widehat{e} - \widehat{b}^\dagger \widehat{b}$ is the orthogonal projection onto the kernel of $\widehat{b}$ (compare, e.g.,[7], Section 4.5, Lemma 5.1). As we have remarked above, then the Moore-Penrose inverse of $\widehat{b}$ is given by

$$\widehat{b}^\dagger = (\widehat{b}^*\widehat{b} + \widehat{k})^{-1}\widehat{b}^*, \tag{3.6}$$

which implies (3.5). It remains to show that k belongs to $\mathcal{T}$, and that k is unique. From (3.5) we conclude that

$$bb^\dagger b = b(b^*b+k)^{-1}b^*b = b(b^*b+k)^{-1}(b^*b+k) - b(b^*b+k)^{-1}k = b - b(b^*b+k)^{-1}k$$

which, together with the first identity in (3.4), yields

$$b(b^*b+k)^{-1}k = 0.$$

The elements b and $(b^*b+k)^{-1}$ are invertible modulo $\mathcal{T}$, hence, $k+\mathcal{I} = 0$ or, equivalently, $k \in \mathcal{T}$. The uniqueness of k follows from the uniqueness of the projection $\widehat{k}$ in (3.6). ∎

LIFTING THEOREM, PART 4. *Let the conditions of Part 3 be satisfied. Then,* (vi) *if the $W_t(\mathcal{I}_t)$ are Moore-Penrose ideals for all t, then the ideal $\mathcal{T}$ is a Moore-Penrose ideal, too.*

Proof. Let $b \in \mathcal{B}$ and $b+\mathcal{T}$ be invertible. Then, by assertion (ii), all cosets $W_t(b)+W_t(\mathcal{I}_t)$ are invertible whence, by our assumption, follows that all elements $W_t(b)$ are Moore-Penrose invertible. Moreover, by assertion (iii), the elements $W_t(b)$ are even invertible for almost all t with only finitely many exceptions, say $t \in \{t_1, t_2, \ldots, t_n\}$. For $t = t_i$, let p_i denote the (by the previous proposition) uniquely determined projection in $W_t(\mathcal{I}_t)$ such that

$$W_{t_i}(b)^\dagger = (W_{t_i}(b)^* W_{t_i}(b) + p_i)^{-1} W_{t_i}(b)^*, \tag{3.7}$$

and let k_i stand for the (by our assumption that W_t is a lifting homomorphism) uniquely determined pre-image of p_i in $\mathcal{I}_t$.

We claim that the element

$$c := b^*b + k_1 + k_2 + \cdots + k_n \tag{3.8}$$

is invertible in $\mathcal{B}$. Indeed, if $t \in T \setminus \{t_1, t_2, \ldots, t_n\}$, then the separation condition entails that

$$W_t(c) = W_t(b)^* W_t(b)$$

is invertible, whereas, in case $t = t_i$, the application of W_{t_i} to both sides of (3.8) just yields

$$W_{t_i}(c) = W_{t_i}(b)^* W_{t_i}(b) + p_i$$

which, by definition, is also invertible. Assertion (iv) of the Lifting theorem says that then c must be invertible.

We are going to show that the element

$$a := c^{-1}b^* = (b^*b + k_1 + k_2 + \cdots + k_n)^{-1}b^* \tag{3.9}$$

is the Moore-Penrose inverse of b. It is evident from our constructionthat, for all $t \in T$, $W_t(a)$ is the Moore-Penrose inverse of $W_t(b)$. Consequently,

$$W_t(bab - b) = W_t(b)W_t(a)W_t(b) - W_t(b) = 0$$

for all $t \in T$, which yields via assertion (v) of the Lifting theorem that $bab = b$. Analogously, the remaining three of the defining relations (3.4) of the Moore-Penrose inverse can be verified. ∎

Let us remark that (3.9) is just the representation of $b^\dagger$ according to (3.5) in Proposition 3.2, i.e. $k_1 + \cdots + k_n$ is a projection in $\mathcal{T}$. This is a consequence of the fact that all elements $W_{t_i}(k_i) = p_i$ are projections and can be easily shown by employing the separation property and assertion (v) of the Lifting theorem.

4. SEQUENCE ALGEBRAS AND FRACTAL HOMOMORPHISMS

4.1. ALGEBRAS OF SEQUENCES. Let A be a bounded linear operator on a Hilbert space H. An *approximation method* for A is (in a very wide sense) simply a sequence (A_n) of bounded linear operators on H which converges strongly to A. The method (A_n) *applies* to A if the equations

$$A_n x_n = y \quad (x_n, y \in H)$$

are uniquely solvable for all sufficiently large n, say $n \geqslant n_0$, and for all right hand sides y, and if their solutions x_n converge in the norm of H to a solution of the equation

$$Ax = y \quad (x, y \in H).$$

It is not hard to see that the method (A_n) applies to A if and only if the operator A is invertible and if the sequence (A_n) is stable. Recall that a sequence (A_n) of operators (or, more generally, of elements of an algebra) is said to be *stable* if there is an n_0 such that A_n is invertible for all $n \geqslant n_0$ and if $\sup_{n \geqslant n_0} \|A_n^{-1}\| < \infty$.

Perhaps, it was Kozak who first pointed out that stability of a sequence can be reinterpreted as invertibility of an element in an accordingly constructed C^*-algebra. For this goal we introduce the set $\mathcal{F}$ of all bounded sequences (A_n) of operators $A_n \in L(H)$ (this set clearly contains all approximation methods since each strongly converging sequence is bounded), and we provide this set with the operations

$$(A_n) + (B_n) = (A_n + B_n) \quad \text{and} \quad (A_n)(B_n) = (A_n B_n)$$

called addition and multiplication, with an involution

$$(A_n)^* = (A_n^*),$$

and with a norm

$$\|(A_n)\| = \sup_{n \geqslant 1} \|A_n\|.$$

It is elementary to show that $\mathcal{F}$ becomes a C^*-algebrawith identity element (I) where I is the identity operator on H and that, moreover, the set $\mathcal{K}$ of all sequences (K_n) with $\|K_n\| \to 0$ as $n \to \infty$ forms a closed two-sided $*$-ideal of $\mathcal{F}$.

PROPOSITION 4.1. *Let $(A_n) \in \mathcal{F}$. The sequence (A_n) is stable if and only if the coset $(A_n) + \mathcal{K}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{K}$.*

For (simple) proof see, e.g., [10], Proposition 1.2.

Well, now we would like to apply the lifting theory to study invertibility in $\mathcal{F}/\mathcal{K}$ and, hence, stability of approximation methods. But there seems to be no theory of everything: the algebra $\mathcal{F}/\mathcal{K}$ is simply too large to deal with successfully. So we shall look for a $*$-subalgebra, $\mathcal{A}$, of $\mathcal{F}$ which contains the identity element and which, on the one hand, is large enough to contain, interesting sequences but, on the other hand, small enough to be accessible to the lifting theorem. The reasonable choice of the *sequence algebra* $\mathcal{A}$ is by no means evident; some examples will be given in the fifth section.

Given a sequence algebra $\mathcal{A}$ we abbreviate the intersection $\mathcal{A} \cap \mathcal{K}$ by $\mathcal{G}$. This set is a closed two-sided$*$-ideal of $\mathcal{A}$, and it does not matter whether one examines invertibility of a coset $(A_n) + \mathcal{K}$ in $\mathcal{F}/\mathcal{K}$ or that of $(A_n) + \mathcal{G}$ in $\mathcal{A}/\mathcal{G}$ for $(A_n) \in \mathcal{A}$. Indeed, we have already remarked that, if for the sequence $(A_n) \in \mathcal{A}$ the coset $(A_n) + \mathcal{K}$ is invertible in $\mathcal{F}/\mathcal{K}$, then it is also invertible in $(\mathcal{A} + \mathcal{K})/\mathcal{K}$, and further, there is a natural isomorphism between the quotient algebras $(\mathcal{A} + \mathcal{K})/\mathcal{K}$ and $\mathcal{A}/(\mathcal{A} \cap \mathcal{K}) = \mathcal{A}/\mathcal{G}$.

4.2. FRACTAL HOMOMORPHISMS. In our concrete examples for the algebra $\mathcal{A}$ (see-Section 5 below), the lifting homomorphisms possess a strong property. Namely, if (A_n) is a sequence in $\mathcal{A}$ then $W(A_n) = W(A_{n_k})$ for *each* infinite subsequence (A_{n_k}) of (A_n). Thus, each subsequence of (A_n) contains the full information about the image of the whole sequence (A_n) under the homomorphism W, and that's why we call these homomorphisms fractal.

For a precise definition, we associate with each strongly monotonically increasing mapping $\eta : \mathbb{Z}^+ \to \mathbb{Z}^+$ the subalgebra $\mathcal{A}_\eta$ of $\mathcal{F}$ consisting of all sequences $(A_{\eta(n)})$ with (A_n) running through the sequence algebra $\mathcal{A}$, and we denote the natural operator $\mathcal{A} \to \mathcal{A}_\eta : (A_n) \mapsto (A_{\eta(n)})$ by T_η. A unital homomorphism W from $\mathcal{A}$ into a C^*-algebra $\mathcal{C}$ is called *fractal* if, for each strongly monotonically increasing mapping $\eta : \mathbb{Z}^+ \to \mathbb{Z}^+$ there is a unital homomorphism W_η mapping $\mathcal{A}_\eta$ into $\mathcal{C}$ such that

$$W_\eta(T_\eta(A_n)) = W(A_n) \quad \text{for all} \quad (A_n) \in \mathcal{A}.$$

To have some examples: the mapping $W : (A_n) \mapsto \text{s-lim}\, A_n$ is a fractal homomorphism acting on the algebra of all strongly converging sequences, and the homomorphisms W_0 and W_1 introduced in the second section are fractal, too. To see this, it is more convenient to consider the approximation operators $P_nT(a)P_n$ as acting on and being invertible on the whole space l^2 rather than on its subspace $\operatorname{Im} P_n$. This can be forced by introducing the operators $Q_n = I - P_n$ and replacing $T_n(a) = P_nT(a)P_n$ by $P_nT(a)P_n + Q_n$. Obviously, both sequences $(T_n(a))$ and $(P_nT(a)P_n + Q_n)$ are stable or not only simultaneously. Moreover, one has to replace the operators W_n by $E_n := W_n + Q_n$. These operators are isometries on l^2, and

$$\text{s-lim}\, W_nA_nW_n = \text{s-lim}\, E_n^{-1}(A_n + Q_n)E_n$$

for all sequences (A_n) belonging to the algebra of the finite section sequences considered in Section 2. With these identifications, we can think of this algebra as a subalgebra of the algebra of all bounded sequences of operators on l^2. Now it is evident that the homomorphism

$$W : (A_n) \mapsto \operatorname*{s-lim}_{n\to\infty} E_n^{-1}(A_n + Q_n)E_n$$

is fractal; given a function η one defines

$$W_\eta : (A_{n_k}) \mapsto \operatorname*{s-lim}_{k\to\infty} E_{n_k}^{-1}(A_{n_k} + Q_{n_k})E_{n_k}.$$

PROPOSITION 4.2. *If $\mathcal{A}$ is a sequence algebra and $W : \mathcal{A} \to \mathcal{C}$ is a fractal homomorphism then the ideal $\mathcal{G}$ belongs to the kernel of W.*

Proof. Let $(G_n) \in \mathcal{G}$. Given $\varepsilon > 0$ there is an n_0 such that $\|G_n\| < \varepsilon$ for all $n \geqslant n_0$. Define $\eta(n) = n + n_0$. Then

$$\|W_\eta(G_{\eta(n)})\| \leqslant \|W_\eta\|\, \|(G_{\eta(n)})\|$$

and hence

$$\|W(G_n)\| \leqslant \varepsilon\|W\| = \varepsilon.$$

Letting ε to go to 0 we arrive at $\|W(G_n)\| = 0$. ∎

Thus, $W(A_n)$ actually depends on the coset $(A_n)+\mathcal{G}$ only. For convenience, we denote the (correctly defined) quotient mapping

$$(A_n)+\mathcal{G}\mapsto W(A_n)$$

by W again.

4.3. APPROXIMATION SEQUENCES: STABILITY. Throughout what follows we suppose $\mathcal{A}$ to be a sequence algebra and $(W_t)_{t\in T}$ to be a family of fractal homomorphisms mapping $\mathcal{A}$ into certain C^*-algebras $\mathcal{C}_t$, respectively. Further we assume the quotient homomorphisms $W_t : \mathcal{A}/\mathcal{G}\to\mathcal{C}_t$ to be ideal-lifting in the sense of Section 3 and that all hypotheses made in the parts 1 to 3 of the Lifting theorem are satisfied. The notations introduced in the third section for certain ideals etc. will be also taken over into the present context of sequence algebras. Finally, we define a mapping Smb : $\mathcal{A}\to\mathcal{S}$, where $\mathcal{S}$ refers to the *symbol algebra* introduced in Subsection 3.3, by $(\text{Smb}(A_n))(t)=W_t(A_n)$. Evidently, $(\text{Smb}(A_n))(t)$ coincides with $(\text{smb}((A_n)+\mathcal{G}))(t)$ for all $t\in T$ where smb is the symbol mapping introduced in 3.3 (which in the present context acts form $\mathcal{A}/\mathcal{G}$ into $\mathcal{S}$), and we shall hence forth speak of $\text{Smb}(A_n)$ as the *stability symbol* of (A_n).

THEOREM 4.3. (i) *Let (A_n) be a sequence in $\mathcal{A}$. This sequence is stable if and only if the elements $W_t(A_n)$ are invertible for all $t\in T$ or, equivalently, if the stability symbol* $\text{Smb}(A_n)$ *is invertible in $\mathcal{S}$.*

(ii) *The sequence (A_n) is stable if and only if one of its infinite subsequences is stable.*

Proof. Assertion (i) is an immediate consequence of Part 1 of the Lifting theorem and Proposition 4.1, and assertion (ii) follows from the fractal property of the homomorphisms: If there is an infinite subsequence $(A_{\eta(n)})$ then all elements $(W_t)_\eta(A_{\eta(n)})$ and, thus, all elements $W_t(A_n)$ are invertible. Then, by assertion(i), the coset $(A_n)+\mathcal{G}$ is invertible. ∎

As we have already remarked, the homomorphisms W_0 and W_1 introduced in Section 2 for elements of the finite section algebra $\mathcal{A}$ satisfy the assumptions made above. Thus, specifying Theorem 4.3 to this context yields exactly Theorem 2.2.

4.4. APPROXIMATION SEQUENCES: STRONG CONVERGENCE. If (A_n) is an approximation sequence for an operator A then this sequence should strongly converge to A. (For example, the finite sections of a Toeplitz operator do so.) The condition of strong convergence can be directly included into the definition of the sequence algebras. Indeed, one considers the subset $\mathcal{F}_c$ of $\mathcal{F}$ consisting of all sequences (A_n) for which there is an operator A on H such that

$$A_n \to A \quad \text{and} \quad A_n^* \to A^* \quad \text{strongly as} \quad n \to \infty.$$

This set $\mathcal{F}_c$ is a $*$-subalgebra of $\mathcal{F}$, and it involves the natural homomorphism

$$W_0 : \mathcal{F}_c \to L(H), \quad (A_n) \mapsto A = \text{s-lim} A_n. \tag{4.1}$$

PROPOSITION 4.4. (i) *The homomorphism W_0 in (4.1) is fractal.*

(ii) *The set $\mathcal{I}_0 := (K) + \mathcal{G}$ with (K) ranging through the constant sequences of compact operators is a closed two-sided $*$-ideal of $\mathcal{F}_c$, and the homomorphism W_0 lifts this ideal.*

Proof. Assertion (i) is obvious. Assertion (ii): In order to verify that $\mathcal{I}_0$ is a left sided ideal, let (A_n) be a sequence in $\mathcal{F}_c$ with strong limit A, and let K be compact and $(G_n) \in \mathcal{G}$. Then

$$(A_n)(K + G_n) = ((A_n - A)K + A_n G_n + AK),$$

and the sequence on the right hand side is in $\mathcal{I}_0$ since the strong convergence of (A_n) to A implies that the sequence $((A_n - A)K + A_n G_n)$ is in $\mathcal{G}$, and since AK is compact again. Analogously one checks that $\mathcal{I}_0$ is a right sided ideal.

It is also easy to see that the image of the ideal $\mathcal{I}_0$ under the homomorphism W_0 is an ideal (the ideal of all compact operators on H), and that the restriction of W_0 onto $\mathcal{I}_0$ is an isomorphism. ∎

THEOREM 4.5. *Suppose besides the assumptions from the previous section that the sequence algebra $\mathcal{A}$ is even a subalgebra of $\mathcal{F}_c$ and that the natural homomorphism W_0 defined in (4.1) is part of the family $(W_t)_{t\in T}$ of homomorphisms figuring in the lifting theorem. If $(A_n) \in \mathcal{A}$, and if all elements $W_t(A_n)$ are invertible, then the approximation method (A_n) applies to the operator $W_0(A_n) = A$, i.e. the sequence (A_n) is stable, and the inverse operators A_n^{-1} converge strongly to $W_0(A_n)^{-1}$.*

4.5. APPROXIMATION SEQUENCES: NORMS AND CONDITION NUMBERS. The following relation between the norms of the approximation operators A_n and the norm of the symbol of the sequence (A_n) was first recognized by A. Böttcher in the particular case considered in Section 2 as a consequence of Theorem 3.

THEOREM 4.6. *Let all hypotheses made in Subsection 4.3 be satisfied. If $(A_n) \in \mathcal{A}$ then the limit $\lim_{n\to\infty} \|A_n\|$ exists, and*

$$\lim_{n\to\infty} \|A_n\| = \sup_{t\in T} \|W_t(A_n)\| = \|\mathrm{Smb}(A_n)\|.$$

Proof. Recall that the norm in $\mathcal{A}$ is given by

$$\|(A_n)\| = \sup_n \|A_n\|.$$

Having this in mind it is easy to see that the norm in $\mathcal{A}/\mathcal{G}$ is just

$$\|(A_n) + \mathcal{G}\| = \limsup_{n\to\infty} \|A_n\|.$$

Thus, by assertion (v) of the Lifting theorem,

$$\limsup_{n\to\infty} \|A_n\| = \sup_{t\in T} \|W_t(A_n)\|. \tag{4.2}$$

It remains to show that the superior limit on the left hand side of (4.2) is actually a limit. Suppose, it is not. Then there is a subsequence $(A_{\eta(n)})$ of (A_n) such that the limit $\lim_{n\to\infty} \|A_{\eta(n)}\|$ exists and

$$\lim_{n\to\infty} \|A_{\eta(n)}\| \neq \limsup_{n\to\infty} \|A_n\|.$$

On the other hand, due to the fractal nature of the homomorphisms W_t, we find via replacing (A_n) in (4.2) by $(A_{\eta(n)})$

$$\lim_{n\to\infty} \|A_{\eta(n)}\| = \limsup_{n\to\infty} \|A_{\eta(n)}\| = \sup_{t\in T} \|W_t(A_n)\|.$$

The obtained contradiction proves our claim. ∎

In case of the finite section method for Toeplitz operators this theorem exactly reproduces Böttcher's observation:

If $(A_n) \in \mathcal{A}$ then $\lim \|A_n\| = \max\{\|W_0(A_n)\|, \|W_1(A_n)\|\}$.

As a by-product, we can describe the asymptotic behaviour of the condition numbers of an approximation sequence in $\mathcal{A}$ as follows.

COROLLARY 4.7. *Let all hypotheses made in Subsection 4.3 be satisfied, and let $(A_n) \in \mathcal{A}$ be a stable sequence. Then the sequence of the condition numbers $\|A_n\|\,\|A_n^{-1}\|$ of A_n is convergent, and its limit is equal to*

$$\sup_{t\in T} \|W_t(A_n)\| \sup_{t\in T} \|W_t(A_n)^{-1}\| = \|\mathrm{Smb}(A_n)\|\,\|\mathrm{Smb}(A_n)^{-1}\|.$$

4.6. APPROXIMATION SEQUENCES: MOORE-PENROSE INVERTIBILITY. In this subsection, we additionally suppose that the ideals $W_t(\mathcal{I}_t)$ are Moore-Penrose ideals for all t. Specifying assertion (vi) of the Lifting theorem to the present contextyields:

THEOREM 4.8. *Let $(A_n) \in \mathcal{A}$. If all cosets $W_t(A_n) + W_t(\mathcal{I}_t)$ with $t \in T$ are invertible then the coset $(A_n) + \mathcal{G}$ is Moore-Penrose invertible in $\mathcal{A}/\mathcal{G}$, i.e. there is a sequence $(B_n) \in \mathcal{A}$ such that*

$$\|A_nB_nA_n - A_n\| \to 0, \quad \|B_nA_nB_n - B_n\| \to 0,$$

$$\|(A_nB_n)^* - A_nB_n\| \to 0, \quad \|(B_nA_n)^* - B_nA_n\| \to 0.$$

Moreover, the sequence (B_n) is unique up to sequences in the ideal $\mathcal{G}$.

One might ask for the relation between the sequences (B_n) established in the preceding theorem, and the sequence $(A_n^\dagger)$ of the Moore-Penrose inverses of A_n (provided they exist). Now, we cannot answer this question in general. The point is that the operators $A_n^\dagger$ of course satisfy the convergence relations in the theorem in place of B_n, but it seems to be very hard to decide whether the sequence $(A_n^\dagger)$ belongs to the algebra $\mathcal{A}$ or at least to the algebra $\mathcal{F}$, i.e. whether this sequence is *bounded*. Only in case of the finite section method for Toeplitz operators, we have the following result, where P_M denotes the operator of orthogonal projection onto a subspace M:

PROPOSITION 4.9. *Let $T(a)$ be a Fredholm operator and suppose there is an n_0 such that $\ker T(a) \subseteq \operatorname{Im} P_{n_0}$ and $\ker T(\tilde{a}) \subseteq \operatorname{Im} P_{n_0}$. Then*

(i) $P_{\ker P_nT(a)P_n} = P_nP_{\ker T(a)}P_n + W_nP_{\ker T(\tilde{a})}W_n$ *for all n large enough;*

(ii) *the Moore-Penrose inverses of $P_nT(a)P_n$ converge strongly to the Moore-Penrose inverse of $T(a)$.*

This observation is a generalization of some recent results by Heinig and Hellinger ([12]) who started the study of stability and strong convergence of sequences formed by the Moore-Penrose inverses of Toeplitz matrices. They derived their results in a completely different way. The above approach is due to one of the authors (see [29]).

Employing the results of Sections 4.3–4.5 to the Moore-Penrose invertibility we obtain:

PROPOSITION 4.10. *Let (B_n) be any of the sequences established in the previous theorem.*

(i) *The sequence of the norms $\|B_n\|$ converges, and its limit is equal to*

$$\sup_{t\in T} \|(W_t(A_n))^\dagger\| = \|(\operatorname{Smb}(A_n))^\dagger\|,$$

i.e. to the norm of the Moore-Penrose inverse of the stability symbol of (A_n).

(ii) *The sequence of the generalized condition numbers* $\|A_n\|\,\|B_n\|$ *converges to*

$$\|\mathrm{Smb}\,(A_n)\|\,\|(\mathrm{Smb}\,(A_n))^\dagger\|,$$

i.e. to the generalized condition number of the stability symbol of (A_n).

(iii) *If, in addition, the sequence algebra is a subalgebra of* $\mathcal{F}_c$ *and if the natural homomorphism* W_0 *defined in* (4.1) *is part of the family* $(W_t)_{t\in T}$ *then the sequence* (B_n) *converges strongly to the Moore-Penrose inverse of the operator* $W_0(A_n)$.

4.7. Approximation sequences: Limiting sets of eigenvalues. Our next topic is the asymptotic behaviour of the eigenvalues of the approximation operators A_n. A first information about this can be obtained in terms of the (partial) limiting set of the sequence $(\sigma(A_n))$ of the spectra of A_n. By definition, the *partial limiting set* $\lim\limits_{n\to\infty} M_n$ of a sequence of subsets M_n of the complexplane is the collection of all complex numbers which are a partial limit of a sequence (t_n) of numbers $t_n \in M_n$.

The limiting set $\lim\sigma(A_n)$ is related with a certain kind of stability of the sequence (A_n) which we call spectral stability (in contrast to the notion of stability of a sequence introduced in Subsection 4.1 which could be referred to as *norm stability*). A sequence (A_n) is *spectral-stable* if the operators A_n are invertible for large n (say, $n \geqslant n_0$) and if the spectral radii $\rho(A_n^{-1})$ of their inverses are uniformly bounded: $\sup\limits_{n\geqslant n_0} \rho(A_n^{-1}) < \infty$.

Proposition 4.11. *A complex number* s *belongs to the limiting set* $\lim\sigma(A_n)$ *if and only if the sequence* $(A_n - sI)$ *is not spectral-stable.*

Proof. Let $(A_n - sI)$ be spectral-stable, i.e.

$$\sup_{n\geqslant n_0} \rho((A_n - sI)^{-1}) < \infty$$

with a certain n_0. Then, for all $n \geqslant n_0$,

$$\infty > \sup_{t\in\sigma((A_n-sI)^{-1})} |t| = \sup_{t\in\sigma(A_n-sI)} \left|\frac{1}{t}\right| = \frac{1}{\inf\limits_{t\in\sigma(A_n-sI)} |t|}$$

whence it follows that

$$0 < \inf_{t\in\sigma(A_n)-s} |t| = \inf_{t\in\sigma(A_n)} |t-s| = \mathrm{dist}\,(s, \sigma(A_n)).$$

The latter estimate shows that s cannot be a partial limit of a sequence (t_n) with $t_n \in \sigma(A_n)$.

Now suppose the sequence $(A_n - sI)$ is not spectral-stable. Then either

– there exists a subsequence $(A_{n_k} - sI)_{k\geqslant 0}$ such that none of the operators $A_{n_k} - sI$ is invertible, or

– all operators $A_n - sI$ are invertible if only n is large enough, but there is a subsequence $(A_{n_k} - sI)_{k\geqslant 0}$ such that

$$\rho((A_{n_k} - sI)^{-1}) \to \infty \quad \text{as} \quad k \to \infty.$$

In the first case one has $s \in \sigma(A_{n_k} - sI)$ for all k, thus, $s \in \lim \sigma(A_n)$. In the second case there are numbers $t_{n_k} \in \sigma(A_{n_k})$ such that $|t_{n_k} - s|^{-1} \to \infty$ or, equivalently, $t_{n_k} - s \to 0$ as $k \to \infty$. Hence, $s \in \lim \sigma(A_n)$ again. ∎

The C^*-algebra approach presented in Subsection 4.1 is suitable and intended for investigation of norm stability rather than for spectral stability. But there is one case where these notions coincide: a sequence of *self-adjoint* operators is norm-stable if and only if it is spectral-stable. Thus, the combination of Theorem 4.3 (i) with Proposition 4.11 yields:

THEOREM 4.12. *Let all hypotheses made in Subsection* 4.3 *be satisfied, and let* $(A_n) \in \mathcal{A}$ *be a sequence of self-adjoint operators. Then*

$$\lim \sigma(A_n) = \bigcup_{t\in T} \sigma(W_t(A_n)) = \sigma(\mathrm{Smb}\,(A_n)). \tag{4.3}$$

Besides the partial limiting set of a sequence (M_n) of subsets of the complex plane one can also consider its uniform limiting set. The *uniform limiting set* $\mathrm{Lim}\,(M_n)$ is the collection of all complex numbers which are the limit of a sequence (t_n) of numbers $t_n \in M_n$.

THEOREM 4.13. *Let the hypotheses of Subsection* 4.3 *be satisfied, and let* $(A_n) \in \mathcal{A}$ *be a sequence of self-adjoint operators. Then*

$$\lim \sigma(A_n) = \mathrm{Lim}\, \sigma(A_n).$$

Proof. Clearly, $\mathrm{Lim}\, \sigma(A_n) \subseteq \lim \sigma(A_n)$. For the reverse direction, suppose there is a $t \in \mathrm{Lim}\, \sigma(A_n) \setminus \lim \sigma(A_n)$. Then there existsan $\varepsilon > 0$ and a strongly monotonically increasing sequence(n_k) such that

$$\mathrm{dist}\,(t, \sigma(A_{n_k})) > \varepsilon$$

for all k. Hence, the sequence $(A_{n_k} - tI)$ is norm-stable whichimplies via Theorem 4.3 (ii) norm stability of $(A_n - tI)$. This contradicts Proposition 4.11. ∎

Let us emphasize that the uniform limiting set $\operatorname{Lim}\sigma(A_n)$ is more uniform than its definition indicates (compare [23]). Concretely: If $(A_n) \in \mathcal{A}$ is a self-adjoint sequence and U is an open and bounded neighborhood of $\operatorname{Lim}\sigma(A_n)$, then there is an n_0 such that $\sigma(A_n) \subseteq U$ for all $n \geqslant n_0$.

Indeed: For t large enough, say outside a certain closed disk V containing U, all operators $A_n - tI$ are invertible for all $n \geqslant n_0 = n_0(t)$. Further, given $t \in V \setminus U$, there is an open neighborhood $U(t)$ of t such that $A_n - sI$ is invertible for all $s \in U(t)$ and $n \geqslant n_0$. The open sets $U(t)$ with $t \in V \setminus U$ cover the compact set $V \setminus U$, hence, one can pick a finite subcovering $U(t_1), U(t_2), \ldots, U(t_m)$ of $V \setminus U$. Clearly

$$n \geqslant n_0 := \max\{n_0(t_1), n_0(t_2), \ldots, n_0(t_m)\},$$

then all operators $A_n - tI$ with $t \in V \setminus U$ are invertible, and the norms of their inverses are uniformly boundedwhich gives the assertion. ∎

4.8. APPROXIMATION SEQUENCES: LIMITING SETS OF s-NUMBERS. The set $\mathrm{s}(A)$ of the *s-numbers* of an operator A is defined by

$$\mathrm{s}(A) = \sigma((A^*A)^{\frac{1}{2}})$$

where $(A^*A)^{1/2}$ refers to the non-negative square root of the non-negative operator A^*A. Since $(A^*A)^{1/2}$ is in self-adjoint, the following result is an immediate consequence of the previous two theorems.

THEOREM 4.14. *Let the hypotheses of Subsection* 4.3 *be satisfied. Then, for all sequences* $(A_n) \in \mathcal{A}$,

$$\lim \mathrm{s}(A_n) = \operatorname{Lim} \mathrm{s}(A_n) = \bigcup_{t \in T} \mathrm{s}(W_t(A_n)) = \mathrm{s}(\operatorname{Smb}(A_n)).$$

In case of the finite section method for Toeplitz operators, the results of the previous two sections are well known. Namely, a classical result by Szegö ([8], 5.2 (b)), states that, for real-valued (even not necessarily piecewise continuous) functions a, both the partial limiting set $\lim\sigma(P_nT(a)P_n)$ and the uniform limiting set $\operatorname{Lim}\sigma(P_nT(a)P_n)$ coincide with the interval $[\operatorname{ess\,inf} a,\ \operatorname{ess\,sup} a]$. The assertions concerning s-numbers (also in a somewhat more general situation) go back to Widom ([31]) who showed that

$$\lim \mathrm{s}(P_nT(a)P_n) = \operatorname{Lim} \mathrm{s}(P_nT(a)P_n) = \mathrm{s}(T(a)) \cup \mathrm{s}(T(\tilde{a})).$$

For further generalizations see Widom ([32]), Roch and Silbermann ([23]) and Silbermann ([28]).

4.9. APPROXIMATION SEQUENCES: LIMITING SETS OF ε-PSEUDOSPECTRA. Let $\varepsilon > 0$ and let B be a C^*-algebra with identity e. The *ε-pseudospectrum* $\sigma_\varepsilon(a)$ of an element $b \in B$ is the set

$$\sigma_\varepsilon(b) = \left\{ t \in C;\ b - te \text{ is not invertible or } \|(b-te)^{-1}\| \geqslant \frac{1}{\varepsilon} \right\}. \tag{4.4}$$

Here is an equivalent characterization of the ε-pseudospectrum.

PROPOSITION 4.15. *The ε-pseudospectrum of $b \in B$ coincides with the set*

$$\{t \in C;\ \text{there is a } p \in B \textit{ with } \|p\| \leqslant \varepsilon \textit{ and } t \in \sigma(b+p)\}. \tag{4.5}$$

In case B is the algebra of $n \times n$-matrices, this result seems to be well-known. Reichel and Trefethen ([24]) claimed without proof that, in the general case, the *closure* of the set (4.5) coincides with (4.4). For this reason, we are going to add a proof here which is due to our collegues Torsten Ehrhardt and Tilo Finck.

Proof. Let S_1 and S_2 abbreviate the sets (4.4) and (4.5), respectively. Our first goal is the inclusion $S_2 \subseteq S_1$. Given $t \in S_2$ choose $p \in B$ such that $\|p\| \leqslant \varepsilon$and that $b + p - te$ is not invertible. If $t \in \sigma(b)$ then, evidently, $t \in S_1$. So we can suppose that the element $b - te$ is invertible. Then the representation

$$b + p - te = (b - te)(e + (b - te)^{-1}p)$$

involves that $e + (b - te)^{-1}p$ cannot be invertible. Hence,$\|(b - te)^{-1}p\| \geqslant 1$ (otherwise invertibility would follow via the Neumann series), and the estimate

$$1 \leqslant \|(b - te)^{-1}p\| \leqslant \|(b - te)^{-1}\|\,\|p\|$$

yields

$$\|(b - te)^{-1}\| \geqslant \frac{1}{\|p\|} \geqslant \frac{1}{\varepsilon},$$

i.e. $t \in S_1$ as desired.

To verify the reverse inclusion $S_1 \subseteq S_2$ we suppose for contrary that there exists a $t \in S_1$ such that $b + p - te$ is invertible for all $p \in B$ with $\|p\| \leqslant \varepsilon$. Setting $p = 0$ we get the invertibility of $b - te$ and consequently that of $b^* - \bar{t}e$, and setting $p = \lambda(b^* - \bar{t}e)^{-1}$ where λ is an arbitrary complex number satisfying

$$0 < |\lambda| \leqslant \frac{\varepsilon}{\|(b^* - \bar{t}e)^{-1}\|} \tag{4.6}$$

we get the invertibility of

$$b - te + \lambda(b^* - \bar{t}e)^{-1} = \lambda(b - te)\left(\frac{1}{\lambda}e + (a - te)^{-1}(b^* - \bar{t}e)^{-1}\right)$$

(observe that in both cases $\|p\| \leqslant \varepsilon$). Thus, $\frac{1}{\lambda}e+(a-te)^{-1}(b^*-\bar{t}e)^{-1}$ is invertible for all λ satisfying (4.6) which implies that

$$\rho((a-te)^{-1}(b^*-\bar{t}e)^{-1}) < \frac{\|(b^*-\bar{t}e)^{-1}\|}{\varepsilon}.$$

The self-adjointness of $(a-te)^{-1}(b^*-\bar{t}e)^{-1}$ yields that

$$\|(a-te)^{-1}(b^*-\bar{t}e)^{-1}\| < \frac{\|(b^*-\bar{t}e)^{-1}\|}{\varepsilon},$$

so

$$\|(a-te)^{-1}\|^2 = \|(b^*-\bar{t}e)^{-1}\|^2 < \frac{\|(b^*-\bar{t}e)^{-1}\|}{\varepsilon}$$

and finally

$$\|(a-te)^{-1}\| = \|(b^*-\bar{t}e)^{-1}\| < \frac{1}{\varepsilon}.$$

This contradicts our assumption $t \in S_1$. ∎

Let us return to the context of sequence algebras now. The following result shows that the asymptotic behaviour of the ε-pseudospectra is (in a certain sense) much better than that of the eigenvalues. Namely, they *always* converge to the ε-pseudospectrum of the symbol of the given approximation sequence.

THEOREM 4.16. *Let the hypotheses of Subsection 4.3 be satisfied and $(A_n) \in \mathcal{A}$. Then, for all $\varepsilon > 0$,*

$$\lim \sigma_\varepsilon(A_n) = \bigcup_{t\in T} \sigma_\varepsilon(W_t(A_n)) = \sigma_\varepsilon(\mathrm{Smb}(A_n)) \tag{4.7}$$

and, moreover,

$$\lim \sigma_\varepsilon(A_n) = \mathrm{Lim}\, \sigma_\varepsilon(A_n).$$

In case of the finite section method for Toeplitz operators this result goes back to Reichel and Trefethen ([24]) (generating functions in the Wiener algebra) and Böttcher ([2]) (piecewise continuous generating functions) and, since both the spectra and the norms of the inverses of $T(a)$ and $T(\tilde{a})$ coincide in these situations, one has

$$\lim \sigma_\varepsilon(P_n T(a) P_n) = \sigma_\varepsilon(T(a)).$$

Our proof uses ideas of Böttcher's; in particular it also bases on the following observation by Daniluk which is proved in [2],Proposition 6.1.

PROPOSITION 4.17. *Let $\mathcal{B}$ be a C^*-algebra with identity e, let $a \in \mathcal{B}$ and suppose that $a - te$ is invertible for all t in some open subset U of the complex plane and that $\|(a-te)^{-1}\| \leqslant C$ for all $t \in U$. Then $\|(a-te)^{-1}\| < C$ for all $t \in U$.*

Proof of Theorem 4.16. We start with verifying that

$$\sigma_\varepsilon(W_t(A_n)) \subseteq \lim \sigma_\varepsilon(A_n)$$

for all $t \in T$.

Let first $s \in \sigma(W_t(A_n))$. Then, by Theorem 4.3, the sequence $(A_n - sI)$ cannot be norm stable which implies that either

– there is a subsequence $(A_{n_k} - sI)_{k\geqslant 0}$ consisting of non-invertible operators only, or

– all operators $A_n - sI$ are invertible if n is large enough, but there is a subsequence $(A_{n_k} - sI)_{k\geqslant 0}$ such that

$$\|(A_{n_k} - sI)^{-1}\| \to \infty \quad \text{as} \quad k \to \infty.$$

In the first case, s is even in $\lim \sigma(A_n)$ and, hence, in $\lim \sigma_\varepsilon(A_n)$, too; in the second case, s belongs to $\sigma_\varepsilon(A_{n_k})$ for all k large enough which, of course, also implies $s \in \lim \sigma_\varepsilon(A_n)$.

Now suppose $s \in \sigma_\varepsilon(W_t(A_n)) \backslash \sigma(W_t(A_n))$. Then $W_t(A_n)) - sI$ is an invertible operator, and $\|(W_t(A_n)) - sI)^{-1}\| \geqslant 1/\varepsilon$. Let U be an open neighborhood of s. Daniluk's result implies that there is an r in U such that

$$\|(W_t(A_n)) - rI)^{-1}\| > \frac{1}{\varepsilon}$$

(otherwise we would have $\|(W_t(A_n)) - rI)^{-1}\| \leqslant 1/\varepsilon$ for all $r \in U$, therefore, $\|(W_t(A_n)) - rI)^{-1}\| < 1/\varepsilon$ for all $r \in U$ including $r = s$). Thus,

$$\|(W_t(A_n)) - rI)^{-1}\| \geqslant \frac{1}{\varepsilon - \frac{1}{k}}$$

for all sufficiently large k. This shows that we can find numbers $s_k \in \sigma_{\varepsilon - 1/k}(W_t(A_n))$ which converge to s. Now Theorem 4.6 entails that

$$\lim_{n\to\infty} \|(A_n - s_k I)^{-1}\| \geqslant \frac{1}{\varepsilon - \frac{1}{k}}$$

(in case an operator B is not invertible we agree upon defining $\|B^{-1}\| = \infty$ so that the previous conclusion remains valid in any case). Consequently,

$$\|(A_n - s_k I)^{-1}\| \geqslant \frac{1}{\varepsilon}$$

for all sufficiently large n which yields that $s_k \in \sigma_\varepsilon(A_n)$ and, thus, $s = \lim s_k \in \lim \sigma_\varepsilon(A_n)$.

In order to prove the reverse inclusion,

$$\lim \sigma_\varepsilon(A_n) \subseteq \bigcup_{t \in T} \sigma_\varepsilon(W_t(A_n)),$$

suppose that $s \notin \bigcup_{t \in T} \sigma_\varepsilon(W_t(A_n))$. Then all operators $W_t(A_n) - sI$ are invertible, and

$$\|(W_t(A_n) - sI)^{-1}\| < \frac{1}{\varepsilon}.$$

The second assertion of part (v) of the Lifting theorem gives that

$$\sup_{t \in T} \|(W_t(A_n) - sI)^{-1}\| = \frac{1}{\varepsilon} - 2\delta < \frac{1}{\varepsilon}$$

and, hence, by Theorem 4.6,

$$\|(A_n - sI)^{-1}\| < \frac{1}{\varepsilon} - \delta$$

for all $n \geqslant n_0$. If $n \geqslant n_0$ and $|s - r| < \varepsilon\delta(1/\varepsilon - \delta)^{-1}$ then

$$\begin{aligned} \|(A_n - r)^{-1}\| &\leqslant \frac{\|(A_n - sI)^{-1}\|}{1 - |r - s|\,\|(A_n - sI)^{-1}\|} \\ &< \frac{\frac{1}{\varepsilon} - \delta}{1 - \varepsilon\delta\left(\frac{1}{\varepsilon} - \delta\right)^{-1}\left(\frac{1}{\varepsilon} - \delta\right)} = \frac{1}{\varepsilon} \end{aligned}$$

and, thus, $r \notin \sigma_\varepsilon(A_n)$.

The second identity in (4.7) follows easily from the second assertion of part (v) of the Lifting theorem, and the equality between the partial and the uniform limiting set is again a consequence of the fractal property of the lifting homomorphisms. ∎

Let us finally remark that the ε-pseudospectra of the approximation operators A_n do not only approximate the ε-pseudospectrum of the stability symbol of the sequence (A_n); they can also be used to approximate the spectrum of the stability symbolitself. Indeed, since evidently

$$\bigcap_{\varepsilon > 0} \sigma_\varepsilon(b) = \sigma(b)$$

for each element b of a C^*-algebra $\mathcal{B}$, one gets immediately from (4.7) that

$$\bigcap_{\varepsilon > 0} \lim \sigma_\varepsilon(A_n) = \sigma(\mathrm{Smb}(A_n))$$

for all sequences $(A_n) \in \mathcal{A}$.

4.10. THE LIMIT SPECTRUM IN SENSE OF NEVANLINNA AND VAINIKKO. Let (A_n) be a sequence of approximation operators. Nevanlinna and Vainikko (see [17]) introduced the sets

$$\Sigma_s(A_n) = \{t \in C;\ \limsup_{n\to\infty} \|(A_n - tI)^{-1}\| = \infty\},$$

$$\Sigma_i(A_n) = \{t \in C;\ \liminf_{n\to\infty} \|(A_n - tI)^{-1}\| = \infty\}$$

where we again define $\|(A_n - tI)^{-1}\| = \infty$ in case $t \in \sigma(A_n)$. If $\Sigma_s(A_n) = \Sigma_i(A_n)$ for a sequence (A_n), then they call (one of) these sets the *limit spectrum* of (A_n) and denote it by $\Sigma(A_n)$.

THEOREM 4.18. *Let the hypotheses of Subsection 4.3 be satisfied and $(A_n) \in \mathcal{A}$. Then the limit spectrum $\Sigma(A_n)$ exists, and it coincides with the spectrum of the symbol of the sequence (A_n).*

Proof. It is easy to see that $t \notin \Sigma_i(A_n)$ if and only if there exists a norm-stable subsequence of $(A_n - tI)$, where as$t \notin \Sigma_s(A_n)$ if and only if the sequence $(A_n - tI)$ is norm-stable itself. Both sets coincide due to the fractal nature of the homomorphisms W_s. ∎

5. EXAMPLES

We are going to mention some examples of algebras of concrete approximation sequences which satisfy the hypotheses made in the forth section.

5.1. THE FINITE SECTION METHOD FOR OPERATORS IN THE TOEPLITZ ALGEBRA. Let $l^2_{\mathbf{Z}}$ stand for the Hilbert space of all two-sided sequences of complex numbers with norm

$$\|(x_n)\| = \Big(\sum_{n\in\mathbf{Z}} |x_n|^2\Big)^{\frac{1}{2}}.$$

Given a piecewise continuous function a with nth Fourier coefficient a_n we define its Laurent operator $T^o(a)$ by

$$T^o(a) : l^2_{\mathbf{Z}} \to l^2_{\mathbf{Z}}, \quad (x_n) \mapsto (y_n) \quad \text{with} \quad y_n = \sum_{k\in\mathbf{Z}} a_{n-k} x_k.$$

The operator $T^o(a)$ is bounded on $l^2_{\mathbf{Z}}$, and its norm is equal to $\operatorname{ess\,sup}_{t\in\mathbf{T}} |a(t)|$. Further we introduce the projection operators

$$P : l^2_{\mathbf{Z}} \to l^2_{\mathbf{Z}}, \quad (x_n) \mapsto (\ldots, 0, 0, x_0, x_1, \ldots)$$

and

$$R_n : l^2_{\mathbb{Z}} \to l^2_{\mathbb{Z}}, \quad (x_n) \mapsto (\ldots, 0, x_{-n}, \ldots, x_{n-1}, 0, \ldots).$$

Clearly, one can identify the range of P with l^2, the operator $PT^o(a)P$ with the Toeplitz operator $T(a)$, and the projections PR_n with the projections P_n introduced in Section 2.

Let $\mathcal{T}$ denote the smallest closed subalgebra of the algebra $L(l^2_{\mathbb{Z}})$ which contains all Laurent operators $T^o(a)$ with piecewise continuous generating function a and the projection P, and let $\mathcal{A}$ stand for the smallest closed subalgebra of the algebra $\mathcal{F}$ of all bounded sequences (A_n) of operators $A_n \in L(l^2_{\mathbb{Z}})$ which contains all constant sequences (A) with $A \in \mathcal{T}$ and the sequence (R_n).

The algebra $\mathcal{A}$ exhibits exactly the afore mentioned structure. In order to introduce the lifting homomorphisms we need some more notations.

Let $L^2_{\mathbb{R}}$ refer to the Lebesgue space of all measurable functionson the real axis $\mathbb{R}$ with norm

$$\|f\| = \Big(\int_{\mathbb{R}} |f(s)|^2 \, ds\Big)^{\frac{1}{2}},$$

and denote its subspace consisting of all functions which are constant on each of the intervals $[k/n, (k+1)/n]$ with k running through the integers and n being fixed by S_n. One can show that the orthogonal projections L_n from L^2 onto S_n converge strongly to the identity operator as $n \to \infty$. Moreover, the characteristic functions χ_k of the intervals $[k/n, (k+1)/n]$ form an orthogonal basis of S_n, and the operators

$$F_{-n} : S_n \to l^2_{\mathbb{Z}}, \quad \sum_{k \in \mathbb{Z}} x_k \chi_k \mapsto \frac{1}{\sqrt{n}} (x_k)_{k \in \mathbb{Z}}$$

are isometries having inverses given by

$$F_n : l^2_{\mathbb{Z}} \to S_n, \quad (x_k) \mapsto \sqrt{n} \sum x_k \chi_k.$$

Finally, given $t \in \mathbb{T}$ and $s \in \mathbb{Z}$, define operators Y_t andU_s by

$$Y_t : l^2_{\mathbb{Z}} \to l^2_{\mathbb{Z}}, \quad (x_n) \mapsto (t^n x_n),$$

$$U_s : l^2_{\mathbb{Z}} \to l^2_{\mathbb{Z}}, \quad (x_n) \mapsto (x_{n-s}).$$

One can show that the strong limits

$$W^0(A_n) := \text{s-lim}\, A_n \quad (\in L(l^2_{\mathbb{Z}})), \tag{5.1}$$

$$W^{\pm 1}(A_n) := \text{s-lim}\, U_{\mp n} A_n U_{\pm n} \quad (\in L(l^2_{\mathbb{Z}})) \tag{5.2}$$

and

$$W_t(A_n) := \text{s-lim}\, F_n Y_{t^{-1}} A_n Y_t F_{-n} L_n \quad (\in L(L^2_{\mathbb{R}})) \tag{5.3}$$

exist for all $t \in \mathbb{T}$ and $(A_n) \in \mathcal{A}$, and that the mappings W^l with $l \in \{-1, 0, 1\}$ and W_t with $t \in \mathbb{T}$ are fractal homomorphisms. Moreover, the homomorphisms W_t lift the ideals

$$\mathcal{I}_t := \{(F_{-n} L_n K F_n);\ K \text{ compact}\} + \mathcal{G},$$

(obviously, the restriction of W_t onto $\mathcal{I}_t$ is an isomorphism between $\mathcal{I}_t$ and the ideal of all compact operators on $L^2_{\mathbb{R}}$) and we further think of the homomorphisms W^l as being lifting the zero ideal. The following theorem states that we then have exactly the situation described in the general lifting theorems.

THEOREM 5.1. *A sequence $(A_n) \in \mathcal{A}$ is stable if and only if all operators W^l with $l \in \{-1, 0, 1\}$ and W_t with $t \in \mathbb{T}$ are invertible.*

This theorem is the result of the efforts by many mathematicians including Böttcher, Verbitski, Rathsfeld, Roch, Silbermann. For a detailed history and complete proofs (using local principles)we refer to the monographs [4], 7.55–7.72, and [10], Section 4.1.

Let us further emphasize that the case of matrix–valued generating functions can also be handled as one of the authors pointed out in [19]. If one replaces the operators F_n, etc. by the corresponding diagonal matrices diag$(F_n, \ldots, F_n)$ etc. then the assertion of the previous theorem remains valid without any changes. Moreover, these results can be even generalized to the case of piecewise continuous generating functions taking values in the algebra of all operators on $l^2_{\mathbb{Z}}$ of the form $cI + K$ with c a complex number and K compact (see [30]).

Hence, in all of the mentioned cases, one can apply the theory developed in the forth section in order to describe the asymptotic behaviour of norms, condition numbers, eigenvalues, pseudospectra etc. of the approximating operators.

5.2. THE FINITE SECTION METHOD FOR TOEPLITZ + HANKEL OPERATORS. The results of Subsection 5.1 can be extended to a larger class of operators including Hankel operators with piecewise continuous generating function. For this goal, let $\mathcal{TH}$ denote the smallest closed subalgebra of $L(l^2_{\mathbb{Z}})$ which besides the Laurent operators $T^o(a)$ (a piecewise continuous) and the projection operator P contains the flip operator

$$J : l^2_{\mathbb{Z}} \to l^2_{\mathbb{Z}}, \quad (x_n) \mapsto (x_{-n-1}).$$

Prominent elements of this algebra are the singular integral operators with Carleman shift,

$$T^o(a)P + T^o(b)Q + (T^o(c)P + T^o(d)Q)J$$

where $Q = I - P$, and the Toeplitz + Hankel operators

$$PT^o(a)P + PT^o(b)QJ$$

or, in earlier notations, $T(a) + H(b)$.

Again we let $\mathcal{F}$ denote the algebra of all bounded sequences, and we shall use the notation $\mathcal{AH}$ now in order to designate the smallest closed subalgebra of $\mathcal{F}$ which contains all constant sequences (A) with $A \in \mathcal{TH}$ as well as the sequence (R_n). Our stability criterion again relates stability of a sequence $(A_n) \in \mathcal{AH}$ with the invertibility of two families of operators, $\widehat{W}^l(A_n)$ with $l \in \{0, 1\}$ and $\widehat{W}_t(A_n)$ with $t \in \mathbf{T}$ and $\operatorname{Im} t \geqslant 0$.

We prepare the definition of these homomorphisms by the following reflection. If $\mathcal{B}$ is any C^*-algebra with identity e, and if p and j are self-adjoint elements of $\mathcal{B}$ such that

$$p^2 = p, \quad j^2 = e, \quad jpj = e - p$$

then the mapping

$$a \mapsto \begin{pmatrix} pap & pa(e-p)j \\ j(e-p)ap & j(e-p)a(e-p)j \end{pmatrix} \tag{5.4}$$

is a natural $*$-isomorphism from $\mathcal{B}$ onto a $*$-subalgebra of the algebra $\mathcal{B}_{2\times 2}$ of 2×2 matrices with entries in $\mathcal{B}$. We are going to apply the mapping (5.4) for the algebra $\mathcal{AH}$ where we choose $p = (T^o(\chi_+))$ with χ_+ referring to the characteristic function of the upper semi-circle $\{t \in \mathbf{T};\ \operatorname{Im} t \geqslant 0\}$ and $j = (J)$. A simple calculation yields for the generating sequences of the algebra $\mathcal{AH}$:

$$(T^o(a)) \mapsto \begin{pmatrix} (T^o(\chi_+ a \chi_+)) & 0 \\ 0 & (T^o(\chi_+ \tilde{a} \chi_+)) \end{pmatrix},$$

$$(P) \mapsto \begin{pmatrix} (T^o(\chi_+)PT^o(\chi_+)) & (JT^o(\chi_-)QT^o(\chi_+)) \\ (JT^o(\chi_-)PT^o(\chi_+)) & (T^o(\chi_+)QT^o(\chi_+)) \end{pmatrix},$$

$$(R_n) \mapsto \begin{pmatrix} (T^o(\chi_+)R_nT^o(\chi_+)) & (JT^o(\chi_-)R_nT^o(\chi_+)) \\ (JT^o(\chi_-)R_nT^o(\chi_+)) & (T^o(\chi_+)R_nT^o(\chi_+)) \end{pmatrix},$$

$$(J) \mapsto \begin{pmatrix} 0 & (T^o(\chi_+)) \\ (T^o(\chi_+)) & 0 \end{pmatrix}$$

where we set $\chi_- = 1 - \chi_+$.

There are two types of sequences which occur as entries of these matrices: sequences which belong to the algebra $\mathcal{A}$ introduced in Subsection 5.1, and sequences of the form (JB_n) with $(B_n) \in \mathcal{A}$. It is easy to see that these sequences (B_n) have an additional property: whenever $t \in \mathbf{T}$ and $\operatorname{Im} t > 0$ then $W_t(B_n) = 0$ where W_t again denotes the strong limit (5.3). Thus, for all $t \in \mathbf{T}$ and $\operatorname{Im} t > 0$, the stronglimits$W_t(JB_n)$ exist and are equal to zero. Since one can moreover show thatthe strong limits $W_{\pm 1}(A_n)$ exist for *each* sequence in$\mathcal{AH}$, it is correct to define now

$$(5.5) \qquad \widehat{W}_{\pm 1}(A_n) := W_{\pm 1}(A_n),$$

and, for $t \in \mathbf{T}$ and $\operatorname{Im} t > 0$,

$$(5.6) \qquad \widehat{W}_t(A_n) := \begin{pmatrix} W_t(T^o(\chi_+)A_nT^o(\chi_+)) & W_t(T^o(\chi_+)A_nT^o(\chi_-)J) \\ W_t(JT^o(\chi_-)A_nT^o(\chi_+)) & W_t(JT^o(\chi_-)A_nT^o(\chi_-)J) \end{pmatrix}.$$

The definition of the homomorphisms $\widehat{W}^l$ with $l \in \{0,1\}$ proceeds similarly. For $l = 0$ one can set

$$(5.7) \qquad \widehat{W}^0(A_n) := W^0(A_n),$$

and for $l = 1$ we apply the mapping (5.4) again, but now with p and j being identified with (P) and (J), respectively, and define

$$(5.8) \qquad \widehat{W}^1(A_n) := \begin{pmatrix} W^1(PA_nP) & W^1(PA_nQJ) \\ W^1(JQA_nP) & W^1(JQA_nQJ) \end{pmatrix}.$$

Now we have the following generalization of Theorem 5.1.

THEOREM 5.2. *A sequence $(A_n) \in \mathcal{AH}$ is stable if and only if all operators $\widehat{W}_t(A_n)$ with $t \in \mathbf{T}$ and $\operatorname{Im} t > 0$ and all operators $\widehat{W}^l(A_n)$ with $l \in \{0,1\}$ are invertible.*

This result goes back to one of the authors ([20]); see also Section 6.2 in [10].

One can show that the homomorphisms $\widehat{W}^l$ and $\widehat{W}_t$ satisfy all assumptions made above. In particular, they are fractal (which is immediate from their definition), and they are ideal-lifting in the following manner: the homomorphisms $\widehat{W}^l$ lift the zero ideal, the homomorphisms $\widehat{W}_t$ with $t = \pm 1$ lift the ideals$\mathcal{I}_t$ and establish an isomorphism between these ideals and the ideal of the compact operators on $L^2_{\mathbb{R}}$, and the homomorphisms $\widehat{W}_t$ with $\operatorname{Im} t > 0$ lift the ideals $\widehat{\mathcal{I}}_t$ generated by the ideals $\mathcal{I}_t$, $\mathcal{I}_{-t}$ and by the sequence (J), and they induce an isomorphism between these ideals and the ideal of all compact operators on $L^2_{\mathbb{R}} \times L^2_{\mathbb{R}}$. Thus, again the whole theory of the forth section applies to the finite section method for operators in $\mathcal{AH}$.

5.3. SPLINE AND WAVELET APPROXIMATION METHODS FOR SINGULAR INTEGRAL AND MELLIN OPERATORS. The spline spaces under consideration are supposed to be of a rather natural structure, namely, we start with a *mother spline*, that is, with a bounded, measurable, and compactly supported function φ satisfying the following conditions:

$$\sum_{k\in\mathbb{Z}} \varphi(x-k) \equiv 1 \quad \text{for } x \in \mathbb{R}, \tag{5.9}$$

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \varphi(t+k)\overline{\varphi(t)}\, dt \cdot z^k \neq 0 \quad \text{for } z \in \mathbb{T}. \tag{5.10}$$

(Observe that the sums in (5.9) and (5.10) are actually finite by thecompactness of the support of φ.) Then we set $\varphi_{kn}(t) := \varphi(nt-k)$ and define the spline space S_n as the smallest closed subspace of $L^2_{\mathbb{R}}$ containing all functions φ_{kn} with$k \in \mathbb{Z}$. For example one can take $\varphi = \chi_{[0,1]}$, the characteristic function of $[0,1]$, or $\varphi = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$, the d-fold convolution of $\chi_{[0,1]}$ by itself. Then(5.9) and (5.10) are satisfied, and S_n is just the space of all L^2-functions which are polynomials of degree d over each interval $[k, k+1]$, and which are $d-1$ times continuously differentiable on $\mathbb{R}$. Further, spaces of compactly supported wavelets are also subject to our conditions (see [10], Section 2.9.5, for examples).

A basic observation which is usually attributed to de Boor is that the spaces $S_n \subseteq L^2$ and $l^2_{\mathbb{Z}}$ are isomorphic : If the function $\sum x_k\varphi_{kn}$ is in S_n, then the coefficient sequence (x_k) is in $l^2_{\mathbb{Z}}$ and conversely and, moreover, the mappings

$$F_n : l^2_{\mathbb{Z}} \to S_n, \quad (x_k) \mapsto \sum x_k \varphi_{kn}$$

and

$$F_{-n} : S_n \to l^2_{\mathbb{Z}}, \quad \sum x_k\varphi_{kn} \mapsto (x_k)$$

are continuous and $\sup_n \|F_n\|\,\|F_{-n}\| < \infty$.

The *Galerkin projection* L_n is the operator mapping L^2 onto $S_n$such that $(L_n f, \varphi_{kn}) = (f, \varphi_{kn})$ for all $f \in L^2$and $k \in \mathbb{Z}$. Condition (5.10) ensures the existence of L_n,whereas (5.9) involves the strong convergence of L_n to the identity operator as $n \to \infty$.

As a typical example, we consider the Galerkin method for solving the singular integral equation

$$(aI + bS_{\mathbf{R}})u = f \tag{5.11}$$

where a and b are piecewise continuous coefficients and $S_{\mathbf{R}}$ is the operator of singular integration,

$$(S_{\mathbf{R}}u)(t) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(s)}{s-t}\,ds, \quad t \in \mathbf{R}.$$

For the *Galerkin method* for solving (5.11) we replace (5.11) by the sequence of approximation equations

$$L_n(aI + bS_{\mathbf{R}})u_n = L_n f, \quad u_n \in S_n. \tag{5.12}$$

Since the operators $L_n(aI + bS_{\mathbf{R}})L_n$ converge strongly to thesingular integral operator $aI + bS_{\mathbf{R}}$ we have to examine the stabilityof the sequence $(L_n(aI + bS_{\mathbf{R}})|S_n)$ of approximation operatorsin order to guarantee the applicability of the method (5.12) to the equation (5.11). This problem can be studied again by embedding the sequences we are interested in into a suitably constructed C^*-algebra.

There is a rather elegant and effective (but not evident) way to define an adequate algebra of approximation sequences over spline spaces. Let $\mathcal{F}$ stand for the set of all bounded sequences (A_n) of operators $A_n : S_n \to S_n$ such that $\|(A_n)\| := \sup_n \|A_n L_n\| < \infty$. Provided with elementwise operations, $\mathcal{F}$ becomes a C^*-algebra, and the subset $\mathcal{K}$ of $\mathcal{F}$ containing all sequences (K_n) with $\|K_n\| \to 0$ forms a closed two-sided ideal in $\mathcal{F}$. Clearly, a sequence $(A_n) \in \mathcal{F}$ is stable if and only if the coset $(A_n) + \mathcal{K}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{K}$.

In what follows we use the notations U_n and Y_t introduced in Subsection 5.1, and for each real number x we let $\{x\}$ stand for the smallest integer which is not less than x. Further, we fix areal number r. Now define $\mathcal{A}_r$ as the smallest closed subalgebra of $\mathcal{F}$ which contains all sequences of the form $(F_n T^o(a) F_{-n})$ where a is an arbitrary piecewise continuous function on $\mathbf{T}$, and all sequences $(F_n U_{\{tn+r\}} P U_{-\{tn+r\}} F_{-n})_{n \in \mathbf{Z}^+}$ with t running through the reals.

One can show for example that $\mathcal{A}_0$ contains the Galerkin approximation sequences (5.12) even with different ansatz space $S_n = S_n^{\varphi}$ and test space S_n^{ψ} both for singular integral operators $aI + bS_{\mathbf{R}}$ (where a and b are assumed to be piecewise continuous on $\mathbf{R}$ and continuoüson $\dot{\mathbf{R}} \setminus \mathbf{Z}$) and for Mellin convolution operators,

$$(Mf)(t) = \int_0^{\infty} k\left(\frac{t}{s}\right) f(s) s^{-1}\,ds,$$

whereas $\mathcal{A}_{-\varepsilon}$ contains ε-collocation sequences for these operators (even with general piecewise continuous coefficients), and the algebra $\mathcal{A}_0$ is also suitable to investigate certain qualocation and quadrature methods. For a detailed treatment of these questions, for an account ofthe history of the topic, and also for somegeneralizations (spline spaces generated by finitely many mother splines, operators on weighted Lebesgue spaces) we refer to the monograph [10].

The algebra $\mathcal{A}_r/(\mathcal{A}_r \cap \mathcal{K})$ again possesses a symbolin the sense of Subsection 3.3 which is constituted by two families of fractal homomorphisms. Here is their definition.

– Let $(A_n) \in \mathcal{A}_r$ and $t \in \mathbf{T}$. Then there is an operator $W_t(A_n) \in L(L^2)$ such that

$$\text{s-}\lim_{n\to\infty} F_n Y_{t^{-1}} F_{-n} A_n F_n Y_t F_{-n} L_n = W_t(A_n), \tag{5.13}$$

and the mapping $W_t : \mathcal{A}_r \to L(L^2)$ is a continuous algebra homomorphism.

– For each $(A_n) \in \mathcal{A}_r$ and $s \in \mathbf{R}$, there is an operator $W^s(A_n) \in L(l^2_{\mathbf{Z}})$ such that

$$\text{s-}\lim_{n\to\infty} U_{-\{sn+r\}} F_{-n} A_n F_n U_{\{sn+r\}} = W^s(A_n), \tag{5.14}$$

and the mapping $W^s : \mathcal{A}_r \to L(l^2_{\mathbf{Z}})$ is a continuous algebra homomorphism.

– The cosets $F_{-n} A_n F_n + K(l^2)$ with $K(l^2)$ referring to the ideal of the compact operators on $l^2_{\mathbf{Z}}$ are independent on n, and if $W^\infty(A_n)$ denotes one of them, then the mapping $W_\infty : \mathcal{A}_r \to L(l^2)/K(l^2)$ is a continuous algebra homomorphism.

THEOREM 5.3. *A sequence $(A_n) \in \mathcal{A}_r$ is stable if and only if the operators $W_t(A_n)$ with $t \in \mathbf{T}$ and $W^s(A_n)$ with $s \in \mathbf{R}$ as well as the coset $W^\infty(A_n)$ are invertible.*

It is easy to check that all homomorphisms appearing in the the oremare fractal and that the homomorphisms W_t lift the ideals

$$\mathcal{I}_t = \{(F_n Y_t F_{-n} L_n K F_n Y_{t^{-1}} F_{-n} | S_n) + (\mathcal{A}_r \cap \mathcal{K})); \ K \text{ compact}\}.$$

Thinking of the homomorphisms W^s with $s \in \mathbf{R} \cup \{\infty\}$ as lifting the zero ideal we again see that the situation considered here is subject to the general lifting theorem and its consequences.

The above mentioned results extend to weighted L^p-spaces and to operators on curves which are even allowed to possess corners, intersections, and endpoints (see for details [10], Chapter 5), to spline and wavelet approximation methods cutting off singularities (see [10], Chapter 4), and also to some classes of multidimensional singular integral operators (compare [9]).

5.4. FURTHER APPLICATIONS. Here we list some further applications for which the "lifting programme"(i.e. construction of algebras, determination of a sufficient number of fractal and lifting homomorphisms) has already been accomplished.

– Algebras of sequences of *paired circulants* (which appear when discretizing singular integral operators on simple closed curves) are studied by Hagen and Silbermann in [11], compare also [18], Sections 10.31–10.42.

– The finite section method for singular integral equations on theinterval [0,1] based on Chebyshev polynomials which are orthogonal with respect to certain Jacobi weights is considered in [13].

– Only recently, Böttcher and Wolf investigated the finite section method for Toeplitz operators on the multidimensional Segal-Bargmann space.

– The machinery developed above evidently also applies to the finitesection method for singular integral operators on the semi-axis (which cuts off the infinity) as well as to the same method for operators on complicated curves (where it is then used to cut off singular points such as intersections), although, in these situations, the approximation operators depend on a continuous parameter, and so one should rather speak about function algebras than about sequence algebras (for these results see [21] and [10], Section 6.1).

– Finally, all things apply to the harmonic approximation of Toeplitz operators as examined in [26] and [4], Chapter 3.

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